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A Covering System with Minimum Modulus 42

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ABSTRACT

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We construct a covering system whose minimum modulus is 42. This improves the previous record of 40 by P. Nielsen.

Keywords: Covering system, minimum modulus, up-arrow notation

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Contents

Chapter 1. Introduction

Covering systems were first introduced by Paul Erdős in his 1950 paper on integers of the form $2^k + p$ [1]. A covering system is a finite set of congruence classes with distinct moduli greater than 1 such that every integer belongs to at least one of the classes. In that same paper, Erdős wrote "It seems likely that for every c there exists such a system all the moduli of which are $> c$." This statement became known as the minimum modulus problem, and it took over sixty years before the problem was resolved in the negative by Bob Hough [2], who obtained the upper bound of 10^{16} on the smallest modulus.

While it is now known that the minimum modulus of a covering system cannot be arbitrarily large, the question of how large the minimum modulus can be remains. The best result to date is that of Pace Nielsen [3], who constructed a covering system with minimum modulus 40. Building on his ideas and methods, in this paper we construct a covering system with minimum modulus 42.

Chapter 2. A Covering Principle and Notation

In this work, it will be useful to use a different notation for congruence classes mod m than is standard. Understanding this paper requires readers to be familiar with the notations used in Nielsen's paper [3], but for clarity we briefly review the main terminology here. Given a prime p, there are exactly p distinct congruence classes modulo p. We denote those p classes as a p-tuple, and we put a 1 in the ith position to signify that $i \pmod{p}$ is in our covering system, and an empty spot in the jth position to denote that the congruence class $j \pmod{p}$ is not in our covering system. We write p at the start of the p-tuple to make the modulus explicit. For example, $3(1, \_, 1)$ represents the classes $1 \pmod{3}$ and $3 \pmod{3}$ but not $2 \pmod{3}$. (Note that this couldn't be part of a covering system because the modulus 3 is repeated.) We will often use the word 'input' to signify positions in p-tuples (informally thinking of such p-tuples as functions with arguments/inputs to be filled). We will also nest tuples to signify higher powers of primes and composite moduli. For example, $A = 3(\_, 1, 2(1, \_))$ denotes a partial cover containing two congruence classes, namely 2 (mod 3) and 3 (mod 3) $\cap$ 1 (mod 2) = 3 (mod 6). In this notation, we will also use x to represent a congruence class that is included in the covering by a previously constructed congruence. For example, if we have already constructed A as above, the x's in $5(\_, \_, 3(1, x, 2(x, 1)), \_, \_)$ denote that some classes are already covered by A. One further bit of notation is the use of the addition symbol when putting multiple congruence classes in the same input. For example, we will write $5(\_, \_, 2(1, \_) + 3(\_, 2(x, 1), \_), \_, \_)$ to denote the fact that both $2(1, \_)$ and $3(\_, 2(x, 1), \_)$ are in the third input of a 5 (which is also why we can put an x in the second set).

Recall that $p^{\uparrow}(c_1, c_2, \ldots, c_{p-1})$ is recursively defined by the formula

$$p(c_1, c_2, \ldots, c_{p-1}, p^{\uparrow}(c_1, c_2, \ldots, c_{p-1})).$$

Thus $3^{\uparrow}(\_, 1)$ represents one class modulo 3^n for each $n \geq 1$. While this notation technically creates an infinite cover, there is a standard procedure to finitize the process, as described in Morikawa [4], Gibson [5], and Nielsen [3]. This finitization is performed in §4.

Finally, it should be noted that we will occasionally leave out inputs in $p^{\uparrow}$ when context tells us what those inputs would be. For example, we will often write $2^{\uparrow}$ instead of $2^{\uparrow}(1)$. As another example, if we are restricted to the congruence classes 1 (mod 3) and 2 (mod 2), then we may write $9^{\uparrow}(1, 2)$ to mean $3(3^{\uparrow}(1, 2(\_, 1)), \_, \_)$. As a final example, if we know from context that we are restricted to the classes 1 (mod 2), 1 (mod 3), and 2 (mod 5), we may write $5 \cdot 3 \cdot 2$ to represent $5(\_, 3(2(1, \_), \_, \_), \_, \_, \_)$.

Chapter 3. The Covering System

We will now construct a covering system with minimum modulus 42. We will work prime by prime, encouraging the reader to verify at each step that the sets we construct do in fact cover the congruence classes in which we are working, and also verify that no modulus is repeated.

The basic principle for constructing the covering system is to start with a covering system with small moduli, and remove the moduli which are too small. This leaves behind holes. We then fill in these holes with moduli involving larger primes. For example, if we wanted a covering system with minimum modulus 4, we would start with $2^{\uparrow}(1)$. Because $2 < 4$ we would remove the class $1 \pmod 2$. We would then move to the prime 3. Since $3 < 4$, we could use the set $3^{\uparrow}(2(1, x), 2(2^{\uparrow}, x))$ to fill in the hole left open when removing $1 \pmod 2$.

