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Ranking triangle and trapezoidal fuzzy numbers based on the relative preference relation

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ABSTRACT

In this paper, we first propose a fuzzy preference relation with membership function representing preference degree to compare two fuzzy numbers. Then a relative preference relation is constructed on the fuzzy preference relation to rank a set of fuzzy numbers. Since the fuzzy preference relation is a total ordering relation satisfying reciprocal and transitive laws on fuzzy numbers, the relative preference relation satisfies a total ordering relation on fuzzy numbers as well. Normally, utilizing preference relation is more reasonable than defuzzification on ranking fuzzy numbers, because defuzzification does not present preference degree between two fuzzy numbers and loses some messages. However, fuzzy pair-wise comparison by preference relation is complex and difficult. To avoid above shortcomings, the relative preference relation adopts the strengths of defuzzification and fuzzy preference relation. That is to say, the relative preference relation expresses preference degrees of several fuzzy numbers over average as similar as the fuzzy preference relation does, and ranks fuzzy numbers by relative crisp values as defuzzification does. Thus utilizing the relative preference relation ranks fuzzy numbers easily and quickly, and is able to reserve fuzzy information.

1. Introduction

Since Jain, Dubois and Prade [1–3] proposed related concept for fuzzy numbers, comparing fuzzy numbers had been an important issue of fuzzy aspect. For instance, Bortolan and Degani [4] reviewed fuzzy ranking methods in 1985, Lee and Li [5] proposed comparison of fuzzy numbers based on the probability measure of fuzzy events in 1988, Chen and Hwang [6] expressed fuzzy multiple attribute decision making in 1992, Choobineh and Li [7] developed an index for ordering fuzzy numbers in 1993, Dias [8] used fuzzy numbers to rank alternatives in 1993, Lee, Cho and Lee-Kwang [9] ranked fuzzy values with satisfaction function in 1994, Requena et al. [10] utilized an artificial neural network to rank fuzzy numbers automatically in 1994, Fortemps and Roubens [11] expressed ranking and defuzzification methods based on area compensation in 1996, Cheng [12] proposed the coefficient of variance(or called CV index) to rank fuzzy numbers in 1998, Raj and Kumar [13] used maximizing and minimizing sets for ranking fuzzy alternatives with fuzzy weights in 1999, and Chu and Tsao [14] ranked fuzzy numbers with an area between centroid and original points in 2002. Besides, some researches [15–21] were helpful to rank fuzzy numbers as well.

Through the previous researches, fuzzy ranking methods are commonly classified into two varied categories. One is defuzzification and the other is comparing fuzzy numbers by preference relation (i.e. fuzzy pair-wise comparison). Defuzzification is

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simpler and easier than fuzzy pair-wise comparison on ranking fuzzy numbers. However, defuzzification loses fuzzy messages. On the other hand, fuzzy pair-wise comparison is complex and difficult, but the work reserves fuzzy messages. Thus, Yuan [22] supposed that a fuzzy ranking method had to present preference relation in fuzzy terms. For instance, comparing two fuzzy numbers A with B does not merely demonstrate that A is preferred or not preferred to B. A fuzzy ranking method desires to express some situations, such as A dominates B, A is slight better than B, or A and B do not have the difference, etc. In short, a fuzzy preference relation with membership function represents preference degree.

Based on above, we first propose a fuzzy preference relation in this paper. The fuzzy preference relation with membership function represents preference degree of two fuzzy numbers to compare the two fuzzy numbers. Then the fuzzy preference relation is modified into a relative preference relation, and the relative preference relation with membership function represents relative preference degrees of a set of fuzzy numbers over average. The fuzzy numbers will be ranked according to their relative preference degrees.

For the sake of clarity, related concepts of fuzzy theory are expressed in Section 2. The fuzzy preference relation is proposed in Section 3, and the relative preference relation constructed on the fuzzy preference relation is presented in Section 4. Some numerical examples of comparing fuzzy numbers based on the fuzzy preference relation and relative preference relation are illustrated in Section 5. Additionally, we compare the proposed method with other similar methods through the illustrated examples in this section. Finally, an extension in general fuzzy numbers is shown in Section 6.

