



Short communication

Note on uncertain linguistic Bonferroni mean operators and their application to multiple attribute decision making[☆]Deqing Li^{a,b}, Wenyi Zeng^{b,*}, Junhong Li^b^a Department of Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, PR China^b College of Information Science and Technology, Beijing Normal University, Beijing 100875, PR China

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ABSTRACT

In this paper, we point out that two theorems in a previous paper by Wei et al. (2013) are incorrect by some counterexamples in detail, and present the modified theorems.

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1. Introduction

In the real world, there are many decision situations in which the information can not be assessed by precisely in a quantitative form but may be in a qualitative one. In such situations, the use of linguistic approach is necessary [1]. Thus, multiple attribute decision making problems under uncertain linguistic environment is an important and interesting topic, and has attracted much attention from some scholars [2]. Wei et al. [3] applied the uncertain linguistic information to Bonferroni mean operator and did some meaningful work. However, two of theorems in Wei et al. [3] were false, and Wei et al. [4] gave a corrigendum to these errors according to our suggestion. But Wei et al. [4] only changed the conditions of the theorems which are much simpler than one in Wei et al. [3]. Therefore, we think that it is significant to further consider whether the results are correct in other cases in order to analyze the errors thoroughly. The aim of this paper is to point out some errors to the Theorems 3 and Theorem 7 in Wei et al. [3] by numerical examples. In this paper, we take the operator with $n = 2$ and investigate the conditions in detail in 6 different cases, show by counterexamples that the results are not true in 5 cases and the result is true only in one case. Finally, we propose the revised theorems and their proof. Briefly we mention some basic concepts given in [3].

Let $S = \{s_i | i = 1, 2, \dots, t\}$ be linguistic term set with odd cardinality, where s_i represents a possible value for a linguistic variable. It satisfies the following characteristics: (1) The set is ordered: $s_i > s_j$ iff $i > j$. (2) There is the negation operator: $neg(s_i) = s_j$ such that $j = t + 1 - i$. (3) Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$. Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$.

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* Corresponding author.

E-mail address: zengwy@bnu.edu.cn (W. Zeng).

To preserve all the given information, the discrete term set S should be extended to a continuous term set $\bar{S} = \{s_\alpha | s_1 \leq s_\alpha \leq s_q, \alpha \in [1, q]\}$, where q is a sufficiently large positive integer in [2].

Definition 1 [2]. Let $\tilde{s} = [s_\alpha, s_\beta]$, where $s_\alpha, s_\beta \in \bar{S}$, s_α and s_β are the lower and the upper limits, respectively, we call \tilde{s} the uncertain linguistic variable, and \tilde{S} stands for the set of all the uncertain linguistic variables.

Consider any three uncertain linguistic variables $\tilde{s} = [s_\alpha, s_\beta]$, $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$, $\tilde{s}, \tilde{s}_1, \tilde{s}_2 \in \tilde{S}$, $\lambda \in [0, 1]$, their operational laws are defined as follows in [2]:

- (1) $\tilde{s}_1 \oplus \tilde{s}_2 = [s_{\alpha_1 + \alpha_2}, s_{\beta_1 + \beta_2}]$;
- (2) $\lambda \tilde{s} = [s_{\lambda \alpha}, s_{\lambda \beta}]$;
- (3) $\tilde{s}_1 \otimes \tilde{s}_2 = [s_{\alpha_1 \alpha_2}, s_{\beta_1 \beta_2}]$;
- (4) $(\tilde{s})^\lambda = [s_{\alpha^\lambda}, s_{\beta^\lambda}]$.