Accompanying some of the congruences in the following sections will be pictorial representations of these holes in the covering system. In the diagrams, white circles will denote completely empty holes in the covering system, grey circles will represent partially filled holes, and black circles will represent completely covered congruences. The diagrams will branch to denote which moduli are being used. For example, $3(1, 2(1, \_), \_)$ could be represented by the following diagram:

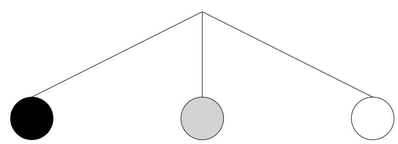

Figure 3.1:

Notice that the center circle is grey because $2(1, \_)$ only covers part of the input. Alternatively, if we wanted to provide more detail, we could draw the following picture:

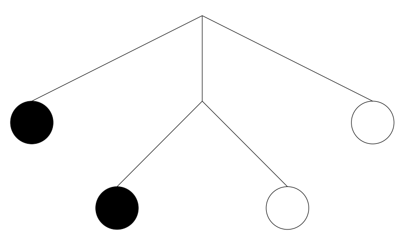

Figure 3.2:

Finally, it will sometimes be helpful to switch the order in which we branch. For example, this same set of congruences could be represented as:

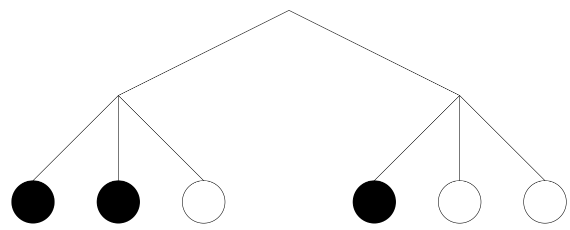

Figure 3.3:

3.1 The Prime 2

For both the prime 2 and the prime 3, we will use exactly the same congruence classes as in the covering system of Nielsen [3]. As they form the framework in which we will construct the rest of the covering, we give them again here.

We begin with $2^{\uparrow}(1)$. Because we want the minimum modulus of our covering system to be 42, we must remove the congruences with moduli 2, 4, 8, 16, and 32. This leaves us with a $64^{\uparrow}$, which covers one class modulo 32. At this point it is not important which class mod 32 we choose, so for concreteness we choose $32 \pmod{32}$.

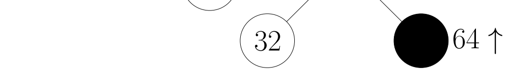

Figure 3.4:

3.2 The Prime 3

For the prime 3 we restrict ourselves to the branch $1 \pmod 2$ and attempt to cover as much as possible. We start with the covering $3^{\uparrow}(2,4^{\uparrow})$ and once again remove the congruences with moduli less than 42, namely 6, 12, 18, 24, and 36. Specifically, this leaves us with the set

$$3(\_,2(2(\_,2(\_,2^{\uparrow})),\_),3(\_,2(2(\_,2^{\uparrow}),\_),3^{\uparrow}(2(1,\_),2(2^{\uparrow},\_)))).$$

We can represent these congruence classes with the following diagram:

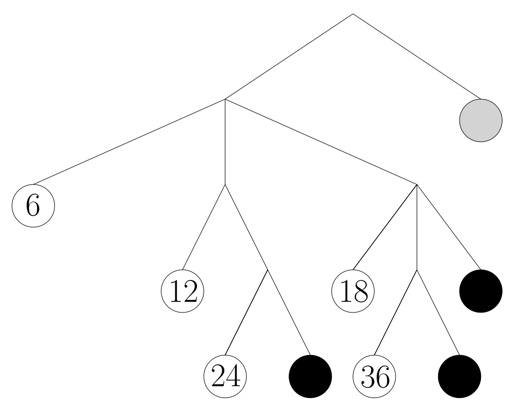

Figure 3.5:

We then add the set $81^{\uparrow}(1,\_)$ in the branch $21 \pmod{27}$, which falls in the hole labeled 18 in Figure 3.5.

3.3 The Prime 5

The prime 5 marks the first, and most significant, departure from the covering system described by Nielsen [3].

We now turn our attention back to the class $2 \pmod 2$. On this branch we only have four classes to fill, one for each of the moduli 4, 8, 16, and 32 (see Figure 3.4). To simplify notation, in this section 4, 8, 16, and 32 will always refer to the classes $2 \pmod 4$, $4 \pmod 8$, $8 \pmod{16}$, and $16 \pmod{32}$ respectively (which are the holes with the corresponding moduli in Figure 3.4), while 2 refers to $2 \pmod 2$. We use the prime 5 to fill as much of these classes as possible. Because the minimum modulus is 42, the smallest power of 2 that we can use in conjunction with the 5 is 16 (since $5 \cdot 8 = 40 < 42$). So in the first of the five inputs of $5(\_,\_,\_,\_,\_)$ we put $16 + 32$. For the next input we insert the set

$$3(\_,\_,3^{\uparrow}(4+8,\_) + 3^{\uparrow}(16,32^{\uparrow})) + 64^{\uparrow}$$

Here (and elsewhere in this section) the $32^{\uparrow}$ inside of the $3^{\uparrow}$ is restricted to the 16 branch and the $64^{\uparrow}$ is restricted to the 32 branch. For the third input we use the set

$$3(64^{\uparrow}, 4+8+16+32, 3^{\uparrow}(1,2)).$$

The order of these inputs might seem arbitrary at this point, but we have chosen the inputs in this order for use in later sections. We will remark on this more at the end of the section. The fourth input we leave blank. In the fifth input we insert the set

$$5(2, 4+8+16+32, 3^{\uparrow}(1,2), 3^{\uparrow}(32^{\uparrow}, 4+8+16) + 64^{\uparrow}, 5^{\uparrow}(1,2,3^{\uparrow}(1,2),4^{\uparrow})).$$

Here the $4^{\uparrow}$ covers all of $2 \pmod 2$.