2. Preliminaries

In this section, we review basic notions of fuzzy numbers [23–28].

Definition 2.1. A fuzzy number A is defined in an interval $[a_l, a_r]$, where a_l and a_r are respectively lower and upper boundaries of A.

Definition 2.2. A triangular fuzzy number A is a fuzzy number with piecewise linear membership function μ_A defined by

$$\mu_A = \begin{cases} \frac{x-a_l}{a_m-a_r}, & a_l \leqslant x \leqslant a_m, \\ \frac{a_r-x}{a_r-a_m}, & a_m \leqslant x \leqslant a_r, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

which can be indicated as a triplet (a_l, a_m, a_r).

Definition 2.3. A trapezoidal fuzzy number A is a fuzzy number with membership function μ_A defined by

$$\mu_A = \begin{cases} \frac{x-a_l}{a_h-a_l}, & a_l \leqslant x \leqslant a_h, \\ 1, & a_h \leqslant x \leqslant a_m, \\ \frac{a_r-x}{a_r-a_m}, & a_m \leqslant x \leqslant a_r, \\ 0, & \text{otherwise}, \end{cases} \tag{2}$$

which can be denoted as a quartet (a_l, a_h, a_m, a_r).

Definition 2.4. Let $\circ$ be an operation on real numbers, such as $+, -, *, \wedge, \vee$, etc. Let A and B be two fuzzy numbers. By extension principle, an extended operation $\circ$ on fuzzy numbers is defined by

$$\mu_{A\circ B}(z) = \sup_{x,y:z=x\circ y} \{\mu_A(x) \wedge \mu_B(y)\}. \tag{3}$$

Definition 2.5. Let A be a fuzzy number. Then A^L_α and A^U_α are respectively defined as

$$A^L_\alpha = \inf_{\mu_A(z)\geqslant\alpha}(z), \tag{4}$$

and

$$A^U_\alpha = \sup_{\mu_A(z)\geqslant\alpha}(z). \tag{5}$$

Definition 2.6. A fuzzy preference relation R is a fuzzy subset of $\Re \times \Re$ with membership function $\mu_R(A,B)$ representing preference degree of fuzzy numbers A over B [27–29].

(i) R is reciprocal iff $\mu_R(A,B) = 1 - \mu_R(B,A)$ for all fuzzy numbers A and B.
(ii) R is transitive iff $\mu_R(A,B) \geqslant \frac{1}{2}$ and $\mu_R(B,C) \geqslant \frac{1}{2} \Rightarrow \mu_R(A,C) \geqslant \frac{1}{2}$ for all fuzzy numbers A, B and C.
(iii) R is a fuzzy total ordering iff R is both reciprocal and transitive.

Comparing A with B by the fuzzy preference relation R, A is preferred to B iff $\mu_R(A,B) > \frac{1}{2}$, and A is equal to B iff $\mu_R(A,B) = \frac{1}{2}$.

Definition 2.7. Let $\succ$ be a binary relation on fuzzy numbers defined by $A \succ B$ iff A is preferred to B (i.e. $\mu_R(A,B) > \frac{1}{2}$).

Definition 2.8. Let A and B be two fuzzy numbers for $A \succ B$. Based on Lee's extended fuzzy preference relation [28], the difference of A and B is defined as $\int_0^1 ((A-B)_\alpha^L + (A-B)_\alpha^U)d\alpha$.

3. The fuzzy preference relation between two fuzzy numbers

A fuzzy preference relation P revised from Lee's extended fuzzy preference relation [28] is proposed to compare two fuzzy numbers, and presented in following definitions.

