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A compact difference scheme for a partial integro-differential equation with a weakly singular kernel ☆

Man Luo [a,*], Da Xu [a,*], Limei Li [b,1]

[a] Department of Mathematics, Hunan Normal University, 410081 Changsha, Hunan, China
[b] Department of Mathematics, Hunan Institute of Science and Technology, 414000 Yueyang, Hunan, China

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ABSTRACT

A compact difference scheme is presented for a partial integro-differential equation. The integral term is treated by means of the product trapezoidal method. The stability and L_2 convergence are proved by the energy method. The convergence order is $O(k^{3/2}+h^4)$. Two numerical examples are given to support the theoretical results.

1. Introduction

This paper is devoted to the study of a compact difference method for the partial integro-differential equation

$$u_t = \mu u_{xx} + \int_0^t (t-s)^{-1/2} u_{xx} ds, \quad 0 < x < 1, \ t \geqslant 0, \tag{1}$$

with $\mu \geqslant 0$ and the boundary conditions

$$u(0,t) = u(1,t) = 0, \quad t \geqslant 0, \tag{2}$$

and the initial condition

$$u(x,0) = u_0(x), \quad 0 \leqslant x \leqslant 1. \tag{3}$$

Equation similar to (1) can be found in the modeling of physical phenomena involving heat flow in materials with memory [1,2], phenomena associated with linear viscoelastic mechanics [3,4]. The integral term in (1) represents the viscosity part of the equation and $\mu \geqslant 0$ in (1) is a Newtonian contribution to the viscosity.

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* Corresponding authors. Tel.: +86 13755112450 (M. Luo), +86 13787121849 (D. Xu).
E-mail addresses: lmlwlx@163.com (M. Luo), daxu@hunnu.ed.cn (D. Xu), limeiliyy@163.com (L. Li).
[1] Tel.: +86 13973057562.

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In the twentieth century decade, many considerable works on theoretical analysis [5–8,13–15] have been carried on. Yan and Fairweather [5] presented orthogonal spline collocation method for some partial integro-differential equations with smooth integral kernels. Xu [14,15] considered backward Euler method in time direction for a parabolic integro-differential equation and derived the stability and convergence properties of the time discretizations. Lopez-Marcos [10] studied the nonlinear partial integro-differential equation which is similar to problem (1)–(3), he used one order full discrete difference scheme and used a convolution quadrature to treat the integral term. A compact difference scheme is presented by Chen and Xu [8] for an evolution equation with a weakly singular kernel with the truncation error of order 3/2 in time and order 4 in space, the convergence and stability were obtained. The Crank–Nicolson scheme in time direction for solving problem (1)–(3) are provided by Tang [6], and the $O(k^{3/2}+h^2)$ order conditional convergence is proved. It is well known that the Crank–Nicolson scheme has $O(k^2)$ order accuracy, but due to the lack of smoothness of the integral kernel, the overall numerical procedure in [6] does not achieve second-order convergence. In this article, we give a compact difference scheme for problem (1)–(3) and proved that the compact difference scheme is stable and convergent in L_2 norm. The convergence order is $O(k^{3/2}+h^4)$.

Throughout the paper, we assume that u_0 in (3) is such that the problem (1)–(3) has a unique solution in $[0,1]\times[0,T]$. Furthermore, we suppose that u_{tt} and u_{ttxx} are continuous for $0\leqslant x\leqslant 1$ and $0\leqslant t\leqslant T$, and we assume that there exists a positive constant C_0 such that

$$|u_{tt}(x,t)|\leqslant C_0t^{-1/2},\quad |u_{ttt}(x,t)|\leqslant C_0t^{-3/2},\quad |u_{xxtt}(x,t)|\leqslant C_0t^{-1/2}. \tag{4}$$

The outline of the paper is organized as follows: a compact finite difference scheme is introduced in Section 2. The analysis of stability and convergence of the scheme is given in Section 3. The numerical results are presented in Section 4. This paper ends with a conclusion.