We note a few things now about our covering so far. First, it will be important later (such as in sections 3.5 and 3.6) that the classes $2 \pmod 3$ and $3 \pmod 9$ are covered on the third input of $5^{\uparrow}$ in each of the holes. It will be useful to visualize what we have done so far on this branch. We do so as follows:

$$\begin{aligned}
&\text{Hole } 4: 5(_,3(_,_,3^{\uparrow}(x,_)),3(_,x,x),_,5(x,x,x,3^{\uparrow}(_,x),x))\\
&\text{Hole } 8: 5(_,3(_,_,3^{\uparrow}(x,_)),3(_,x,x),_,5(x,x,x,3^{\uparrow}(_,x),x))\\
&\text{Hole } 16: 5(x,3(_,_,x),3(_,x,x),_,x)\\
&\text{Hole } 32: 5(x,x,x,_,x)
\end{aligned}$$

We note that there are still some unused moduli. We almost completely cover the last empty input in the 8 hole, with the sets

$$125^{\uparrow}(3^{\uparrow}(4,x),3^{\uparrow}(8,x),3^{\uparrow}(16^{\uparrow},x),3^{\uparrow}(\_,x))))$$

leaving just a single $125^{\uparrow} \cdot 3^{\uparrow}$ to fill.

Pictorially, on both of the branches 2 (mod 4) and 4 (mod 8) we have:

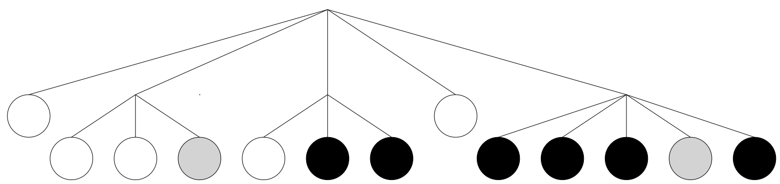

Figure 3.6:

On the branch 8 (mod 16) we have:

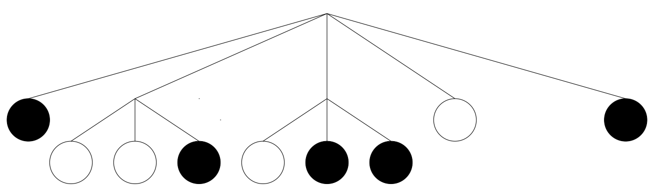

Figure 3.7:

Finally, on the branch 16 (mod 32) we have:

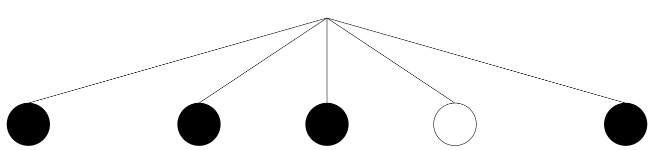

Figure 3.8:

3.4 The Prime 7

Starting with the prime $p = 7$ and hereafter, we will be filling the entries of $p^{\uparrow}$, so that we need only $p-1$ inputs, in this case 6. With the prime 7 we will almost completely cover the holes in our covering of the 8 and 16 branches from the previous section (pictured in Figures 3.6 and 3.7). Specifically we will fill all but the first input on the 5 on the 8 hole (which is the leftmost white circle in Figure 3.6), and we will completely fill the 16 hole. To this end, we fill the first input of the $7^{\uparrow}$ with the set $8+16$. The second input is filled by the set $3^{\uparrow}(8, 16^{\uparrow}) + 32^{\uparrow}$ (of course, $16^{\uparrow}$ is used to cover the hole 8 and $32^{\uparrow}$ is used to cover the hole 16). The third input is filled by $3(2, 4, 3^{\uparrow}(1, 2))$ (where here 4 corresponds to $4 \pmod 4$ so that it covers both the 8 and 16 holes).

Next we consider the fourth, fifth, and sixth inputs. For the moment we will be working only with the middle three classes modulo 5 because both the first and the final classes modulo 5 are already covered on the 16 hole. We will fill the last class modulo 5 on the 8 hole later in the section. We thus write $5(\_, \_, \_)$ with only three inputs to simplify the notation. The fourth set in $7^{\uparrow}$ is then $5(3(4, 8+16, 3^{\uparrow}(x, 4)), 3(1, x, x), 2)$. The fifth set is $5(8+16, 3(3^{\uparrow}(1, 2), x, x), 5^{\uparrow}(1, 2, 3^{\uparrow}(1, 2), 4))$. The sixth and final set is more complex. Let

$$A = 32^{\uparrow} + 3(3^{\uparrow}(8, \_), \_, 3^{\uparrow}(x, 16^{\uparrow})) + 5(8, 16^{\uparrow}, 3(3^{\uparrow}(x, 4), 4, x), 3(3^{\uparrow}(x, 8), 8, x), 3(3^{\uparrow}(x, 16^{\uparrow}), 16^{\uparrow}, x)).$$

Here, as usual, $32^{\uparrow}$ covers the 16 hole, whereas $16^{\uparrow}$ covers the 8 hole. Noting that A fills the 8 hole, in the second input of 5, we take as our sixth set $5(A, 3(2, x, x), 4)$.

We now notice that we have yet to use $125 \cdot 3^{\uparrow} \cdot 4$, $125^{\uparrow} \cdot 3^{\uparrow} \cdot 8$, or $125^{\uparrow} \cdot 3^{\uparrow} \cdot 16^{\uparrow}$. We can add these sets to our covering to cover the last input of 5 on the 8 hole in the fourth, fifth, and sixth inputs of $7^{\uparrow}$. This leaves just a hole in the first input on 5 in the 8 hole.

