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Short communication

Global residue harmonic balance method for large-amplitude oscillations of a nonlinear system

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ABSTRACT

A new approach, namely global residue harmonic balance method, was used to deal with large-amplitude oscillations of a nonlinear system with inertia and static nonlinearities. Unlike other harmonic balance methods, all the former residual errors are introduced in the present approximation to improve the accuracy. Comparison of the result obtained using this approach with the exact one and existing results reveals that the high accuracy, simplicity and efficiency of the proposed procedure. The methods are valid for small as well as large amplitudes of oscillation, and can be easily extended to other strongly nonlinear oscillators.

1. Introduction

The problems related to large-amplitude oscillations of non-linear engineering structures have received considerable attention in the past years, where all the references cited in this section are only a small sample of the available literature on this topic [1–13]. There are several methods for approximating solutions for nonlinear problems with large parameters, such as harmonic based methods [5,6,13], coupled homotopy–variational formulation [7], variational approach [7,8], amplitude–frequency formulation [7,9], optimal homotopy asymptotic method [10,11] have been used to find approximate solutions to nonlinear problems.

In this paper, we put forward a novel approximate method, namely the global residue harmonic balance method, to determine the periodic solutions of free vibrations of cantilever beam. This oscillator is a conservative non-linear large-amplitude oscillatory system having inertia and static non-linearities. To obtain higher-order analytical approximations, all the residual errors are considered in the process of every order approximation. Excellent agreement of the approximate frequencies with the exact ones has been demonstrated and discussed. As can be seen, the results obtained in this paper revel that the method is very effective and convenient for conservative nonlinear oscillators.

2. Basic idea of global residue harmonic balance approach

For simplicity, we consider systems governed by equations of the form

$$\Phi(\ddot{u}, \dot{u}, u) = 0, \quad u(0) = A, \quad \dot{u}(0) = 0, \tag{1}$$

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where the over-dot denotes differentiation with respect to time t and A is the amplitude of the oscillations. For convenience, we assume Eq. (1) is a conservative system (i.e. $\Phi(-\ddot{u}, -\dot{u}, -u) = -\Phi(\ddot{u}, \dot{u}, u)$).

Eq. (1) describes a system oscillating with an unknown angular frequency ω. To determine the unknown frequency, we introduce a new independent variable $\tau = \omega t$. Then Eq. (1) becomes

$$\Phi(\omega^2 u'', \omega u', u) = 0, \quad u(0) = A, \ u'(0) = 0, \tag{2}$$

where prime denotes the derivative with respect to τ.

Considering the periodic solution does exist, it may be better to approximate the solution $u(\tau)$ by such a set of base functions

$$\{cos((2k-1)\tau)|\ k = 1, 2, 3, \ldots\}. \tag{3}$$

According to Eq. (3), the initial approximate periodic solution satisfying initial conditions in Eq. (2) is

$$u_{(0)}(\tau) = Acos(\tau), \quad \tau = \omega_{(0)} t, \tag{4}$$

where $\omega_{(0)}$ is an unknown constant to be determined later.

To improve the accuracy, we will use the residual of the initial approximation. Substituting Eq. (4) into Eq. (2), we obtain the initial residual

$$R_0(\tau) = \Phi\left(\omega_{(0)}^2 u''_{(0)}, \omega_{(0)} u'_{(0)}, u_{(0)}\right). \tag{5}$$

If $R_0(\tau) = 0$, then $u_0(\tau)$ happens to be the exact solution. Generally, such case will not arise for nonlinear problems.

Eq. (5) should not contain secular terms of $cos(\tau)$. Equating its coefficients to zero, we can determine the unknown constant $\omega_{(0)}$ and taking it as the approximation ω_0. Then, the zero-order approximation u_0 is also obtained of the form

$$u_0(\tau) = Acos(\tau), \quad \tau = \omega_0 t, \tag{6}$$

This yields the initial residual

$$R_0(\tau) = \Phi(\omega_0^2 u''_0, \omega_0 u'_0, u_0). \tag{7}$$

In the following, we consider an iterative method by expanding $u(\tau)$ in a series with respect to the embedding parameter p of the form

$$u(\tau) = u_{(k-1)}(\tau) + p u_k(\tau), \quad \omega^2 = \omega_{(k-1)}^2 + p\omega_k, \quad k = 1, 2, 3, \ldots, \tag{8}$$

