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A note on single-machine scheduling with sum-of-processing-time-based learning and forgetting effects

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ABSTRACT

In this note, we investigate some single-machine problems with both the learning and forgetting effects. A model based on sum-of-processing-time-based learning and forgetting effects are proposed, including some existing models as special cases of our model. Under the proposed model, we derive the optimal solutions for some single-machine problems. Several helpful lemmas are provided to prove various objective functions: the makespan, the total completion time, the total weighted completion time, the total tardiness, and the maximum lateness.

1. Introduction

In production and manufacturing, workers can improve their skills by the repeated processing of similar tasks such as handling raw materials and components, software operations, and machine settings. Biskup [1] firstly introduced single-machine scheduling problems with learning effects.

In the last decade, scheduling with learning effects has received substantial research attention. Kuo and Yang [2] studied single-machine scheduling problems under the Biskup [1] model. Furthermore, Wang et al. [3] presented some algorithms to construct the optimal schedule for problems with various objective functions under the same model. Wang [4] extended the model of Kuo and Yang [2] to the case of time-dependent setup times. Wang [5] investigated the model of Koulamas and Kyparisis [6]. He presented some counterexamples to demonstrate that the total weighted completion time minimization, the maximum lateness minimization, and the number of tardy jobs minimization problems cannot be optimally solved by the corresponding classical scheduling rules. Later, Cheng et al. [7] and Low and Lin [8] generalized Wang's [5] results to models with job-position-based learning effects. Biskup [9] gave a comprehensive overview on the work of scheduling with learning effects.

Recently, Cheng et al. [10] proposed a logarithm-processing-times-based learning effects model. Wu and Lee [11] firstly proposed a learning model including both machine and human learning effects simultaneously. Later, Zhang and Yan [12] also proposed another learning model. Wang et al. [13] considered the combined model stemming from Mosheiov [14] and Kuo and Yang [2]. They proposed heuristic algorithms utilizing the V-shaped property, and performed some computational experiments to evaluate the algorithms. Yang [15] proposed three models with group technology, learning and deterioration effects. Bai et al. [16] also proposed a learning model with general deterioration effects to analyze the

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single-machine scheduling problem. Lai and Lee [17] considered a model where the actual job processing time is a function of its scheduled position and the processing time of jobs processed. Under the same model, Wang and Wang [18] studied the single machine scheduling problems with past-sequence-dependent setup times and the general effects of deterioration and learning. Lu et al. [19] combined the existing models by addition and multiplication operations to obtain two new single-machine learning effect models. Kuo [20] investigated the model with sum-of-processing-time-based learning for job processing time and position-based learning for setup time. Bai et al. [21] discussed a general exponential learning effects model with setup times. Rudek [22] provided a survey on sum-of-processing-time-based learning effects. Moreover, he proved that the weighted completion time problem is at least NP-hard and gave several efficient algorithms.

Lately, Behdin et al. [23] considered scheduling problems with availability constraints and learning effects simultaneously. They presented a new binary integer programming model, then developed three algorithms including branch-and-bound, genetic algorithm and simulated annealing to minimize the total completion time under multiple availability constraints. Rudek [24] analyzed the single processor scheduling problems with learning effects dependent on the number of processed jobs and proved that the problems are at least NP-hard. Rudek [25] showed that the maximum lateness problem is strongly NP-hard even with simple position-based learning effect. Wang et al. [26] proposed a single-machine scheduling model with the effects of exponential learning and general deterioration. They investigated the scheduling problems with various objective functions and showed some of them remained polynomially solvable. Shen and Wu [27] investigated some scheduling problems with delivery times on a single machine. Later, Wang and Wang [28] proposed a learning model and proved that some special cases for flow shop scheduling problems are polynomially solvable under the proposed model. Rudek [29] proved that the makespan problem with position-based learning is strongly NP-hard even for two machines flowshop, whereas Rudek and Rudek [30] provided efficient algorithms for problems with position-based learning effects. Jiang et al. [31] proved that scheduling problems with sum-of-processing-time-based-learning effects and job-dependent learning (power) functions were proved to be NP-hard for the objectives such as the makespan, the maximum lateness, the weighted completion time, whereas Rudek [32] showed that the makespan problem with job release dates is strongly NP-hard.

In practice, forgetting effects may occur along with the learning effects. As suggested in Jaber et al. [33], the learning and forgetting effects are images on the mirror of each other. Jaber and Sikström [34] mentioned that to learn faster than its competitors might be the only sustainable advantage in the future. They provided a numerical comparison of three potential learning and forgetting models. Jaber and Bonney [35] pointed out that the forgetting curves depend on the learning curve, the quantity produced to date, and the minimum break at which total forgetting occurs. They studied the impact of learning and forgetting to the production quantity and inventory system cost. Furthermore, Jaber and Bonney [36] claimed that unlike the learning curves, a full understanding of the behavior and factors affecting the forgetting process have not yet been developed. They discussed the similarities and differences between three learning and forgetting models. Jaber [37] provided an extensive review of learning and forgetting models and their applications. For more comprehensive results of models and applications, reader can refer to Jaber [38]. Although the learning and forgetting models have been widely discussed, it is seldom studied in the scheduling field. Yang and Chand [39] constructed three single-machine models to investigate the learning and forgetting effects on the total completion time. Lai and Lee [40] proposed a model with both the effects. They also presented an example to illustrate that the forgetting effect might exist in some situations.