In summary, we use the following sets:

$$
\begin{aligned}
&7^{\uparrow}(8+16,\\
&\quad 3^{\uparrow}(8,16^{\uparrow})+32^{\uparrow},\\
&\quad 3(2,4,3^{\uparrow}(1,2)),\\
&\quad 5(\_+x,3(4,8+16,3^{\uparrow}(x,4)),3(1,x,x),2,125^{\uparrow}\cdot 3^{\uparrow}\cdot 4),\\
&\quad 5(\_+x,8+16,3(3^{\uparrow}(1,2),x,x),5^{\uparrow}(1,2,3^{\uparrow}(1,2),4),125^{\uparrow}\cdot 3^{\uparrow}\cdot 8),\\
&\quad 5(\_+x,A,3(2,x,x),4,125^{\uparrow}\cdot 3^{\uparrow}\cdot 16^{\uparrow}).
\end{aligned}
$$

Here we use $\_+x$ to signify that we are not covering the first input modulo 5 on the 8 hole, but that this class is already filled on the 16 hole.

Our diagram from the previous section is now:

$$
\begin{aligned}
\text{Hole } 4:&\ 5(\_,3(\_,\_,3^{\uparrow}(x,\_)),3(\_,x,x),\_,5(x,x,x,3^{\uparrow}(\_,x),x))\\
\text{Hole } 8:&\ 5(\_,x,x,x,x)\\
\text{Hole } 16:&\ 5(x,x,x,x,x)\\
\text{Hole } 32:&\ 5(x,x,x,\_,x)
\end{aligned}
$$

Pictorially, we have (where we use grey holes only when they are mostly filled):

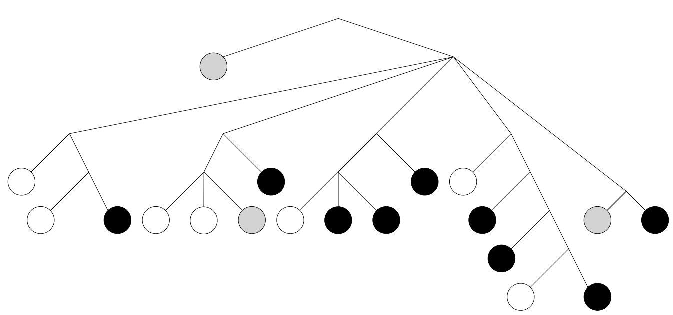

Figure 3.9:

We note at this point that we have yet to use $125^{\uparrow} \cdot 8^{\uparrow}$ or $9^{\uparrow} \cdot 4$. First, restrict to the fifth input of the $7^{\uparrow}$, in the fourth input of a 5. Here, on the entirety of the 4 hole we are only missing one input in a $7^{\uparrow} \cdot 25^{\uparrow}$. We use $125^{\uparrow} \cdot 8^{\uparrow}$ to reduce this hole to only missing a single input in a $7^{\uparrow} \cdot 25$. We will use this in our work with the primes 61 and 67 in Section 3.18. Finally, we use $9 \cdot 4$ to fill the third entry of a $7^{\uparrow}$ on the intersection $1 \pmod 4 \cap 6 \pmod 9$; this will be used when dealing with the prime 31 in Section 3.11.

3.5 The Prime 11

For the prime 11 we use the same congruences constructed by Nielsen [3], with the exception of moving the class modulo $11 \cdot 5$ to fall in $4 \pmod 5$ instead of $1 \pmod 5$. This simple permutation of the inputs does not affect the combinatorial approach we take to constructing the sets, as long as we make the same shift in all of the congruences cited from Nielsen's paper. This change will be vital in later sections, beginning with 3.14 .

3.6 The Prime 13

For the prime 13 we use the same congruences constructed by Nielsen [3], again permuting the inputs so that the class modulo $13 \cdot 5$ falls in $2 \pmod 5$. This will make work in Section 3.13 easier.

We note that the congruence sets of this and the previous section taken together cover the holes left over (from our work with the prime 3) when we removed the moduli 6 and 18 (see Figure 3.5).

3.7 The Prime 17

For the prime 17 we will focus on the hole left when we removed the congruence class with modulus 12 during our work with the prime 3 (see Figure 3.5). In his paper, Nielsen used the prime 19 to partially fill this hole. Specifically, he constructed seventeen sets to use in filling inputs on a $19^{\uparrow}$. As only one of these sets involved the prime 17, we may use the other sixteen sets to completely fill the desired $17^{\uparrow}$, again making the necessary changes to accommodate the rearranged inputs.

3.8 The Prime 19

We now return to the branch $2 \pmod 4$. We will use $19^{\uparrow}$ to completely fill the first input in the 5 on the 4 hole. (This is the left-most empty hole in Figure 3.9.) Any use of the prime 5 in this section will be restricted to that branch. Beginning in this section and for the rest of our construction, we will suppress the idea of inputs being filled, and instead just construct sets which can be used to fill these inputs. This is because the order of the sets used to fill each $p^{\uparrow}$ will not matter to us. So, we begin with the sets 1, 2, 4 and $8^{\uparrow}$ (remembering that we will need to drop 1 and 2 as inputs in the $19^{\uparrow}$, due to our minimum modulus restriction). We can then take these four sets and use them in conjunction with 5 and $25^{\uparrow}$ to make five more sets. Because the sixth input in $11^{\uparrow}$ is already covered on this branch, we can fill a copy of $11^{\uparrow}$. We now have ten sets, so we can fill five copies of $3^{\uparrow}$ (using these sets in sequence). We then fill $13^{\uparrow}$, and $17^{\uparrow}$. This gives us seventeen sets. Because $7^{\uparrow}$ needs only one input of a 3 in its third entry on this branch, we can fill three copies of $7^{\uparrow}$ (making sure one of the sets is $7^{\uparrow}(1, 2, 3(x, 1, x), 4, 8^{\uparrow}, 3^{\uparrow}(2, 4)))$. This gives us a total of twenty sets, which after removing 1 and 2 gives us the eighteen needed inputs for $19^{\uparrow}$. We note that five of these inputs apply to the entire 4 hole, so that on the 4 hole any future $19^{\uparrow}$ will need only thirteen inputs filled.