Definition 2 [2]. Let $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}]$ and $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}]$ be two uncertain linguistic variables, and let $\text{len}(\tilde{s}_1) = \beta_1 - \alpha_1$, $\text{len}(\tilde{s}_2) = \beta_2 - \alpha_2$, then the degree of possibility of $\tilde{s}_1 \geq \tilde{s}_2$ is defined as

$$p(\tilde{s}_1 \geq \tilde{s}_2) = \frac{\max(0, \text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2) - \max(\beta_2 - \alpha_1, 0))}{\text{len}(\tilde{s}_1) + \text{len}(\tilde{s}_2)} \quad (1)$$

Definition 3 (See Wei Definition 6). Let $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}]$ ($j = 1, 2, \dots, n$) be uncertain linguistic variables, and let $p, q \geq 0$. If

$$\begin{aligned} \text{ULBM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) &= \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^n (\tilde{s}_i^p \otimes \tilde{s}_j^q) \right)^{\frac{1}{p+q}} \\ &= \left[\left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^n ((s_{\alpha_i})^p \otimes (s_{\alpha_j})^q) \right)^{\frac{1}{p+q}}, \left(\frac{1}{n(n-1)} \bigoplus_{\substack{i,j=1 \\ i \neq j}}^n ((s_{\beta_i})^p \otimes (s_{\beta_j})^q) \right)^{\frac{1}{p+q}} \right] \end{aligned} \quad (2)$$

Then $\text{ULBM}^{p,q}$ is called the uncertain linguistic Bonferroni mean (ULBM) operator.

Definition 4 (See Wei Definition 9). Let $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}]$ ($j = 1, 2, \dots, n$) be uncertain linguistic variables, and let $p, q \geq 0$. If

$$\begin{aligned} \text{ULGBM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) &= \frac{1}{p+q} \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^n (p\tilde{s}_i \oplus q\tilde{s}_j) \right)^{\frac{1}{n(n-1)}} \\ &= \left[\frac{1}{p+q} \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^n (ps_{\alpha_i} \oplus qs_{\alpha_j}) \right)^{\frac{1}{n(n-1)}}, \frac{1}{p+q} \left(\bigotimes_{\substack{i,j=1 \\ i \neq j}}^n (ps_{\beta_i} \oplus qs_{\beta_j}) \right)^{\frac{1}{n(n-1)}} \right] \end{aligned} \quad (3)$$

Then $\text{ULGBM}^{p,q}$ is called the uncertain linguistic geometric Bonferroni mean (ULGBM) operator.

2. Counterexample

In the following, we will show by counterexamples that two theorems in Wei et al. [3] are incorrect.

Theorem 1 (Monotonicity (See Wei Theorem 3)). Let $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}]$ and $\tilde{s}'_j = [s'_{\alpha_j}, s'_{\beta_j}]$ ($j = 1, 2, \dots, n$) be two set of uncertain linguistic variables, if $\tilde{s}_j \leq \tilde{s}'_j$, for all j , then

$$\text{ULBM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) \leq \text{ULBM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n). \quad (4)$$

Theorem 2 (Monotonicity (See Wei Theorem 7)). Let $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}]$ and $\tilde{s}'_j = [s'_{\alpha_j}, s'_{\beta_j}]$ ($j = 1, 2, \dots, n$) be two set of uncertain linguistic variables, if $\tilde{s}_j \leq \tilde{s}'_j$, for all j , then

$$\text{ULGBM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) \leq \text{ULGBM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n) \quad (5)$$

For convenience, we first let $n = 2$, and assume that $\tilde{s}_1 = [s_{\alpha_1}, s_{\beta_1}] \leq \tilde{s}'_1 = [s'_{\alpha_1}, s'_{\beta_1}]$, $\tilde{s}_2 = [s_{\alpha_2}, s_{\beta_2}] \leq \tilde{s}'_2 = [s'_{\alpha_2}, s'_{\beta_2}]$. Then, the position relationships between \tilde{s}_j and \tilde{s}'_j ($j = 1, 2$) can be classified into three cases, shown by the indices as

- (i) $\alpha_j \leq \alpha'_j, \beta_j \leq \beta'_j$,
- (ii) $\alpha_j \leq \alpha'_j, \beta'_j \leq \beta_j$,
- (iii) $\alpha'_j \leq \alpha_j, \beta_j \leq \beta'_j$.

Considering the symmetry of \tilde{s}_1 and \tilde{s}_2 , or \tilde{s}'_1 and \tilde{s}'_2 in the formulas of ULBM and ULGBM, we only need to investigate the following six cases.