2. The derivation of the compact difference scheme

We introduce a grid $w_h=\{x|x_j=jh, j=0,1,\ldots,J\}$, $t_n=nk, n=0,1,\ldots,N$ with $h=1/J, k=1/N$ and J,N are positive integers. Moreover, we let $t_{n+1/2}=(n+1/2)k, u_j^n=u(x_j,t_n)$, $0\leqslant j\leqslant J, 0\leqslant n\leqslant N$.

We first introduce the following product trapezoidal method to approximate $I(f,t)=\int_0^t(t-s)^{-1/2}f(s)ds$ which is introduced by Tang [6]:

$$I(f,t_n)=A_nf(t_0)+\sum_{p=0}^{n}\beta_pf(t_{n-p})+O(k^{3/2}),\quad 1\leqslant n\leqslant N, \tag{5}$$

where

$$A_n=2\left[t_n^{1/2}-\frac{1}{k}\int_{t_n}^{t_{n+1}}\theta^{1/2}d\theta\right],\quad \beta_0=\frac{2}{k}\int_0^{t_1}\theta^{1/2}d\theta+\frac{4\sqrt{k}}{3}\beta,\quad \beta_1=\frac{2}{k}\left[\int_{t_1}^{t_2}\theta^{1/2}d\theta-\int_{t_0}^{t_1}\theta^{1/2}d\theta\right]-\frac{4\sqrt{k}}{3}\beta,$$
$$\beta_p=\frac{2}{k}\left[\int_{t_p}^{t_{p+1}}\theta^{1/2}d\theta-\int_{t_{p-1}}^{t_p}\theta^{1/2}d\theta\right],\quad p\geqslant 2. \tag{6}$$

where β is a nonnegative constant and is dependent of k and h. i.e., $\beta\geqslant 0$ and $\beta=O(1)$.

The following lemma will be used in the derivation of the compact difference scheme.

Lemma 2.1 ([9,11,12]). *Suppose $g(x)\in C^6[x_{i-1},x_{i+1}]$. Then*

$$\frac{1}{12}[g''(x_{i-1})+10g''(x_i)+g''(x_{i+1})]-\frac{1}{h^2}[g(x_{i-1})-2g(x_i)+g(x_{i+1})]=\frac{h^4}{240}g^{(6)}(w_i),\quad w_i\in(x_{i-1},x_{i+1}). \tag{7}$$

Lemma 2.2 [6]. *Let $I(f,t)=\int_0^t(t-s)^{-1/2}f(s)ds$, then*

$$I(f,t_{n+1/2})=\frac{1}{2}[I(f,t_n)+I(f,t_{n+1})]+O(k^2t_n^{-3/2}),\quad n\geqslant 1. \tag{8}$$

Let

$$V(x,t)=\frac{\partial u(x,t)}{\partial t}, \tag{9}$$

then (1) becomes

$$V = \mu u_{xx} + I(u_{xx}, t). \tag{10}$$

Considering (10) at point (x_j, t), we have

$$V(x_j, t) = \mu \frac{\partial^2 u}{\partial x^2}(x_j, t) + I\left(\frac{\partial^2 u}{\partial x^2}(x_j, t), t\right). \tag{11}$$

For $g = (g_0, g_1, \ldots, g_J)$, introduce the operator $\mathscr{A}$ as follows:

$$\mathscr{A} g_j = \begin{cases} g_0, & j = 0, \\ \frac{1}{12}(g_{j-1} + 10g_j + g_{j+1}), & 1 \leqslant j \leqslant J-1, \\ g_J & j = J. \end{cases} \tag{12}$$

Acting $\mathscr{A}$ on the both sides of (11), we obtain

$$\mathscr{A} V(x_j, t) = \mathscr{A} \mu \frac{\partial^2 u}{\partial x^2}(x_j, t) + \mathscr{A} I\left(\frac{\partial^2 u}{\partial x^2}(x_j, t), t\right). \tag{13}$$

According to Lemma 2.1, we have

$$\mathscr{A} \frac{\partial^2 u(x_j, t)}{\partial x^2} = \frac{1}{h^2}\left[u(x_{j-1}, t) - 2u(x_j, t) + u(x_{j+1}, t)\right] + O(h^4). \tag{14}$$