3.9 The Prime 23

We use the same congruences (again up to rearrangement) constructed in Nielsen's paper to completely cover the hole that arose from the removal of the congruence class with modulus 24 during our work with the prime 3 (see Figure 3.5). In particular, we replace the entry $5^{\uparrow}(1, 2, 4, 8)$ with $5^{\uparrow}(2, 1, 4, 8)$ so that on the fourth from the left empty hole in Figure 3.9, we only need nineteen inputs in $23^{\uparrow}$. This will be used in Section 3.13.

3.10 The Prime 29

We will now fill two more holes left over from our work with the prime 5. We will restrict ourselves to the first input of a 3, and also restrict ourselves to both the second and third inputs of a 5, on the 4 hole. (These are the third and fifth from the left empty holes in Figure 3.9.) We begin with the sets 1, 2, 4, $8^{\uparrow}$, $3 \cdot 1$, $3 \cdot 2$, $3 \cdot 4$, $3 \cdot 8^{\uparrow}$, $9^{\uparrow}(1, 2)$, and $9^{\uparrow}(4, 8^{\uparrow})$. We then have ten sets to use in conjunction with 5, so we can create five more sets (remember that we are filling two different inputs in the 5), bringing us up to fifteen sets. The sixteenth set is $25^{\uparrow}(1, 2, 4, 8^{\uparrow}) + 25^{\uparrow}(3 \cdot 1, 3 \cdot 2, 3 \cdot 4, 3 \cdot 8^{\uparrow})$. We use these sixteen sets to fill a $17^{\uparrow}$. There are only two inputs in the $7^{\uparrow}$ that we need to fill on the third input of the 5, whereas we need five entries filled for the second input of the 5. Thus, if we use congruences which apply to both inputs on the 5, we can maximize the effectiveness of the 7. We can thus construct four more sets as follows:

$$7^{\uparrow}(\_, \_, x, 5 \cdot 1, 5 \cdot 2, 5 \cdot 4) + 7^{\uparrow}(1, 2, x, x, x, x)$$

$$7^{\uparrow}(\_, \_, x, 5 \cdot 8^{\uparrow}, 5 \cdot 3 \cdot 1, 5 \cdot 3 \cdot 2) + 7^{\uparrow}(4, 8^{\uparrow}, x, x, x, x)$$

$$7^{\uparrow}(\_, \_, x, 5 \cdot 3 \cdot 4, 5 \cdot 3 \cdot 8^{\uparrow}, 25^{\uparrow}(1, 2, 4, 8^{\uparrow})) + 7^{\uparrow}(3 \cdot 1, 3 \cdot 2, x, x, x, x)$$

$$\begin{aligned} &25^{\uparrow}(9^{\uparrow}(1, 2), 9^{\uparrow}(4, 8^{\uparrow}), \_, \_) \\ &\quad + 7^{\uparrow}(\_, \_, x, 25^{\uparrow}(x, x, 3 \cdot 1, 3 \cdot 2), 25^{\uparrow}(x, x, 3 \cdot 4, 3 \cdot 8^{\uparrow}), 25^{\uparrow}(x, x, 9^{\uparrow}(1, 2), 9^{\uparrow}(4, 8^{\uparrow}))) \\ &\quad + 7^{\uparrow}(3 \cdot 4, 3 \cdot 8^{\uparrow}, x, x, x, x) \end{aligned}$$

We have now constructed 21 sets which we use to fill two copies of $11^{\uparrow}$. We can then fill a $23^{\uparrow}$. This gives us 24 sets which we can use to fill two copies of $13^{\uparrow}$. This gives us 26 sets, which we can use to fill two copies of $19^{\uparrow}$, since five of the inputs from our work in Section 3.8 still apply. This brings our total number of sets to twenty-eight; however, we cannot use the set 1 because $29 < 42$, so we need one more set. We now create the set $7^{\uparrow}(\_, \_, x, 5 \cdot 9^{\uparrow}(1, 2), 5 \cdot (9^{\uparrow}(4, 8^{\uparrow})), B) + 7^{\uparrow}(9^{\uparrow}(1, 2), 9^{\uparrow}(4, 8^{\uparrow}), x, x, x, x)$. Here, B is the set $17^{\uparrow}$ filled with the first sixteen sets.

3.11 The Prime 31

We now return to the branch 1 (mod 2). We are going to use $31^{\uparrow}$ to fill the hole left open when we removed the congruence with modulus 36 in our work with the prime 3. Specifically, we are restricted to the branches 1 (mod 4) and 6 (mod 9). We can easily make fourteen sets to start off with, namely 1, 2, 4, $8^{\uparrow}$, $3 \cdot 1$, $3 \cdot 2$, $3 \cdot 4$, $3 \cdot 8^{\uparrow}$, $9 \cdot 1$, $9 \cdot 2$, $9 \cdot 4$, $9 \cdot 8^{\uparrow}$, $27^{\uparrow}(1, 2)$, and $27^{\uparrow}(4, 8^{\uparrow})$. Using the first twelve of these fourteen sets we can fill three copies of $5^{\uparrow}$, making sure that one of the sets is $5^{\uparrow}(2, 4, 8^{\uparrow}, 1)$ (which will be used in Section 3.16). We also create the set $C = 5^{\uparrow}(27^{\uparrow}(1, 2), 27^{\uparrow}(4, 8^{\uparrow}), \_, \_)$ to be used shortly. This gives us seventeen complete sets, the first fifteen of which we use to fill three copies of $7^{\uparrow}$ (noting, that $7^{\uparrow}$ only needs five entries filled on this branch). We can then create one more set by using C plus