where

$$\begin{aligned} & u_{(k-1)}(\tau) = u_{(k-2)}(\tau) + u_{k-1}(\tau), \quad \omega_{(k-1)}^2 = \omega_{(k-2)}^2 + \omega_{k-1}, \\ & u_k(\tau) = \sum_{i=1}^{k} a_{2i+1,k}(cos(\tau) - cos((2i+1)\tau)), \quad k = 2, 3, \ldots, \end{aligned} \tag{9}$$

where p is the order parameter with values in the interval $[0, 1]$, and the kth-order approximate solutions of $u(\tau)$ and ω can be obtained by taking $p = 1$.

Given the zero-order approximation equation (6) and the residual equation (7), then the first-order approximate periodic solution and frequency can be written as

$$u(\tau) = u_0(\tau) + p u_1(\tau), \quad \omega^2 = \omega_0^2 + p\omega_1. \tag{10}$$

Substituting Eq. (6) into Eq. (2) and collecting the coefficients of p, we can get

$$F_1(\tau, \omega_1, u_1(\tau)) \triangleq \left(\omega_1 \frac{\partial}{\partial(\omega^2)} + u''_1 \frac{\partial}{\partial u''} + u'_1 \frac{\partial}{\partial u'} + u_1 \frac{\partial}{\partial u}\right)\Phi_0, \tag{11}$$

where $\partial\Phi_0/\partial u$ denotes that $\partial\Phi/\partial u$ is to be evaluated at the zero-order approximation after differentiation etc. It is noted that Eq. (11) is linear with respect to ω_1 and u_1.

Considering the solution has the form of Eq. (3), we choose

$$u_1(\tau) = a_{3,1}(cos(\tau) - cos(3\tau)). \tag{12}$$

Substituting Eq. (12) into Eq. (11), we consider the following equation

$$F_1(\tau, \omega_1, u_1(\tau)) + R_0(\tau) = 0. \tag{13}$$

In this way, all the residual errors of the zero-order approximation $R_0(\tau)$ are introduced into Eq. (13) which would improve the accuracy.

The left hand side of Eq. (13) should not contain the terms $cos(\tau)$ and $cos(3\tau)$ based on Galerkin technique. Letting their coefficients be zeros, we obtain two linear equations containing two unknowns ω_1 and $a_{3,1}$. Then the two unknown constants can be solved easily. Thus, we get the first-order approximation

$$u_{(1)}(\tau) = u_0(\tau) + u_1(\tau), \quad \omega_{(1)}^2 = \omega_0^2 + \omega_1, \quad \tau = \omega_{(1)} t, \tag{14}$$

where $u_0(\tau)$ and $u_1(\tau)$ are given by Eqs. (6) and (12) respectively.

For high order approximation can be obtained by the iterate method in Eqs. (8) and (9).

To determine the unknown parameters $a_{2i+1,k}$ $(i = 2, \ldots, k)$ and ω_k, substituting Eq. (8) into Eq. (2) and collecting the coefficients of the p, one yields

$$F_k(\tau, \omega_k, u_k(\tau)) \triangleq \left(\omega_k \frac{\partial}{\partial(\omega^2)} + u_k'' \frac{\partial}{\partial u''} + u_k' \frac{\partial}{\partial u'} + u_k \frac{\partial}{\partial u}\right) \Phi_{k-1}. \tag{15}$$

Eq. (15) is linear with respect to ω_k and u_k.

Substituting the $u_{(k-1)}(\tau)$ and $\omega_{(k-1)}$ of Eq. (9) into Eq. (2), one yields the following residual

$$R_{k-1}(\tau) = \Phi\left(\omega_{(k-1)}^2 u_{(k-1)}'', \omega_{(k-1)} u_{(k-1)}', u_{(k-1)}\right). \tag{16}$$

Considering the following equation

$$F_k(\tau, \omega_k, u_k(\tau)) + R_{k-1}(\tau) = 0. \tag{17}$$

Avoiding the presence of the secular terms $cos(\tau), cos(3\tau), \ldots,$ and $cos((2k+1)\tau)$, there are the same number of linear equations for the same number of unknowns $a_{3,k}, a_{5,k}, \ldots, a_{2k+1,k}$ and ω_k.