Definition 3.1. Let A and B be two fuzzy numbers, where A is in an interval $[a_l, a_r]$ and B is in an interval $[b_l, b_r]$. Let T^+ be in an interval $[t_l^+, t_r^+]$ = [max$\{a_l, b_l\}$, max$\{a_r, b_r\}$] (i.e. the maximum fuzzy number) and T^- be in an interval $[t_l^-, t_r^-]$ = [min$\{a_l, b_l\}$, min$\{a_r, b_r\}$] (i.e. the minimum fuzzy number). We define an extending difference $\|(A,B)\|$ between A and B to be

$$\|(A,B)\| = \|T\| = \begin{cases} \int_0^1 \left((T^+ - T^-)_\alpha^L + (T^+ - T^-)_\alpha^U\right)d\alpha & \text{if } t_l^+ \geqslant t_r^- \\ \int_0^1 \left((T^+ - T^-)_\alpha^L + (T^+ - T^-)_\alpha^U + 2(t_r^- - t_l^+)\right)d\alpha & \text{if } t_l^+ < t_r^- \end{cases}. \tag{6}$$

The extending difference is a difference between two fuzzy numbers if the intersection of the two fuzzy numbers is $\emptyset$ or a point, and it is a important base for fuzzy preference relation to ensure that the fuzzy preference relation is in the interval [1]. In addition, the extending difference is used not only in two fuzzy numbers but also in three fuzzy numbers or more. Further, the extending difference will be the smallest range for a set of fuzzy numbers if the intersection of the minimum and the maximum fuzzy numbers is $\emptyset$ or a point. Otherwise, the two fuzzy numbers must be moved to make the intersection of the two moving values being $\emptyset$ or a point.

Definition 3.2. Let A and B be two fuzzy numbers, where A is in an interval $[a_l, a_r]$ and B is in an interval $[b_l, b_r]$. A fuzzy preference relation P is a fuzzy subset of $\Re \times \Re$ with membership function $\mu_P(A,B)$ representing preference degree of A over B. Define

$$\mu_P(A,B) = \frac{1}{2}\left(\frac{\int_0^1((A-B)_\alpha^L + (A-B)_\alpha^U)d\alpha}{\|T\|} + 1\right), \tag{7}$$

where

$$\|T\| = \begin{cases} \int_0^1((T^+ - T^-)_\alpha^L + (T^+ - T^-)_\alpha^U)d\alpha & \text{if } t_l^+ \geqslant t_r^- \\ \int_0^1((T^+ - T^-)_\alpha^L + (T^+ - T^-)_\alpha^U + 2(t_r^- - t_l^+))d\alpha & \text{if } t_l^+ < t_r^- \end{cases},$$

T^+ is in an interval $[t_l^+, t_r^+]$, T^- is in an interval $[t_l^-, t_r^-]$, t_l^+ = max$\{a_l, b_l\}$, t_r^+ = max$\{a_r, b_r\}$, t_l^- = min$\{a_l, b_l\}$, t_r^- = min$\{a_r, b_r\}$.

Obviously, $\mu_R(A,B) \geqslant \frac{1}{2}$ iff $\int_0^1((A-B)_\alpha^L + (A-B)_\alpha^U)d\alpha \geqslant 0$, where $\mu_P(A,B) = \frac{1}{2}$ as $A = B$.

Lemma 3.1. *The preference relation P is reciprocal iff $\mu_P(A,B) = 1 - \mu_P(B,A)$.*

Proof.

$$\mu_P(A,B) = \frac{1}{2}\left(\frac{\int_0^1\left((A-B)_\alpha^L + (A-B)_\alpha^U\right)d\alpha}{\|T\|} + 1\right) = \frac{1}{2}\left(\frac{-\left(\int_0^1((B-A)_\alpha^L + (B-A)_\alpha^U)d\alpha\right)}{\|T\|} + 1\right)$$
$$= \frac{1}{2}\left(\frac{-\left(\int_0^1((B-A)_\alpha^L + (B-A)_\alpha^U)d\alpha\right)}{\|T\|} - 1\right) + 1 = 1 - \frac{1}{2}\left(\frac{\int_0^1\left((B-A)_\alpha^L + (B-A)_\alpha^U\right)d\alpha}{\|T\|} + 1\right) = 1 - \mu_P(B,A).$$

□

Lemma 3.2. *The preference relation P is transitive iff* $\mu_P(A,B) \geqslant \frac{1}{2}$ *and* $\mu_P(B,C) \geqslant \frac{1}{2} \Rightarrow \mu_P(A,C) \geqslant \frac{1}{2}$.