$$7^{\uparrow}(5^{\uparrow}(x, x, 3 \cdot 1, 3 \cdot 2),\ 5^{\uparrow}(x, x, 3 \cdot 4, 3 \cdot 8^{\uparrow}),\ x,\ 5^{\uparrow}(x, x, 9 \cdot 1, 9 \cdot 2),\ 5^{\uparrow}(x, x, 9 \cdot 4, 9 \cdot 8^{\uparrow}),$$
$$5^{\uparrow}(x, x, 27^{\uparrow}(1, 2),\ 27^{\uparrow}(4, 8^{\uparrow}))).$$

This gives us a total of twenty-one sets.

Because we are working with the branch 2 (mod 4), some of the congruences from or work with the prime 11 carries over. In particular, we only need to fill eight entries, one of which needs three inputs in a $5^{\uparrow}$ and one needs just one input in a $5^{\uparrow}$. Thus, our previous sets can easily fill three copies of $11^{\uparrow}$. These twenty-four sets fill two copies of $13^{\uparrow}$. The twenty-six constructed sets fill two copies of $17^{\uparrow}$ (noting we only needed thirteen entries). Finally, we fill one copy each of $19^{\uparrow}$, $23^{\uparrow}$, and $29^{\uparrow}$. This gives us thirty-one sets, which after dropping 1, fills our $31^{\uparrow}$.

3.12 The Prime 37

With 1 (mod 2) completely filled, we return now to the branch 2 (mod 2). We work to fill in the hole left in the first input of 5 on the 8 hole, which is the second empty hole (from the left) in Figure 3.9. We will restrict to the first two inputs of a 3, and fill the third input later (in Section 3.16 using the prime 53). Start with the sets 1, 2, 4, 8, and $16^{\uparrow}$. The next four sets will be $3(1,2)$, $3(4,8)$, $3(3^{\uparrow}(1,2),3^{\uparrow}(4,8))$, and $3(16^{\uparrow},\_)+5\cdot 3(\_,16^{\uparrow})$. We can create eight more sets by multiplying each of the first eight of these sets by 5. We can also use them to fill two copies of $25^{\uparrow}$, giving us 19 sets to work with. From our work in Section 3.4, we only need three inputs to fill a $7^{\uparrow}$ on this branch, thus giving us six more sets bringing our total number of sets to 25. We can use these sets to fill two copies of $13^{\uparrow}$, then two copies of $19^{\uparrow}$ noting that we need only thirteen inputs because of our work with 19 earlier. This gives us twenty-nine sets which we use to fill a $29^{\uparrow}$, then a $31^{\uparrow}$, three copies of $11^{\uparrow}$, two copies of $17^{\uparrow}$ and finally a $23^{\uparrow}$. This gives us a total of thirty-seven sets, which means we can completely fill the thirty-six inputs of the $37^{\uparrow}$, remembering that we cannot use the set 1 because $37 < 42$.

3.13 The Prime 41

Now turn to the fourth open hole (from the left) in Figure 3.9. We use the prime 41 to fill this hole, which is the second input in a 3, in the second input of a 5, on the 4 hole. We begin with the sets 1, 2, 4, and $8^{\uparrow}$. We use these, in conjunction with 3 and $9^{\uparrow}$ to get ten sets. Next fill an $11^{\uparrow}$ and then a $13^{\uparrow}$ (noting that $13^{\uparrow}$ only requires eleven inputs on this branch). With the twelve sets thus constructed, we create another twelve using the prime 5, and another three with $25^{\uparrow}$. These twenty-seven sets fill $19^{\uparrow}$ twice over (since we only need thirteen inputs), and then we fill a $29^{\uparrow}$. The thirty constructed sets fill six copies of $7^{\uparrow}$. Then we sequentially fill a $31^{\uparrow}$, two copies of $17^{\uparrow}$, a $37^{\uparrow}$, and finally two copies of $23^{\uparrow}$ (noting that on this hole, only nineteen inputs are needed). This results in forty-one sets, which after dropping 1 fills the $41^{\uparrow}$.

3.14 The Prime 43

We will use the prime 43 to partially cover the hole in the fourth input of a 5 on the 4 hole. (This is the second-to-last empty hole in Figure 3.9.) Specifically we will cover the middle input of a 3 on that branch. We start with the sets 1, 2, 4, and $8^\uparrow$. Using 5 and $25^\uparrow$ we fill five more inputs, bringing us to nine sets. Since one input on $11^\uparrow$ is already covered on this branch, we can fill $11^\uparrow$ bringing us to ten sets. We then use 3 and $9^\uparrow$, bringing us to twenty-five sets. Because $7^\uparrow$ needs only five inputs on this branch, we can fill five copies of $7^\uparrow$, bringing us to thirty sets. We then fill $31^\uparrow$, $29^\uparrow$, and two copies of $17^\uparrow$, bringing us to thirty-four sets. We then fill $23^\uparrow$ and $37^\uparrow$ (noting it is partially filled here) to bring us to thirty-six sets. Then fill three copies of $13^\uparrow$, bringing us to thirty-nine sets. Finally, since $19^\uparrow$ only needs thirteen inputs on this branch, we fill three copies of $19^\uparrow$, bringing us to the needed 42 sets.