Then, the kth-order approximate periodic solution and frequency can be obtained in the form

$$u_{(k)}(\tau) = u_{(k-1)}(\tau) + u_k(\tau), \quad \omega_{(k)}^2 = \omega_{(k-1)}^2 + \omega_k, \tag{18}$$

where $u_{(k-1)}(\tau)$ and $u_k(\tau)$ are given by Eq. (9).

3. Solution method

We consider free vibrations of a slender inextensible cantilever beam carrying an intermediate lumped mass with a rotary inertia, described by the nonlinear equation

$$\ddot{u} + u + \alpha u^2 \ddot{u} + \alpha u \dot{u}^2 + \beta u^3 = 0, \quad u(0) = A, \quad \dot{u}(0) = 0. \tag{19}$$

The third and fourth terms in Eq. (19) represent inertia-type cubic nonlinearity arising from the inextensibility assumption. The last term is a static-type nonlinearity associated with the potential energy stored in bending.

With a new independent variable $\tau = \omega t$, Eq. (19) becomes

$$\omega^2\left[(1 + \alpha u^2)u'' + \alpha u u'^2\right] + u + \beta u^3 = 0 \tag{20}$$

and

$$u(0) = A, \quad u'(0) = 0, \tag{21}$$

where the prime denotes differentiation with respect to τ.

3.1. Zeroth-order approximation

Following Eq. (6), the initial approximation with initial conditions Eq. (21) is

$$u_0(\tau) = Acos(\tau). \tag{22}$$

Substituting Eq. (22) into Eq. (20), yields

$$\left(-\omega + 1 - \frac{1}{2}\alpha\omega A^2 + \frac{3}{4}\beta A^2\right) Acos(\tau) + \left(\frac{1}{4}\beta - \frac{1}{2}\alpha\omega\right) A^3 cos(3\tau) = 0. \tag{23}$$

Equating the coefficient of $cos(\tau)$ to zero, we obtain the zero-order analytical approximate frequency ω_0:

$$\omega_0 = \sqrt{\frac{1}{2} \cdot \frac{4 + 3\beta A^2}{2 + \alpha A^2}} \tag{24}$$

agreeing with Refs. [6,7].

At this time, we have the zeroth-order analytical approximate periodic solution in the form Eq. (22), where $\tau = \omega_0 t$.

Taking $u_0(\tau)$ and ω_0 into the left-hand side of Eq. (19), we have the residual error $R_0(\tau)$.

$$R_0(\tau) = \left(\frac{-2\alpha + \beta - \alpha A^2 \beta}{4 + 2\alpha A^2}\right) A^3 cos(3\tau). \tag{25}$$

3.2. First-order approximation

To obtain the first-order analytical approximation, substituting Eqs. (10) and (12) into Eq. (20), one gets all the coefficients of p

$$F_1(\tau,\omega_1,u_1) \triangleq \frac{F_{11}cos(\tau)+F_{13}cos(3\tau)+F_{15}cos(5\tau)}{4+2\alpha A^2}, \tag{26}$$

where

$$\begin{aligned} F_{11} &= 2\alpha A^2 a_{3,1} - \alpha^2 A^5\omega_1 + 3\alpha A^4 a_{3,1}\beta - 4\omega_1 A^3\alpha - 4\omega_1 A + 3A^2 a_{3,1}\beta, \\ F_{13} &= -2\omega_1 A^3\alpha - \alpha^2 A^5\omega_1 + 12\alpha A^2 a_{3,1} + 24A^2 a_{3,1}\beta + 9\alpha A^4 a_{3,1}\beta + 32a_{3,1}, \\ F_{15} &= 18\alpha A^2 a_{3,1} + 12\alpha A^4 a_{3,1}\beta - 3A^2 a_{3,1}\beta. \end{aligned} \tag{27}$$

