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Short communication

Modelling and stabilization of a nonlinear hybrid system of elasticity

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ABSTRACT

In this article, we briefly present a model which consists of a non-homogeneous flexible beam clamped at its left end to a rigid disk and free at the right end, where another rigid body is attached. We assume that the disk rotates with a non-uniform angular velocity while the beam is supposed to rotate with the disk in another plane perpendicular to that of the disk. Thereafter, we propose a wide class of feedback laws depending on the assumptions made on the physical parameters. In each case, we show that whenever the angular velocity is not exceeding a certain upper bound, the beam vibrations decay exponentially to zero and the disk rotates with a desired angular velocity.

1. Introduction

This work is concerned with a non-homogeneous elastic beam clamped to the center of a disk and free at the other end where a body with mass m is attached. Such systems arise in the study of large scale flexible space structures. The disk is supposed to rotate around the x-axis (see Fig. 1) without friction. In turn, the beam is clamped at the left-end $x = 0$, constrained to the x–y plane and all the deflections are assumed to be parallel to the y-axis (see Fig. 1). Consequently, it follows from [1] that

$$\begin{cases} \rho(x)y_{tt} + (EI(x)y_{xx})_{xx} = \rho(x)\,\omega^2(t)y, & (x,t) \in (0,\ell) \times (0,\infty), \\ y(0,t) = y_x(0,t) = 0, & t > 0, \\ \frac{d}{dt}\left\{\omega(t)\left(I_d + \int_0^\ell \rho(x)\,y^2(x,t)\,dx\right)\right\} = \beta\mathcal{T}(t), & t > 0. \end{cases} \tag{1.1}$$

Here x denotes the position and t represents the time. Moreover, y is the beam's displacement, ω is the angular velocity of the disk, ℓ is the length of the beam, I_d is the disk's moment of inertia and $EI(x),\ \rho(x)$ are respectively the flexural rigidity and the mass per unit length of the beam satisfying

$$0 < \rho_0 < \rho(x) \in C^4[0,\ell], \quad 0 < EI_0 < EI(x) \in C^4[0,\ell]. \tag{1.2}$$

Furthermore, β is a positive feedback gain and $\mathcal{T}(t)$ is the control torque.

Now, we turn to the dynamics of the rigid body attached to the other end of the beam. We have [2]:

$$y(\ell,t) = \varsigma(t), \quad y_x(\ell,t) = \vartheta(t),$$

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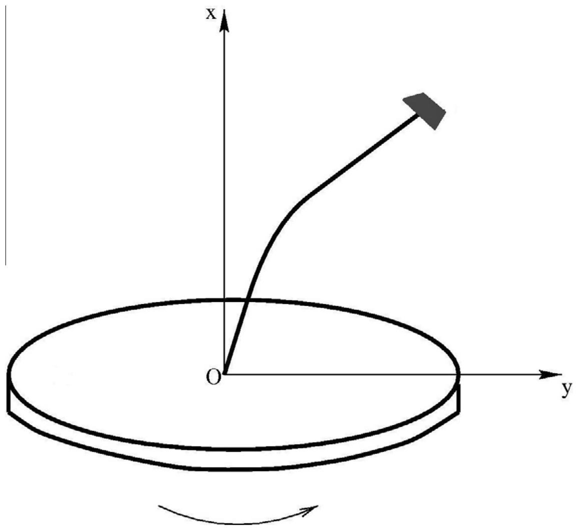

Fig. 1. Disk-beam-body system.

in which $\varsigma(t)$ is the transverse displacement of the centroid of this rigid body, while $\vartheta(t)$ gives the direction its normal makes with the x-axis. Next, we shall neglect, as in [2], the effect of the non-inertial force terms on the object of mass m. Then based on the Newton–Euler principles, the dynamics of the rigid body of mass m are given by (see [2] for more details)

$$\begin{cases} m\ddot{\varsigma}(t) = my_{tt}(\ell,t) = (EI(x)y_{xx})_x(\ell,t) + \alpha_1\Theta_1(t), & t > 0, \\ J\ddot{\vartheta}(t) = Jy_{xtt}(\ell,t) = -(EI(x)y_{xx})(\ell,t) + \alpha_2\Theta_2(t), & t > 0, \end{cases} \tag{1.3}$$

where J is the moment of inertia of the rigid body attached at the right end of the beam; α_1 and α_2 are nonnegative constant feedback gains such that $\boldsymbol{\alpha_1 + \alpha_2 \neq 0}$ and $\Theta_1(t)$, $\Theta_2(t)$ are respectively the control force and the control moment. This, together with (1.1) and (1.3), allows us to claim that the dynamics of motion of our global system are given by the following system

$$\begin{cases} \rho(x)y_{tt} + (EI(x)y_{xx})_{xx} = \rho(x)\,\omega^2(t)y, & (x,t) \in (0,\ell)\times(0,\infty), \\ y(0,t) = y_x(0,t) = 0, & t > 0, \\ my_{tt}(\ell,t) - (EI(x)y_{xx})_x(\ell,t) = \alpha_1\Theta_1(t), & t > 0, \\ Jy_{xtt}(\ell,t) + (EI(x)y_{xx})(\ell,t) = \alpha_2\Theta_2(t), & t > 0, \\ \frac{d}{dt}\left\{\omega(t)\left(I_d + \int_0^\ell \rho(x)y^2(x,t)\,dx\right)\right\} = \beta\mathcal{T}(t), & t > 0. \end{cases} \tag{1.4}$$

In this work, we shall provide several feedback control laws and then show the exponential stability of the closed loop system. To be more precise, the stabilization result will be established in a number of situations depending on the smallness of the dynamical terms $my_{tt}(\ell,t)$ and $Jy_{xtt}(\ell,t)$. This extends the results available in literature in two directions. First, our results generalize those of [3–9] where neither the acceleration term $my_{tt}(\ell,t)$ nor the moment of inertia term $Jy_{xtt}(\ell,t)$ is present in the system (1.4). Secondly, we are also able to extend the stability results of [10–12,2] and related works to the case of the presence of a nonlinear coupling term $\rho(x)\omega^2(t)y(x,t)$. The crucial tool of the proof of our main results, namely, the exponential stability of the closed loop system is the utilization of the principal theorem in [13] for an uncoupled system. This strategy, due to Laousy et al. [7], has been adopted in many previous works [3–6].