- (a) $\alpha_1 \leq \alpha'_1, \beta_1 \leq \beta'_1$, and $\alpha_2 \leq \alpha'_2, \beta_2 \leq \beta'_2$;
- (b) $\alpha_1 \leq \alpha'_1, \beta_1 \leq \beta'_1$, and $\alpha_2 \leq \alpha'_2, \beta'_2 \leq \beta_2$;
- (c) $\alpha_1 \leq \alpha'_1, \beta_1 \leq \beta'_1$, and $\alpha'_2 \leq \alpha_2, \beta_2 \leq \beta'_2$;
- (d) $\alpha_1 \leq \alpha'_1, \beta'_1 \leq \beta_1$, and $\alpha_2 \leq \alpha'_2, \beta'_2 \leq \beta_2$;
- (e) $\alpha_1 \leq \alpha'_1, \beta'_1 \leq \beta_1$, and $\alpha'_2 \leq \alpha_2, \beta_2 \leq \beta'_2$;
- (f) $\alpha'_1 \leq \alpha_1, \beta_1 \leq \beta'_1$, and $\alpha'_2 \leq \alpha_2, \beta_2 \leq \beta'_2$.

In the following, we will show by counterexamples that Theorems 1 and 2 are incorrect except the case (a).

Case (b)

Example 1.

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_4, s_{74}], \tilde{s}_2 = [s_1, s_{50}], \tilde{s}'_1 = [s_5, s_{77}], \tilde{s}'_2 = [s_{31}, s_{32}]$. Then known by 1, we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.4859, p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.38$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, by Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2) = [s_2, s_{60.8276}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{12.4499}, s_{49.6387}]$$

Since $p([s_2, s_{60.8276}] \geq [s_{12.4499}, s_{49.6387}]) = 0.5038$, it implies that $\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2) \geq \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2)$. Hence, Theorem 3 in Wei et al. [3] is incorrect.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{0.7281}, s_{8.0636}], \tilde{s}_2 = [s_{2.1436}, s_{7.9376}], \tilde{s}'_1 = [s_{0.7409}, s_{8.1415}], \tilde{s}'_2 = [s_{5.0120}, s_{5.2065}]$. Then known by 1, we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.4969, p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4885$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, by formula (3), we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2) = [s_{1.3915}, s_{8.0005}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{2.6709}, s_{6.6335}]$$

Since $p([s_{1.3915}, s_{8.0005}] \geq [s_{2.6709}, s_{6.6335}]) = 0.5041$, it implies that $\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2) \geq \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2)$. Hence, Theorem 7 in Wei et al. [3] is incorrect.

Case (c)

Example 2.

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_{17}, s_{28}], \tilde{s}_2 = [s_8, s_{30}], \tilde{s}'_1 = [s_{25}, s_{31}], \tilde{s}'_2 = [s_1, s_{40}]$. Then known by 1, we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.1765, p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4754$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, known by Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2) = [s_{11.6619}, s_{28.9828}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_5, s_{35.2136}]$$

Since $p([s_{11.6619}, s_{28.9828}] \geq [s_5, s_{35.2136}]) = 0.5045$. It also implies that Theorem 3 in Wei et al. [3] is incorrect.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{3.5243}, s_{4.1831}], \tilde{s}_2 = [s_{2.1485}, s_{6.0694}], \tilde{s}'_1 = [s_{3.594}, s_{4.2569}], \tilde{s}'_2 = [s_{0.1857}, s_{8.2064}]$. Then known by 1, we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.4457, p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4927$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, by formula (3), we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2) = [s_{2.8154}, s_{5.1045}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{1.6869}, s_{6.1529}]$$

Since $p([s_{2.8154}, s_{5.1045}] \geq [s_{1.6869}, s_{6.1529}]) = 0.5059$, it implies that Theorem 7 in Wei et al. [3] is incorrect.

Case (d)

Example 3.

