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Optimal pricing and ordering policies for non-instantaneously deteriorating items under order-size-dependent delay in payments

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ABSTRACT

In today's competitive business transactions, the supplier may permit his/her retailers a delay in payment in order to encourage the retailers to buy more. During the permissible delay period, the retailer is allowed to postpone paying for the products bought without incurring any interest. In this study, we consider an inventory system with non-instantaneously deteriorating items in circumstances where the supplier provides the retailer with various trade credits linked to order quantity. First, we develop a mathematical model to identify the optimal pricing and ordering policies for maximizing the retailer's total profit. This followed by a discussion of the characteristics of the optimal solution. We then propose some algorithms for finding the optimal solutions. Finally, numerical examples are presented and a sensitivity analysis is undertaken to illustrate the proposed model.

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1. Introduction

Researchers assume, in the classical inventory model, that the value of inventory items is unaffected by the duration of time. In practice, however, many items deteriorate during the normal storage period. Chemicals, volatile liquids, blood stored in blood banks, and electronic components deteriorate significantly. Deterioration is defined as the decay, damage, spoilage, evaporation, or drying out of products. Thus, the ideal case envisioned by the classical model remains an ideal one. The effects of deterioration are significant in many inventory systems, making the problem of how to control and maintain inventories of deteriorating items a major issue for decision makers in modern organizations. In addressing this issue, Ghare and Schrader [1] first proposed a model for an exponentially decaying inventory, which Covert and Philip [2] extended to a two-parameter Weibull distribution. Goyal and Giri [3] classified the previous studies and provided a detailed review of the literature on deteriorating inventory. Jaber et al. [4] developed a mathematical model that determines batch sizes for deteriorating items while minimizing entropy. Other interesting articles that cover the topic include Shah and Jaiswal [5], Aggarwal [6], Dave and Patel [7], Sachan [8], Hariga [9], Skouri and Papachristos [10], Chang [11], Liao [12] and Jaber et al. [13].

In the existing literature, all the models assume that the deterioration of items in an inventory starts from the moment of their arrival in stock. However, in real life there is a time span during which most commodities maintain their quality or

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original condition, that is, during which no deterioration occurs. Beyond this period, however, some of the items will start to decay. Wu et al. [14] defined this phenomenon as “non-instantaneous deterioration.” It exists commonly among medicines, firsthand vegetables, and fruits, all of which can maintain their fresh quality for a short span of time. During this initial time period, there is almost no spoilage. For these kinds of items, the assumption that the deterioration begins to occur as soon as the retailer receives the items may cause retailers to adopt inappropriate replenishment policies as a result of overvaluing the total relevant inventory cost. Chang et al. [15] proposed optimal replenishment policies for non-instantaneously deteriorating items with stock-dependent demand. Their model set a maximum inventory level to reflect the limited shelf space of most retail outlets. Yang et al. [16] developed a model in which shortages are accepted and partially backlogged with a variable backlogging rate dependent on the waiting time for the next replenishment. Geetha and Uthayakumar [17] developed an economic order quantity (EOQ) model for non-instantaneously deteriorating items with permissible delay in payments in which model shortages are allowed and partially backlogged. Maihmi and Kamalabadi [18] presented a joint pricing and inventory model for non-instantaneously deteriorating items with a price-and-time-dependent demand function. Our study demonstrates the importance of taking into consideration the inventory problems associated with non-instantaneously deteriorating items in the inventory management system.

In addition to the inventory problem, this article addresses the issue of payment to suppliers. The traditional EOQ model tacitly assumes that payment must be made to the supplier immediately after retailers receive the items. In reality, suppliers, hoping to promote the sale of their products, are willing to offer retailers a payment delay period, known as a trade credit period. The trade credit is the largest use of capital for the majority of business-to-business (B2B) and business-to-customer (B2C) sellers and is a critical source of capital for a majority of all businesses. This is an important and popular topic because it characterizes the real situation in the market. During the trade credit period, retailers can actually gain interest from non-payment and sales income, while the supplier loses interest income during this period. Thus, the delay in payment to suppliers serves as a kind of price discount. Because paying later indirectly reduces the purchase cost, retailers are motivated to increase their order quantity.

Issues related to trade credits have been considered by several researchers. Goyal [19] first developed the EOQ model with a permissible delay in payment to determine the optimal order quantity. Aggarwal and Jaggi [20] extended Goyal's model to allow for deteriorating items, which Jamal et al. [21] generalized to allow for shortages. Hwang and Shinn [22] considered demand, which is a function of retail price, and developed the optimal pricing and lot-sizing policy for the retailer in the case of a permissible delay in payments. Teng [23] modified Goyal's model, assuming that the selling price is not equal to the purchasing price, to find that it is economically viable for a well-established buyer to frequently order a lower quantity and take advantage of the benefits of a permissible delay. Teng et al. [24] combined the approaches of Hwang and Shinn [22] and Teng [23], and presented a pricing and lot-sizing model for retailers in which the supplier provides a permissible delay in payments. Urban [25] proposed an extension of inventory models, incorporating financing agreements with both suppliers and customers. Ouyang et al. [26] developed an inventory model for non-instantaneous deteriorating items with a permissible delay in payments. Based on this model, they provided theorems that characterize the optimal solution and a straightforward method for finding the optimal replenishment cycle time and order quantity under various circumstances. Some other studies of permissible delay in payments are Davis and Gaither [27], Arcelus and Srinivasan [28], Shah [29], Khouja and Mehrez [30], Sarker et al. [31], Chang and Wu [32], Chang [33], Ouyang et al. [34,35], Chang et al. [36], Ho, et al. [37], Sana and Chaudhuri [38], Chang et al. [39], Teng and Chang [40], Chen and Kang [41], Liang and Zhou [42], Roy and Samanta [43], Jaber [44], Lou and Wang [45], and Jaber and Osman [46].

Most of the earlier studies dealing with inventory problems in circumstances of permissible delay in payments discuss a case in which the delay in payments is independent of the quantity ordered. However, in today's business transactions, in order to encourage the retailer to order large quantities, the supplier may offer a permissible delay of payment for large quantities but require immediate payment for small quantities. Hence, the supplier may set a predetermined order quantity below which delay in payment is not permitted and payments must be made immediately. For order quantities above this threshold, the trade credit period is permitted. Khouja and Mehrez [29] investigated the effect of supplier credit policies on the optimal order quantity. They addressed two types of supplier credit policies: the first type is one in which credit terms are independent of the quantity ordered, and the second type is one in which the credit terms are linked to the order quantity. Shinn and Hwang [47] analyzed the problem of the retailer who has to decide his/her sale price and order quantity simultaneously in the case of an order-size-dependent delay in payments. Chang et al. [48] developed an EOQ model with deteriorating items where suppliers link credit to order quantity. Chung and Liao [49] discussed the optimal replenishment cycle time for an exponentially deteriorating product under the condition that the delay in payments depends on the quantity ordered. Other researchers who address this topic include Chang [32], Chung et al. [50], Liao [51], Ouyang et al. [52,53], Chang et al. [54], and Yang et al. [55].

In a competitive market, suppliers may offer different trade credit periods with different predetermined quantities to increase retailers' choices and encourage retailers to order higher quantities. Hence, in this article, we will develop an appropriate inventory model for non-instantaneously deteriorating items where suppliers provide a permissible payment delay schedule linked to order quantity. The rest of the article is organized as follows. The assumptions and notations used in this study are presented in Section 2. In Section 3, a mathematical model is developed to show the pricing and ordering policies that will maximize profits in various trade credit situations. We then discuss the necessary and sufficient conditions for an optimal solution and develop the solution algorithms. In Section 4, numerical examples are provided to illustrate the

proposed model and the sensitivity analysis of the optimal solution with respect to the parameters of the system. Finally, we draw our conclusions in Section 5.

2. Assumptions and notation

The following assumption and notations are used in this study.

1. The demand rate for the item is assumed to be retail-price sensitive and is given by $D(p) = \alpha p^{-\beta}$, where p is the selling price per unit, $\alpha > 0$ is a scaling factor, and $\beta > 1$ is a price-elasticity coefficient. For notational simplicity, $D(p)$ and D will be used interchangeably in this paper.
2. The supplier offers a permissible delay schedule M which links to the order quantity Q as follows:

$$M = \begin{cases} M_1, & q_1 \leq Q < q_2 \\ M_2, & q_2 \leq Q < q_3 \\ \vdots & \vdots \\ M_K, & q_K \leq Q < q_{K+1} \end{cases}$$

where $0 < q_1 < q_2 < \dots < q_K < q_{K+1}$ and $0 \leq M_1 < M_2 < \dots < M_K$.

3. The product has no deterioration during the time interval $[0, t_d]$, after which, the on-hand stocks deteriorate at a constant rate, θ , where $0 < \theta < 1$. Following assumptions (4) and (5) in Ouyang et al. [26], t_d is a given constant in this paper.
4. A denotes the ordering cost per order, h denotes the holding cost per unit time excluding interest charges, c , the purchasing cost per unit, I_c , the capital opportunity cost of stock per dollar per unit time, and I_e , the interest earned per dollar per unit time. All of the parameters are positive.
5. T_j is the length of replenishment cycle when the permissible delay period is M_j .
6. $Z(T_j, p)$ is the total profit per unit time which consists of (a) the sales revenue, (b) the cost of purchasing, (c) the cost of placing orders, (d) the cost of carrying inventory (excluding interest charges), (e) the capital opportunity cost after the grace period M_j (this cost is incurred only if $T_j > M_j$), and (f) the interest earned from sales revenue during the interval $[0, M_j]$.

3. Mathematical formulation

For a given delay in payment time M_j , to determine the inventory level, $I(t)$, at time $t \in [0, T_j]$, we consider the following two situations: (i) $T_j \leq t_d$, and (ii) $T_j \geq t_d$.