Proof. Let A, B and C be three fuzzy numbers, where $A = [a_l, a_r]$, $B = [b_l, b_r]$ and $C = [c_l, c_r]$. Since

$$\mu_P(A,B) \geqslant \frac{1}{2} \text{ and } \mu_P(B,C) \geqslant \frac{1}{2},$$

$$\int_0^1 ((A-B)_\alpha^L + (A-B)_\alpha^U) d\alpha \geqslant 0 \text{ and } \int_0^1 ((B-C)_\alpha^L + (B-C)_\alpha^U) d\alpha \geqslant 0.$$

Then

$$\int_0^1 ((A-B)_\alpha^L + (A-B)_\alpha^U) d\alpha + \int_0^1 ((B-C)_\alpha^L + (B-C)_\alpha^U) d\alpha \geqslant 0.$$

In addition,

$$\begin{aligned}\int_0^1 ((A-B)_\alpha^L + (A-B)_\alpha^U) d\alpha + \int_0^1 ((B-C)_\alpha^L + (B-C)_\alpha^U) d\alpha &= \int_0^1 ((A-B)_\alpha^L + (B-C)_\alpha^L + (A-B)_\alpha^U + (B-C)_\alpha^U) d\alpha \\ &= \int_0^1 (A_\alpha^L - B_\alpha^U + B_\alpha^L - C_\alpha^U + A_\alpha^U - B_\alpha^L + B_\alpha^U - C_\alpha^L) d\alpha = \int_0^1 ((A_\alpha^L - C_\alpha^U) + (A_\alpha^U - C_\alpha^L)) d\alpha \\ &= \int_0^1 ((A-C)_\alpha^L + (A-C)_\alpha^U) d\alpha \geqslant 0.\end{aligned}$$

Based on above, $\int_0^1 ((A-C)_\alpha^L + (A-C)_\alpha^U) d\alpha \geqslant 0$ iff $\mu_P(A,C) \geqslant \frac{1}{2}$. □

According to Lemma 3.1 and Lemma 3.2, the preference relation P is a total ordering relation [27,29].

Lemma 3.3. *Let A and B be two fuzzy numbers. By the fuzzy preference relation P, A is preferred to B iff* $\mu_P(A,B) > \frac{1}{2}$.

Lemma 3.4. $A \succ B$ *iff* $\mu_P(A,B) > \frac{1}{2}$, *where* $\succ$ *is a binary relation on fuzzy numbers presented on* Definition 2.7.

Lemma 3.5. *A and B are two triangular fuzzy numbers, where* $A = (a_l, a_m, a_r)$ and $B = (b_l, b_m, b_r)$. *Let* T^+ *be* $(\max\{a_l, b_l\}, \max\{a_m, b_m\}, \max\{a_r, b_r\})$ *and* T^- *be* $(\min\{a_l, b_l\}, \min\{a_m, b_m\}, \min\{a_r, b_r\})$. *We define an extending difference* $\|(A,B)\|$ *between A and B to be*

$$\|(A,B)\| = \|T\| = \begin{cases} \frac{(t_l^+ - t_r^-) + 2(t_m^+ - t_m^-) + (t_r^+ - t_l^-)}{2} & \text{if } t_l^+ \geqslant t_r^- \\ \frac{(t_l^+ - t_r^-) + 2(t_m^+ - t_m^-) + (t_r^+ - t_l^-)}{2} + 2(t_r^- - t_l^+) & \text{if } t_l^+ < t_r^- \end{cases}. \tag{8}$$

Likewise, the extending difference $\|(A,B)\|$ is used in fuzzy preference relation between triangular fuzzy numbers A and B to ensure that the fuzzy preference relation must be in the interval [1]. Then we illustrate a situation to demonstrate the extending difference $\|(A,B)\|$ for triangular fuzzy numbers. The cause of utilizing triangular fuzzy numbers demonstration is that they can be presented by certain figures, whereas fuzzy numbers of Definition 3.1 are merely expressed in intervals. In the illustrated situation, the membership functions of A and B are presented blow.

Based on Lemma 3.5, the membership functions of T^- and T^+ are derived and shown in Fig. 2.