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_{1.1541}, s_{8.9037}], \tilde{s}_2 = [s_{2.2178}, s_{8.9665}], \tilde{s}'_1 = [s_{1.7382}, s_{8.5599}], \tilde{s}'_2 = [s_{5.1552}, s_{6.415}]$. Then known by 1, we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.4918, p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4759$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, by Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2) = [s_{1.5999}, s_{8.9351}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{2.9935}, s_{7.4103}]$$

Since $p([s_{1.5999}, s_{8.9351}] \geq [s_{2.9935}, s_{7.4103}]) = 0.5056$, hence, Theorem 3 in Wei et al. [3] is incorrect in case d.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{0.8167}, s_{9.1657}], \tilde{s}_2 = [s_{0.162}, s_{6.8553}], \tilde{s}'_1 = [s_{4.6387}, s_{5.3485}], \tilde{s}'_2 = [s_{0.7527}, s_{6.2768}]$. Then known by (1), we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.4997, p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4995$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, known by formula (3), then we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2) = [s_{0.4611}, s_{7.9896}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{2.5146}, s_{5.8080}]$$

Since $p([s_{0.4611}, s_{7.9896}] \geq [s_{2.5146}, s_{5.8080}]) = 0.5059$, it also implies that Theorem 7 in Wei et al. [3] is incorrect.

Case (e)

Example 4.

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_{3.2452}, s_{8.3422}]$, $\tilde{s}_2 = [s_{2.7434}, s_{6.8357}]$, $\tilde{s}'_1 = [s_{6.7928}, s_{7.4841}]$, $\tilde{s}'_2 = [s_{0.292}, s_{9.6681}]$. Then known by (1), we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.2677$, $p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4859$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, by Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2) = [s_{2.9838}, s_{7.5515}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{1.4083}, s_{8.5063}]$$

Since $p([s_{2.9838}, s_{7.5515}] \geq [s_{1.4083}, s_{8.5063}]) = 0.5266$, it means that Theorem 3 in Wei et al. [3] is incorrect.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{2.0387}, s_{9.1595}]$, $\tilde{s}_2 = [s_{1.8736}, s_{8.5259}]$, $\tilde{s}'_1 = [s_{5.4431}, s_{5.9009}]$, $\tilde{s}'_2 = [s_{1.0698}, s_{9.3457}]$. Then known by (1), we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.4904$, $p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4995$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, known by formula (3), then we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2) = [s_{1.9557}, s_{8.8413}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{3.0674}, s_{7.5745}]$$

Since $p([s_{1.9557}, s_{8.8413}] \geq [s_{3.0674}, s_{7.5745}]) = 0.5068$, it also implies that Theorem 7 in Wei et al. [3] is incorrect.

Case (f)

Example 5.

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_{9.3212}, s_{9.9692}]$, $\tilde{s}_2 = [s_{6.0619}, s_{6.1673}]$, $\tilde{s}'_1 = [s_{8.4929}, s_{10.9197}]$, $\tilde{s}'_2 = [s_{2.0828}, s_{10.3694}]$. Then known by (1), we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.4801$, $p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4867$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, known by formula (2), then we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2) = [s_{7.5169}, s_{7.8411}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{4.2058}, s_{10.6410}]$$

Since $p([s_{7.5169}, s_{7.8411}] \geq [s_{4.2058}, s_{10.6410}]) = 0.5378$, it implies that Theorem 3 in Wei et al. [3] is incorrect.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{8.6998}, s_{9.3480}]$, $\tilde{s}_2 = [s_{4.0877}, s_{7.5232}]$, $\tilde{s}'_1 = [s_{8.6897}, s_{9.5147}]$, $\tilde{s}'_2 = [s_{1.0082}, s_{10.6228}]$. Known by (1), we have $p(\tilde{s}_1 \geq \tilde{s}'_1) = 0.4469$, $p(\tilde{s}_2 \geq \tilde{s}'_2) = 0.4992$, thus, $\tilde{s}_1 \leq \tilde{s}'_1$, and $\tilde{s}_2 \leq \tilde{s}'_2$. However, known by formula (3), then we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2) = [s_{6.2889}, s_{8.4232}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2) = [s_{4.4525}, s_{10.0649}]$$

Since $p([s_{6.2889}, s_{8.4232}] \geq [s_{4.4525}, s_{10.0649}]) = 0.5126$, it also implies that Theorem 7 in Wei et al. [3] is incorrect.