When $T_j \leq t_d$, the replenishment cycle is shorter than or equal to the length of time in which the product does not deteriorate; thus, no deterioration occurs during the replenishment cycle. In this situation, the order quantity per order is $Q_j = DT_j$, and the inventory level decreases only owing to the demand during the time interval $[0, T_j]$. Hence, the inventory level, $I(t)$, at time $t \in [0, T_j]$ is given by

$$I(t) = Q_j - Dt = D(T_j - t), \quad 0 \leq t \leq T_j. \quad (1)$$

When $T_j \geq t_d$, during the time interval $[0, t_d]$, the inventory level decreases only owing to demand, thus, the inventory level, $I_1(t)$, at time $t \in [0, t_d]$ is given by

$$I_1(t) = Q_j - Dt, \quad 0 \leq t \leq t_d. \quad (2)$$

Another, during the time interval $[t_d, T_j]$, the inventory level, $I_2(t)$, decreases owing to demand and deterioration. Hence, the change of inventory level can be represented by the following differential equation:

$$\frac{dI_2(t)}{dt} + \theta I_2(t) = -D, \quad t_d < t < T_j, \quad (3)$$

with the boundary condition $I_2(T_j) = 0$. The solution of Eq. (3) is

$$I_2(t) = \frac{D}{\theta} [e^{\theta(T_j-t)} - 1], \quad t_d \leq t \leq T_j. \quad (4)$$

Considering continuity of $I_1(t)$ and $I_2(t)$ at time $t = t_d$, i.e., $I_1(t_d) = I_2(t_d)$, it follows from Eqs. (2) and (4) that $Q_j - Dt_d = (D/\theta)[e^{\theta(T_j-t_d)} - 1]$, which implies that the order quantity for each cycle is

$$Q_j = Dt_d + \frac{D}{\theta} [e^{\theta(T_j-t_d)} - 1]. \quad (5)$$

Substituting Eq. (5) into Eq. (2), we obtain

$$I_1(t) = D(t_d - t) + \frac{D}{\theta} [e^{\theta(T_j-t_d)} - 1], \quad 0 \leq t \leq t_d. \quad (6)$$

The total profit per unit time consists of the following six elements:

(a) Sales revenue (denoted by SR) is

$$SR = pD. \quad (7)$$

(b) Cost of purchasing (denoted by CP) is

$$CP = cQ_j/T_j = \begin{cases} cD, & \text{if } T_j \leq t_d, \\ \frac{cD}{T_j} \left\{ t_d + \frac{1}{\theta} [e^{\theta(T_j-t_d)} - 1] \right\}, & \text{if } T_j \geq t_d, \end{cases} \quad (8)$$

(c) Cost of placing orders (denoted by OC) is

$$OC = A/T_j. \quad (9)$$

(d) Cost of carrying inventory (denoted by HC)

When $T_j \leq t_d$, the stock holding cost per unit time is $hDT_j/2$.

When $T_j \geq t_d$, the stock holding cost per unit time is given by

$$\frac{h}{T_j} \int_0^{T_j} I(t)dt = \frac{h}{T_j} \left[\int_0^{t_d} I_1(t)dt + \int_{t_d}^{T_j} I_2(t)dt \right] = \frac{hD}{T_j} \left\{ \frac{t_d}{\theta} [e^{\theta(T_j-t_d)} - 1] + \frac{t_d^2}{2} + \frac{1}{\theta^2} [e^{\theta(T_j-t_d)} - \theta(T_j - t_d) - 1] \right\}.$$

Therefore, cost of carrying inventory per unit time is given by

$$HC = \begin{cases} \frac{hDT_j}{2}, & \text{if } T_j \leq t_d, \\ \frac{hD}{T_j} \left\{ \frac{t_d}{\theta} [e^{\theta(T_j-t_d)} - 1] + \frac{t_d^2}{2} + \frac{1}{\theta^2} [e^{\theta(T_j-t_d)} - \theta(T_j - t_d) - 1] \right\}, & \text{if } T_j \geq t_d. \end{cases} \quad (10)$$

(e) Capital opportunity cost (denoted by IC)

Based on the values of M_j and T_j , there are two possible situations as follows: (i) $M_j \leq T_j$, and (ii) $M_j \geq T_j$.

(i) $M_j \leq T_j$ When the permissible delay M_j is shorter than or equal to the replenishment cycle T_j , payment for goods is settled and the retailer starts paying the capital opportunity cost for the items in stock with rate l_c . Thus, the opportunity cost per unit time is

$$IC = \frac{cI_c}{T_j} \int_{M_j}^{T_j} I(t)dt = \begin{cases} \frac{cI_c}{T_j} \int_{M_j}^{T_j} I_2(t)dt, & \text{if } t_d \leq M_j \leq T_j, \\ \frac{cI_c}{T_j} \left[\int_{M_j}^{t_d} I_1(t)dt + \int_{t_d}^{T_j} I_2(t)dt \right], & \text{if } M_j \leq t_d \leq T_j, \\ \frac{cI_c}{T_j} \int_{M_j}^{T_j} I(t)dt, & \text{if } M_j \leq T_j \leq t_d, \end{cases} \quad (11)$$

$$= \begin{cases} \frac{cI_c D}{T_j} \left\{ \frac{1}{\theta^2} [e^{\theta(T_j-M_j)} - \theta(T_j - M_j) - 1] \right\}, & \text{if } t_d \leq M_j \leq T_j, \\ \frac{cI_c D}{T_j} \left\{ \frac{(t_d-M_j)}{\theta} [e^{\theta(T_j-t_d)} - 1] + \frac{(t_d-M_j)^2}{2} + \frac{1}{\theta^2} [e^{\theta(T_j-t_d)} - \theta(T_j - t_d) - 1] \right\}, & \text{if } M_j \leq t_d \leq T_j, \\ \frac{cI_c D}{T_j} \left[\frac{(T_j-M_j)^2}{2} \right], & \text{if } M_j \leq T_j \leq t_d. \end{cases}$$

(ii) $M_j \geq T_j$ When the permissible delay M_j is longer than or equal to the replenishment cycle T_j , there is no opportunity cost, hence, $IC = 0$.

(f) Interest earned from sales revenue (denoted by IE)

$$IE = \begin{cases} \frac{pI_e}{T_j} \int_0^{M_j} Dtdt = \frac{pI_e DM_j^2}{2T_j}, & \text{if } M_j \leq T_j, \\ \frac{pI_e}{T_j} \left[\int_0^{T_j} Dtdt + DT_j(M_j - T_j) \right] = pI_e D \left(M_j - \frac{T_j}{2} \right), & \text{if } M_j \geq T_j. \end{cases} \quad (12)$$

Therefore, for a given delay in payment time M_j , according to (i) $M_j \leq t_d$ and (ii) $M_j \geq t_d$, we can obtain the total profit per unit time as follows:

$$Z_j(T_j, p) = \begin{cases} Z_{1j}(T_j, p), & \text{if } M_j \leq t_d, \\ Z_{2j}(T_j, p), & \text{if } M_j \geq t_d, \end{cases} = SR - CP - OC - HC - IC + IE.$$

Case 1: $M_j \leq t_d$

$$Z_{1j}(T_j, p) = \begin{cases} Z_{11j}(T_j, p), & \text{if } T_j \leq M_j \leq t_d, \\ Z_{12j}(T_j, p), & \text{if } M_j \leq T_j \leq t_d, \\ Z_{13j}(T_j, p), & \text{if } M_j \leq t_d \leq T_j, \end{cases} \quad (13)$$

where

$$Z_{11j}(T_j, p) = (p - c)D - \frac{A}{T_j} - \frac{hDT_j}{2} + pI_e D \left(M_j - \frac{T_j}{2} \right), \tag{13a}$$

$$Z_{12j}(T_j, p) = (p - c)D - \frac{A}{T_j} - \frac{hDT_j}{2} - \frac{cI_c D}{2T_j} (T_j - M_j)^2 + \frac{pI_e DM_j^2}{2T_j}, \tag{13b}$$

and

$$Z_{13j}(T_j, p) = pD - \frac{A + cDt_d}{T_j} - \frac{D[c + ht_d + cI_c(t_d - M_j)]}{\theta T_j} [e^{\theta(T_j - t_d)} - 1] - \frac{D(h + cI_c)}{\theta^2 T_j} [e^{\theta(T_j - t_d)} - \theta(T_j - t_d) - 1] - \frac{hDt_d^2 + cI_c D(t_d - M_j)^2}{2T_j} + \frac{pI_e DM_j^2}{2T_j}. \tag{13c}$$

It is obvious that $Z_{11j}(M_j, p) = Z_{12j}(M_j, p)$ and $Z_{12j}(t_d, p) = Z_{13j}(t_d, p)$. Hence, for fixed p , $Z_{1j}(T_j, p)$ is a continuous function on $T_j > 0$.