Using Taylor expansion, it follows from 9 that

$$\frac{\partial u}{\partial t}(x_j, t_{n+1/2}) = \delta_t u_j^n + (\overline{R}_1)_j^n, (\overline{R}_1)_j^n \leqslant O(k^2), \tag{15}$$

then substituting (8), (14), and (15) into (13), we obtain

$$\mathscr{A} \delta_t u_j^n = \mu \delta_x^2 \overline{u}_j^n + \frac{1}{2}\left(A_n \delta_x^2 u_j^0 + \sum_{p=0}^{n} \beta_p \delta_x^2 u_j^{n-p} + A_{n+1} \delta_x^2 u_j^0 + \sum_{p=0}^{n+1} \beta_p \delta_x^2 u_j^{n+1-p}\right) + (\overline{R}_0)_j^n, \tag{16}$$

and there exists a constant C_1 such that

$$|\overline{R}_0|_j^n \leqslant C_1(k^2 + h^4 + k^{3/2} + k^2 t_n^{-3/2}). \tag{17}$$

Noticing the initial and boundary condition (2) and (3), we have

$$u_0^n = u_J^n = 0, \quad 0 \leqslant n \leqslant N. \tag{18}$$

$$u_j^0 = u_0(x_j), \quad 1 \leqslant j \leqslant J-1. \tag{19}$$

Omitting the small term $(\overline{R}_0)_j^n$ in (16) and replacing u_j^n with U_j^n, and constructing the compact difference scheme as follows:

$$\mathscr{A} \delta_t U_j^n = \mu \delta_x^2 \overline{U}_j^n + \widetilde{A}_n \delta_x^2 U_j^0 + \frac{1}{2}\sum_{p=0}^{n} \beta_p \left(\delta_x^2 U_j^{n-p} + \delta_x^2 U_j^{n+1-p}\right), \quad \widetilde{A}_n = \frac{A_n + A_{n+1} + \beta_{n+1}}{2}. \tag{20}$$

Furthermore

$$U_0^n = U_J^n = 0, \quad n = 0, 1, \ldots, N. \tag{21}$$

$$U_j^0 = u_0(x_j), \quad j = 0, 1, \ldots, J. \tag{22}$$

3. Analysis of the compact difference scheme

In this section we will show that the proposed method is convergent with the convergence order $O(k^{3/2} + h^4)$. However, Tang [6] analysed a Crank–Nicolson and finite difference method for the time and space discretizations. He used the product trapezoidal method to deal the integral term, which only obtained $O(k^{3/2} + h^2)$ order convergence.

Let

$$V_h = \{U | U = (U_0, U_1, \ldots, U_{J-1}, U_J),\ U_0 = U_J = 0\}, \tag{23}$$

be the space of grid functions.

For any grid functions $U, W \in V_h$, denote

$$\delta_t U_j^n = \frac{1}{k}(U_j^{n+1} - U_j^n), \quad \delta_x U_j = \frac{1}{h}(U_{j+1} - U_j), \tag{24}$$

$$\delta_x^2 U_j = \frac{1}{h^2}(U_{j+1} - 2U_j + U_{j-1}), \quad \overline{U}_j^n = \frac{U_j^n + U_j^{n+1}}{2}, \tag{25}$$

$$\|U\|_\infty = \max_{1 \leqslant j \leqslant J-1} |U_j|, \quad \langle U, W\rangle = h\sum_{j=1}^{J-1} U_j W_j, \quad \|U\|^2 = \langle U, U\rangle. \tag{26}$$

Before we prove the stability and convergence, we give the following lemmas.

Lemma 3.1 [8, Lemma 3.1].

(1) *Let $U, W \in V_h$, we have*

$$\langle \delta_x^2 U, W\rangle = -h\sum_{j=0}^{J-1}(\delta_x U_j)(\delta_x W_j)$$

(2) *Let $U^m, U^n \in V_h$, we have*

$$|\langle \delta_x^2 U^m, U^n\rangle| \leqslant \frac{4}{h^2}\|U^m\| \cdot \|U^n\|.$$

The following results is concerned with the nonnegative character of certain real quadratic forms with convolution structure is due to Lopez-Marcos [10].