Based on the above works, we can easily present counterexamples to show that both Theorems 3 and 7 in Wei et al. [3] are incorrect for any n . The simplest method is extending $\{\tilde{s}_1, \tilde{s}_2\}$ and $\{\tilde{s}'_1, \tilde{s}'_2\}$ by adding an uncertain linguistic variable \tilde{s} to $\{\tilde{s}_1, \tilde{s}_2, \tilde{s}, \dots, \tilde{s}\}$ and $\{\tilde{s}'_1, \tilde{s}'_2, \tilde{s}, \dots, \tilde{s}\}$ for every case, where $\{\tilde{s}_1, \tilde{s}_2\}$ and $\{\tilde{s}'_1, \tilde{s}'_2\}$ are the uncertain linguistic variables discussed in the above examples. For example, suppose $n = 10$,

Case (b)

Example 6.

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_4, s_{74}]$, $\tilde{s}_2 = [s_1, s_{50}]$, $\tilde{s}'_1 = [s_5, s_{77}]$, $\tilde{s}'_2 = [s_{31}, s_{32}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.0579}, s_{0.3529}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . However, by Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{0.3774}, s_{9.4916}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{1.9537}, s_{7.8531}]$$

Since $p([s_{0.3774}, s_{9.4916}] \geq [s_{1.9537}, s_{7.8531}]) = 0.5021$, it implies that Theorem 3 in Wei et al. [3] is incorrect.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{0.7281}, s_{8.0636}]$, $\tilde{s}_2 = [s_{2.1436}, s_{7.9376}]$, $\tilde{s}'_1 = [s_{0.7409}, s_{8.1415}]$, $\tilde{s}'_2 = [s_{5.0120}, s_{5.2065}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.0153}, s_{0.7468}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By formula (3), we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{0.0605}, s_{1.4271}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{0.0715}, s_{1.3353}]$$

Since $p([s_{0.0605}, s_{1.4271}] \geq [s_{0.0715}, s_{1.3353}]) = 0.5154$, it implies that Theorem 7 in Wei et al. [3] is incorrect.

Case (c)**Example 7.**

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_{17}, s_{28}]$, $\tilde{s}_2 = [s_8, s_{30}]$, $\tilde{s}'_1 = [s_{25}, s_{31}]$, $\tilde{s}'_2 = [s_1, s_{40}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.0099}, s_{0.8132}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{1.7510}, s_{5.2405}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{0.7754}, s_{6.1831}]$$

Since $p([s_{1.7510}, s_{5.2405}] \geq [s_{0.7754}, s_{6.1831}]) = 0.5019$, it also implies that Theorem 3 in Wei et al. [3] is incorrect.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{3.5243}, s_{4.1831}]$, $\tilde{s}_2 = [s_{2.1485}, s_{6.0694}]$, $\tilde{s}'_1 = [s_{3.594}, s_{4.2569}]$, $\tilde{s}'_2 = [s_{0.1857}, s_{8.2064}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.5252}, s_{0.8462}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By formula (3), we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{0.8006}, s_{1.3402}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{0.6293}, s_{1.4118}]$$

Since $p([s_{0.8006}, s_{1.3402}] \geq [s_{0.6293}, s_{1.4118}]) = 0.5377$, it implies that Theorem 7 in Wei et al. [3] is incorrect.