Case 2: $M_j \geq t_d$

$$Z_{2j}(T_j, p) = \begin{cases} Z_{21j}(T_j, p), & \text{if } T_j \leq t_d \leq M_j, \\ Z_{22j}(T_j, p), & \text{if } t_d \leq T_j \leq M_j, \\ Z_{23j}(T_j, p), & \text{if } t_d \leq M_j \leq T_j, \end{cases} \tag{14}$$

where

$$Z_{21j}(T_j, p) = (p - c)D - \frac{A}{T_j} - \frac{hDT_j}{2} + pI_e D \left(M_j - \frac{T_j}{2} \right), \tag{14a}$$

$$Z_{22j}(T_j, p) = pD - \frac{A + cDt_d}{T_j} - \frac{D(ht_d + c)}{\theta T_j} [e^{\theta(T_j - t_d)} - 1] - \frac{Dh}{\theta^2 T_j} [e^{\theta(T_j - t_d)} - \theta(T_j - t_d) - 1] - \frac{hDt_d^2}{2T_j} + pI_e D \left(M_j - \frac{T_j}{2} \right), \tag{14b}$$

and

$$Z_{23j}(T_j, p) = pD - \frac{A + cDt_d}{T_j} - \frac{D(ht_d + c)}{\theta T_j} [e^{\theta(T_j - t_d)} - 1] - \frac{Dh}{\theta^2 T_j} [e^{\theta(T_j - t_d)} - \theta(T_j - t_d) - 1] - \frac{cI_c D}{\theta^2 T_j} [e^{\theta(T_j - M_j)} - \theta(T_j - M_j) - 1] - \frac{hDt_d^2}{2T_j} + \frac{pI_e DM_j^2}{2T_j}. \tag{14c}$$

It is obvious that $Z_{21j}(t_d, p) = Z_{22j}(t_d, p)$ and $Z_{22j}(M_j, p) = Z_{23j}(M_j, p)$. Hence, for fixed p , $Z_{2j}(T_j, p)$ is a continuous function on $T_j > 0$.

3.1. Determination of the optimal replenishment time T_j for any given price p

For low deterioration rate (i.e., $\theta \ll 1$), we can assume

$$e^{\theta(T_j - t_d)} \approx 1 + \theta(T_j - t_d) + [\theta(T_j - t_d)]^2/2, \tag{15}$$

and

$$e^{\theta(T_j - M_j)} \approx 1 + \theta(T_j - M_j) + [\theta(T_j - M_j)]^2/2. \tag{16}$$

Hence, Eqs. (13c), (14b) and (14c) can be rewritten as follows:

$$Z_{13j}(T_j, p) \approx pD - \frac{A + cDt_d}{T_j} - D[c + ht_d + cI_c(t_d - M_j)] \left[1 - \frac{t_d}{T_j} + \frac{\theta(T_j - t_d)^2}{2T_j} \right] - D(h + cI_c) \frac{(T_j - t_d)^2}{2T_j} - \frac{hDt_d^2 + cI_c D(t_d - M_j)^2}{2T_j} + \frac{pI_e DM_j^2}{2T_j}, \tag{17}$$

$$Z_{22j}(T_j, p) \approx pD - \frac{A + cDt_d}{T_j} - D(ht_d + c) \left[1 - \frac{t_d}{T_j} + \frac{\theta(T_j - t_d)^2}{2T_j} \right] - \frac{Dh(T_j - t_d)^2}{2T_j} - \frac{hDt_d^2}{2T_j} + pI_e D \left(M_j - \frac{T_j}{2} \right), \tag{18}$$

and

$$Z_{23j}(T_j, p) \approx pD - \frac{A + cDt_d}{T_j} - D(ht_d + c) \left[1 - \frac{t_d}{T_j} + \frac{\theta(T_j - t_d)^2}{2T_j} \right] - \frac{Dh(T_j - t_d)^2}{2T_j} - \frac{cl_c D(T_j - M_j)^2}{2T_j} - \frac{hDt_d^2}{2T_j} + \frac{pl_e DM_j^2}{2T_j}, \quad (19)$$

respectively.

Note that the purpose of this approximation is to find the unique closed-form solution for the optimal value of T_j . This approximation retains the properties of the continuity.

Case 1: $M_j \leq t_d$.

For fixed p and M_j , let $T_{1kj}^*(p)$ denote the optimal value of T_j which maximizes $Z_{1kj}(T_j, p)$, $k = 1, 2, 3$.

Sub-case 1-1. $T_j \leq M_j \leq t_d$.

By taking the first and second order derivatives of $Z_{11j}(T_j, p)$ in Eq. (13a) with respect to $T_j \in (0, M_j)$, we obtain

$$\frac{\partial Z_{11j}(T_j, p)}{\partial T_j} = \frac{A}{T_j^2} - \frac{hD}{2} - \frac{pl_e D}{2}, \quad (20)$$

and

$$\frac{\partial^2 Z_{11j}(T_j, p)}{\partial T_j^2} = -\frac{2A}{T_j^3} < 0. \quad (21)$$

Hence, $Z_{11j}(T_j, p)$ is a concave function of $T_j \in (0, M_j]$, the value of T_j (denoted by $T_{11j}(p)$) which maximizes $Z_{11j}(T_j, p)$ can be obtained by solving $\partial Z_{11j}(T_j, p)/\partial T_j = 0$ and is given as

$$T_{11j}(p) = \sqrt{\frac{2A}{D(h + pl_e)}}. \quad (22)$$

To ensure the inequality $T_{11j}(p) \leq M_j$ holds, we substitute $T_{11j}(p)$ in Eq. (22) into this inequality, and obtain

$$\text{if } 2A \leq DM_j^2(h + pl_e), \quad \text{then } T_{11j}(p) \leq M_j. \quad (23)$$

On the other hand, if $2A > DM_j^2(h + pl_e)$, then we have

$$\frac{\partial Z_{11j}(T_j, p)}{\partial T_j} = \frac{2A - T_j^2 D(h + pl_e)}{2T_j^2} > \frac{D(h + pl_e)(M_j^2 - T_j^2)}{2T_j^2} > 0, \quad \text{for } T_j \in (0, M_j).$$

Thus, $Z_{11j}(T_j, p)$ is a strictly increasing function of $T_j \in (0, M_j]$, which implies $Z_{11j}(T_j, p)$ has a maximum value at the boundary point $T_j = M_j$. For convenience, let

$$\Delta_{1j} \equiv DM_j^2(h + pl_e). \quad (24)$$

Then, from the above results, we obtain the following lemma.

Lemma 1. For any given p and M_j , the optimal value of T_j which maximizes $Z_{11j}(T_j, p)$ is given by

$$T_{11j}^*(p) = \begin{cases} T_{11j}(p), & \text{if } 2A \leq \Delta_{1j}, \\ M_j, & \text{if } 2A > \Delta_{1j} \end{cases}$$

Sub-case 1-2. $M_j \leq T_j \leq t_d$.

Similarly, by taking the first and second order derivatives of $Z_{12j}(T_j, p)$ in Eq. (13b) with respect to $T_j \in (M_j, t_d)$, we obtain

$$\frac{\partial Z_{12j}(T_j, p)}{\partial T_j} = \frac{1}{2T_j^2} [2A - DT_j^2(h + cl_c) + DM_j^2(cl_c - pl_e)], \quad (25)$$

and

$$\frac{\partial^2 Z_{12j}(T_j, p)}{\partial T_j^2} = -\frac{1}{T_j^3} [2A + DM_j^2(cl_c - pl_e)]. \quad (26)$$

By solving $\partial Z_{12j}(T_j, p)/\partial T_j = 0$, we obtain the value of T_j (denoted by $T_{12j}(p)$) as

$$T_{12j}(p) = \sqrt{\frac{2A + DM_j^2(cl_c - pl_e)}{D(h + cl_c)}}. \quad (27)$$

To ensure $M_j \leq T_{12j}(p) \leq t_d$, substituting Eq. (27) into this inequality, we get

$$\text{if } \Delta_{1j} \leq 2A \leq \Delta_{2j}, \quad \text{then } M_j \leq T_{12j}(p) \leq t_d, \quad (28)$$

where Δ_{1j} is defined as in Eq. (24), and

$$\Delta_{2j} \equiv D[ht_d^2 + cl_c(t_d^2 - M_j^2) + pl_e M_j^2]. \tag{29}$$

Note that when $2A \geq \Delta_{1j}$ holds, then

$$2A + DM_j^2(cl_c - pl_e) \geq DM_j^2(h + pl_e) + DM_j^2(cl_c - pl_e) = DM_j^2(h + cl_c) > 0, \tag{30}$$

which implies $T_{12j}(p)$ in Eq. (27) is well-defined. Further, we obtain $\partial^2 Z_{12j}(T_j, p)/\partial T_j^2 < 0$. Hence, $T_{12j}(p) \in [M_j, t_d]$ in Eq. (27) is a unique value which maximizes $Z_{12j}(T_j, p)$. Conversely, if $2A < \Delta_{1j}$, then we have

$$\frac{\partial Z_{12j}(T_j, p)}{\partial T_j} < \frac{1}{2T_j^2} [DM_j^2(h + pl_e) - DT_j^2(h + cl_c) + DM_j^2(cl_c - pl_e)] = \frac{1}{2T_j^2} D(h + cl_c)(M_j^2 - T_j^2) < 0, \quad \text{for } T_j \in (M_j, t_d). \tag{31}$$

Thus, $Z_{12j}(T_j, p)$ is a strictly decreasing function of $T_j \in [M_j, t_d]$, which implies $Z_{12j}(T_j, p)$ has a maximum value at the boundary point $T_j = M_j$. On the other hand, if $2A > \Delta_{2j}$, we have

$$\frac{\partial Z_{12j}(T_j, p)}{\partial T_j} > \frac{1}{2T_j^2} \{D[ht_d^2 + cl_c(t_d^2 - M_j^2) + pl_e M_j^2] - DT_j^2(h + cl_c) + DM_j^2(cl_c - pl_e)\} = \frac{1}{2T_j^2} D(h + cl_c)(t_d^2 - T_j^2) > 0, \quad \text{for } T_j \in (M_j, t_d). \tag{32}$$

Thus, $Z_{12j}(T_j, p)$ is a strictly increasing function of $T_j \in [M_j, t_d]$, which implies $Z_{12j}(T_j, p)$ has a maximum value at the boundary point $T_j = t_d$. Then, from the above results and the fact that $\Delta_{2j} > \Delta_{1j}$, we obtain the following lemma.

Lemma 2. For any given p and M_j , the optimal value of T_j which maximizes $Z_{12j}(T_j, p)$ is given by

$$T_{12j}^*(p) = \begin{cases} M_j, & \text{if } 2A < \Delta_{1j}, \\ T_{12j}(p), & \text{if } \Delta_{1j} \leq 2A \leq \Delta_{2j}, \\ t_d, & \text{if } 2A > \Delta_{2j}. \end{cases}$$

Sub-case 1-3. $M_j \leq t_d \leq T_j$.