The paper is organized as follows. In the next section, we shall assume that $my_{tt}(\ell,t)$ is not neglected and accordingly, we provide two different feedback laws depending on the smallness of the other dynamical term $Jy_{xtt}(\ell,t)$. It is worth mentioning that, in this case, the main difference of the controls resides in the simplicity feature. In fact, the first controls are of higher order whereas the second ones are known to be simple. Section 3 will be devoted to a thorough analysis of the system (1.4) without the acceleration term $my_{tt}(\ell,t)$. Once again, in such a situation, two feedback laws are proposed to stabilize the system. Finally, this note closes with conclusions and discussions.

2. The acceleration term $my_{tt}(\ell,t)$ is not negligible

Throughout this section, it will be assumed that the dynamical term $my_{tt}(\ell,t)$ cannot be neglected. Then, we will deal with the stabilization problem of the system (1.4) when the other dynamical term $Jy_{xtt}(\ell,t)$ has a significant value as well as when it is too small to be taken into consideration.

2.1. The dynamical term $Jy_{xtt}(\ell,t)$ cannot be neglected

In this subsection, we further suppose that neither the moment of inertia J nor its dynamical term $y_{xtt}(\ell,t)$ can be omitted. Subsequently, for $t>0$, we propose the feedback law:

$$\begin{cases}\Theta_1(t)=y_{xxxt}(\ell,t),\\ \Theta_2(t)=-y_{xxt}(\ell,t),\\ \mathcal{T}(t)=-(\omega(t)-\Omega),\quad \Omega\in\mathbb{R}.\end{cases} \tag{2.1}$$

Note that the boundary controls are of higher order (the reader is referred to the work [14] for more details about this type of controls, their physical interpretation and implementations).

In order to study in details the closed loop system (1.4) and (2.1), let

$$L^2(0,\ell)=\{\mathrm{f}:(0,\ell)\to\mathbb{R};\ \mathrm{f}\text{ is measurable and }\int_0^\ell |f(x)|^2\,dx<\infty\}$$

and the Sobolev space

$$H^k(0,\ell)=\{\mathrm{f}:(0,\ell)\to\mathbb{R};\ \mathrm{f}^{(k)}\in L^2(0,\ell),\ \text{for } k\in\mathbb{N}\}$$

equipped with their usual norms. Moreover, let

$$H_c^n(0,\ell)=\left\{f\in H^n(0,\ell); f(0)=f_x(0)=0\right\}\quad\text{for}\quad n=2,3,\ldots \tag{2.2}$$

and the state space

$$\mathcal{X}_1\times\mathbb{R}=\left(H_c^2(0,\ell)\times L^2(0,\ell)\times\mathbb{R}^2\right)\times\mathbb{R}$$

equipped with the following inner product

$$\begin{aligned}\langle(y,z,r_1,r_2,\omega),(\tilde y,\tilde z,\tilde r_1,\tilde r_2,\tilde\omega)\rangle_{\mathcal{X}_1\times\mathbb{R}}&=\langle(y,z,r_1,r_2),(\tilde y,\tilde z,\tilde r_1,\tilde r_2)\rangle_{\mathcal{X}_1}+\omega\tilde\omega\\ &=\int_0^\ell\left[EI(x)\,y_{xx}\tilde y_{xx}-\Omega^2\rho(x)\,y\tilde y+\rho(x)\,z\tilde z\right]dx+\frac{1}{m}r_1\tilde r_1+\frac{1}{J}r_2\tilde r_2+\omega\tilde\omega.\end{aligned} \tag{2.3}$$

It is known that $\mathcal{X}_1$ is a Hilbert space (see [5]) provided that

$$|\Omega|<\frac{2}{\ell^2}\sqrt{3EI_0/\|\rho\|_\infty}, \tag{2.4}$$

which will be assumed to be true throughout the paper.

Clearly, the system (1.4) and (2.1) writes

$$\frac{d}{dt}\begin{pmatrix}\chi_1(t)\\ \omega(t)\end{pmatrix}=\left[\begin{pmatrix}A_{1\Omega}&0\\0&0\end{pmatrix}+\mathcal{E}_1\right]\begin{pmatrix}\chi_1(t)\\ \omega(t)\end{pmatrix}, \tag{2.5}$$

where $\chi_1(t)=(y(\cdot,t),y_t(\cdot,t),\ my_t(\ell,t)-\alpha_1y_{xxx}(\ell,t),\ Jy_{xt}(\ell,t)+\alpha_2y_{xx}(\ell,t))$ and the unbounded linear operator $A_{1\Omega}$ is defined, on the Hilbert space $\mathcal{X}_1=H_c^2(0,\ell)\times L^2(0,\ell)\times\mathbb{R}^2$, as follows

$$\begin{aligned}\mathcal{D}(A_{1\Omega})\ &=\Big\{\chi_1=(y,z,r_1,r_2)\in H_c^4(0,\ell)\times H_c^2(0,\ell)\times\mathbb{R}^2;\\ &\qquad r_1=mz(\ell)-\alpha_1y_{xxx}(\ell);\ r_2=Jz_x(\ell)+\alpha_2y_{xx}(\ell)\},\\ A_{1\Omega}\chi_1\ &=\left(z,-\frac{1}{\rho(x)}(EI(x)y_{xx})_{xx}+\Omega^2y,(EI(x)y_{xx})_x(\ell),-(EI(x)y_{xx})(\ell)\right).\end{aligned} \tag{2.6}$$