According to Eq. (13), yields

$$F_1(\tau,\omega_1,u_1) + R_0(\tau) = 0. \tag{28}$$

Equating the coefficients of $cos(\tau),\ cos(3\tau)$ to zero in Eq. (28), one can obtain

$$\begin{aligned} &\frac{-4\omega_1 A - \alpha^2 A^5\omega_1 + 3a_{3,1}\beta A^2 - 4\omega_1 A^3\alpha + 2\alpha A^2 a_{3,1} + 3\alpha A^4 a_{3,1}\beta}{4+2\alpha A^2} = 0, \\ &\frac{1}{4+2\alpha A^2}\left(-2\alpha A^3 - 2\omega_1 A^3\alpha - \alpha^2 A^5\omega_1 + 24a_{3,1}\beta A^2 + 12\omega_1 A^2 a_{3,1} + 9\alpha A^4 a_{3,1}\beta + \beta A^3 - A^5\alpha\beta + 32a_{3,1}\right) = 0. \end{aligned} \tag{29}$$

From Eq. (29), we obtain the constants $a_{3,1}$ and ω_1 as below

$$\begin{aligned} a_{3,1} &= A^3\frac{4\alpha + \alpha A^2\beta + \alpha^2 A^4\beta + 2\alpha^2 A^2 - 2\beta}{\Gamma_1}, \\ \omega_1 &= A^4\frac{(2\alpha + \alpha A^2\beta - \beta)(3\alpha A^2\beta + 3\beta + 2\alpha)}{(2+\alpha A^2)\Gamma_1}, \end{aligned} \tag{30}$$

where $\Gamma_1 = 48\beta A^2 + 56\alpha A^2 + 39\alpha A^4\beta + 64 + 10A^4\alpha^2 + 6\alpha^2 A^6\beta$.

Then, we obtain the first-order approximate frequency and periodic solution from Eq. 14

$$\begin{aligned} u_{(1)}(\tau) &= Acos(\tau) + \frac{A^3(4\alpha + \alpha A^2\beta + \alpha^2 A^4\beta + 2\alpha^2 A^2 - 2\beta)}{\Gamma_1}(cos(\tau) - cos(3\tau)), \\ \omega_{(1)} &= \sqrt{\frac{1}{2}\cdot\frac{4+3\beta A^2}{2+\alpha A^2} + \frac{A^4(2\alpha + \alpha A^2\beta - \beta)(3\alpha A^2\beta + 3\beta + 2\alpha)}{(2+\alpha A^2)\Gamma_1}}, \end{aligned} \tag{31}$$

where $\tau = \omega_{(1)}t,\ \Gamma_1 = 48\beta A^2 + 56\alpha A^2 + 39\alpha A^4\beta + 64 + 10A^4\alpha^2 + 6\alpha^2 A^6\beta$.

The further higher-order approximation can be obtained by the same technique.

4. Comparison and discussion

In order to illustrate the applicability, accuracy and effectiveness of the proposed approach, we compare the analytical approximate frequency and periodic solution with the exact ones.

The exact frequency is given by [6]

$$\omega_e = \frac{\pi}{2\int_0^{\pi/2}\left[2(1+\alpha A^2cos(\theta)^2)/(2+\beta A^2(1+cos(\theta)^2))\right]^{1}/2d\theta}. \tag{32}$$

Wu et al. [6] using the harmonic balance method with linearization obtained

$$\omega_{W1} = \sqrt{\frac{40 + (18\alpha + 31\beta)A^2 + 15\alpha\beta A^4 + \sqrt{\Delta}}{72 + 68\alpha A^2 + 14\alpha^2 A^4}}. \tag{33}$$

$$u_{W1} = [A + x_1(A)]cos(\tau) - x_1(A)cos(3\tau), \quad \tau = \omega_{W1}t, \tag{34}$$

where $\Delta = 1024 + (896\alpha + 1472\beta)A^2 + (212\alpha^2\beta + 1364\alpha\beta + 421\beta^2)A^4 + (344\alpha^2\beta + 420\alpha\beta^2)A^6 + 120\alpha^2\beta^2 A^8$.

$$x_1(A) = -\frac{(4A + 3\beta A^3) - (4A + 2\alpha A^3)\omega_{W1}^2}{-4\omega_{W1}^2 + 4 + 6\beta A^2}. \tag{35}$$

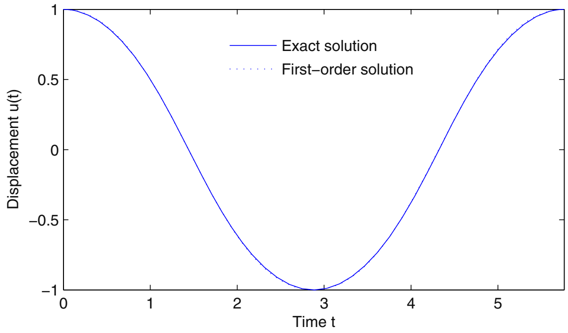

Fig. 1. Comparison between analytical approximate solutions and exact solution for $A = 1$.