Likewise, by taking the first and second order derivatives of $Z_{13j}(T_j, p)$ in Eq. (17) with respect to $T_j \in (t_d, \infty)$, we obtain

$$\frac{\partial Z_{13j}(T_j, p)}{\partial T_j} = \frac{1}{2T_j^2} [2A - Dt_d^2(h + cl_c) + DM_j^2(cl_c - pl_e) - \delta_{1j}(T_j^2 - t_d^2)], \tag{33}$$

and

$$\frac{\partial^2 Z_{13j}(T_j, p)}{\partial T_j^2} = -\frac{1}{T_j^3} [2A - Dt_d^2(h + cl_c) + DM_j^2(cl_c - pl_e) + \delta_{1j}t_d^2], \tag{34}$$

where $\delta_{1j} = D\theta[ht_d + cl_c(t_d - M_j)] + D(h + c\theta + cl_c) > 0$.

By solving $\partial Z_{13j}(T_j, p)/\partial T_j = 0$, we obtain the value of T_j (denoted by $T_{13j}(p)$) as

$$T_{13j}(p) = \sqrt{\frac{2A - Dt_d^2(h + cl_c) + DM_j^2(cl_c - pl_e) + \delta_{1j}t_d^2}{\delta_{1j}}}. \tag{35}$$

To ensure $T_{13j}(p) \geq t_d$, substituting Eq. (35) into this inequality, we get

$$\text{if } 2A \geq \Delta_{2j}, \quad \text{then } T_{13j}(p) \geq t_d, \tag{36}$$

where Δ_{2j} is defined as in Eq. (29).

Note that when $2A \geq \Delta_{2j}$, then we have

$$2A - Dt_d^2(h + cl_c) + DM_j^2(cl_c - pl_e) + \delta_{1j}t_d^2 \geq \delta_{1j}t_d^2 > 0,$$

which implies $T_{13j}(p)$ in Eq. (35) is well-defined. Further, we obtain $\partial^2 Z_{13j}(T_j, p)/\partial T_j^2 < 0$. Hence, $T_{13j}(p) \in [t_d, \infty)$ in Eq. (35) is a unique value which maximizes $Z_{13j}(T_j, p)$. Conversely, if $2A < \Delta_{2j}$, then we have

$$\frac{\partial Z_{13j}(T_j, p)}{\partial T_j} < \frac{-\delta_{1j}}{2T_j^2} (T_j^2 - t_d^2) < 0, \quad \text{for } T_j \in (t_d, \infty). \tag{37}$$

Thus, $Z_{13j}(T_j, p)$ is a strictly decreasing function of $T_j \in [t_d, \infty)$, which implies $Z_{13j}(T_j, p)$ has a maximum value at the boundary point $T_j = t_d$.

Then, from the above results, we obtain the following lemma.

Lemma 3. For any given p and M_j , the optimal value of T_j which maximizes $Z_{13j}(T_j, p)$ is given by

$$T_{13j}^*(p) = \begin{cases} T_{13j}(p), & \text{if } 2A \geq \Delta_{2j}, \\ t_d, & \text{if } 2A < \Delta_{2j}. \end{cases}$$

Combining Lemmas 1–3, we obtain the optimal replenishment cycle length (denoted by $T_{1j}^*(p)$) in case 1 as follows:

Lemma 4. For any given p and M_j ,

$$T_{1j}^*(p) = \begin{cases} T_{11j}(p), & \text{if } 0 < 2A \leq \Delta_{1j}, \\ T_{12j}(p), & \text{if } \Delta_{1j} \leq 2A \leq \Delta_{2j}, \\ T_{13j}(p), & \text{if } 2A \geq \Delta_{2j}. \end{cases}$$

Proof. It is immediately obtained from the facts that $Z_{11j}(M_j, p) = Z_{12j}(M_j, p)$, $Z_{12j}(t_d, p) = Z_{13j}(t_d, p)$, Lemmas 1–3.

From Lemma 4, when p and M_j are given, we can get the maximum total profit per unit time for case 1 as follows:

$$Z_{1j}(p) \equiv Z_{1j}(T_{1j}^*(p), p) = \begin{cases} Z_{11j}(T_{11j}(p), p), & \text{if } 0 < 2A \leq \Delta_{1j}, \\ Z_{12j}(T_{12j}(p), p), & \text{if } \Delta_{1j} \leq 2A \leq \Delta_{2j}, \\ Z_{13j}(T_{13j}(p), p), & \text{if } 2A \geq \Delta_{2j}, \end{cases} \quad (38)$$

where

$$Z_{11j}(T_{11j}(p), p) = (p - c)D - \sqrt{2DA(h + pI_e)} + pI_eDM_j, \quad (38a)$$

$$Z_{12j}(T_{12j}(p), p) = (p - c)D + cl_cDM_j - \sqrt{D(h + cl_c)[2A + DM_j^2(cl_c - pI_e)]}, \quad (38b)$$

$$Z_{13j}(T_{13j}(p), p) = (p - c)D - \sqrt{\delta_{1j}[2A + \delta_{1j}t_d^2 - Dt_d^2(h + cl_c) + DM_j^2(cl_c - pI_e)]} - D[ht_d + cl_c(t_d - M_j)] + \delta_{1j}t_d, \quad (38c)$$

and δ_{1j} is defined as above.

Case 2: $M_j \geq t_d$.

For fixed p and M_j , let $T_{2kj}^*(p)$ denote the optimal value of T_j which maximizes $Z_{2kj}(T_j, p)$, for $k = 1, 2$ and 3. By using the similar approach as in case 1, let $\Delta_{3j} = Dt_d^2(h + pI_e)$ and $\Delta_{4j} = D\theta(M_j^2 - t_d^2)(ht_d + c) + DM_j^2(h + pI_e)$. The fact that $\Delta_{4j} > \Delta_{3j}$ is known. Hence, the following lemmas can be easily obtained. The proofs are omitted.

Lemma 5. For any given p and M_j , the optimal value of T_j which maximizes $Z_{21j}(T_j, p)$ is given by

$$T_{21j}^*(p) = \begin{cases} T_{21j}, & \text{if } 2A \leq \Delta_{3j}, \\ t_d, & \text{if } 2A > \Delta_{3j}, \end{cases}$$

where

$$T_{21j}(p) = \sqrt{\frac{2A}{D(h + pI_e)}}. \quad (39)$$

Lemma 6. For any given p and M_j , the optimal value of T_j which maximizes $Z_{22j}(T_j, p)$ is given by

$$T_{22j}^*(p) = \begin{cases} t_d, & \text{if } 0 < 2A < \Delta_{3j}, \\ T_{22j}(p), & \text{if } \Delta_{3j} \leq 2A < \Delta_{4j}, \\ M_j, & \text{if } 2A > \Delta_{4j}, \end{cases}$$

$$\text{where } T_{22j}(p) = \sqrt{\frac{2A + D\theta(ht_d + c)t_d^2}{D(h\theta t_d + h + c\theta + pI_e)}}. \quad (40)$$

Lemma 7. For any given p and M_j , the optimal value of T_j which maximizes $Z_{23j}(T_j, p)$ is given by

$$T_{23j}^*(p) = \begin{cases} T_{23j}(p), & \text{if } 2A \geq \Delta_{4j}, \\ M_j, & \text{if } 2A < \Delta_{4j}, \end{cases}$$

$$\text{where } T_{23j}(p) = \sqrt{\frac{2A + D\theta(ht_d + c)t_d^2 + DM_j^2(cl_c - pl_e)}{D(h\theta t_d + h + c\theta + cl_c)}}. \tag{41}$$

Combining Lemmas 5–7, we obtain the optimal replenishment cycle length (denoted by $T_{2j}^*(p)$) in case 2 as follows.

Lemma 8. For any given p and M_j ,

$$T_{2j}^*(p) = \begin{cases} T_{21j}(p), & \text{if } 0 < 2A \leq \Delta_{3j}, \\ T_{22j}(p), & \text{if } \Delta_{3j} \leq 2A \leq \Delta_{4j}, \\ T_{23j}(p), & \text{if } 2A \geq \Delta_{4j}. \end{cases}$$

Proof. It is immediately obtained from the facts that $Z_{21j}(t_d, p) = Z_{22j}(t_d, p)$, $Z_{22j}(M_j, p) = Z_{23j}(M_j, p)$, Lemmas 5–7.

From Lemma 8, when p and M_j are given, we can get the maximum total profit per unit time for case 2 as follows:

$$Z_{2j}(p) \equiv Z_{2j}(T_{2j}^*(p), p) = \begin{cases} Z_{21j}(T_{21j}(p), p), & \text{if } 0 < 2A \leq \Delta_{3j}, \\ Z_{22j}(T_{22j}(p), p), & \text{if } \Delta_{3j} \leq 2A \leq \Delta_{4j}, \\ Z_{23j}(T_{23j}(p), p), & \text{if } 2A \geq \Delta_{4j}, \end{cases} \tag{42}$$

where

$$Z_{21j}(T_{21j}(p), p) = (p - c)D - \sqrt{2DA(h + pl_e)} + pl_eDM_j, \tag{42a}$$

$$Z_{22j}(T_{22j}(p), p) = (p - c)D + pl_eDM_j + D\theta(ht_d + c)t_d - \sqrt{D[2A + Dt_d^2\theta(ht_d + c)](h\theta t_d + h + c\theta + pl_e)}, \tag{42b}$$

$$Z_{23j}(T_{23j}(p), p) = (p - c)D + D\theta t_d(ht_d + c) + cl_eDM_j - \sqrt{\delta_2[2A + D\theta t_d^2(ht_d + c) + DM_j^2(cl_c - pl_e)]}, \tag{42c}$$

and $\delta_2 = D(h\theta t_d + h + c\theta + cl_e)$.