As mentioned earlier, very few researchers have discussed the scheduling problem with both the effects. Forgetting effects occur, particularly for products with short cycle times in which workers must learn new skills without repeated practice [41]. Moreover, the forgetting effects in Lai and Lee [40] is position dependent. However, the forgetting effects might be dependent on the elapsed time, this motivates us to investigate a model with both the effects where both the effects are elapsed-time dependent. The paper is organized as follows. The problem is described in Section 2. The optimal solutions for some problems are derived in Section 3, and the conclusion is presented at last.

2. Notations and problem formulation

There are n jobs to be scheduled in which job j has a processing time p_j, a weight w_j, and a due date d_j. More notations and assumptions are summarized as follows:

n	number of jobs
S_1, S_2	sequences of jobs
p_j	normal processing time of job j, $j = 1, 2, \ldots, n$
$p_{[k]}$	normal processing time of the job scheduled in the kth position in a sequence
k_0	the threshold that the forgetting effect first occurs, $k_0 \geqslant 0$
w_j	weight of job j, $j = 1, 2, \ldots, n$
d_j	due date of job j, $j = 1, 2, \ldots, n$
$f(x)$	a non-negative, non-increasing learning effect function defined on $[0, \infty)$ satisfying $\int_0^\infty f(x) \leqslant 1$
$F(y)$	the cumulative function of learning effect, i.e., $F(y) = \int_0^y f(x)dx$, $y \geqslant 0$
$g(x)$	a non-negative, forgetting effect function defined on $[0, \infty)$

$G(y)$	the cumulative function of forgetting effect, i.e., $G(y) = \int_0^y g(x)dx$, $y \geqslant 0$
C_j	completion time of job j
$C_{\max}$	makespan
L_j	lateness of job j, i.e., $L_j = C_j - d_j$
$L_{\max}$	maximum lateness
T_j	tardiness of job j, i.e., $T_j = \max(C_j - d_j, 0)$

In the following, we propose a scheduling model with sum-of-processing-time-based learning and forgetting effects. If job j is scheduled in the rth position in a sequence for $r = 1, 2, \ldots, n$, its actual processing time is

$$p^A_{j[r]} = \begin{cases} p_j\left[1 - \int_0^{\sum_{k=1}^{r-1} p_{[k]}} f(x)\mathrm{d}x\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} \leqslant k_0, \\ p_j\left[1 - \int_0^{\sum_{k=1}^{r-1} p_{[k]}} f(x)\mathrm{d}x + \int_0^{\sum_{k=1}^{r-1} p_{[k]} - k_0} g(x)\mathrm{d}x\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} > k_0, \end{cases} \tag{1}$$

where $f(x) \geqslant g(x - k_0)$ for $x \geqslant k_0$, x is the sum of processing times of job already processed, and the threshold value k_0 can be estimated from the empirical data or from the experts. That is, the marginal learning effect always dominates the marginal forgetting effect. It is noted that the models considered in Janiak and Rudek [42], Koulamas and Kyparisis [6], Kuo and Yang [2], and Wang et al. [43] can be included in our proposed model by choosing appropriate settings of $f(x)$, which is tabulated in Table 1. Based on the notations and assumptions described in Section 2, the cumulative function of the forgetting effect G satisfies

$$G(y - k_0) = \int_0^{y-k_0} g(x)dx = \int_{k_0}^{y} g(t - k_0)dt \leqslant \int_{k_0}^{y} f(t)dt \leqslant \int_0^y f(t)dt = F(y) \leqslant 1, \quad y \geqslant k_0. \tag{2}$$

Define $H(y) = F(y) - G(y - k_0)$ for $y \geqslant k_0$, we have $0 \leqslant H(y) \leqslant 1$ and $H(y)$ is a non-decreasing function since $H'(y) = f(y) - g(y - k_0) \geqslant 0$. It is assumed that $H'(y)$ is a non-increasing function. Next, we present a figure to illustrate the behavior of the learning and forgetting curves numerically. Suppose $p_j = 1,\ j = 1, \ldots, 100$ and the marginal and cumulative learning effect functions are, $f(x) = 0.033(1 + 0.05x)^{-2}$ and $F(y) = 0.667y(20 + y)^{-1}$, $x \geqslant 0$, respectively. The marginal and cumulative forgetting effect functions are $g(x) = 0.016(1 + 0.05x)^{-2}$ and $G(y) = 0.333y(20 + y)^{-1}$ $x \geqslant 0$. Since $f(x)$ is non-increasing, $F(M) \leqslant 1$ for a sufficiently large number M and $f(x) \geqslant g(x - 2)$ for $\forall x \geqslant k_0 = 2$. We denote $p^A_{j[r]}$ as a function of integer variable by $P(r)$, the actual processing time of the rth position job with both the effects. The figure of numerical values of $P(r)$, the learning and forgetting effect function $1 - F(r)$ and $G(r)$ for $1 \leqslant r \leqslant 100$ is present in Fig. 1.