Lemma 3.2 [6, Lemma 1]. *Let $\{a_n\}_{n=0}^\infty$ be a sequence of real numbers with the properties*

$$a_n \geqslant 0, \quad a_{n+1} - a_n \leqslant 0, \quad a_{n+1} - 2a_n + a_{n+1} \geqslant 0. \tag{27}$$

Then for any positive integer M, and real vector $(V_1, V_2, \ldots, V_M)$ with M real entries,

$$\sum_{n=0}^{M-1}\left(\sum_{p=0}^{n} a_p V_{n+1-p}\right) V_{n+1} \geqslant 0. \tag{28}$$

Lemma 3.3 [6, Lemma 4]. *If β satisfies*

$$\frac{-3\sqrt{3} + 8\sqrt{2} - 6}{3} \leqslant \beta \leqslant 4 - 12\sqrt{3} + 12\sqrt{2}, \tag{29}$$

then the sequence $\{\beta_p\}_{p=0}^\infty$ defined by (6) satisfies (27).

Now we begin to prove the stability and convergence.

3.1. Stability

First we will give the proof of stability.

Theorem 3.1. *Assume $\{\beta_p\}_{p=0}^\infty$ in (6) satisfies (27), let $U^n = (U_1^n, U_2^n, \ldots, U_{J-1}^n)$ be the solution of*

$$\mathscr{A}\delta_t U_j^n = \mu\delta_x^2 \overline{U}_j^n + \widetilde{A}_n \delta_x^2 U_j^0 + \sum_{p=0}^{n} \beta_p \delta_x^2 \overline{U}_j^{n-p}, \quad j = 1, \ldots, J-1, \ n = 1, \ldots, N, \tag{30}$$

where

$$U_0^n = U_J^n = 0, \quad n = 0, 1, \ldots, N, \tag{31}$$

$$U^0 = (u_0(x_0), u_0(x_1), \ldots, u_0(x_J)) \quad given. \tag{32}$$

Then for $N \geqslant 1$, we have

$$\|U^N\| \leqslant \frac{3}{2}\|U^0\| + \frac{C\sqrt{Tk}^{3/2}}{h^2}\|U^0\|. \tag{33}$$

Proof. Multiplying (30) by $h\overline{U}_j^n$ and summing up j from 1 to $J-1$, and n from 0 to m, we can obtain

$$\sum_{n=0}^{m}\langle \mathscr{A}\delta_t U^n, \overline{U}^n\rangle = \mu\sum_{n=0}^{m}\langle \delta_x^2\overline{U}^n, \overline{U}^n\rangle + \sum_{n=0}^{m}\sum_{p=0}^{n}\beta_p\langle \delta_x^2\overline{U}^{n-p}, \overline{U}^n\rangle + \sum_{n=0}^{m}\langle \widetilde{A}_n\delta_x^2 U^0, \overline{U}^n\rangle. \tag{34}$$

Now we estimate each term in the above Eq. (34). First, we have

$$k\langle \mathscr{A}\delta_t U^n, \overline{U}^n\rangle = \frac{1}{2}\left[\left(\|U^{n+1}\|^2 - \frac{h^2}{12}\|\delta_x U^{n+1}\|^2\right) - \left(\|U^n\|^2 - \frac{h^2}{12}\|\delta_x U^n\|^2\right)\right]$$

(see [8], Eq. 3.9), where

$$\|\delta_x U^n\|^2 = h\sum_{j=1}^{J-1}\left|\frac{U_{j+1}^n - U_j^n}{h}\right|^2 \leqslant \frac{1}{h}\left(\sum_{j=1}^{J-1}2(|U_{j+1}^n|^2 + |U_j^n|^2)\right) = \frac{4}{h^2}\|U^n\|^2. \tag{35}$$

Because

$$\|U^n\|^2 - \frac{h^2}{12}\|\delta_x U^n\|^2 \geqslant \|U^n\|^2 - \frac{h^2}{12}\cdot\frac{4}{h^2}\|U^n\|^2,$$

so

$$\frac{2}{3}\|U^n\|^2 \leqslant \|U^n\|^2 - \frac{h^2}{12}\|\delta_x U^n\|^2 \leqslant \|U^n\|^2. \tag{36}$$