Case (d)**Example 8.**

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_{1.1541}, s_{8.9037}]$, $\tilde{s}_2 = [s_{2.2178}, s_{8.9665}]$, $\tilde{s}'_1 = [s_{1.7382}, s_{8.5599}]$, $\tilde{s}'_2 = [s_{5.1552}, s_{6.415}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.1509}, s_{0.6979}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{0.4018}, s_{2.0723}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{0.6310}, s_{1.8388}]$$

Since $p([s_{0.4018}, s_{2.0723}] \geq [s_{0.6310}, s_{1.8388}]) = 0.5007$, hence, Theorem 3 in Wei et al. [3] is incorrect in case d.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{0.8167}, s_{9.1657}]$, $\tilde{s}_2 = [s_{0.162}, s_{6.8553}]$, $\tilde{s}'_1 = [s_{4.6387}, s_{5.3485}]$, $\tilde{s}'_2 = [s_{0.7527}, s_{6.2768}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.1730}, s_{0.9797}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By formula (3), we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{0.2098}, s_{1.7085}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{0.3386}, s_{1.5469}]$$

Since $p([s_{0.2098}, s_{1.7085}] \geq [s_{0.3386}, s_{1.5469}]) = 0.5061$, it also implies that Theorem 7 in Wei et al. [3] is incorrect.

Case (e)**Example 9.**

- (1) Let $p = q = 1$, and $\tilde{s}_1 = [s_{3.2452}, s_{8.3422}]$, $\tilde{s}_2 = [s_{2.7434}, s_{6.8357}]$, $\tilde{s}'_1 = [s_{6.7928}, s_{7.4841}]$, $\tilde{s}'_2 = [s_{0.292}, s_{9.6681}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.0118}, s_{0.1365}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{0.4587}, s_{1.2834}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{0.2428}, s_{1.4268}]$$

Since $p([s_{0.4587}, s_{1.2834}] \geq [s_{0.2428}, s_{1.4268}]) = 0.5180$, it means that Theorem 3 in Wei et al. [3] is incorrect.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{2.0387}, s_{9.1595}]$, $\tilde{s}_2 = [s_{1.8736}, s_{8.5259}]$, $\tilde{s}'_1 = [s_{5.4431}, s_{5.9009}]$, $\tilde{s}'_2 = [s_{1.0698}, s_{9.3457}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.0648}, s_{0.9883}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By formula (3), we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{0.1776}, s_{1.7793}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{0.1935}, s_{1.6847}]$$

Since $p([s_{0.1776}, s_{1.7793}] \geq [s_{0.1935}, s_{1.6847}]) = 0.5127$, it also implies that Theorem 7 in Wei et al. [3] is incorrect.

Case (f)

Example 10. Let $p = q = 1$, and $\tilde{s}_1 = [s_{9.3212}, s_{9.9692}]$, $\tilde{s}_2 = [s_{6.0619}, s_{6.1673}]$, $\tilde{s}'_1 = [s_{8.4929}, s_{10.9197}]$, $\tilde{s}'_2 = [s_{2.0828}, s_{10.3694}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.1663}, s_{0.8289}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By Eq. (2), we have

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{1.3144}, s_{2.0424}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{0.8502}, s_{2.4659}]$$

Since $p([s_{1.3144}, s_{2.0424}] \geq [s_{0.8502}, s_{2.4659}]) = 0.5087$, it implies that Theorem 3 in Wei et al. [3] is incorrect.

- (2) Let $p = 3, q = 1$, and $\tilde{s}_1 = [s_{8.6998}, s_{9.3480}]$, $\tilde{s}_2 = [s_{4.0877}, s_{7.5232}]$, $\tilde{s}'_1 = [s_{8.6897}, s_{9.5147}]$, $\tilde{s}'_2 = [s_{1.0082}, s_{10.6228}]$, $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{6.5164}, s_{7.2180}]$, $i = 3, 4, \dots, 10$. Then $\tilde{s}_i \leq \tilde{s}'_i$, for all i . By formula (3), we have

$$\text{ULGBM}^{3,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{6.4416}, s_{7.4471}], \quad \text{ULGBM}^{3,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{5.9440}, s_{7.7417}]$$

Since $p([s_{6.4416}, s_{7.4471}] \geq [s_{5.9440}, s_{7.7417}]) = 0.5362$, it also implies that Theorem 7 in Wei et al. [3] is incorrect.