In the following section, several properties are presented for objective functions of the makespan ($C_{\max}$), the total completion time ($\sum C_j$), the total weighted completion time ($\sum w_j C_j$), the total tardiness ($\sum T_j$), and the maximum lateness ($L_{\max}$). The properties will be proved using the pairwise interchange technique. Let S_1 and S_2 be two job schedules and the difference between S_1 and S_2 is a pairwise interchange of two adjacent jobs i and j. That is, $S_1 = (\pi, j, i, \pi')$ and $S_2 = (\pi, i, j, \pi')$, where π and π' each denote a partial sequence. It is assumed that there are $r - 1$ scheduled jobs in π. In addition, let A denote the completion time of the last job in π. Then, the completion times of jobs j and i in S_1 are

$$C_j(S_1) = \begin{cases} A + p_j\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} \leqslant k_0, \\ A + p_j\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} > k_0, \end{cases} \tag{12}$$

and

Table 1
A list of the special cases (existing models) of our model.

Models	$f(x)$	References	Remarks
$p^A_{j[r]} = p_j\left(1 + \sum_{k=1}^{r-1} p_{[k]}\right)^a$	$-a(1 + x)^{a-1}$	Kuo and Yang [2]	$a < 0$
$p^A_{j[r]} = p_j\left(1 - \frac{\sum_{k=1}^{r-1} p_{[k]}}{\sum_{k=1}^{n} p_{[k]}}\right)^a$	$\frac{a}{\sum_{k=1}^{n} p_{[k]}}\left(1 - \frac{x}{\sum_{k=1}^{n} p_{[k]}}\right)^{a-1}$	Koulamas and Kyparisis [6]	$a \geqslant 1$
$p^A_{j[r]} = p_j\left(1 - \frac{1}{k}\sum_{k=1}^{r-1} p_{[k]}\right)^a$	$\frac{a}{k}\left(1 - \frac{x}{k}\right)^{a-1}$	Janiak and Rudek [42]	$a > 0$
$p^A_{j[r]} = p_j\left(\alpha a^{\sum_{k=1}^{r-1} p_{[k]}} + \beta\right)$	$-\alpha a^x \ln a$	Wang et al. [43]	$0 < a \leqslant 1,\ \alpha + \beta = 1$

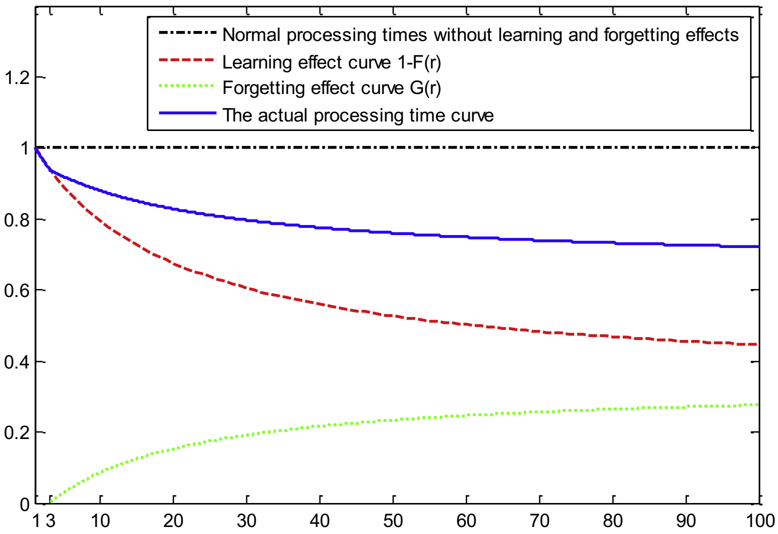

Fig. 1. The figure of a numerical values where the solid line is the actual processing time, the dashed line is the learning effect function $1 - F(r)$ and the dotted line is the forgetting effect function $G(r)$.

$$C_i(S_1) = \begin{cases} A + p_j\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_i\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]} + p_j\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} + p_j \leqslant k_0, \\ A + p_j\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_i\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_j\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} \leqslant k_0 < \sum_{k=1}^{r-1} p_{[k]} + p_j, \\ A + p_j\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_i\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_j\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} > k_0. \end{cases} \tag{13}$$

On the other hand, the completion times of jobs i and j in S_2 are

$$C_i(S_2) = \begin{cases} A + p_i\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} \leqslant k_0, \\ A + p_i\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} > k_0, \end{cases} \tag{14}$$

and

$$C_j(S_2) = \begin{cases} A + p_i\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_j\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]} + p_i\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} + p_i \leqslant k_0, \\ A + p_i\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_j\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_i\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} \leqslant k_0 < \sum_{k=1}^{r-1} p_{[k]} + p_i, \\ A + p_i\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_j\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_i\right)\right], & \text{if } \sum_{k=1}^{r-1} p_{[k]} > k_0. \end{cases} \tag{15}$$

3. Single-machine problems

In this section, we first prove the following lemmas.