Furthermore, for any $T>0$ the operator $\mathcal{E}_1:[0,T]\times\mathcal{X}_1\times\mathbb{R}\longrightarrow\mathcal{X}_1\times\mathbb{R}$ is nonlinear and defined by

$$\mathcal{E}_1(t,\chi_1,\omega)=\left(0,\ (\omega^2(t)-\Omega^2)y,\ 0,\ 0,\ \frac{-\beta(\omega(t)-\Omega)-2\omega(t)\,\langle\rho y,z\rangle_{L^2(0,\ell)}}{I_d+\|\sqrt{\rho}y\|^2_{L^2(0,\ell)}}\right). \tag{2.7}$$

For sake of simplicity and without loss of generality, we shall assume throughout this section that $\alpha_1=0$ (only the control moment is applied to the system). Indeed, the case $\alpha_2=0$ (only the force control is exerted) can be treated in a similar way.

It is well-known that the theory of semigroups of linear (and nonlinear) operators plays an important role when dealing with such systems. This theory is extensively exposed in many books, from the classic Hille-Phillips monograph [15] to the most recent textbooks of Engel and Nagel [16,17]. Nevertheless, we shall provide a very brief introduction to this theory in Hilbert spaces and present some basic results.

Definition 1. Let H be a Hilbert space and $S(t)$ be a bounded linear operator $S(t):H\longrightarrow H$. Then, $S(t)$ is a C_0-semigroup of contractions if

- $S(0) = I$;
- $S(t+s) = S(t)S(s)$, $\forall t, s \geqslant 0$;
- $\lim_{t\to 0^+} S(t)x = x$, for all $x \in H$;
- $\|S(t)\| \leqslant 1$, for any $t \geqslant 0$.

The linear operator A defined by

$$D(A) = \left\{ x \in H, \lim_{t\to 0^+} \frac{S(t)x - x}{t} \text{ exists} \right\}$$

and

$$Ax = \lim_{t\to 0^+} \frac{S(t)x - x}{t} = \frac{d^+ S(t)x}{dt}\bigg|_{t=0}, \quad \forall x \in D(A)$$

is called infinitesimal generator of $S(t)$.

Definition 2. A real-valued linear operator A on a Hilbert space H is dissipative if $\langle Ax, x\rangle \leqslant 0,\ \forall x \in D(A)$.

The following theorem will be systematically used throughout this paper.

Theorem 1 (*Lumer-Phillips [18]*). *Let A be a linear operator on a Hilbert space H with $\overline{D(A)} = H$. If A is dissipative and there is a scalar λ_0 such that the range $R(\lambda_0 I - A) = H$, then A is the infinitesimal generator of a C_0 semigroup of contractions on H.*

We also need to provide the definitions of the stability of semigroups.

Definition 3. A semigroup $S(t)$ is said to be strongly stable on if $\lim_{t\to\infty}\|S(t)x\| = 0$, for any $x \in H$. In turn, the semigroup is exponentially stable if there exist positive constants M and m such that $\|S(t)\| \leqslant Me^{-mt}$ for all $t \geqslant 0$.

Finally, we close this review part by recalling Huang's result:

Theorem 2 [19]. *Let A be the infinitesimal generator of a C_0-semigroup $S(t)$ in H with compact resolvent. Then, $S(t)$ is strongly stable if and only if it is uniformly bounded and $Re\lambda < 0$, for any λ in the spectrum of A.*

Now, we are ready to deal with the well-posedness problem of our system.

Lemma 1. *Assume that (2.4) holds. Then, for any initial data $\Upsilon_0 \in \mathcal{X}_1 \times \mathbb{R}$, the closed loop system (2.5) has a unique mild global bounded solution $\Upsilon(t) \in \mathcal{X}_1 \times \mathbb{R}$. In return, if $\Upsilon_0 \in \mathcal{D}(A_{1\Omega}) \times \mathbb{R}$, there exits a unique classical global solution $\Upsilon(t) \in \mathcal{D}(A_{1\Omega}) \times \mathbb{R}$.*

Proof. A straightforward computation shows that the operator $A_{1\Omega}$ defined by (2.6) is dissipative. To be more specific, using (2.3), (2.6) and integrating by parts, we have

$$\begin{aligned}
\langle A_{1\Omega}\chi_1, \chi_1\rangle_{\mathcal{X}_1} &= \int_0^\ell \left(EI(x)y_{xx}z_{xx} - (EI(x)y_{xx})_{xx}z\right)dx + \frac{1}{m}(EI(x)y_{xx})_x(\ell)r_1 - \frac{1}{J}(EI(x)y_{xx})(\ell)r_2 \\
&= -\frac{\alpha_2}{J}(EI(x)y_{xx}^2)(\ell) \leqslant 0
\end{aligned} \tag{2.8}$$

for any $\chi_1 = (y, z, r_1, r_2) \in \mathcal{D}(A_{1\Omega})$.