Remark 1. If $p(\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2) \geq \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2)) \geq 0.5$, when we extend $\{\tilde{s}_1, \tilde{s}_2\}$ and $\{\tilde{s}'_1, \tilde{s}'_2\}$ by adding an uncertain linguistic variable \tilde{s} to $\{\tilde{s}_1, \tilde{s}_2, \tilde{s}, \dots, \tilde{s}\}$ and $\{\tilde{s}'_1, \tilde{s}'_2, \tilde{s}, \dots, \tilde{s}\}$, we cannot guarantee $p(\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) \geq \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n)) \geq 0.5$. For example, in case b), suppose $n = 10$ and $\tilde{s}_i = \tilde{s}'_i = \tilde{s} = [s_{0.6154}, s_{0.7919}]$, $i = 3, 4, \dots, 10$. Then

$$\text{ULBM}^{1,1}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_{10}) = [s_{0.9336}, s_{10.0035}], \quad \text{ULBM}^{1,1}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_{10}) = [s_{2.7602}, s_{8.3959}]$$

$p([s_{0.9336}, s_{10.0035}] \geq [s_{2.7602}, s_{8.3959}]) = 0.4926$. For $\text{ULGBM}^{p,q}$, we have the same result.

3. Modification

By the previous analysis, we modify Theorems 1 and 2 as follows.

Theorem 3 (Monotonicity). $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}]$ and $\tilde{s}'_j = [s_{\alpha'_j}, s_{\beta'_j}]$ ($j = 1, 2, \dots, n$) be two set of uncertain linguistic variables, if $\alpha_j \leq \alpha'_j$, and $\beta_j \leq \beta'_j$ for all j , then

$$\text{ULBM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) \leq \text{ULBM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n)$$

Proof. By Eq. (2), we have

$$\text{ULBM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) = [s_{\alpha}, s_{\beta}]$$

$$\text{ULBM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n) = [s_{\alpha'}, s_{\beta'}]$$

where

$$\alpha = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n \alpha_i^p \alpha_j^q \right)^{\frac{1}{p+q}}, \quad \beta = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n \beta_i^p \beta_j^q \right)^{\frac{1}{p+q}}$$

$$\alpha' = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n \alpha_i'^p \alpha_j'^q \right)^{\frac{1}{p+q}}, \quad \beta' = \left(\frac{1}{n(n-1)} \sum_{\substack{i,j=1 \\ i \neq j}}^n \beta_i'^p \beta_j'^q \right)^{\frac{1}{p+q}}$$

Since $\alpha_j \leq \alpha'_j$, and $\beta_j \leq \beta'_j$ for all j , then $\alpha \leq \alpha'$, and $\beta \leq \beta'$. Hence, we have $p([s_{\alpha}, s_{\beta}] \geq [s_{\alpha'}, s_{\beta'}]) \leq 0.5$. It implies that $\text{ULBM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) \leq \text{ULBM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n)$.

Hence, we complete the proof of Theorem 3. \square

Theorem 4 (Monotonicity). Let $\tilde{s}_j = [s_{\alpha_j}, s_{\beta_j}]$ and $\tilde{s}'_j = [s_{\alpha'_j}, s_{\beta'_j}]$ ($j = 1, 2, \dots, n$) be two set of uncertain linguistic variables, if $\alpha_j \leq \alpha'_j$, and $\beta_j \leq \beta'_j$ for all j , then

$$\text{ULGBM}^{p,q}(\tilde{s}_1, \tilde{s}_2, \dots, \tilde{s}_n) \leq \text{ULGBM}^{p,q}(\tilde{s}'_1, \tilde{s}'_2, \dots, \tilde{s}'_n)$$

The proof is similar to Theorem 3.

4. Conclusion

In this study we investigate the property of monotonicity for $\text{ULBM}^{p,q}$ and $\text{ULGBM}^{p,q}$ in detail, and point out that Theorems 3 and 7 in Wei et al. [3] are incorrect by some counterexamples. Finally we present some new conditions and propose the modifications of these Theorems.

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