Lemma 1. *$I_1(\rho, t) = (\rho - 1)[1 - H(\alpha)] + [1 - H(\alpha + \rho t)] - \rho[1 - H(\alpha + t)] \geqslant 0$ for $\rho > 1$, $t > 0$, $\alpha \geqslant k_0 \geqslant 0$.*

Proof. Taking the derivative of $I_1(\rho, t)$ with respect to ρ and t, we have

$$\begin{cases} \frac{\partial I_1}{\partial \rho} = H(\alpha+t) - H(\alpha) - tH'(\alpha+\rho t), \\ \frac{\partial I_1}{\partial t} = \rho\left[H'(\alpha+t) - H'(\alpha+\rho t)\right]. \end{cases} \tag{4}$$

It is noted that $\partial I_1/\partial t \geqslant 0$ since H' is a non-increasing function and $\alpha + \rho t > \alpha + t$. Furthermore, the Mean Value Theorem (MVT) implies that there exists $\alpha \leqslant \delta_H \leqslant \alpha + t$ such that

$$H(\alpha+t) - H(\alpha) = t\left[\frac{F(\alpha+t) - F(\alpha)}{t}\right] = tH'(\delta_H). \tag{5}$$

Therefore, the derivative is

$$\frac{\partial I_1}{\partial \rho} = H(\alpha+t) - H(\alpha) - tH'(\alpha+\rho t) = t\left[H'(\delta_H) - H'(\alpha+\rho t)\right] \geqslant 0, \tag{6}$$

since $\alpha \leqslant \delta_H \leqslant \alpha + t$ and H' is a non-increasing function. Since $I_1(1,0) = 0$ and I_1 is an non-decreasing function of ρ and t, we complete the proof of Lemma 1. $\square$

Lemma 2. *$I_2(\rho, t) = (\rho - 1)[1 - F(\alpha)] + [1 - H(\alpha + \rho t)] - \rho[1 - H(\alpha + t)] \geqslant 0$ for $\rho > 1$, $t > 0$, $\alpha + t \geqslant k_0 > \alpha > 0$.*

Proof. Since $\alpha < k_0$, we have

$$G(\alpha - k_0) = \int_0^{\alpha - k_0} g(x)dx = -\int_{\alpha-k_0}^{0} g(x)dx = 0. \tag{7}$$

Hence,

$$\begin{aligned} I_2(\rho,t) &= (\rho-1)[1-F(\alpha)] + [1-H(\alpha+\rho t)] - \rho[1-H(\alpha+t)] \\ &= (\rho-1)[1-H(\alpha)] + [1-H(\alpha+\rho t)] - \rho[1-H(\alpha+t)], \end{aligned} \tag{8}$$

which is the same as $I_1(\rho, t)$. Taking the derivative of $I_2(\rho, t)$ with respect to ρ and t, we have

$$\begin{cases} \frac{\partial I_2}{\partial \rho} = H(\alpha+t) - H(\alpha) - tH'(\alpha+\rho t), \\ \frac{\partial I_2}{\partial t} = \rho[H'(\alpha+t) - H'(\alpha+\rho t)]. \end{cases} \tag{9}$$

Similar to $\partial I_2/\partial t$ in Lemma 1, $\partial I_2/\partial t \geqslant 0$ since H' is a non-increasing function. By MVT, there exists $\alpha \leqslant \delta_H \leqslant \alpha + t$ such that

$$\frac{\partial I_2}{\partial \rho} = H(\alpha+t) - H(\alpha) - tH'(\alpha+\rho t) = t[H'(\delta_H) - H'(\alpha+\rho t)] \geqslant 0. \tag{10}$$

That is, $I_2(\rho, t)$ is a non-decreasing function with respect to parameters ρ and t. Finally, we complete the proof of Lemma 2 since $I_2(1, t) = 0$ and

$$\begin{aligned} I_2(\rho, k_0 - \alpha) &= (\rho-1)[1-H(\alpha)] + [1-H(\alpha+\rho(k_0-\alpha))] - \rho[1-H(k_0)] \\ &= \rho[H(k_0) - H(\alpha)] + [H(\alpha) - H(\alpha+\rho(k_0-\alpha))] \\ &\geqslant \rho[H(k_0) - H(\alpha)] + [H(\alpha) - H(k_0)] = (\rho-1)H(k_0) \geqslant 0. \qquad \square \end{aligned} \tag{11}$$