Next, one could easily solve the resolvent equation $(\lambda I - A_{1\Omega})(y, z, r_1, r_2) = (f, g, \xi, \eta)$ for sufficiently small $\lambda > 0$ by means of Lax–Milgram Theorem [20]. Indeed, we firstly solve the above resolvent equation with $\lambda = 0$. This gives

$$\begin{cases}
(EI(x)y_{xx})_{xx} - \Omega^2\rho(x)y = \rho(x)g, \\
y(0) = y_x(0) = 0, \\
(EI(x)y_{xx})_x(\ell) = -\xi, \\
(EI(x)y_{xx})(\ell) = \eta, \\
z = -f, \\
r_1 = -mf(\ell) - \alpha_1 y_{xxx}(\ell), \\
r_2 = -Jf_x(\ell).
\end{cases} \tag{2.9}$$

Obviously, it suffices to seek y. To proceed, let $\phi \in H_c^2(0,\ell)$. A simple integration by parts yields

$$\int_0^\ell \left(EI(x)y_{xx}\phi_{xx} - \Omega^2\rho(x)y\phi\right)dx = \int_0^\ell \rho(x)g\phi dx + \eta\phi_x(\ell) + \xi\phi(\ell), \tag{2.10}$$

which can be written as $a(y,\phi) = J(\phi)$, where

$$a : H_c^2(0,\ell) \times H_c^2(0,\ell) \longrightarrow \mathbb{R}$$
$$(y,\phi) \mapsto a(y,\phi) = \int_0^\ell \Big(EI(x)\Omega^2 y_{xx}\phi_{xx} - \Omega^2 \rho(x) y \phi \Big) dx$$

and

$$J : H_c^2(0,\ell) \longrightarrow \mathbb{R}$$
$$\phi \mapsto J(\phi) = \int_0^\ell \rho(x) g \phi dx + \eta \phi_x(\ell) + \xi \phi(\ell).$$

Subsequently, invoking (2.4) we can show that $a(y,\psi)$ is a continuous coercive bilinear form on $H_c^2(0,\ell) \times H_c^2(0,\ell)$ and $J(\phi)$ is a continuous coercive linear form on $H_c^2(0,\ell)$. Thus, it follows from Lax–Milgram that there exists a unique solution $y \in H_c^2(0,\ell)$ of (2.10). Then, by standard argument used for solving elliptic linear equations we can recover the boundary conditions in (2.9). Hence $(A_{1\Omega})^{-1} \in \mathcal{L}(\mathcal{X}_1)$ and so finally $(\lambda I - A_{1\Omega})(y,z,r_1,r_2) = (f,g,\xi,\eta)$ for sufficiently small $\lambda > 0$ [21]. Thereafter, using the compactness of the canonical embedding $i : \mathcal{D}(A_{1\Omega}) \to \mathcal{X}_1$ [20], we have $(A_{1\Omega})^{-1} \in \mathcal{L}(\mathcal{X}_1)$ is compact.

As a direct consequence of semigroups theory, the operator $A_{1\Omega}$ is densely defined closed in $\mathcal{X}_1$ and generates a C_0-semigroup of contractions $S(t)$ on $\mathcal{X}_1$ [18].

In order to deal with the well-posedness of the global system (2.5), one can verify that the operator $\mathcal{E}_1$ is continuously differentiable [22] in the sense of Fréchet (see [9] for a simpler case). In this light and using the fact that $A_{1\Omega}$ generates a C_0-semigroup of contractions $S(t)$ on $\mathcal{X}_1$, it follows that for any $\Upsilon_0 = (\chi_0, \omega_0) \in \mathcal{X}_1 \times \mathbb{R}$, there exists a unique local mild solution $\Upsilon(\cdot) = (\chi_1(\cdot), \omega(\cdot)) \in C([0,T]; \mathcal{X}_1 \times \mathbb{R})$ of the system (2.5), for some $T > 0$ [18]. In turn, any local mild solution of (2.5) stemmed from a smooth initial data in $\mathcal{D}(A_{1\Omega}) \times \mathbb{R}$ is a strong one [18]. Finally, with regard to the global existence, it can be verified, thanks to (2.4), that the following functional

$$\mathcal{V}_1(\chi_1(t),\omega(t)) = \frac{1}{2}(\omega(t)-\Omega)^2\left(I_d + \int_0^\ell \rho(x) y^2(x,t)dx\right) + \frac{1}{2} y_t^2(\ell,t) + \frac{1}{2J}(J y_{xt}(\ell,t) + \alpha_2 y_{xx}(\ell,t))^2$$
$$+ \frac{1}{2}\int_0^\ell (\rho(x) y_t^2(x,t) + EI(x) y_{xx}^2(x,t))dx - \frac{\Omega^2}{2}\int_0^\ell \rho(x) y^2(x,t)dx$$

is a Lyapunov function. Indeed, it follows from Poincaré inequality that

$$2\mathcal{V}_1(\chi_1(t),\omega(t)) \geqslant (\omega(t)-\Omega)^2\left(I_d + \int_0^\ell \rho(x) y^2(x,t)dx\right) + y_t^2(\ell,t) + \frac{1}{J}(J y_{xt}(\ell,t) + \alpha_2 y_{xx}(\ell,t))^2$$
$$+ \int_0^\ell \Big(\rho(x) y_t^2(x,t) + \Big(EI(x) - \rho(x)\ell^2\Omega^2/12\Big) y_{xx}^2(x,t)\Big)dx - \Omega^2 \int_0^\ell \rho(x) y^2(x,t)dx,$$