Consequently,

$$\sum_{n=0}^{m}k\langle \mathscr{A}\delta_t U^n, \overline{U}^n\rangle = \frac{1}{2}\left[\left(\|U^{m+1}\|^2 - \frac{h^2}{12}\|\delta_x U^{m+1}\|^2\right) - \left(\|U^0\|^2 - \frac{h^2}{12}\|\delta_x U^0\|^2\right)\right] \geqslant \frac{1}{3}\|U^{m+1}\|^2 - \frac{1}{2}\|U^0\|^2. \tag{37}$$

In addition, the first term of the right equality (34) will be estimated:

$$k\mu\langle \delta_x^2\overline{U}^n, \overline{U}^n\rangle = k\mu h\sum_{j=1}^{J-1}(\delta_x^2\overline{U}_j^n)(\overline{U}_j^n) = -k\mu h\sum_{j=1}^{J-1}(\delta_x\overline{U}_j^n)(\delta_x\overline{U}_j^n) = -k\mu\|\delta_x\overline{U}^n\|^2 \leqslant 0. \tag{38}$$

In the second equality above, we have used Lemma 3.1(1).

For the second term of the right equality (34), we have

$$\begin{aligned} k\sum_{n=0}^{m}\sum_{p=0}^{n}\beta_p\langle \delta_x^2\overline{U}^{n-p}, \overline{U}^n\rangle &= kh\sum_{n=0}^{m}\sum_{p=0}^{n}\sum_{j=1}^{J-1}\beta_p\left(\delta_x^2\overline{U}_j^{n-p}\right)\left(\overline{U}_j^n\right) = -kh\sum_{n=0}^{m}\sum_{p=0}^{n}\sum_{j=1}^{J-1}\beta_p\left(\delta_x\overline{U}_j^{n-p}\right)\left(\delta_x\overline{U}_j^n\right) \\ &= -k\sum_{j=1}^{J-1}\sum_{n=0}^{m}h\left(\sum_{p=0}^{n}\beta_p\delta_x\left(\frac{U_j^{n+1-p} + U_j^{n-p}}{2}\right)\right)\left(\delta_x\left(\frac{U_j^{n+1} + U_j^n}{2}\right)\right). \end{aligned} \tag{39}$$

for $0 \leqslant m \leqslant N-1$. It follows from Lemma 3.2 and Lemma 3.3 that the above term of (39) is nonpositive.

For the last term of the right equality (34), using Cauchy–Schwarz inequality and Lemma 3.1(2).

$$k\sum_{n=0}^{m}\langle \widetilde{A}_n\delta_x^2 U^0, \overline{U}^n\rangle = k\sum_{n=0}^{m}\sum_{j=1}^{J-1}h\widetilde{A}_n\left(\delta_x^2 U_j^0\right)\left(\overline{U}_j^n\right) \leqslant k\left|\sum_{n=0}^{m}\widetilde{A}_n\right|\langle \delta_x^2 U^0, \overline{U}^n\rangle \leqslant \frac{4k}{h^2}\left|\sum_{n=0}^{m}\widetilde{A}_n\right|\|U^0\|\,\|\overline{U}^n\|. \tag{40}$$

In the second inequality above, we have used Lemma 3.1(2).

According to (6), and using the mean value theorem of integrals and Rolle theorem, we have

$$\begin{aligned} k\sum_{n=0}^{m}A_n &= k\sum_{n=0}^{m}2\left(t_n^{1/2} - \frac{1}{k}\int_{t_n}^{t_{n+1}}\theta^{1/2}d\theta\right) \approx 2k\sum_{n=0}^{m}\left(t_n^{1/2} - \theta_1^{1/2}\right) \leqslant 2k\sum_{n=0}^{m}\frac{1}{2}kt_n^{-1/2} \\ &= k^2\frac{k}{k^{1/2}}\left[\frac{1}{\sqrt{1k}} + \frac{1}{\sqrt{2k}} + \cdots + \frac{1}{\sqrt{mk}}\right] \leqslant k^{3/2}\int_0^{mk}\frac{dx}{\sqrt{x}} \leqslant 2k^{3/2}\sqrt{T}, \end{aligned} \tag{41}$$