Lemma 3. *$I_3(\rho, t) = (\rho - 1)[1 - F(\alpha)] + [1 - H(\alpha + \rho t)] - \rho[1 - F(\alpha + t)] \geqslant 0$ for $\rho > 1$, $t > 0$, $\alpha > 0$, $\alpha + \rho t \geqslant k_0 > 0$.*

Lemma 4. *$I_4(\rho, t) = (\rho - 1)[1 - F(\alpha)] + [1 - F(\alpha + \rho t)] - \rho[1 - F(\alpha + t)] \geqslant 0$ for $\rho > 1$, $t > 0$, $\alpha > 0$, $k_0 > \alpha + \rho t$.*

The proof of Lemmas 3 and 4 are similar to that of Lemma 1. Next, we present the proofs of the properties.

Property 1. *The optimal sequence for $1|\mathrm{L+F}|C_{\max}$ is the SPT sequence.*

Proof. By contradiction, suppose there is an optimal sequence S_1 that does not follow the SPT rule. Thus, there are job j followed by job i, such that $p_j > p_i$. We now interchange only the positions of jobs i and j to obtain a new sequence S_2. It suffices to show that $C_j(S_2) \leqslant C_i(S_1)$ to prove S_2 dominates S_1. For various values of k_0, four cases are considered as follows:

(1) $\sum_{k=1}^{r-1} p_{[k]} + p_j > \sum_{k=1}^{r-1} p_{[k]} + p_i > \sum_{k=1}^{r-1} p_{[k]} \geqslant k_0$

(2) $\sum_{k=1}^{r-1} p_{[k]} + p_j > \sum_{k=1}^{r-1} p_{[k]} + p_i \geqslant k_0 > \sum_{k=1}^{r-1} p_{[k]}$

(3) $\sum_{k=1}^{r-1} p_{[k]} + p_j \geqslant k_0 > \sum_{k=1}^{r-1} p_{[k]} + p_i > \sum_{k=1}^{r-1} p_{[k]}$

(4) $k_0 > \sum_{k=1}^{r-1} p_{[k]} + p_j > \sum_{k=1}^{r-1} p_{[k]} + p_i > \sum_{k=1}^{r-1} p_{[k]}$

Let $\alpha = \sum_{k=1}^{r-1} p_{[k]}$, $t = p_i$, and $\rho = p_j/p_i > 1$. The four cases are simplified as

(1) $\alpha + \rho t > \alpha + t > \alpha \geqslant k_0$

(2) $\alpha + \rho t > \alpha + t \geqslant k_0 > \alpha$

(3) $\alpha + \rho t \geqslant k_0 > \alpha + t > \alpha$

(4) $k_0 > \alpha + \rho t > \alpha + t > \alpha$

For cases (1–4), taking the difference between Eqs. (13) and (15) yields

$$C_i(S_1) - C_j(S_2) = \begin{cases} (p_j - p_i)\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_i\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_j\right)\right] - p_j\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_i\right)\right] \\ (p_j - p_i)\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_i\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_j\right)\right] - p_j\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_i\right)\right] \\ (p_j - p_i)\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_i\left[1 - H\left(\sum_{k=1}^{r-1} p_{[k]} + p_j\right)\right] - p_j\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]} + p_i\right)\right] \\ (p_j - p_i)\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]}\right)\right] + p_i\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]} + p_j\right)\right] - p_j\left[1 - F\left(\sum_{k=1}^{r-1} p_{[k]} + p_i\right)\right] \end{cases}. \quad (16)$$

Substituting $\alpha = \sum_{k=1}^{r-1} p_{[k]}$, $t = p_i$, and $\rho = p_j/p_i > 1$ into Eq. (16), we have $C_i(S_1) \geqslant C_j(S_2)$ from Lemmas 1–4. This contradicts the optimality of S_1 and proves Property 1. $\square$

Property 2. *The optimal sequence for* $1|\mathrm{L} + \mathrm{F}|\sum C_j$ *is obtained by the SPT rule.*

Next, we show that the WSPT rule provides an optimal solution for the total weighted completion time problem under the proposed model if the processing times and the weights are agreeable, i.e., $p_i \leqslant p_j$ *implies* $w_i \geqslant w_j$ *for all jobs i and j.*

Property 3. *The optimal sequence for* $1|\mathrm{L} + \mathrm{F}|\sum w_j C_j$ *is the WSPT sequence i* $p_i \leqslant p_j$ *implies* $w_i \geqslant w_j$ *for all jobs i and j.*

Proof. By contradiction, using the notations $\alpha = \sum_{k=1}^{r-1} p_{[k]}$, $t = p_i$, and $\rho = p_j/p_i > 1$ as mentioned earlier, we consider the following four cases:

Case 1. For $\alpha + \rho t > \alpha + t > \alpha \geqslant k_0$, we have

$$\begin{aligned} &[w_j C_j(S_1) + w_i C_i(S_1)] - [w_i C_i(S_2) + w_j C_j(S_2)] \\ &= w_j \rho t[1 - H(\alpha)] + w_i\{\rho t[1 - H(\alpha)] + t[1 - H(\alpha + \rho t)]\} \\ &\quad - w_i t[1 - H(\alpha)] - w_j\{t[1 - H(\alpha)] + \rho t[1 - H(\alpha + t)]\} \\ &= (w_i + w_j)t\{(\rho - 1)[1 - H(\alpha)] + \xi[1 - H(\alpha + \rho t)] - (1 - \xi)\rho[1 - H(\alpha + t)]\}, \end{aligned} \quad (17)$$

where $\xi = w_i/(w_i + w_j)$. Because $w_i \geqslant w_j$, it implies $1/2 \leqslant \xi \leqslant 1$. Thus, Eq. (17) is reduced to

$$\begin{aligned} &(w_i + w_j)t\{(\rho - 1)[1 - H(\alpha)] + \xi[1 - H(\alpha + \rho t)] - (1 - \xi)\rho[1 - H(\alpha + t)]\} \\ &\geqslant (w_i + w_j)t\{(\rho - 1)[1 - H(\alpha)] + [1 - H(\alpha + \rho t)]/2 - \rho[1 - H(\alpha + t)]/2\} \\ &= (w_i + w_j)t\{(\rho - 1)[1 - H(\alpha)]/2 + I_1(\rho, t)/2\} \geqslant 0. \end{aligned} \quad (18)$$

Case 2. For $\alpha + \rho t > \alpha + t \geqslant k_0 > \alpha$, we have

$$\begin{aligned}
&[w_jC_j(S_1) + w_iC_i(S_1)] - [w_iC_i(S_2) + w_jC_j(S_2)] \\
&= (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)] + \xi[1 - H(\alpha + \rho t)] - (1 - \xi)\rho[1 - H(\alpha + t)]\} \\
&\geqslant (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)] + [1 - H(\alpha + \rho t)]/2 - \rho[1 - H(\alpha + t)]/2\} \\
&= (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)]/2 + I_2(\rho, t)/2\} \geqslant 0.
\end{aligned} \tag{19}$$

Case 3. For $\alpha + \rho t \geqslant k_0 > \alpha + t > \alpha$, we have

$$\begin{aligned}
&[w_jC_j(S_1) + w_iC_i(S_1)] - [w_iC_i(S_2) + w_jC_j(S_2)] \\
&= (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)] + \xi[1 - H(\alpha + \rho t)] - (1 - \xi)\rho[1 - F(\alpha + t)]\} \\
&\geqslant (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)] + [1 - H(\alpha + \rho t)]/2 - \rho[1 - F(\alpha + t)]/2\} \\
&= (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)]/2 + I_3(\rho, t)/2\} \geqslant 0.
\end{aligned} \tag{20}$$

Case 4. For $k_0 > \alpha + \rho t > \alpha + t > \alpha$, we have

$$\begin{aligned}
&[w_jC_j(S_1) + w_iC_i(S_1)] - [w_iC_i(S_2) + w_jC_j(S_2)] \\
&= (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)] + \xi[1 - F(\alpha + \rho t)] - (1 - \xi)\rho[1 - F(\alpha + t)]\} \\
&\geqslant (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)] + [1 - F(\alpha + \rho t)]/2 - \rho[1 - F(\alpha + t)]/2\} \\
&= (w_i + w_j)t\{(\rho - 1)[1 - F(\alpha)]/2 + I_4(\rho, t)/2\} \geqslant 0.
\end{aligned} \tag{21}$$

From Cases 1–4, we have shown that $w_jC_j(S_1) + w_iC_i(S_1) \geqslant w_iC_i(S_2) + w_jC_j(S_2)$. This proves Property 3.
Next, we provide the optimal solution for the total tardiness problem under the proposed models.

Property 4. *The optimal sequence for the problem* $1|\mathrm{L} + \mathrm{F}|\sum T_j$ *is the EDD sequence if* p_i *and* d_i *are agreeable.*

Proof. By contradiction, suppose that there is an optimal sequence S_1 that does not follow the EDD rule. That is, there are job j followed by job i, such that $d_j > d_i$. Let S_2 be a new sequence derived by performing an adjacent pairwise interchange of jobs i and j without changing the positions of other jobs. By Property 1, $\sum_{j\in\pi'} T_j(S_1) \geqslant \sum_{j\in\pi'} T_j(S_2)$. Thus, to show that $T_i(S_1) + T_j(S_1) \geqslant T_j(S_2) + T_i(S_2)$ is sufficient to show the contradiction. To prove this property, four cases are considered:

Case 1. For $\alpha + \rho t > \alpha + t > \alpha \geqslant k_0$, the tardiness of jobs j and i in S_1 are

$$T_j(S_1) = \max\{A + \rho t[1 - H(\alpha)] - d_j, 0\},$$

and

$$T_i(S_1) = \max\{A + \rho t[1 - H(\alpha)] + t[1 - H(\alpha + \rho t)] - d_i, 0\}.$$