3.15 The Prime 47

We will use the prime 47 to completely fill the first input of a 3 in the fourth input of a 5 on the 4 hole. We begin with the same twenty-five sets constructed in the section on the prime 43 (where here the 3 and $9^\uparrow$ refer now to the first class modulo 3). Because $7^\uparrow$ only needs four inputs on this branch, one of which only needs a 25, we can use $7^\uparrow$ to bring us to thirty-two sets (the $25^\uparrow$ breaks into five sets, giving us twenty-nine sets to work with in filling the four inputs of the $7^\uparrow$, making sure that the 25's are used in the correct input). We then fill two copies of $17^\uparrow$, $29^\uparrow$, $31^\uparrow$, and three copies of $13^\uparrow$, bringing us to thirty-nine sets. Because $19^\uparrow$ needs only thirteen inputs on this branch, we can fill three copies of $19^\uparrow$, bringing us to forty-two sets. We then fill $41^\uparrow$, $43^\uparrow$, and two copies of $23^\uparrow$, bringing us to the forty-six sets we need.

3.16 The Prime 53

We will use the prime 53 to fill the last input of a 5 on the 4 hole. Note that we only need to fill one hole mod $25 \cdot 3^\uparrow \cdot 4$. We begin with the sets 1, 2, 4, and $8^\uparrow$. We then use these in conjunction with the prime 3 to bring our total number of sets to eight. We can use these eight sets in conjunction with 5 and 25, along with filling two copies of $125^\uparrow$ to bring our total number of sets to twenty-six. Because $19^\uparrow$ needs only thirteen inputs, we can fill two copies of $19^\uparrow$, bringing us to twenty-eight sets. We can now fill a $29^\uparrow$. Noting that one of the inputs from our work with the prime 31 carries over on this branch, we can fill $31^\uparrow$, bringing us to thirty sets. Since the third input in $7^\uparrow$ is already covered on this branch from our earlier work, we can use these 30 sets to make six copies of $7^\uparrow$, bringing us to thirty-six sets. We then sequentially fill three copies of $13^\uparrow$, $37^\uparrow$, four copies of $11^\uparrow$, $41^\uparrow$, and $43^\uparrow$, giving us forty-six sets. We can then fill $47^\uparrow$, two copies of $23^\uparrow$, and three copies of $17^\uparrow$, bringing us to the fifty-two sets we need.

3.17 The Prime 59

We will now fill the last input on the 3 in the first input of the 5 on the 8 hole. We begin with the sets 1, 2, 4, 8, and $16^\uparrow$. We then use these in conjunction with 3 and $9^\uparrow$ to bring our total to twelve sets, with a half-filled $9^\uparrow$ still at our disposal. We can use these twelve sets with 5 to bring our total to twenty-five sets, the last set being $5 \cdot 9^\uparrow(16^\uparrow, \_) + 9^\uparrow(\_, 16^\uparrow)$. We can also fill three copies of $25^\uparrow$. This gives us a total of twenty-eight sets. We then fill $29^\uparrow$ and $23^\uparrow$ giving us thirty sets. Because we only need three inputs on a $7^\uparrow$ on this branch, we can then fill ten copies of $7^\uparrow$, bringing us to forty sets. We then fill $41^\uparrow$, four copies of $11^\uparrow$, $43^\uparrow$, $47^\uparrow$, $37^\uparrow$, three copies of $17^\uparrow$, four copies of $13^\uparrow$, and three copies of $19^\uparrow$, bringing us to the fifty-eight sets needed to fill $59^\uparrow$.

3.18 The Primes 61, 67, and 89

We will use the primes 61, 67, and 89 to fill the third input in the 3 in the fourth input of the 5 on the 4 hole, by splitting the input in half as was done for 11 and 13. We begin by creating twenty-eight sets, in the same manner as in the previous section. We then fill a $29^\uparrow$ and $31^\uparrow$ (noting the latter needs only twenty-nine inputs). On this branch, we only need to fill $7^{\uparrow}(\_, \_, x, x, 25(x, x, x, \_, x), \_)$. Using the previous sets, we can easily do this nine times, bringing our total to thirty-nine sets. We then fill $19^{\uparrow}$ three times, $37^{\uparrow}$, $41^{\uparrow}$, $43^{\uparrow}$, $23^{\uparrow}$ twice, $47^{\uparrow}$ $17^{\uparrow}$ three times, $13^{\uparrow}$ four times, $53^{\uparrow}$, $11^{\uparrow}$ six times (because it only needs nine inputs), and $59^{\uparrow}$. This totals in sixty-three sets, easily filling the $61^{\uparrow}$.

Using the same construction as above (but restricted to a different class modulo 8), we get sixty-three sets. We then can use $61^{\uparrow}$ to fill two more entries, using the doubling idea from Nielsen's work with the prime 13. This leaves one entry in a $67^{\uparrow}$, which is then easily filled using the prime 89.

3.19 The Primes 71 and 73

We will use these two primes to fill the hole remaining in the last input of a 3 in the second input of a 5 on the branch 2 (mod 4). We note that we need only fill one input in a $9^{\uparrow}$ on this branch, and we will split the hole in two as was done with 11 and 13 (and again with 61 and 67). We begin with the sets 1, 2, 4, 8, and $16^{\uparrow}$. Since $7^{\uparrow}$ needs only five inputs we get another set, and then in conjunction with 3 and $9^{\uparrow}$ we have eighteen sets. Now fill a $17^{\uparrow}$, $19^{\uparrow}$, $11^{\uparrow}$ twice, and $13^{\uparrow}$ twice (since we only need eleven inputs), giving a total of twenty-four sets. Using 5 and $25^{\uparrow}$, we then have fifty-four sets. Sequentially fill $53^{\uparrow}$, $47^{\uparrow}$, $43^{\uparrow}$, $41^{\uparrow}$, $59^{\uparrow}$, $37^{\uparrow}$, $31^{\uparrow}$ twice, $61^{\uparrow}$, $23^{\uparrow}$ three times (since it only needs twenty-one inputs), $29^{\uparrow}$ three times (since it only needs twenty-two inputs), and a $67^{\uparrow}$. This yields the seventy sets to fill a $71^{\uparrow}$.