Now, for any given p, M_j and the optimal replenishment cycle $T_{ij}^*(p), i = 1, 2$ and $j = 1, 2, \dots, K$, we can obtain the corresponding order quantity

$$Q_{ij} = \begin{cases} D(p)T_{ij}^*(p), & \text{if } T_{ij}^*(p) \leq t_d, \\ D(p)t_d + \frac{D(p)}{\theta} [e^{\theta(T_{ij}^*(p) - t_d)} - 1], & \text{if } T_{ij}^*(p) \geq t_d. \end{cases} \tag{43}$$

Therefore, from Lemmas 4 and 8, assumption 2 and Eq. (43), we obtain the following result. The proof is trivial, hence, we omit it here.

Theorem 1. For any given p, M_j and $i = 1, 2$,

- (1) if $q_j \leq Q_{ij} < q_{j+1}$, then $T_{ij}^*(p)$ is the optimal replenishment cycle length.
- (2) if $Q_{ij} \geq q_{j+1}$, then $T_{ij}^*(p)$ is not a feasible solution.
- (3) if $Q_{ij} < q_j$, then $T_{ij}^*(p)$ is not a feasible solution. Further,
 - (i) when $T_{ij}^*(p) \leq t_d$, the optimal replenishment cycle length is $T_j^*(p) = q_j/D(p)$,
 - (ii) when $T_{ij}^*(p) \geq t_d$, the optimal replenishment cycle length is

$$T_j^*(p) = t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - D(p)t_d)}{D(p)} \right].$$

Next, we will establish the corresponding total profit per unit time for the following two scenarios: (A) $T_j^*(p) = q_j/D(p)$ and (B) $T_j^*(p) = t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - D(p)t_d)}{D(p)} \right]$.

(A) When the optimal replenishment cycle is $T_j^*(p) = q_j/D(p)$.

Because $D(p) = \alpha p^{-\beta}$, then

$$T_j^*(p) < t_d \quad \text{if and if } p < \hat{p}_j, \quad \text{where } \hat{p}_j = (\alpha t_d/q_j)^{1/\beta}. \tag{44}$$

Substituting $T_j^*(p) = q_j/(\alpha p^{-\beta})$ into Eqs. (13a), (13b) and (14a), respectively, we can get the corresponding total profit per unit time as follows:

$$Z_{11j}(p) \equiv Z_{11j}(T_j^*(p), p) = (p - c)\alpha p^{-\beta} - \frac{\alpha A}{q_j p^\beta} - \frac{h\alpha p^{-\beta} q_j p^\beta}{2\alpha} + pI_e \alpha p^{-\beta} \left(M_j - \frac{q_j p^\beta}{2\alpha} \right), \quad (45)$$

$$Z_{12j}(p) \equiv Z_{12j}(T_j^*(p), p) = (p - c)\alpha p^{-\beta} - \frac{\alpha A}{q_j p^\beta} - \frac{\alpha p^{-\beta} q_j p^\beta}{2\alpha} (h + cI_c) + cI_c \alpha p^{-\beta} M_j^2 + \frac{\alpha^2 p^{-\beta} M_j^2}{2q_j p^\beta} (pI_e - cI_c), \quad (46)$$

and

$$Z_{21j}(p) \equiv Z_{21j}(T_j^*(p), p) = (p - c)\alpha p^{-\beta} - \frac{\alpha A}{q_j p^\beta} - \frac{h\alpha p^{-\beta} q_j p^\beta}{2\alpha} + pI_e \alpha p^{-\beta} \left(M_j - \frac{q_j p^\beta}{2\alpha} \right). \quad (47)$$

(B) When the optimal replenishment cycle is $T_j^*(p) = t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - D(p)t_d)}{D(p)} \right]$

Because $D(p) = \alpha p^{-\beta}$, then

$$T_j^*(p) \geq t_d \quad \text{if and only if } p \geq \hat{p}_j, \quad \text{where } \hat{p}_j \text{ is defined as in Eq. (44)}. \quad (48)$$

Substituting $T_j^*(p) = t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} \right]$ into Eqs. (13c), (14b) and (14c), respectively, we can get the corresponding total profit per unit time as follows:

$$\begin{aligned} Z_{13j}(p) &\equiv Z_{13j}(T_j^*(p), p) \\ &= p\alpha p^{-\beta} - \frac{A + c\alpha p^{-\beta} t_d}{t_d + (1/\theta) \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} - \frac{\alpha p^{-\beta} [c + ht_d + cI_c(t_d - M_j)]}{\theta t_d + \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} \cdot \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} \\ &\quad - \frac{\alpha p^{-\beta} (h + cI_c)}{\theta^2 t_d + \theta \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} \left\{ \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} - \ln \left[1 + \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} \right] \right\} \\ &\quad - \frac{\alpha p^{-\beta} [ht_d^2 + cI_c(t_d - M_j)^2 - pI_e M_j^2]}{2 \{ t_d + (1/\theta) \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right] \}}, \end{aligned} \quad (49)$$

$$\begin{aligned} Z_{22j}(p) &\equiv Z_{22j}(T_j^*(p), p) \\ &= p\alpha p^{-\beta} - \frac{A + c\alpha p^{-\beta} t_d}{t_d + (1/\theta) \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} - \frac{\alpha p^{-\beta} (ht_d + c)}{\theta t_d + \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} \cdot \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} \\ &\quad - \frac{\alpha p^{-\beta} h}{\theta^2 t_d + \theta \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} \left\{ \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} - \ln \left[1 + \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} \right] \right\} \\ &\quad - \frac{h\alpha p^{-\beta} t_d^2}{2 \{ t_d + (1/\theta) \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right] \}} + pI_e \alpha p^{-\beta} \left\{ M_j - \frac{t_d + (1/\theta) \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]}{2} \right\}, \end{aligned} \quad (50)$$

and

$$\begin{aligned} Z_{23j}(p) &\equiv Z_{23j}(T_j^*(p), p) = p\alpha p^{-\beta} - \frac{A + c\alpha p^{-\beta} t_d}{t_d + (1/\theta) \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} - \frac{\alpha p^{-\beta} (ht_d + c)}{\theta t_d + \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} \\ &\quad \cdot \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} - \frac{\alpha p^{-\beta} h}{\theta^2 t_d + \theta \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} \left\{ \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} - \ln \left[1 + \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} \right] \right\} \\ &\quad - \frac{\alpha p^{-\beta} cI_c}{\theta^2 t_d + \theta \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right]} \left\{ \exp \left\{ \theta t_d + \ln \left[1 + \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} \right] - \theta M_j \right\} - \theta t_d - \ln \left[1 + \frac{\theta(q_j - \alpha p^{-\beta} t_d)}{\alpha p^{-\beta}} \right] + \theta M_j - 1 \right\} \\ &\quad - \frac{(ht_d^2 - pI_e M_j^2) \alpha p^{-\beta}}{2 \{ t_d + (1/\theta) \ln \left[1 + \theta(q_j - \alpha p^{-\beta} t_d)/(\alpha p^{-\beta}) \right] \}}. \end{aligned} \quad (51)$$

3.2. Determination of the optimal price p

Now, for any given $M_j, j = 1, 2, \dots, K$, we want to find the optimal price p which maximize $Z_{ij}(p), i = 1, 2$, respectively.

Theorem 1 indicates that when the optimal replenishment cycle exists, then it is $T_{ij}^*(p), q_j/D(p)$ or $t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - D(p)t_d)}{D(p)} \right]$.

The remaining part of this subsection will discuss these three possible situations in detail.

Situation 1. The optimal replenishment cycle is $T_{ij}^*(p)$.

Case 1: $M_j \leq t_d$.

Let

$$f_{1j}(p) \equiv 2A - \Delta_{1j} = 2A - \alpha p^{-\beta} M_j^2 (h + pI_e), \quad (52)$$

and

$$f_{2j}(p) \equiv 2A - \Delta_{2j} = 2A - \alpha p^{-\beta} [t_d^2(h + cl_c) - M_j^2(cl_c - pl_e)]. \tag{53}$$

Because $\Delta_{2j} > \Delta_{1j}$, we can see that $f_{2j}(p) < f_{1j}(p)$, for all $p \in (0, \infty)$. Further, we can show that $f_{1j}(p)$ and $f_{2j}(p)$ are strictly increasing functions of $p \in (0, \infty)$. Also, $\lim_{p \rightarrow \infty} f_{1j}(p) = \lim_{p \rightarrow \infty} f_{2j}(p) = 2A$, and $\lim_{p \rightarrow 0^+} f_{1j}(p) = \lim_{p \rightarrow 0^+} f_{2j}(p) = -\infty$. By the Intermediate Value Theorem, we can find a unique value \tilde{p}_{1j} and \tilde{p}_{2j} such that

$$f_{1j}(\tilde{p}_{1j}) = 2A - \alpha \tilde{p}_{1j}^{-\beta} M_j^2(h + \tilde{p}_{1j}l_e) = 0, \tag{54}$$

and

$$f_{2j}(\tilde{p}_{2j}) = 2A - \alpha \tilde{p}_{2j}^{-\beta} [t_d^2(h + cl_c) - M_j^2(cl_c - \tilde{p}_{2j}l_e)] = 0, \tag{55}$$

respectively. Due to the properties of $f_{1j}(p)$ and $f_{2j}(p)$, we obtain the following lemma.

Lemma 9.

- (a) $0 < 2A \leq \Delta_{1j}$, if and only if $p \leq \tilde{p}_{1j}$,
- (b) $\Delta_{1j} \leq 2A \leq \Delta_{2j}$, if and only if $\tilde{p}_{1j} \leq p \leq \tilde{p}_{2j}$,
- (c) $2A \geq \Delta_{2j}$, if and only if $p \geq \tilde{p}_{2j}$.