which in turn implies by means of (2.4) that $\mathcal{V}_1(\chi_1(t),\omega(t)) \geqslant K\|(\chi_1(t),\omega(t))\|^2_{\mathcal{X}_1\times\mathbb{R}}$, for some positive constant K and for all $(\chi_1(t),\omega(t)) \in \mathcal{X}_1 \times \mathbb{R}$. Furthermore, arguing as for (2.8) we get for any solution stemmed from an initial data in $\mathcal{D}(A_{1\Omega}) \times \mathbb{R}$

$$\dot{\mathcal{V}}_1(\chi_1(t),\omega(t)) = -\beta(\omega-\Omega)^2 - (\alpha_2/J)(EI(x)y_{xx}^2)(\ell,t) \leqslant 0. \tag{2.11}$$

Consequently, the solution of (2.5) corresponding to the initial condition $\Upsilon_0 \in \mathcal{D}(A_{1\Omega}) \times \mathbb{R}$ exists globally in a classical sense and is bounded. At last, using Theorem 1.4 in [18], one can conclude that each weak solution exists globally and is bounded.

Remark 1.

(i) It is worth mentioning that Lemma 1 remains valid even if the assumption (1.2) is relaxed. To be more precise, one could suppose that $0 < \rho_0 < \rho(x) \in L^\infty(0,\ell)$, $0 < EI_0 < EI(x) \in L^\infty(0,\ell)$. However, (1.2) is required for the stability result (see the proof of Theorem 3).

(ii) It follows from (2.5), Lemma 1 and the fact that $\mathcal{V}$ is a Lyapunov functional that the solution $(\chi_1(t),\omega(t))$ is bounded in $\mathcal{X}_1 \times \mathbb{R}$, $\int_0^{+\infty}(\omega(t)-\Omega)^2 dt$ converges and $\omega(t)-\Omega$ as well as its derivative $\frac{d}{dt}(\omega(t)-\Omega)$ are bounded (see [7] for more details). This implies thanks to Barbalat's lemma [23] that $\lim_{t\to+\infty}\omega(t) = \Omega$ and hence for all $\epsilon > 0$, there exists τ sufficiently large such that for any $t \geqslant \tau$

$$|\omega^2(t) - \Omega^2| < \epsilon. \tag{2.12}$$

The main result of this section is:

Theorem 3. *For each angular velocity Ω satisfying (2.4) and for each initial data $\Upsilon_0 \in \mathcal{D}(A_{\Omega 1}) \times \mathbb{R}$, the solution $\Upsilon(t) = (\chi_1(t),\omega(t))$ of the system (2.5) exponentially converges to the equilibrium point $(0_{\mathcal{X}_1},\Omega)$ in $\mathcal{X}_1 \times \mathbb{R}$ as $t \to \infty$.*

Proof. Define the linear compact operator $\mathcal{K}_1$ on $\mathcal{X}_1$ as follows: $\mathcal{K}_1(y,z,r_1,r_2)=(0,y,0,0)$. In the light of the results obtained in [12] under the condition (1.2), the operator $A_{1\Omega}-\Omega^2\mathcal{K}_1$ generates a uniformly exponentially stable semigroup. This, together with the compactness of $\mathcal{K}_1$, implies that the strong stability of the semigroup $S(t)$, generated by the operator $A_{1\Omega}$, will allow us to deduce its exponential stability [13]. Therefore, our immediate objective now is to establish the strong stability of the semigroup $S(t)$. To achieve this, it follows from the proof of Lemma 1 that the spectrum $\sigma(A_{1\Omega})$ of $A_{1\Omega}$ consists of only isolated eigenvalues with finite multiplicity [21] and $\mathrm{Re}\lambda \leqslant 0$ for each $\lambda\in\sigma(A_{1\Omega})$. Consequently, the strong stability of the semigroup $S(t)$ is showed as long as one establishes the following resolvent set property $\rho(A_{1\Omega})\supset\{i\omega;\ \omega\in\mathbb{R}\}$ (see Theorem 2). Were this not true, there would be a nonzero $\zeta\in\mathbb{R}$ such that $i\zeta\in\sigma(A_{1\Omega})$, that is, there exists $\chi_0=(y,z,r_1,r_2)\in\mathcal{D}(A_{1\Omega})$ with $\|\chi_0\|_{\mathcal{X}_1}\neq 0$ such that $A_{1\Omega}\chi_0=i\zeta\chi_0$. This immediately yields

$$\begin{cases}(EI(x)y_{xx})_{xx}-\rho(x)(\zeta^2+\Omega^2)y=0,\\ y(0)=y_x(0)=0\\ -(EI(x)y_{xx})(\ell)=-J\zeta^2y_x(\ell)+i\alpha_2\zeta y_{xx}(\ell),\\ (EIy_{xx})_x(\ell)=-m\zeta^2y(\ell)\\ z=i\zeta y,\\ r_1=im\zeta y(\ell),\quad r_2=iJ\zeta y_x(\ell)+\alpha_2y_{xx}(\ell).\end{cases}\tag{2.13}$$

Then, let $\chi(t)=e^{i\zeta t}\chi_0$. Obviously, $\frac{d}{dt}\left(\|\chi(t)\|^2_{\mathcal{X}_1}\right)=0$, and thus by (2.8), we have $y_{xx}(\ell)=0$. Hence $r_2=y_x(\ell)=0$ and y is solution of the following system:

$$\begin{cases}(EI(x)y_{xx})_{xx}-\rho(x)(\zeta^2+\Omega^2)y=0,\\ y(0)=y_x(0)=y_x(\ell)=y_{xx}(\ell)=0,\\ (EIy_{xx})_x(\ell)=-m\zeta^2y(\ell).\end{cases}\tag{2.14}$$

Arguing as in the proof of Theorem 4.2 in [12], we have $y=0$ which will lead to a contradiction with the fact that $\|\chi_0\|_{\mathcal{X}_1}\neq 0$. Therefore the operator $A_{1\Omega}$ has no purely imaginary eigenvalues and hence the semigroup $S(t)$ generated by $A_{1\Omega}$ is uniformly exponentially stable.