where $\theta_1 \in (t_n, t_{n+1})$, and

$$k\sum_{n=0}^{m}\beta_{n+1} = k\sum_{n=0}^{m}\frac{2}{k}\left(\int_{t_{n+1}}^{t_{n+2}}\theta^{1/2}d\theta - \int_{t_n}^{t_{n+1}}\theta^{1/2}d\theta\right) \approx Ck\sum_{n=0}^{m}\frac{2}{k}\left[\theta_2^{1/2}k - \theta_3^{1/2}k\right] \leqslant Ck\sum_{n=0}^{m}\left(t_{n+2}^{1/2} - t_n^{1/2}\right) \leqslant Ck^2\sum_{n=0}^{m}t_n^{-1/2}$$
$$\leqslant Ck^{3/2}\int_0^{mk}\frac{dx}{\sqrt{x}} \leqslant Ck^{3/2}\sqrt{T}, \tag{42}$$

where $\theta_2 \in (t_{n+1}, t_{n+2}), \theta_3 \in (t_n, t_{n+1})$.

Since $\widetilde{A}_n = \frac{1}{2}(A_n + A_{n+1} + \beta_{n+1})$, substituting (37)–(42) into (34), we obtain

$$\frac{1}{3}\|U^{m+1}\|^2 - \frac{1}{2}\|U^0\|^2 \leqslant \sum_{n=0}^{m}\langle \mathscr{A}\delta_t U^n, \overline{U}^n\rangle \leqslant \sum_{n=0}^{m}\langle \widetilde{A}_n\delta_x^2 U^0, \overline{U}^n\rangle \leqslant \frac{4k}{h^2}\left|\sum_{n=0}^{m}\widetilde{A}_n\right|\|U^0\|\,\|\overline{U}^n\|$$
$$\leqslant \left(2k^{3/2}\sqrt{T} + 2k^{3/2}\sqrt{T} + Ck^{3/2}\sqrt{T}\right)\cdot\frac{4}{2h^2}\|U^0\|\,\|\overline{U}^n\|, \tag{43}$$

so we have

$$\|U^{m+1}\|^2 \leqslant 3\left(4k^{3/2}\sqrt{T} + Ck^{3/2}\sqrt{T}\right)\cdot\frac{2}{h^2}\|U^0\|\,\|\overline{U}^n\| + \frac{3}{2}\|U^0\|^2.$$

Choosing M so that $\|U^M\| = \max_{0\leqslant n\leqslant N}\|U^n\|$. So

$$\|U^M\|^2 \leqslant 3(4k^{3/2}\sqrt{T} + Ck^{3/2}\sqrt{T})\cdot\frac{4}{h^2}\|U^0\|\,\|U^M\| + \frac{3}{2}\|U^0\|^2 \leqslant \frac{C\sqrt{T}k^{3/2}}{h^2}\|U^0\|\,\|U^M\| + \frac{3}{2}\|U^0\|\,\|U^M\|. \tag{44}$$

Therefore, for $N \geqslant 1$, we obtain

$$\|U^N\| \leqslant \|U^M\| \leqslant \frac{3}{2}\|U^0\| + \frac{C\sqrt{T}k^{3/2}}{h^2}\|U^0\|. \tag{45}$$

This completes the proof. □

3.2. Convergence

We can derive the convergence of the numerical method (20)–(22) as similarly as proving Theorem 3.1. Denote

$$e_j^n = u_j^n - U_j^n, \quad 0 \leqslant j \leqslant J, \quad 0 \leqslant n \leqslant N.$$