The tardiness of jobs i and j in S_2 are

$$T_i(S_2) = \max\{A + t[1 - H(\alpha)] - d_i, 0\},$$

and

$$T_j(S_2) = \max\{A + t[1 - H(\alpha)] + \rho t[1 - H(\alpha + t)] - d_j, 0\}.$$

Furthermore, we divide it into four sub-cases as:

Case 1.1. For $A + \rho t[1 - H(\alpha)] > d_j$ and $A + t[1 - H(\alpha)] > d_i$, it is obvious that

$$\begin{aligned}
&T_j(S_1) = A + \rho t[1 - H(\alpha)] - d_j > 0, \\
&T_i(S_1) = A + \rho t[1 - H(\alpha)] + t[1 - H(\alpha + \rho t)] - d_i > d_j + t[1 - H(\alpha + \rho t)] - d_i > 0, \\
&T_i(S_2) = A + t[1 - H(\alpha)] - d_i > 0, \\
&T_j(S_2) = \max\{A + t[1 - H(\alpha)] + \rho t[1 - H(\alpha + t)] - d_j, 0\}.
\end{aligned}$$

If $T_j(S_2) = A + t[1 - H(\alpha)] + \rho t[1 - H(\alpha + t)] - d_j > 0$, it implies

$$\begin{aligned}
&[T_j(S_1) + T_i(S_1)] - [T_i(S_2) + T_j(S_2)] \\
&= 2(\rho - 1)t[1 - H(\alpha)] + t[1 - H(\alpha + \rho t)] - \rho t[1 - H(\alpha + t)] \\
&= t\{(\rho - 1)[1 - H(\alpha)] + I_1(\rho, t)\} \geqslant 0.
\end{aligned}$$

Thus, $[T_j(S_1) + T_i(S_1)] - [T_i(S_2) + T_j(S_2)]$ is also non-negative as $T_j(S_2) = 0$.

Case 1.2. For $A+\rho t[1-H(\alpha)] \leqslant d_j$ and $A+t[1-H(\alpha)] > d_i$, it is obvious that

$$\begin{aligned}
&T_j(S_1)=0,\\
&T_i(S_1)=A+\rho t[1-H(\alpha)]+t[1-H(\alpha+\rho t)]-d_i > A+t[1-H(\alpha)]-d_i > T_i(S_2)>0,\\
&T_i(S_2)=A+t[1-H(\alpha)]-d_i>0,\\
&T_j(S_2)=\max\{A+t[1-H(\alpha)]+\rho t[1-H(\alpha+t)]-d_j,0\}.
\end{aligned}$$

Since $T_i(S_1) > T_i(S_2) > 0$, we have $[T_j(S_1)+T_i(S_1)]-[T_i(S_2)+T_j(S_2)]>0$ as $T_j(S_2)=0$. If $T_j(S_2)>0$, we have

$$\begin{aligned}
&[T_j(S_1)+T_i(S_1)]-[T_i(S_2)+T_j(S_2)]\\
&= d_j - A + \rho t[1-H(\alpha)] + t[1-H(\alpha+\rho t)] - 2t[1-H(\alpha)] - \rho t[1-H(\alpha+t)]\\
&= tI_1(\rho,t)+d_j-A-t[1-H(\alpha)] \geqslant 0.
\end{aligned}$$

Case 1.3. For $A+\rho t[1-H(\alpha)] > d_j$ and $A+t[1-H(\alpha)] \leqslant d_i$, it is obvious that

$$\begin{aligned}
&T_j(S_1)=A+\rho t[1-H(\alpha)]-d_j>0,\\
&T_i(S_1)=A+\rho t[1-H(\alpha)]+t[1-H(\alpha+\rho t)]-d_i > d_j+t[1-H(\alpha+\rho t)]-d_i>0,\\
&T_i(S_2)=0,\\
&T_j(S_2)=\max\{A+t[1-H(\alpha)]+\rho t[1-H(\alpha+t)]-d_j,0\}.
\end{aligned}$$

Thus, if $T_j(S_2)=0$, $[T_j(S_1)+T_i(S_1)]-[T_i(S_2)+T_j(S_2)]>0$. When $T_j(S_2)>0$, we have

$$\begin{aligned}
&[T_j(S_1)+T_i(S_1)]-[T_i(S_2)+T_j(S_2)]\\
&= A-d_i+2\rho t[1-H(\alpha)]-t[1-H(\alpha)]+t[1-H(\alpha+\rho t)]-\rho t[1-H(\alpha+t)]\\
&= tI_1(\rho,t)+A+\rho t[1-H(\alpha)]-d_i\\
&> tI_1(\rho,t)+(d_j-d_i)>0.
\end{aligned}$$