With regard to stability of the global system (2.5), let $\Upsilon(t)=(\chi_1(t),\omega(t))=(y,y_t,r_1,r_2,\omega)$ be the unique solution of the closed loop system (2.5) subject to a smooth initial data $\Upsilon_0\in\mathcal{D}(A_{1\Omega})\times\mathbb{R}$. Then, it is easy to check that

$$\dot\chi_1(t)=\left[A_{1\Omega}+(\omega^2(t)-\Omega^2)\mathcal{K}_1\right]\chi_1(t)\tag{2.15}$$

and

$$\dot\omega(t)=\frac{-\beta(\omega-\omega_\Omega)-2\,\omega(t)\,\langle\rho(x)y,\ y_t\rangle_{L^2(0,\ell)}}{I_d+\|\sqrt{\rho}(x)y\|^2_{L^2(0,\ell)}},\tag{2.16}$$

in which $\mathcal{K}_1(y,z,r_1,r_2)=(0,y,0,0)$ is defined above. Recall that we have already established, at the beginning of this proof, that the semigroup $S(t)$, generated by the operator $A_{1\Omega}$, is exponentially stable. On the other hand, $\omega(t)$ verifies (2.12). Taking into account these properties together with (2.15) and proceeding as in [9] (see also [3–5]), one can use Gronwall's inequality to show that $\chi_1(t)$ is exponentially stable in $\mathcal{X}_1$. Finally, returning to (2.16) we conclude that for all $t\geqslant 0,\ |\omega(t)-\Omega|\leqslant Me^{-mt}$, where M and m are positive constants. Thus $(\omega(t)-\Omega)\to 0$ exponentially in $\mathbb{R}$.

2.2. The dynamical term $Jy_{xtt}(\ell,t)$ is neglected

Suppose now that the dynamical term $Jy_{xtt}(\ell,t)$ in (1.4) is small so that one can neglect it. In physical terms, this corresponds for instance to the case where the rigid body, attached to the flexible beam, has high density. In view of this assumption, we propose one and only one ($\alpha_1=0$) simple control moment applied on the beam, that is, $\Theta_2(t)=-y_{xt}(\ell,t)$ combined with a torque control $\mathcal{T}(t)=-(\omega(t)-\Omega)$, for $\Omega\in\mathbb{R}$. As a consequence, the closed loop system writes

$$\begin{cases}\rho(x)y_{tt}(x,t)+(EI(x)y_{xx})_{xx}(x,t)=\rho(x)\omega^2(t)y, & (x,t)\in(0,\ell)\times(0,\infty),\\ y(0,t)=y_x(0,t)=0, & t>0,\\ my_{tt}(\ell,t)-(EI(x)y_{xx})_x(\ell,t)=0, & t>0,\\ (EI(x)y_{xx})(\ell,t)=-\alpha_2y_{xt}(\ell,t), & t>0,\\ \frac{d}{dt}\left\{\omega(t)\left(I_d+\int_0^\ell\rho(x)y^2(x,t)\,dx\right)\right\}=-\beta(\omega(t)-\Omega), & t>0.\end{cases}\tag{2.17}$$

Next, define the state space

$$\mathcal{X}_2\times\mathbb{R}=\left(H_c^2(0,\ell)\times L^2(0,\ell)\times\mathbb{R}\right)\times\mathbb{R}$$

equipped with the following inner product:

$$\langle (y,z,r_1,\omega),(\tilde{y},\tilde{z},\tilde{r}_1,\tilde{\omega})\rangle_{\mathcal{X}_2\times\mathbb{R}} = \langle (y,z,r_1),(\tilde{y},\tilde{z},\tilde{r}_1)\rangle_{\mathcal{X}_2} + \omega\tilde{\omega} = \int_0^\ell \left[EI(x)y_{xx}\tilde{y}_{xx} - \Omega^2\rho(x)y\tilde{y} + \rho(x)z\tilde{z}\right]dx + mr_1\tilde{r}_1 + \omega\tilde{\omega},$$

which is a Hilbert space as long as $|\Omega| < \frac{2}{\ell^2}\sqrt{3EI_0/\|\rho\|_\infty}$. Then, the system (2.17) has the abstract form

$$\frac{d}{dt}\begin{pmatrix}\chi_2(t)\\ \omega(t)\end{pmatrix} = \left[\begin{pmatrix}A_{2\Omega} & 0\\ 0 & 0\end{pmatrix} + \mathcal{E}_2\right]\begin{pmatrix}\chi_2(t)\\ \omega(t)\end{pmatrix}, \tag{2.18}$$

where $\chi_2(t) = (y(\cdot,t), y_t(\cdot,t), y_t(\ell,t))$ and $A_{2\Omega}$ is an unbounded linear operator defined as follows