Theorem 3.2. *Assume that $\{u_j^n | 0 \leqslant j \leqslant J, 0 \leqslant n \leqslant N\}$ be the solution of problem (1)–(3) and they satisfies the smoothness requirements stated in (4). Let $(U^0, \ldots, U^N)$ be the solution of compact difference scheme (20)–(22). If β in (6) satisfies (29), when h and k tend to zero independently, we have*

$$\max_{1\leqslant n\leqslant N}\|e^n\| = O(k^{3/2} + h^4). \tag{46}$$

Proof. Subtracting (16)–(19) from (20)–(22), respectively, we obtain the error equations

$$\mathscr{A}\delta_t e_j^n = \mu\delta_x^2\overline{e}_j^n + \sum_{p=0}^{n}\beta_p\delta_x^2\overline{e}_j^{n-p} + (\overline{R}_0)_j^n, \quad 1 \leqslant j \leqslant J-1,\ 1 \leqslant n \leqslant N-1, \tag{47}$$

$$e_0^n = e_J^n = 0, \quad n = 0, 1, \ldots, N. \tag{48}$$

$$e_j^0 = 0, \quad 1 \leqslant j \leqslant J-1. \tag{49}$$

Multiplying (47) by $kh\overline{e}_j^n$ and summing up j from 1 to $J-1$ and N from 0 to m, we can obtain

$$k\sum_{n=0}^{m}\langle \mathscr{A}\delta_t e^n, \overline{e}^n\rangle = k\mu\sum_{n=0}^{m}\langle \delta_x^2\overline{e}^n, \overline{e}^n\rangle + k\sum_{n=0}^{m}\sum_{p=0}^{n}\beta_p\langle \delta_x^2\overline{e}^{n-p}, \overline{e}^n\rangle + k\sum_{n=0}^{m}\langle (\overline{R}_0)^n, \overline{e}^n\rangle. \tag{50}$$

Just as the proof of Theorem 3.1, denote $\|e^M\| = \max_{1\leqslant n\leqslant N}\|e^n\|$, we can get

$$\frac{1}{3}\|e^M\|^2 - \frac{1}{2}\|e^0\|^2 \leqslant k\sum_{n=0}^{m}\langle (\overline{R}_0)^n, \overline{e}^n\rangle \leqslant Ck\sum_{n=0}^{m}\|(\overline{R}_0)^n\|_\infty\cdot\|e^M\| \leqslant \left[C\sum_{n=0}^{m}\left(k^3 t_{n+1}^{-3/2} + k^{5/2} + kh^4\right)\right]\|e^M\|$$
$$\leqslant \left[Ck^{3/2}\sum_{n=0}^{m}(n+1)^{-3/2} + Ck^{3/2} + Ch^4 + O\left(k^{3/2} + h^4\right)\right]\|e^M\|, \tag{51}$$

therefore

$$\|e^M\|^2 \leqslant 3C(k^{3/2} + h^4)\|e^M\|. \tag{52}$$

The above inequality implies that $\|e^M\| \leqslant C(k^{3/2} + h^4)$. Hence, we have obtained the convergence result (46). □

4. Numerical experiment

In order to demonstrate the effectiveness of our schemes, we compute the following examples.

Example 1.

$$u_t = \int_0^t (t-s)^{-1/2} u_{xx}(x,s)ds, \tag{53}$$

$$u(0,t) = u(1,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{54}$$

$$u(x,0) = sin(\pi x), \quad 0 \leqslant x \leqslant 1. \tag{55}$$

The exact solution of this problem is $u(x,t) = M(\pi^{5/2}t^{3/2})sin(\pi x)$, where M denotes the entire function

$$M(z) = \sum_{n=0}^{\infty}(-1)^n \Gamma\left(\frac{3}{2}n+1\right)^{-1} z^n. \tag{56}$$

In the calculation we set $\beta = 0.1, J = 10$ or $J = N^{1/3}$, take $h = 1/J, k = 1/N$, Tables 1 and 2 present the maximum errors of numerical solutions obtained with different step-size, and presents rates of convergence in time for $T = 0.5$ and $T = 1$, respectively. In Tables 1 and 2 we take $J = 10$ for calculation. The numerical results reflect a convergence rate $\approx \frac{3}{2}$ in time. Also in Tables 1 and 2 we list the errors and rates for numerical solutions with $\beta = 0$. Table 3 we present the maximum errors of numerical solutions obtained with different step-size, and presents rates of convergence in space for $T = 0.5$. In order to calculation the convergence order in space, we have to take $k^{3/2} \leqslant h^4$, so we set $N = J^3$ in Table 3. We also list the errors and rates for numerical solutions with $\beta = 0$ in Table 3.