Case 1.4. For $A+\rho t[1-H(\alpha)] < d_j$ and $A+t[1-H(\alpha)] < d_i$, it is obvious that

$$\begin{aligned}
&T_j(S_1)=0,\\
&T_i(S_1)=\max\{A+\rho t[1-H(\alpha)]+t[1-H(\alpha+\rho t)]-d_i,0\},\\
&T_i(S_2)=0,\\
&T_j(S_2)=\max\{A+t[1-H(\alpha)]+\rho t[1-H(\alpha+t)]-d_j,0\}.
\end{aligned}$$

Because $I_1(\rho,t)=(\rho-1)[1-H(\alpha)]+[1-H(\alpha+\rho t)]-\rho[1-H(\alpha+t)] \geqslant 0$, we have

$$\begin{aligned}
&\rho t[1-H(\alpha)]+t[1-H(\alpha+\rho t)] > t[1-H(\alpha)]+\rho t[1-H(\alpha+t)]\\
&\Rightarrow A+\rho t[1-H(\alpha)]+t[1-H(\alpha+\rho t)]-d_i > A+t[1-H(\alpha)]+\rho t[1-H(\alpha+t)]-d_j.
\end{aligned}$$

Therefore, $T_j(S_2)$ must be zero, if $T_i(S_1)$ equals to zero. When $T_i(S_1)>0$, the difference $[T_j(S_1)+T_i(S_1)]-[T_i(S_2)+T_j(S_2)]$ is either $T_i(S_1)$ or $tI_1(\rho,t)+(d_j-d_i)$ which are both non-negative. From Cases 1.1–1.4, we have $[T_j(S_1)+T_i(S_1)] > [T_i(S_2)+T_j(S_2)]$ in Case 1. This contradicts the optimality of S_1 and completes the proof of Case 1. The proofs of Cases 2–4 are similar thus omitted.

Corollary 1. *The optimal sequence for* $1|\mathrm{L+F}|L_{\max}$ *is the EDD sequence if* p_i *and* d_i *are agreeable.*

Example 1. We provide an example of five jobs to illustrate the optimal schedules for problems $1|\mathrm{L+F}|C_{\max}$, $1|\mathrm{L+F}|\sum C_j$, $1|\mathrm{L+F}|\sum w_jC_j$, $1|\mathrm{L+F}|\sum T_j$ and $1|\mathrm{L+F}|L_{\max}$. The normal processing times, weights and due dates of jobs are listed in Table 2.

Suppose the marginal and cumulative learning effect functions are, $f(x)=0.033(1+0.05x)^{-2}$ and $F(y)=0.667y(20+y)^{-1}$, $x \geqslant 0$, respectively. The marginal and cumulative forgetting effect functions are $g(x)=0.016(1+0.05x)^{-2}$ and $G(y)=0.333y(20+y)^{-1}$ $x \geqslant 0$. Since $f(x)$ is non-increasing, $F(M) \leqslant 1$ for a sufficiently large number M and $f(x) \geqslant g(x-2)$ for $\forall x \geqslant k_0 = 2$. The optimal solution for each problem can be obtained by the SPT rule or the EDD rule, which is $(J_5, J_2, J_1, J_3, J_4)$ in this example. The data and calculation for each job are tabulated in Table 3.

Table 2
The normal processing time, weight and due date for each job.

Jobs	J_1	J_2	J_3	J_4	J_5
Normal processing times (days)	16	14	20	28	10
Weights	3	4	2	1	5
Due dates (days)	24	22	30	40	15

Table 3
The actual processing time and completion time for each job.

Scheduled jobs	J_5	J_2	J_1	J_3	J_4
Normal processing times	10	14	16	20	28
Actual processing times	10	12.444	13.090	15.556	21
Completion times	10	22.444	35.535	51.091	72.091
Weighted completion times	50	89.776	106.605	102.182	72.091
Tardiness	0	0.444	11.535	21.091	32.091
Lateness	−5	0.444	11.535	21.091	32.091

Thus, for the optimal schedule $(J_5, J_2, J_1, J_3, J_4)$, the maximum completion time is 72.091, the total completion time is 191.162, the total weighted completion time is 420.654, the total tardiness is 65.161 and the maximum lateness is 32.091.

4. Conclusions

Forgetting effects occur, particularly for products with short cycle times in which workers must constantly learn new skills without repeated practice. In this paper, we proposed a general model with learning and forgetting effects which are expressed as the functions of the sum of processing times of jobs already processed for this situation. For some single-machine problems, we proved that the optimal sequence for the makespan and the total completion time criteria is the SPT sequence. In addition, we proved that the optimal sequence for the total weighted completion time criterion is the WSPT sequence under an agreeable condition. Finally, we proved that the optimal sequence for the single-machine total tardiness and the maximum lateness problems is the EDD sequence under an agreeable condition. Jaber and Bonney [35,36] showed that forgetting is dependent on the length of the interruption with respect to the time for total forgetting, the learning exponent and the cumulative knowledge. Considering other general models that are capable of capturing the length of the interruption with respect to the time for total forgetting could be interesting topics for future research.

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