$$\mathcal{D}(A_{2\Omega}) = \left\{\chi_2 = (y,z,r_1) \in H_c^4(0,\ell)\times H_c^2(0,\ell)\times\mathbb{R};\ r_1 = z(\ell);\ (EI(x)y_{xx})(\ell) + \alpha_2 r_{1x} = 0\right\},$$
$$A_{2\Omega}\chi_2 = \left(z, -\frac{1}{\rho(x)}(EI(x)y_{xx})_{xx} + \Omega^2 y, \frac{1}{m}(EI(x)y_{xx})_x(\ell)\right) \tag{2.19}$$

and the operator $\mathcal{E}_2$ is defined on $\mathcal{X}_2\times\mathbb{R}$ by

$$\mathcal{E}_2(t,\chi_2,\omega) = \left(0,\ (\omega^2(t)-\Omega^2)y,\ 0,\ \frac{-\beta(\omega(t)-\Omega) - 2\omega(t)\langle \rho y, z\rangle_{L^2(0,\ell)}}{I_d + \|\sqrt{\rho}y\|^2_{L^2(0,\ell)}}\right). \tag{2.20}$$

Now, it can be verified by retracing the proof of Lemma 1 that the above system has a unique global mild or classical solution depending on the regularity of the initial condition. Only one modification has to be made, namely,

$$\mathcal{V}_2(\chi_2(t),\omega(t)) = \frac{1}{2}(\omega(t)-\Omega)^2\left(I_d + \int_0^\ell \rho(x)y(x,t)dx\right) - \frac{\Omega^2}{2}\int_0^\ell \rho(x)y^2(x,t)dx + \frac{1}{2}\int_0^\ell \left[\rho(x)y_t^2(x,t) + EI(x)y_{xx}^2(x,t)\right]dx$$
$$+ \frac{m}{2}y_t^2(\ell,t).$$

Then, consider the following compact operator $\mathcal{K}_2$ on $\mathcal{X}_2 : \mathcal{K}_2(y,z,r_1) = (0,y,0)$. Next, as the operator $A_{2\Omega} - \Omega^2\mathcal{K}_2$ generates a uniformly exponentially stable semigroup [11], one can use the same arguments as in the proof of Theorem 3 to end up with a uniqueness problem of exactly the same system (2.14). Thus, as expected we have the following counterpart to Theorem 3.

Theorem 4. *Assume that the desired angular velocity Ω satisfies the condition (2.4). Then, the solution $\Upsilon(t) = (\chi_2(t),\omega(t))$ of the system (2.18) stemmed from the initial data $\Upsilon_0 \in \mathcal{D}(A_{2\Omega})\times\mathbb{R}$ exponentially tends to $(0_{\mathcal{X}_2},\Omega)$ in $\mathcal{X}_2\times\mathbb{R}$ as $t\to\infty$.*

3. The acceleration term $my_{tt}(\ell,t)$ is neglected

The objective of this section is to treat the case where the dynamical term $my_{tt}(1,t)$ can be neglected. This assumption can be explained in practice in the situation of a very light body attached to the beam. Furthermore, the reader has surely noticed in the previous section that the non-homogeneous property of the beam had no effective impact on the stabilization result of the system (1.4) whenever the conditions (1.2) are fulfilled. Consequently, we are going to consider the constant spatial version of system (1.4) with unit length, or specifically

$$\begin{cases}\rho y_{tt} + EIy_{xxxx} = \rho\omega^2(t)y, & (x,t)\in(0,1)\times(0,\infty),\\ y(0,t) = y_x(0,t) = 0, & t>0,\\ Jy_{xtt}(1,t) + y_{xx}(1,t) = \alpha_1\Theta_1(t), & t>0,\\ -y_{xxx}(1,t) = \alpha_2\Theta_2(t), & t>0,\\ \frac{d}{dt}\left\{\omega(t)\left(I_d + \int_0^1\rho(x)y^2(x,t)\,dx\right)\right\} = \beta\mathcal{T}(t), & t>0.\end{cases} \tag{3.1}$$

In turn, we suggest in this section the feedback law

$$\Theta_1(t) = -y_{xt}(1,t),\quad \Theta_2(t) = -y_t(1,t), \mathcal{T}(t) = -(\omega(t)-\Omega),\ \Omega\in\mathbb{R}. \tag{3.2}$$

Furthermore, the assumption (2.4) writes

$$|\Omega| < 2\sqrt{3EI/\rho} \tag{3.3}$$

and the closed loop system (3.1), (3.2) can be formulated in an abstract form in the state space

$$\mathcal{X}_3\times\mathbb{R} = \left(H_c^2(0,1)\times L^2(0,1)\times\mathbb{R}\right)\times\mathbb{R},$$

equipped with the following inner product

$$\langle (y,z,r_2,\omega),(\tilde{y},\tilde{z},\tilde{r}_2,\tilde{\omega})\rangle_{\mathcal{X}_3\times\mathbb{R}} = \langle (y,z,r_2),(\tilde{y},\tilde{z},\tilde{r}_2)\rangle_{\mathcal{X}_3} + \omega\tilde{\omega} = \int_0^1\left[EIy_{xx}\tilde{y}_{xx} - \Omega^2\rho y\tilde{y} + \rho z\tilde{z}\right]dx + +JEIr_2\tilde{r}_2 + \omega\tilde{\omega},$$

as follows

$$\frac{d}{dt}\begin{pmatrix}\chi_3(t)\\ \omega(t)\end{pmatrix}=\left[\begin{pmatrix}A_{3\Omega} & 0\\ 0 & 0\end{pmatrix}+\mathcal{E}_2\right]\begin{pmatrix}\chi_3(t)\\ \omega(t)\end{pmatrix}. \tag{3.4}$$

