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Global stability of VEISV propagation modeling for network worm attack ☆

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ABSTRACT

In this paper, using the Li–Muldowney geometric approach, we establish the global stability of the worm-epidemic equilibrium for a VEISV network worm attack model. This improves the related results presented in Toutonji et al. (2012) [1].

1. Introduction

In this paper, we consider a VEISV network worm attack model, which is appropriate for measuring the effects of security countermeasures on worm propagation that was investigated by Toutonji et al. in [1]. Their model is formulated as follows:

$$\begin{cases} \frac{dV}{dt} = -f\,EV - \psi_1 V + \phi S, \\ \frac{dE}{dt} = f\,EV - (\alpha + \psi_2)E, \\ \frac{dI}{dt} = \alpha E - (\gamma + \theta)I, \\ \frac{dS}{dt} = \mu N + \psi_1 V + \psi_2 E + \gamma I - \phi S, \end{cases} \tag{1}$$

where $V(t), E(t), I(t)$ and $S(t)$ denote the number of vulnerable hosts, exposed hosts, infectious hosts and secured hosts at time t, respectively. The parameter β is the contact rate. $\alpha, \psi_1, \psi_2, \gamma$ and ϕ are the state transition rates from E to I, V to S, E to S, I to S and S to V, respectively. θ represents the dysfunctional rate which is a constant. μ is the replacement rate. N is the total number of hosts, which is fixed and defined by $N = V(t) + E(t) + I(t) + S(t)$. $f = \frac{\alpha\beta}{N}$ is the force of incident.

Due to the physical restrictions the states of the system must be non-negative and it is easy to see that

$$\Gamma = \left\{(V, E, I, S) \in \mathbb{R}_+^4 : V + E + I + S = N\right\}$$

is positively invariant with respect to system (1). Thus, we focus on the reduced system

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$$\begin{cases} \frac{dV}{dt} = \phi N - f\,EV - (\psi_1 + \phi)V - \phi E - \phi I, \\ \frac{dE}{dt} = f\,EV - (\alpha + \psi_2)E, \\ \frac{dI}{dt} = \alpha E - (\gamma + \theta)I. \end{cases} \tag{2}$$

By calculation, we conclude that the reproduction rate of system (2) is

$$\mathcal{R}_0 = \frac{\alpha\beta\phi}{(\psi_1 + \phi)(\alpha + \psi_2)},$$

System (2) has two equilibria: the worm-free equilibrium

$$EQ_{wf} = (V_1^*, E_1^*, I_1^*) = \left(\frac{\phi N}{\psi_1 + \phi}, 0, 0\right)$$

and the worm-epidemic equilibrium

$$EQ_{we} = (V_2^*, E_2^*, I_2^*) = \left(\frac{(\alpha + \psi_2)N}{\alpha\beta}, \frac{\alpha\beta\phi - (\psi_1 + \phi)(\alpha + \psi_2)}{\alpha\beta\left(\alpha + \psi_2 + \phi\left(1 + \frac{\alpha}{\gamma+\theta}\right)\right)}N, \frac{\alpha}{\gamma + \theta}E_2^*\right).$$

Toutonji et al. [1] established the following results:

Theorem 1.1. *If $\mathcal{R}_0 \le 1$, the worm-free equilibrium EQ_{wf} of system (2) is globally asymptotically stable.*

Theorem 1.2. *If $\mathcal{R}_0 > 1$, the worm-epidemic equilibrium EQ_{we} of system (2) is locally asymptotically stable.*

Remark 1.1. We point out that the Jacobian matrix at EQ_{we} is not correct in [1]. We correct it as follows:

$$J(EQ_{we}) = \begin{pmatrix} -(f\,E_2^* + \psi_1 + \phi) & -(\alpha + \psi_2 + \phi) & -\phi \\ f\,E_2^* & 0 & 0 \\ 0 & \alpha & -(\gamma + \theta) \end{pmatrix}.$$

Thus, the characteristic polynomial of $J(EQ_{we})$ is

$$h(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

where

$$\begin{cases} a_3 = 1, \\ a_2 = f\,E_2^* + \psi_1 + \phi + \gamma + \theta, \\ a_1 = (f\,E_2^* + \psi_1 + \phi)(\gamma + \theta) + f\,E_2^*(\alpha + \psi_2 + \phi), \\ a_0 = f\,E_2^*[(\alpha + \psi_2 + \phi)(\gamma + \theta) + \alpha\phi]. \end{cases}$$

It is obvious that $a_0 > 0, a_1 > 0, a_2 > 0, a_3 > 0$ and $a_1a_2 > a_0a_3$ hold. Based on the Routh–Hurwitz theorem, the worm-epidemic equilibrium EQ_{we} of system (2) is locally asymptotically stable when $\mathcal{R}_0 > 1$.

The main purpose of this paper is to study the global stability of the worm-epidemic equilibrium EQ_{we} for system (2) when $\mathcal{R}_0 > 1$. We use the Li–Muldowney geometric approach (see [2]), which was adopted by many researchers (see, for example, [3–13]), to obtain our main result.