Repeat for the other class modulo 8, to get seventy sets. Then $71^{\uparrow}$ can be filled twice (using the doubling trick) and we thus can completely fill the $73^{\uparrow}$.

3.20 The Primes 79 and 83

We will now fill the final empty hole, which is in the fourth input of a 5 on the 32 hole. (This is the right-most empty hole in Figure 3.9.) We will split this hole into two parts, using $79^{\uparrow}$ to fill the first input of a $3^{\uparrow}$ and $83^{\uparrow}$ to fill the second. We begin with the sets 1, 2, 4, 8, 16, 32, and $64^{\uparrow}$. We then use these with $3^{\uparrow}$ to bring our total number of sets to fourteen. Since $7^{\uparrow}$ only needs two inputs on this branch, we can fill seven copies of $7^{\uparrow}$, bringing us to twenty-one sets. We can then use 5 and $25^{\uparrow}$ to bring us to forty-seven sets. We then fill $47^{\uparrow}$, three copies of $17^{\uparrow}$, five copies of $11^{\uparrow}$, two copies of $29^{\uparrow}$, $59^{\uparrow}$, $53^{\uparrow}$, $61^{\uparrow}$, two copies of $31^{\uparrow}$, five copies of $13^{\uparrow}$, $67^{\uparrow}$, three copies of $23^{\uparrow}$, four copies of $19^{\uparrow}$, $71^{\uparrow}$, $73^{\uparrow}$. This gives us enough sets to fill a $79^{\uparrow}$.

Now using $79^{\uparrow}$, $43^{\uparrow}$, and two copies of $41^{\uparrow}$, this gives us a total of eighty-two sets, which fills the $83^{\uparrow}$, and finishes the construction.

Chapter 4. Conclusion

Now that we have an infinite cover, we can utilize any large (unused) prime to finitize the covering, giving us the desired covering system with minimum modulus 42. The final covering system can be visually represented by Figure 4.1, which shows where the primes are used in relation to each other.

How would one go about improving this result? In Nielsen's covering system, there were no small primes left over to be able to fill the hole created by removing the congruence class with the modulus 40. Raising the minimum modulus of the covering system required fundamentally changing the approach used in constructing the covering system, most notable in the change of placement of the prime 5. In the case of this covering system, there is a similar lack of freedom among the placement of small primes. So if we remove the class with modulus 42, it would be difficult to fill in the resulting hole. To improve the cover further, one would likely need to make a similar dramatic change early on in the process of constructing the covering system, perhaps in the sets used with the prime 3, or perhaps by choosing a better place to put the extra $81^{\uparrow}$.

It is worth noting that Nielsen's covering system with minimum modulus 40 used a scheme with all of the primes up to 103. My covering system with minimum modulus 42 only uses a scheme utilizing all of the primes up to 89. A natural question for future research is whether one can use fewer primes still. If so, can a covering system with minimum modulus 43 or higher be constructed? It is likely that it would require the use of at least the primes used in constructing this covering system, and would likely require even more large primes.

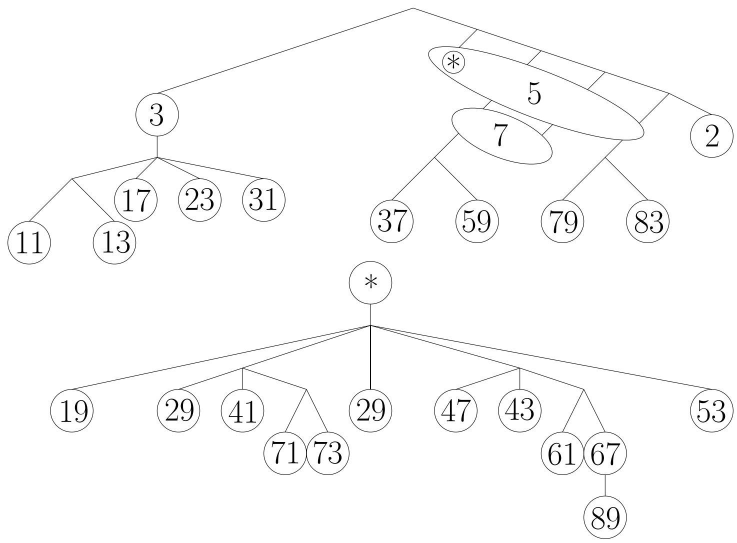

Figure 4.1: The Primes Used in Constructing the Covering System

Bibliography

[1] P. Erdős. On integers of the form $2^k + p$ and some related problems. *Summa Brasil. Math.*, 2:113–123, 1950.

[2] Bob Hough. Solution of the minimum modulus problem for covering systems. *Ann. of Math. (2)*, 181(1):361–382, 2015.

[3] Pace P. Nielsen. A covering system whose smallest modulus is 40. *J. Number Theory*, 129(3):640–666, 2009.

[4] Ryozo Morikawa. On a method to construct covering sets. *Bull. Fac. Liberal Arts Nagasaki Univ.*, 22(1):1–11, 1981.

[5] D.J. Gibson. Covering systems. *Ph.D. thesis, University of Illinois at Urbana-Champaign*, 2006.