From Lemmas 4 and 9, we can obtain the following result. The proof is omitted here.

Theorem 2. For any given p and M_j , the optimal replenishment cycle length

$$T_{1j}^*(p) = \begin{cases} T_{11j}(p), & \text{if } p \leq \tilde{p}_{1j}, \\ T_{12j}(p), & \text{if } \tilde{p}_{1j} \leq p \leq \tilde{p}_{2j}, \\ T_{13j}(p), & \text{if } p \geq \tilde{p}_{2j}. \end{cases}$$

Now, we want to find the optimal selling price p which maximize $Z_{1j}(p)$ in Eq. (38). That is, to find the value of p which satisfies both $\partial Z_{1j}(p)/\partial p = 0$ and $\partial^2 Z_{1j}(p)/\partial p^2 < 0$. From Eqs. (38a)–(38c) and Theorem 2, we can obtain the following result. The proof is trivial, hence, we omit it here.

Theorem 3. For any given M_j ,

- (a) if there exists a value p_{11j} which satisfies $\partial Z_{11j}(T_{11j}(p), p)/\partial p = 0$, $\partial^2 Z_{11j}(T_{11j}(p), p)/\partial p^2 < 0$ and $p_{11j} \leq \tilde{p}_{1j}$, then p_{11j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{11j}(T_{11j}(p_{11j}), p_{11j})$.
- (b) if there exists a value p_{12j} which satisfies $\partial Z_{12j}(T_{12j}(p), p)/\partial p = 0$, $\partial^2 Z_{12j}(T_{12j}(p), p)/\partial p^2 < 0$ and $\tilde{p}_{1j} \leq p_{12j} \leq \tilde{p}_{2j}$, then p_{12j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{12j}(T_{12j}(p_{12j}), p_{12j})$.
- (c) if there exists a value p_{13j} which satisfies $\partial Z_{13j}(T_{13j}(p), p)/\partial p = 0$, $\partial^2 Z_{13j}(T_{13j}(p), p)/\partial p^2 < 0$ and $p_{13j} \geq \tilde{p}_{2j}$, then p_{13j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{13j}(T_{13j}(p_{13j}), p_{13j})$.

Case 2: $M_j \geq t_d$.

Similarly, let

$$f_{3j}(p) \equiv 2A - \Delta_{3j} = 2A - \alpha p^{-\beta} t_d^2(h + pl_e), \tag{56}$$

and

$$f_{4j}(p) \equiv 2A - \Delta_{4j} = 2A - \alpha p^{-\beta} [(M_j^2 - t_d^2)(ht_d + c) + M_j^2(h + pl_e)]. \tag{57}$$

Because $\Delta_{4j} > \Delta_{3j}$, we can see that $f_{4j}(p) < f_{3j}(p)$, for all $p \in (0, \infty)$. Further, we can show that $f_{3j}(p)$ and $f_{4j}(p)$ are strictly increasing functions of $p \in (0, \infty)$. Also, $\lim_{p \rightarrow \infty} f_{3j}(p) = \lim_{p \rightarrow \infty} f_{4j}(p) = 2A$, and $\lim_{p \rightarrow 0^+} f_{3j}(p) = \lim_{p \rightarrow 0^+} f_{4j}(p) = -\infty$. By the Intermediate Value Theorem, we can find a unique value \tilde{p}_{3j} and \tilde{p}_{4j} such that

$$f_{3j}(\tilde{p}_{3j}) = 2A - \alpha \tilde{p}_{3j}^{-\beta} t_d^2(h + \tilde{p}_{3j}l_e) = 0, \tag{58}$$

and

$$f_{4j}(\tilde{p}_{4j}) = 2A - \alpha \tilde{p}_{4j}^{-\beta} [(M_j^2 - t_d^2)(ht_d + c) + M_j^2(h + \tilde{p}_{4j}l_e)] = 0, \tag{59}$$

respectively. Due to the properties of $f_{3j}(p)$ and $f_{4j}(p)$, we obtain the following lemma.

Lemma 10.

- (a) $0 < 2A \leq \Delta_{3j}$, if and only if $p \leq \tilde{p}_{3j}$,
- (b) $\Delta_{3j} \leq 2A \leq \Delta_{4j}$, if and only if $\tilde{p}_{3j} \leq p \leq \tilde{p}_{4j}$,
- (c) $2A \geq \Delta_{4j}$, if and only if $p \geq \tilde{p}_{4j}$.

From Lemmas 8 and 10, we can obtain the following result. The proof is omitted here.

Theorem 4. For any given p and M_j , the optimal replenishment cycle length

$$T_{2j}^*(p) = \begin{cases} T_{21j}(p), & \text{if } p \leq \tilde{p}_{3j}, \\ T_{22j}(p), & \text{if } \tilde{p}_{3j} \leq p \leq \tilde{p}_{4j}, \\ T_{23j}(p), & \text{if } p \geq \tilde{p}_{4j}. \end{cases}$$

Now, we want to find the optimal selling price p which maximize $Z_{2j}(p)$ in Eq. (42). That is, to find the value of p which satisfies both $\partial Z_{2j}(p)/\partial p = 0$ and $\partial^2 Z_{2j}(p)/\partial p^2 < 0$ for concavity. From Eqs. (42a)–(42c) and Theorem 4, we can obtain the following result. The proof is trivial, hence, we omit it here.

Theorem 5. For any given M_j ,

- (a) if there exists a value p_{21j} which satisfies $\partial Z_{21j}(T_{21j}(p), p)/\partial p = 0$, $\partial^2 Z_{21j}(T_{21j}(p), p)/\partial p^2 < 0$ and $p_{21j} \leq \tilde{p}_{3j}$, then p_{21j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{21j}(T_{21j}(p_{21j}), p_{21j})$.
- (b) if there exists a value p_{22j} which satisfies $\partial Z_{22j}(T_{22j}(p), p)/\partial p = 0$, $\partial^2 Z_{22j}(T_{22j}(p), p)/\partial p^2 < 0$ and $\tilde{p}_{3j} \leq p_{22j} \leq \tilde{p}_{4j}$, then p_{22j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{22j}(T_{22j}(p_{22j}), p_{22j})$.
- (c) if there exists a value p_{23j} which satisfies $\partial Z_{23j}(T_{23j}(p), p)/\partial p = 0$, $\partial^2 Z_{23j}(T_{23j}(p), p)/\partial p^2 < 0$ and $p_{23j} \geq \tilde{p}_{4j}$, then p_{23j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{23j}(T_{23j}(p_{23j}), p_{23j})$.

Situation 2. The optimal replenishment cycle length is $q_j/D(p)$ (i.e., $T_{ij}^*(p) \leq t_d$).

From Eqs. (44)–(47), we obtain the following result.

Theorem 6. For any given M_j ,

- (a) if there exists a value p_{11j} which satisfies $\partial Z_{11j}(p)/\partial p = 0$, $\partial^2 Z_{11j}(p)/\partial p^2 < 0$ and $p_{11j} \leq \hat{p}_j$, then p_{11j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{11j}(p_{11j})$.
- (b) if there exists a value p_{12j} which satisfies $\partial Z_{12j}(p)/\partial p = 0$, $\partial^2 Z_{12j}(p)/\partial p^2 < 0$ and $p_{12j} \leq \hat{p}_j$, then p_{12j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{12j}(p_{12j})$.
- (c) if there exists a value p_{21j} which satisfies $\partial Z_{21j}(p)/\partial p = 0$, $\partial^2 Z_{21j}(p)/\partial p^2 < 0$ and $p_{21j} \leq \hat{p}_j$, then p_{21j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{21j}(p_{21j})$.

Situation 3. The optimal replenishment cycle length is $t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - D(p)t_d)}{D(p)} \right]$ (i.e., $T_{ij}^*(p) \geq t_d$).

From Eqs. (48)–(51), we obtain the following result.

Theorem 7. For any given M_j ,

- (a) if there exists a value p_{13j} which satisfies $\partial Z_{13j}(p)/\partial p = 0$, $\partial^2 Z_{13j}(p)/\partial p^2 < 0$ and $p_{13j} \geq \hat{p}_j$, then p_{13j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{13j}(p_{13j})$.
- (b) if there exists a value p_{22j} which satisfies $\partial Z_{22j}(p)/\partial p = 0$, $\partial^2 Z_{22j}(p)/\partial p^2 < 0$ and $p_{22j} \geq \hat{p}_j$, then p_{22j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{22j}(p_{22j})$.
- (c) if there exists a value p_{23j} which satisfies $\partial Z_{23j}(p)/\partial p = 0$, $\partial^2 Z_{23j}(p)/\partial p^2 < 0$ and $p_{23j} \geq \hat{p}_j$, then p_{23j} is the optimal selling price and the corresponding optimal total profit per unit time is $Z_{23j}(p_{23j})$.

Summarizing the above arguments and Theorems 1, 3 and 5, we establish the following algorithm to find the optimal solution (T^*, p^*)

3.3. Algorithm

For given t_d and $0 \leq M_1 < M_2 < \dots < M_K$,

- (a) if $M_K \leq t_d$, then for $j = 1, 2, \dots, K$, find \tilde{p}_{1j} and \tilde{p}_{2j} from Eqs. (54) and (55), respectively. Go to Algorithm 1.

- (b) if $M_1 \geq t_d$, then for $j = 1, 2, \dots, K$, find \tilde{p}_{3j} and \tilde{p}_{4j} from Eqs. (58) and (59), respectively. Go to Algorithm 2.
- (c) if there exists an integer $n \in \{1, 2, \dots, K\}$ such that $0 \leq M_1 < M_2 < \dots < M_n \leq t_d \leq M_{n+1} < \dots < M_K$, then
 - (i) for $j = 1, 2, \dots, n$, find \tilde{p}_{1j} and \tilde{p}_{2j} from Eqs. (54) and (55), respectively;
 - (ii) for $j = n + 1, n + 2, \dots, K$, find \tilde{p}_{3j} and \tilde{p}_{4j} from Eqs. (58) and (59), respectively.