Here $\chi_3(t)=(y(\cdot,t),y_t(\cdot,t),y_{xt}(1,t))$ and the unbounded linear operator $A_{3\Omega}$ is defined on the Hilbert space $\mathcal{X}_3=H_c^2(0,1)\times L^2(0,1)\times\mathbb{R}^2$ as follows

$$\mathcal{D}(A_{3\Omega})=\left\{\chi_3=(y,z,r_2)\in H_c^4(0,1)\times H_c^2(0,1)\times\mathbb{R};\alpha_2z(1)-y_{xxx}(1)=0;\ r_2=z_x(1)\right\},$$
$$A_{3\Omega}\chi_3=\left(z,-\frac{EI}{\rho}y_{xxxx}+\Omega^2y,-\frac{1}{J}(y_{xx}(1)+\alpha_1r_2)\right). \tag{3.5}$$

Moreover, the operator $\mathcal{E}_2$ is defined, as in the previous subsection, by (2.20).

The well-posedness result of the above problem can be established in a very similar way as in the previous section. We leave the details to the reader. Nevertheless, let us focus on the stability property of the closed loop system.

3.1. Both moment and force boundary controls are exerted

We suppose that $\alpha_1\alpha_2\neq 0$ and hence both moment and force controls are acting on the right end of the beam. Then, we have the following result:

Theorem 5. *Suppose that (3.3) holds. Then, the solution $\Upsilon(t)=(\chi_3(t),\omega(t))$ of the system (3.4) stemmed from the initial data $\Upsilon_0\in\mathcal{D}(A_{3\Omega})\times\mathbb{R}$ exponentially goes to $(0_{\mathcal{X}_3},\Omega)$ in $\mathcal{X}_3\times\mathbb{R}$ as $t\to\infty$.*

Proof. Define the compact operator $\mathcal{K}_3$ on $\mathcal{X}_3:\mathcal{K}_3(y,z,r_2)=(0,y,0)$. Then, thanks to the exponential result of the semigroup generated by the operator $A_{3\Omega}-\Omega^2\mathcal{K}_3$ (see Theorem 1 in [10]), it suffices to verify the strong stability in $\mathcal{X}_3$ of the system

$$\begin{cases}\rho y_{tt}+EIy_{xxxx}=\rho\Omega^2y, & (x,t)\in(0,1)\times(0,\infty),\\ y(0,t)=y_x(0,t)=0, & t>0,\\ Jy_{xtt}(1,t)+y_{xx}(1,t)=-\alpha_1y_{xt}(1,t), & t>0,\\ y_{xxx}(1,t)=\alpha_2y_t(1,t), & t>0,\end{cases}$$

which is a simple task thanks to Theorem 2 and by proceeding as for the stability of $S(t)$ (see the proof of Theorem 3).

Finally, the rest of the proof of the theorem can be completed in the same way as we did previously.

3.2. Only a force boundary control is applied

This corresponds to the case $\alpha_1=0$. At the expense of this restriction, an additional condition on the physical parameters of the system is required. Indeed, we have

Theorem 6. *Assume that the angular velocity Ω satisfies (3.3). If for each natural integer k,*

$$J\left(\frac{EI}{\rho}-\frac{\Omega^2}{(k\pi)^4}\right)\neq\frac{\sinh k\pi}{(k\pi)^3\left(\cosh k\pi-(-1)^k\right)}, \tag{3.6}$$

then the solution $\Upsilon(t)=(\chi_3(t),\omega(t))$ of the system (3.4) (with $\alpha_1=0$) corresponding to the initial data $\Upsilon_0\in\mathcal{D}(A_{3\Omega})\times\mathbb{R}$ exponentially tends to $(0_{\mathcal{X}_3},\Omega)$ in $\mathcal{X}_3\times\mathbb{R}$ as $t\to\infty$.

Proof. Under the condition (3.6), the operator $A_{3\Omega}-\Omega^2\mathcal{K}_2$, where $A_{3\Omega}$ is defined by (3.5) (with $\alpha_1=0$), is uniformly exponentially stable (Theorem 4 of [10]). Consequently, one has to check the strong stability of the system

$$\begin{cases}\rho y_{tt}+EIy_{xxxx}=\rho\Omega^2y, & (x,t)\in(0,1)\times(0,\infty),\\ y(0,t)=y_x(0,t)=0, & t>0,\\ Jy_{xtt}(1,t)+y_{xx}(1,t)=0, & t>0,\\ y_{xxx}(1,t)=\alpha_2y_t(1,t), & t>0,\end{cases}$$

to conclude the exponential stability of the semigroup generated by $A_{3\Omega}$ with $\alpha_1=0$. Once again this desired result can be obtained by utilizing Theorem 2. Then, arguing as in the proof of Theorem 3, one can achieve the proof of Theorem 6.

4. Conclusions and discussion

In this paper, we considered a new model which describes the dynamics of a rotating disk where a beam, with a rigid body, is attached. Our main concern was the stabilization of the system via a torque control exerted on the disk and boundary controls applied on the beam. Different physical situations are treated and accordingly appropriate feedback control laws are provided. Thereafter, exponential stability results of the closed loop systems are showed under a restriction on the angular velocity of the disk.

We point out that those results have been established under the assumption of neglecting the effects of rotational non-inertial forces. Hence, it would be desirable to investigate the case where these forces are taken into consideration. Finally, it would be interesting to study the stabilization of each system considered in this work but under the presence of a time delay. This will be the subject of future works.

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