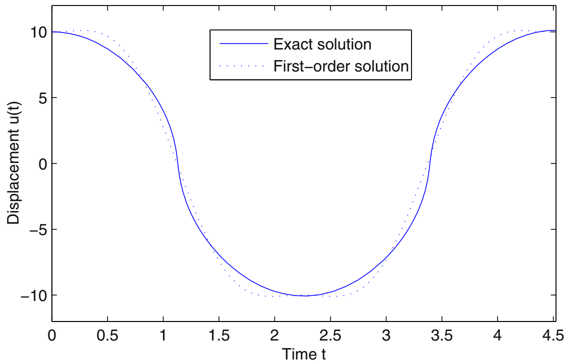

Fig. 2. Comparison between analytical approximate solutions and exact solution for $A = 10$.

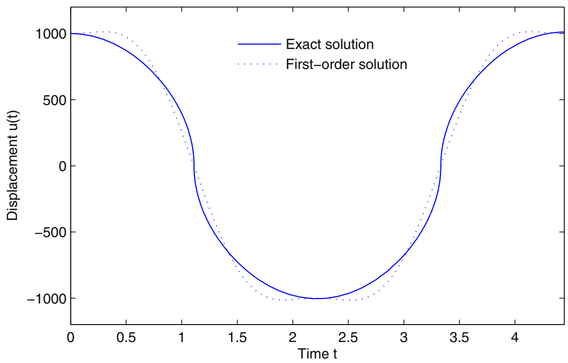

Fig. 3. Comparison between analytical approximate solutions and exact solution for $A = 1000$.

For $\alpha = 1$ and $\beta = 1$, the periodic solution achieved by numerical integration of Eq. (19) using a fourth-order Runge–Kutta scheme and the approximate periodic solution given by Eq. (31) are plotted in Figs. 1–3. These figures represent, respectively, three different amplitudes $A = 1, 10$ and 1000. They show that the approximate periodic solution provides relatively good approximation comparing to the exact periodic solution for small as well as for large amplitude of oscillation.

Furthermore, for any $\alpha > 0$ and $\beta > 0$,

$$\lim_{A\to\infty} \omega_e = \frac{\pi\sqrt{2(\beta/\alpha)}}{4\int_0^{\pi/2}(cost/\sqrt{1+cos^2t})dt}. \tag{36}$$

Wu et al. [6] obtained

$$\lim_{A\to\infty}\frac{\omega_{W1}}{\omega_e}=\frac{2\sqrt{105+14\sqrt{30}}}{7\pi}\int_0^{\pi/2}\frac{cost}{\sqrt{1+cos^2t}}dt\approx 0.96278. \tag{37}$$

Akbarzade and Khan [7] using the coupled homotopy–variational formulation obtained

$$\lim_{A\to\infty}\frac{\omega_{CHV2}}{\omega_e}\approx 0.9993. \tag{38}$$

Furthermore, we have the following equation

$$\lim_{A\to\infty}\frac{\omega_{(1)}}{\omega_e}=\frac{4}{\pi}\int_0^{\pi/2}\frac{cost}{\sqrt{1+cost^2}}dt=1. \tag{39}$$

It is important to point out that the exact behavior of the approximate frequency when A tends to infinity is not obtained when other approximate methods as used including the energy balance method [5], the method of harmonic balance with linearization [6], the coupled homotopy–variational formulation [7], the He's amplitude–frequency formulation [8] and He's variational approach method [9].

5. Conclusions

In this paper, a novel analytical technique, namely the global residue harmonic balance method, has been presented to determine accurate analytical approximate periodic solutions of a conservative system having inertia and static non-linearities. Having taken all the residual errors into the process of solving the approximation, the new method has great difference with the residue harmonic balance approach. Excellent agreement between approximate periods and the exact one has been demonstrated and discussed. Finally, we can see that the method considered here is very simple in its principle and we think that the method has great potential and can be applied to other strongly nonlinear oscillators.

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