Go to Algorithm 3.

Algorithm 1.

Step 1.

- (a) if there exists a p_{11j} such that $p_{11j} \leq \tilde{p}_{1j}$ and p_{11j} satisfies both $\partial Z_{11j}(T_{11j}(p), p) / \partial p = 0$, and $\partial^2 Z_{11j}(T_{11j}(p), p) / \partial p^2 < 0$, then find $T_{11j}(p_{11j})$ from Eq. (22), and then determine $Q_{11j} = D(p_{11j}) \cdot T_{11j}(p_{11j})$; otherwise, let $Z_{11j}(T_j, p) = 0$.
- (b) if there exists a p_{12j} such that $\tilde{p}_{1j} \leq p_{12j} \leq \tilde{p}_{2j}$ and p_{12j} satisfies both $\partial Z_{12j}(T_{12j}(p), p) / \partial p = 0$ and $\partial^2 Z_{12j}(T_{12j}(p), p) / \partial p^2 < 0$, then find $T_{12j}(p_{12j})$ from Eq. (27), and then determine $Q_{12j} = D(p_{12j}) \cdot T_{12j}(p_{12j})$; otherwise, let $Z_{12j}(T_j, p) = 0$.
- (c) if there exists a p_{13j} such that $p_{13j} \geq \tilde{p}_{2j}$ and p_{13j} satisfies both $\partial Z_{13j}(T_{13j}(p), p) / \partial p = 0$ and $\partial^2 Z_{13j}(T_{13j}(p), p) / \partial p^2 < 0$, then find $T_{13j}(p_{13j})$ from Eq. (35), and then determine $Q_{13j} = D(p_{13j})t_d + \frac{D(p_{13j})}{\theta} [e^{\theta(T_{13j}(p_{13j})-t_d)} - 1]$; otherwise, let $Z_{13j}(T_j, p) = 0$.

Step 2.

- (a) (i) If $q_j \leq Q_{11j} < q_{j+1}$, then $T_{11j}(p_{11j})$ is a feasible solution, using Eq. (38a) to get $Z_{11j}(T_j, p) = Z_{11j}(T_{11j}(p_{11j}), p_{11j})$.
 (ii) If $Q_{11j} \geq q_{j+1}$, then $T_{11j}(p_{11j})$ is not a feasible solution, set $Z_{11j}(T_j, p) = 0$.
 (iii) If $Q_{11j} < q_j$, then find \hat{p}_j from Eq. (44).
 If there exists a p_{11j} such that $p_{11j} < \hat{p}_j$ and p_{11j} satisfies both $\partial Z_{11j}(p) / \partial p = 0$, and $\partial^2 Z_{11j}(p) / \partial p^2 < 0$, then set $T_j^*(p_{11j}) = q_j / D(p_{11j})$. And using Eq. (45) to get $Z_{11j}(T_j, p) = Z_{11j}(T_j^*(p_{11j}), p_{11j})$. Otherwise, set $Z_{11j}(T_j, p) = 0$.
- (b) (i) If $q_j \leq Q_{12j} < q_{j+1}$, then $T_{12j}(p_{12j})$ is a feasible solution, using Eq. (38b) to get $Z_{12j}(T_j, p) = Z_{12j}(T_{12j}(p_{12j}), p_{12j})$.
 (ii) If $Q_{12j} \geq q_{j+1}$, then $T_{12j}(p_{12j})$ is not a feasible solution, set $Z_{12j}(T_j, p) = 0$.
 (iii) If $Q_{12j} < q_j$, then find \hat{p}_j from Eq. (44).
 If there exists a p_{12j} such that $p_{12j} < \hat{p}_j$ and p_{12j} satisfies both $\partial Z_{12j}(p) / \partial p = 0$, and $\partial^2 Z_{12j}(p) / \partial p^2 < 0$, then set $T_j^*(p_{12j}) = q_j / D(p_{12j})$. And using Eq. (46) to get $Z_{12j}(T_j, p) = Z_{12j}(T_j^*(p_{12j}), p_{12j})$. Otherwise, set $Z_{12j}(T_j, p) = 0$.
- (c) (i) If $q_j \leq Q_{13j} < q_{j+1}$, then $T_{13j}(p_{13j})$ is a feasible solution, using Eq. (38c) to get $Z_{13j}(T_j, p) = Z_{13j}(T_{13j}(p_{13j}), p_{13j})$.
 (ii) If $Q_{13j} \geq q_{j+1}$, then $T_{13j}(p_{13j})$ is not a feasible solution, set $Z_{13j}(T_j, p) = 0$.
 (iii) If $Q_{13j} < q_j$, then find \hat{p}_j from Eq. (44).
 If there exists a p_{13j} such that $p_{13j} < \hat{p}_j$ and p_{13j} satisfies both $\partial Z_{13j}(p) / \partial p = 0$, and $\partial^2 Z_{13j}(p) / \partial p^2 < 0$, then set $T_j^*(p_{13j}) = t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - D(p_{13j})t_d)}{D(p_{13j})} \right]$. And using Eq. (49) to get $Z_{13j}(T_j, p) = Z_{13j}(T_j^*(p_{13j}), p_{13j})$. Otherwise, set $Z_{13j}(T_j, p) = 0$.

Step 3. Find $\max \{Z_{1kj}(T_j, p) | k = 1, 2, 3; j = 1, 2, \dots, K\}$.

If $Z_1(T^*, p^*) = \max \{Z_{1kj}(T_j, p) | k = 1, 2, 3; j = 1, 2, \dots, K\}$, then (T^*, p^*) is the optimal solution.

Step 4. Stop.

Algorithm 2.

Step 1.

- (a) If there exists a p_{21j} such that $p_{21j} \leq \tilde{p}_{3j}$ and p_{21j} satisfies both $\partial Z_{21j}(T_{21j}(p), p) / \partial p = 0$, and $\partial^2 Z_{21j}(T_{21j}(p), p) / \partial p^2 < 0$, then find $T_{21j}(p_{21j})$ from Eq. (39), and then determine $Q_{21j} = D(p_{21j}) \cdot T_{21j}(p_{21j})$; otherwise, let $Z_{21j}(T_j, p) = 0$.
- (b) If there exists a p_{22j} such that $\tilde{p}_{3j} \leq p_{22j} \leq \tilde{p}_{4j}$ and p_{22j} satisfies both $\partial Z_{22j}(T_{22j}(p), p) / \partial p = 0$ and $\partial^2 Z_{22j}(T_{22j}(p), p) / \partial p^2 < 0$, then find $T_{22j}(p_{22j})$ from Eq. (40), and then determine $Q_{22j} = D(p_{22j})t_d + \frac{D(p_{22j})}{\theta} [e^{\theta(T_{22j}(p_{22j})-t_d)} - 1]$; otherwise, let $Z_{22j}(T_j, p) = 0$.
- (c) If there exists a p_{23j} such that $p_{23j} \geq \tilde{p}_{4j}$ and p_{23j} satisfies both $\partial Z_{23j}(T_{23j}(p), p) / \partial p = 0$ and $\partial^2 Z_{23j}(T_{23j}(p), p) / \partial p^2 < 0$, then find $T_{23j}(p_{23j})$ from Eq. (41), and then determine $Q_{23j} = D(p_{23j})t_d + \frac{D(p_{23j})}{\theta} [e^{\theta(T_{23j}(p_{23j})-t_d)} - 1]$; otherwise, let $Z_{23j}(T_j, p) = 0$.

Step 2.

- (a) (i) If $q_j \leq Q_{21j} < q_{j+1}$, then $T_{21j}(p_{21j})$ is a feasible solution, using Eq. (42a) to get $Z_{21j}(T_j, p) = Z_{21j}(T_{21j}(p_{21j}), p_{21j})$.
 (ii) If $Q_{21j} \geq q_{j+1}$, then $T_{21j}(p_{21j})$ is not a feasible solution, set $Z_{21j}(T_j, p) = 0$.
 (iii) If $Q_{21j} < q_j$, then find \hat{p}_j from Eq. (44).
 If there exists a p_{21j} such that $p_{21j} < \hat{p}_j$ and p_{21j} satisfies both $\partial Z_{21j}(p)/\partial p = 0$, and $\partial^2 Z_{21j}(p)/\partial p^2 < 0$, then set $T_j^*(p_{21j}) = q_j/D(p_{21j})$. And using Eq. (47) to get $Z_{21j}(T_j, p) = Z_{21j}(T_j^*(p_{21j}), p_{21j})$. Otherwise, set $Z_{21j}(T_j, p) = 0$.
- (b) (i) If $q_j \leq Q_{22j} < q_{j+1}$, then $T_{22j}(p_{22j})$ is a feasible solution, using Eq. (42b) to get $Z_{22j}(T_j, p) = Z_{22j}(T_{22j}(p_{22j}), p_{22j})$.
 (ii) If $Q_{22j} \geq q_{j+1}$, then $T_{22j}(p_{22j})$ is not a feasible solution, set $Z_{22j}(T_j, p) = 0$.
 (iii) If $Q_{22j} < q_j$, then find \hat{p}_j from Eq. (44).
 If there exists a p_{22j} such that $p_{22j} \geq \hat{p}_j$ and p_{22j} satisfies both $\partial Z_{22j}(p)/\partial p = 0$, and $\partial^2 Z_{22j}(p)/\partial p^2 < 0$, then set $T_j^*(p_{22j}) = t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - D(p_{22j})t_d)}{D(p_{22j})} \right]$. And using Eq. (50) to get $Z_{22j}(T_j, p) = Z_{22j}(T_j^*(p_{22j}), p_{22j})$. Otherwise, set $Z_{22j}(T_j, p) = 0$.
- (c) (i) If $q_j \leq Q_{23j} < q_{j+1}$, then $T_{23j}(p_{23j})$ is a feasible solution, using Eq. (42c) to get $Z_{23j}(T_j, p) = Z_{23j}(T_{23j}(p_{23j}), p_{23j})$.
 (ii) If $Q_{23j} \geq q_{j+1}$, then $T_{23j}(p_{23j})$ is not a feasible solution, set $Z_{23j}(T_j, p) = 0$.
 (iii) If $Q_{23j} < q_j$, then find \hat{p}_j from Eq. (44).
 If there exists a p_{23j} such that $p_{23j} \geq \hat{p}_j$ and p_{23j} satisfies both $\partial Z_{23j}(p)/\partial p = 0$, and $\partial^2 Z_{23j}(p)/\partial p^2 < 0$, then set $T_j^*(p_{23j}) = t_d + \frac{1}{\theta} \ln \left[1 + \frac{\theta(q_j - D(p_{23j})t_d)}{D(p_{23j})} \right]$. And using Eq. (51) to get $Z_{23j}(T_j, p) = Z_{23j}(T_j^*(p_{23j}), p_{23j})$. Otherwise, set $Z_{23j}(T_j, p) = 0$.