2. Global stability of worm-epidemic equilibrium

In this section, we study the global stability of the worm-epidemic equilibrium EQ_{we} for system (2) by using the Li–Muldowney geometric approach.

Similar to the proof of Theorem 3.2 in [14], system (2) is uniformly persistent, which implies that there exists a constant $c > 0$ such that any solution $(V(t), E(t), I(t))$ of system (2) with $(V(0), E(0), I(0)) \in \text{int}\Gamma$ satisfies

$$\min\{\liminf_{t\to\infty} V(t), \quad \liminf_{t\to\infty} E(t), \quad \liminf_{t\to\infty} I(t)\} \geqslant c.$$

Then, the boundedness of solutions and the uniform persistence of system (2), imply the existence of a compact absorbing set $K \subset \Gamma$ (see [15]).

Now, we briefly outline the Li–Muldowney geometric approach. Consider the autonomous dynamical system:

$$\dot{x}(t) = f(x), \tag{3}$$

where $f : D \to \mathbb{R}^n, D \in \mathbb{R}^n$ open set and $f \in C^1(D)$. Assume the following hypotheses hold:

(H_1) System (3) has a unique equilibrium $\bar{x}$ in D;
(H_2) There exists a compact absorbing set $K \subset D$.

Let $P(x)$ be a $\binom{n}{2} \times \binom{n}{2}$ matrix-valued function that is C^1 on D. Assume that $P^{-1}(x)$ exists and is continuous for $x \in K$. Set

$$B = P_f P^{-1} + PJ^{[2]}P^{-1},$$

where the matrix P_f is

$$(p_{ij}(x))_f = \left(\frac{\partial p_{ij}}{\partial x}\right)^{\mathrm{T}} \cdot f(x) = \nabla p_{ij} \cdot f(x),$$

and $J^{[2]}$ is the second additive compound matrix of the Jacobian matrix J, i.e., $J = \frac{\partial f}{\partial x}$.

Define a quantity $\bar{q}_2$ as

$$\bar{q}_2 = \limsup_{t\to\infty} \sup_{x_0 \in K} \frac{1}{t}\int_0^t \mu(B(x(s,x_0)))ds,$$

where $\mu(B)$ is the Lozinskiǐ measure with respect to a vector norm $|\cdot|$ in $\mathbb{R}^m$, $m = \binom{n}{2}$ and

$$\mu(B) = \lim_{h\to 0^+} \frac{|I + hB| - 1}{h}.$$

The following theorem is proved in [2].

Theorem 2.1. *Assume that D is simply connected and that the assumptions* (H_1) *and* (H_2) *hold, then the unique equilibrium* $\bar{x}$ *of system* (3) *is globally asymptotically stable in D if* $\bar{q}_2 < 0$.

Next, we discuss the global stability of the worm-epidemic equilibrium EQ_{we} for system (2).

Theorem 2.2. *When* $\mathcal{R}_0 > 1$ *and one of the following condition is satisfied:*

(i) $\psi_1 > \alpha + \psi_2$ *and* $\phi + \gamma + \theta > \alpha$;
(ii) $\psi_1 \leqslant \alpha + \psi_2$ *and* $\psi_1 + \phi + \gamma + \theta > 2\alpha + \psi_2$,

the worm-epidemic equilibrium EQ_{we} *of system* (2) *is globally asymptotically stable.*

Proof. The Jacobian matrix of system (2) is

$$J = \begin{pmatrix} -fE - (\psi_1 + \phi) & -fV - \phi & -\phi \\ fE & fV - (\alpha + \psi_2) & 0 \\ 0 & \alpha & -(\gamma + \theta) \end{pmatrix},$$

and its second additive compound matrix is

$$J^{[2]} = \begin{pmatrix} fV - fE - n_1 & 0 & \phi \\ \alpha & -fE - n_2 & -fV - \phi \\ 0 & fE & fV - n_3 \end{pmatrix},$$

where

$$n_1 = \alpha + \psi_1 + \psi_2 + \phi, \quad n_2 = \psi_1 + \phi + \gamma + \theta \quad \text{and} \quad n_3 = \alpha + \gamma + \psi_2 + \theta.$$

Choose the function $P(x) = P(V,E,I) = \mathrm{diag}(\frac{V}{E}, \frac{V}{E}, \frac{V}{E})$, then

$$P_f P^{-1} = \mathrm{diag}\left(\frac{\dot{V}}{V} - \frac{\dot{E}}{E}, \frac{\dot{V}}{V} - \frac{\dot{E}}{E}, \frac{\dot{V}}{V} - \frac{\dot{E}}{E}\right)$$

and

$$PJ^{[2]}P^{-1} = \begin{pmatrix} fV - fE - n_1 & 0 & \phi \\ \alpha & -fE - n_2 & -fV - \phi \\ 0 & fE & fV - n_3 \end{pmatrix}.$$