Example 2. We consider the partial integro-differential equation

$$u_t = u_{xx} + \int_0^t (t-s)^{-1/2} u_{xx}(x,s)ds, \tag{57}$$

$$u(0,t) = u(1,t) = 0, \quad 0 \leqslant t \leqslant T. \tag{58}$$

$$u(x,0) = sin(\pi x), \quad 0 \leqslant x \leqslant 1. \tag{59}$$

In the calculation we also set $T = 0.5$, take $h = 1/J, k = 1/N, J = 10$. We use the numerical solution of u corresponding to $N \times J = 640 \times 10$ is used as the "exact" reference solution. Table 4 presents the maximum errors of numerical solutions obtained with different step-size, and presents rates of convergence in time for Example 4 with $\beta = 0.1$ and $\beta = 0$, respectively. Because the value of x_j are difference in different grid, and they have not relationship with each other directly, so we can not calculate the convergence order in space.

This work is a supplement of Tang's work [6] by using the compact difference method in space. The numerical results from Table 3 reflect the convergence rate in space is 4. Our results are similar to the numerical solution in [8] because we use the same method, but the authors in paper [8] did not compute the rates in space. Comparing the results in Tables 3 and 4 to the tables in papers [6,8], we can see our computational solution in this paper is much better. For example, when $N = 40$, the error in paper [6] is 3.05e−003, the error in paper [8] is 6.62e−003, but the errors in this paper is 3.52e−004 in Table 4.

Table 1
Errors and convergence rates for $T = 0.5$.

N	Error	Rate
20	8.93117e−002(8.71410e−002)	–
40	3.21568e−002(3.13501e−002)	1.47372(1.47488)
80	1.15715e−002(1.13353e−002)	1.47455(1.46764)
160	4.16970e−003(4.10482e−003)	1.47256(1.46544)
320	1.50417e−003(1.48438e−003)	1.47098(1.46745)
640	5.46742e−004 (5.37661e−004)	1.46004(1.46510)

Table 2
Errors and convergence rates for $T = 1$.

N	Error	Rate
20	2.49400e−001(2.40633e−001)	–
40	8.93117e−002(8.71410e−002)	1.48154(1.46541)
80	3.21568e−002(3.13501e−002)	1.47372(1.47488)
160	1.15715e−002(1.13353e−002)	1.47455(1.46764)
320	4.16970e−003(4.10482e−003)	1.47256(1.46544)
640	1.50417e−003(1.48438e−003)	1.47098(1.46745)

Table 3
Errors and convergence rates for $T = 0.5$.

$N = J^3$	J	Error	Rate
729	9	4.30775e−004(4.06896e−004)	–
1728	12	1.63014e−004(1.51780e−004)	4.35482(4.41924)
3375	15	7.72578e−005(7.08549e−005)	4.09544(4.17832)
5832	18	4.22009e−005(3.82675e−005)	3.92284(3.99631)
9261	21	2.54077e−005(2.28692e−005)	3.79976(3.85536)

Table 4
Errors and convergence rates at $T = 0.5$.

N	Error	Rate
10	2.87067e−003(1.85121e−003)	–
20	1.00346e−003(6.50154e−004)	1.51641(1.50962)
40	3.52049e−004(2.27852e−004)	1.51113(1.51268)
80	1.21334e−004(7.84763e−005)	1.53679(1.53777)

5. Conclusion

In this article, we constructed a compact difference scheme for the partial integro-differential equation with a weakly singular kernel and we proved the stability and L_2 convergence by energy method. In this article a Crank–Nicolson time-stepping is used to approximate the differential term and a product trapezoidal method is used to approximate the integral term. The convergence order in L_2 is 3/2 in time and four in space. Two numerical example about $\mu = 0$ and $\mu = 1$ supported the theoretical results.

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