Step 3. Find $\max \{Z_{2kj}(T_j, p) | k = 1, 2, 3; j = 1, 2, \dots, K\}$.

If $Z_2(T^*, p^*) = \max \{Z_{2kj}(T_j, p) | k = 1, 2, 3; j = 1, 2, \dots, K\}$, then (T^*, p^*) is the optimal solution.

Step 4. Stop.

Algorithm 3.

- Step 1. (a) For $j=1, 2, \dots, n$, operate Step 1-Step 2 in Algorithm 1, then find $Z_1(T_1, p_1) = \max \{Z_{1kj}(T_j, p) | k = 1, 2, 3; j = 1, 2, \dots, n\}$.
 (b) For $j = n+1, n+2, \dots, K$, operate Step 1-Step 2 in Algorithm 2, then find $Z_2(T_2, p_2) = \max \{Z_{2kj}(T_j, p) | k = 1, 2, 3; j = n+1, n+2, \dots, K\}$.
- Step 2. Find $\max \{Z_1(T_1, p_1), Z_2(T_2, p_2)\}$.
 If $Z(T^*, p^*) = \max \{Z_1(T_1, p_1), Z_2(T_2, p_2)\}$, then (T^*, p^*) is the optimal solution.
- Step 3. Stop.

4. Numerical examples

The following numerical examples are given to illustrate the above solution procedure and investigate the effect of changes in some main parameter values on the optimal solution in our models.

Example 1. We consider an inventory system with the data: $\alpha = 10^5$, $\beta = 1.5$, $\theta = 0.05$, $A = \$100/\text{order}$, $h = \$4/\text{unit}/\text{year}$, $c = \$20/\text{unit}$, $I_c = 0.09$, $I_e = 0.05$, $t_d = 50$ days and a permissible delay schedule M offered by the supplier are listed in Table 1.

Following the algorithms, we obtain the optimal selling price, $p^* = \$63.1761$ per unit, the optimal length of replenishment cycle, $T_{131} = 0.385334$ years, the annual demand rate, $D(p^*) = 199.146$ units, the optimal order quantity, $Q^* = 76.7377$ units and the annual total profit, $Z(T^*, p^*) = \$8131.66$.

Example 2. We discuss the effects of changing the values of the parameters, the ordering cost A and the holding cost h , on the optimal solution. The remainder parameter values are identical to those in Example 1. The sensitivity analysis is performed by changing $A \in \{50, 100, 150, 200\}$ and $h \in \{2, 4, 6, 8\}$ (one at a time), while keeping the remaining parameters unchanged. Following the algorithms, the computational results are shown in Tables 2 and 3.

Table 2 reveals that a higher ordering cost value A results in a higher optimal selling price p^* , the optimal replenishment cycle time T^* , and the optimal order quantity Q^* , but lower values for the annual demand rate $D(p^*)$ and the annual total profit, $Z(T^*, p^*)$. This indicates that the retailer needs to increase the replenishment cycle time to reduce the number of orders and increase the order quantity in each cycle when the ordering cost is higher. In addition, Table 3 shows that a higher holding cost value h results in a higher optimal selling price p^* , but lower values for the optimal replenishment cycle time T^* , the optimal order quantity Q^* , the annual demand rate $D(p^*)$, and the annual total profit $Z(T^*, p^*)$. This implies that the retailers need to reduce their replenishment cycle time and their order quantity to avoid higher holding costs.

Table 1
Credit terms for Example 1.

Credit period (days)	Total amount of purchase (units/order)
$M_1 = 30$	$1 \leq Q < 100$
$M_2 = 45$	$100 \leq Q < 200$
$M_3 = 60$	$200 \leq Q$

Table 2
Computation results for different A .

A (\$/order)	p^* (\$/unit)	T^* (years)	$D(p^*)$ (units/year)	Q^* (units/order)	$Z(T^*, p^*)$
50	62.0384	$T_{131} = 0.270309$	204.649	55.3185	8284.37
100	63.1761	$T_{131} = 0.385334$	199.146	76.7377	8131.66
150	64.0884	$T_{131} = 0.476158$	194.908	92.8072	8015.51
200	64.6422	$T_{132} = 0.551433$	192.409	106.101	7934.86

Table 3
Computation results for different h .

h (\$/unit/year)	p^* (\$/unit)	T^* (years)	$D(p^*)$ (units/year)	Q^* (units/order)	$Z(T^*, p^*)$
2	62.5077	$T_{131} = 0.455223$	202.348	92.1136	8215.67
4	63.1761	$T_{131} = 0.385334$	199.146	76.7377	8131.66
6	63.7590	$T_{131} = 0.340975$	196.421	66.9747	8059.93
8	64.2847	$T_{131} = 0.309636$	194.017	60.0746	7996.44

Table 4
Computation results for different θ .

θ	p^* (\$/unit)	T^* (years)	$D(p^*)$ (units/year)	Q^* (units/order)	$Z(T^*, p^*)$
0.01	62.9976	$T_{232} = 0.498905$	199.992	100.000	8148.59
0.03	63.0916	$T_{232} = 0.497803$	199.546	100.000	8130.30
0.05	63.4120	$T_{231} = 0.383599$	198.036	76.5982	8114.88
0.07	63.4995	$T_{231} = 0.373183$	197.626	74.5844	8101.93
0.10	63.6253	$T_{231} = 0.359092$	197.040	71.8516	8083.28

Table 5
Computation results for different I_c .

I_c	p^* (\$/unit)	T^* (years)	$D(p^*)$ (units/year)	Q^* (units/order)	$Z(T^*, p^*)$
0.03	63.1635	$T_{232} = 0.496452$	199.205	100.000	8145.82
0.06	63.3572	$T_{231} = 0.400643$	198.292	80.1394	8129.43
0.09	63.4120	$T_{231} = 0.383599$	198.036	76.5982	8114.88
0.12	63.4610	$T_{231} = 0.368686$	197.806	73.5077	8101.25

Example 3. We further discuss the effects of changing the values of the parameters, the deterioration rate θ and the capital opportunity cost in stock per dollar I_c , on the optimal solution. The remainder parameter values are identical to those in Example 1 except $t_d = 10$ days. The sensitivity analysis is performed by changing $\theta \in \{0.01, 0.03, 0.05, 0.07, 0.10\}$ and $I_c \in \{0.03, 0.06, 0.09, 0.12\}$, taking one parameter at a time and keeping the remaining parameters unchanged. Following the algorithms, the computational results are shown in Tables 4,5.

Table 4 shows that a higher value of the deterioration rate θ results in a higher value for the optimal selling price p^* but lower values for the optimal length of replenishment cycle T^* , the optimal order quantity Q^* , the annual demand rate $D(p^*)$ and the annual total profit, $Z(T^*, p^*)$. This implies that the retailers will reduce their order quantity to avoid the items deteriorating when the deterioration rate θ increases. Moreover, the computational results in Table 5 demonstrate that a higher value of the capital opportunity cost in stock per dollar per year I_c results in a higher value for the optimal selling price p^* but lower values for the optimal length of replenishment cycle T^* , the optimal order quantity Q^* , the annual demand rate $D(p^*)$ and the annual total profit, $Z(T^*, p^*)$. Consequently, a higher value of I_c implies a lower value of the total profit. A simple management interpretation is that the retailers should reduce their order quantity and take advantage of the permissible delay more frequently.

5. Conclusion

A few authors discuss the fact that there is a time span during which items maintain their quality or original condition. To reflect the real-life situation, it is therefore important to consider non-instantaneously deteriorating items in the inventory system. In addition, use of a trade credit is a common payment feature in B2B and B2C transactions. In this paper, we develop an appropriate inventory model for non-instantaneously deteriorating items in circumstances where the supplier provides the retailer various trade credits linked to order quantity. Some mathematical results and algorithms are established to identify the optimal pricing and ordering policies for maximizing the retailer's total profit. Furthermore, we provide numerical examples and conduct a sensitivity analysis to illustrate the proposed model. The results of the sensitivity analysis show that the retailer needs to increase the order quantity and selling price if the ordering cost is higher. However, retailers tend to reduce not only the replenishment cycle time to avoid higher holding costs but also the order quantities to avoid the deterioration of items. A higher value of the capital opportunity cost in stock implies a lower value of the total profit. Research on this problem can be extended in several ways. For instance, it could be of interest to relax the restriction on the constant deterioration rate. In addition, we may generalize the model to allow for shortages, quantity discounts, and other factors.

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