The matrix $B = P_f P^{-1} + PJ^{[2]}P^{-1}$ can be written as

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where

$$B_{11} = \frac{\dot{V}}{V} - \frac{\dot{E}}{E} + fV - fE - n_1, \quad B_{12} = (0, \phi), \quad B_{21} = (\alpha, 0)^{\mathrm{T}},$$

$$B_{22} = \begin{pmatrix} \frac{\dot{V}}{V} - \frac{\dot{E}}{E} - fE - n_2 & -fV - \phi \\ fE & \frac{\dot{V}}{V} - \frac{\dot{E}}{E} + fV - n_3 \end{pmatrix}.$$

Consider the following norm in $\mathbb{R}^3$:

$$|(u, v, w)| = \max\{|u|, |v| + |w|\},$$

where (u, v, w) is the vector in $\mathbb{R}^3$. Let $\mu(B)$ be the Lozinskiĭ measure with respect to this norm. Then, from [16], we have

$$\mu(B) \leqslant \sup\{g_1, g_2\},$$

where

$$g_i = \mu_1(B_{ii}) + |B_{ij}| \quad \text{for} \quad i = 1, 2 \quad \text{and} \quad i \neq j,$$

where $|B_{12}|, |B_{21}|$ are matrix norms with respect to the l_1 vector norm, and μ_1 denotes the Lozinskiĭ measure with respect to l_1 norm. Moreover, we obtain

$$\mu_1(B_{11}) = \frac{\dot{V}}{V} - \frac{\dot{E}}{E} + fV - fE - n_1, \quad |B_{12}| = \phi, \quad |B_{21}| = \alpha,$$

and

$$\mu_1(B_{22}) = \max\left\{\frac{\dot{V}}{V} - \frac{\dot{E}}{E} - n_2, \quad \frac{\dot{V}}{V} - \frac{\dot{E}}{E} - \phi - n_3\right\} = \frac{\dot{V}}{V} - \frac{\dot{E}}{E} - \min\{\psi_1 + \phi + \gamma + \theta, \alpha + \gamma + \phi + \psi_2 + \theta\}.$$

From system (2), we have

$$\frac{\dot{E}}{E} = fV - (\alpha + \psi_2).$$

Thus, we get

$$g_1 = \frac{\dot{V}}{V} - \frac{\dot{E}}{E} + fV - fE - n_1 + \phi = \frac{\dot{V}}{V} - fE - \psi_1 \leqslant \frac{\dot{V}}{V} - \psi_1,$$

$$\begin{aligned} g_2 &= \frac{\dot{V}}{V} - \frac{\dot{E}}{E} - \min\{\psi_1 + \phi + \gamma + \theta, \alpha + \gamma + \phi + \psi_2 + \theta\} + \alpha \\ &= \frac{\dot{V}}{V} - fV - \min\{\psi_1 + \phi + \gamma + \theta, \alpha + \gamma + \phi + \psi_2 + \theta\} + 2\alpha + \psi_2 \\ &\leqslant \frac{\dot{V}}{V} - (\min\{\psi_1 + \phi + \gamma + \theta, \alpha + \gamma + \phi + \psi_2 + \theta\} - 2\alpha - \psi_2). \end{aligned}$$

Case 1: If $\psi_1 > \alpha + \psi_2$ and $\phi + \gamma + \theta > \alpha$, then we have

$$\min\{\psi_1 + \phi + \gamma + \theta, \alpha + \gamma + \phi + \psi_2 + \theta\} - 2\alpha - \psi_2 = \phi + \gamma + \theta - \alpha > 0.$$

Case 2: If $\psi_1 \leqslant \alpha + \psi_2$ and $\psi_1 + \phi + \gamma + \theta > 2\alpha + \psi_2$, then we get

$$\min\{\psi_1 + \phi + \gamma + \theta, \alpha + \gamma + \phi + \psi_2 + \theta\} - 2\alpha - \psi_2 = \psi_1 + \phi + \gamma + \theta - 2\alpha - \psi_2 > 0.$$

Combining these two cases, we let

$$\eta = \min\{\psi_1, \quad \psi_1 + \phi + \gamma + \theta - 2\alpha - \psi_2, \quad \gamma + \phi + \theta - \alpha\}.$$

Therefore,

$$\mu(B) \leqslant \frac{\dot{V}}{V} - \eta.$$

We have

$$\frac{1}{t}\int_0^t \mu(B)ds \leqslant \frac{1}{t}\int_0^t \left(\frac{\dot{V}}{V}-\eta\right)ds = \frac{1}{t}\ln\frac{V(t)}{V(0)}-\eta,$$

which implies that $\bar{q}_2 \leqslant -\frac{\eta}{2} < 0$. Then from Theorem 2.1, the worm-epidemic equilibrium EQ_{we} is globally asymptotically stable. This completes the proof. □

3. Conclusions

The purpose of this paper is to investigate the global stability of the worm-epidemic equilibrium for a VEISV network worm attack model which proposed by Toutonji et al. in [1]. By using the Li–Muldowney geometric approach, we establish that the worm-epidemic equilibrium EQ_{we} is globally asymptotically stable under some conditions.

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