Since $t_l^+ < t_r^-$, the extending difference relationship of T^- and T^+ are displayed in Fig. 3. It is viewed that T^+ is moved $(t_r^- - t_l^+)$ units into T^{+*} by right. Then the difference between T^- and T^{+*} will be the extending difference between A and B. Thus the extending difference $\|T\|$ will be $\frac{(t_l^+ - t_r^-) + 2(t_m^+ - t_m^-) + (t_r^+ - t_l^-)}{2} + 2(t_r^- - t_l^+)$. Further, $t_l^+ + (t_r^- - t_l^+) = t_r^-$, so the left boundary of T^{+*} is t_r^-.

Additionally, yielding the extending difference for A and B on $t_l^+ \geqslant t_r^-$ is as similar as that of Fig. 3 because the intersection of A and B is $\emptyset$ or a point. Therefore, its demonstration is omitted. Based the figures above, the fuzzy preference relation of A and B will be viewed as the difference ratio of Fig. 1 over Fig. 3.

Lemma 3.6. *A and B are two triangular fuzzy numbers, where* $A = (a_l, a_m, a_r)$ *and* $B = (b_l, b_m, b_r)$. *Then*

$$\mu_P(A,B) = \frac{1}{2}\left(\frac{(a_l - b_r) + 2(a_m - b_m) + (a_r - b_l)}{2\|T\|} + 1\right), \tag{9}$$

where

$$\|T\| = \begin{cases} \frac{(t_l^+ - t_r^-) + 2(t_m^+ - t_m^-) + (t_r^+ - t_l^-)}{2} & \text{if } t_l^+ \geqslant t_r^- \\ \frac{(t_l^+ - t_r^-) + 2(t_m^+ - t_m^-) + (t_r^+ - t_l^-)}{2} + 2(t_r^- - t_l^+) & \text{if } t_l^+ < t_r^- \end{cases},$$

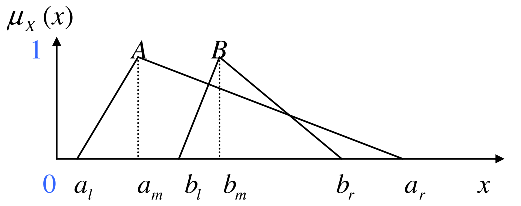

Fig. 1. The membership functions of triangular fuzzy numbers *A* and *B*.

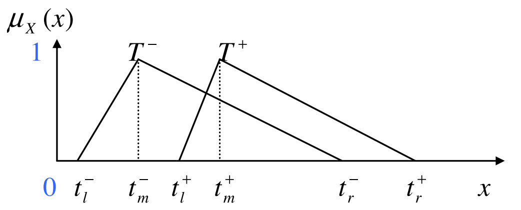

Fig. 2. The membership functions of triangular fuzzy numbers T^- and T^+.

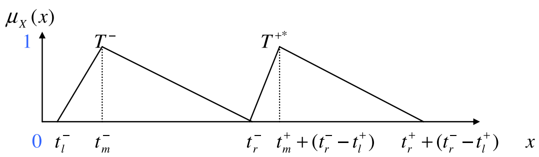

Fig. 3. The membership functions of T^- and T^{+*}.

$$t_l^+ = \max\{a_l, b_l\},$$

$$t_m^+ = \max\{a_m, b_m\},$$

$$t_r^+ = \max\{a_r, b_r\},$$

$$t_l^- = \min\{a_l, b_l\},$$

$$t_m^- = \min\{a_m, b_m\},$$

$$t_r^- = \min\{a_r, b_r\}.$$

Proof. As $A = (a_l, a_m, a_r)$ and $B = (b_l, b_m, b_r)$, $A - B = (a_l - b_r, a_m - b_m, a_r - b_l)$. Thus

$$\int_0^1 (A-B)_\alpha^L d\alpha = \int_0^1 (((a_m - b_m) - (a_l - b_r))\alpha + (a_l - b_r))d\alpha = \frac{(a_m - b_m) + (a_l - b_r)}{2},$$

and

$$\int_0^1 (A-B)_\alpha^U d\alpha = \int_0^1 (((a_m - b_m) - (a_r - b_l))\alpha + (a_r - b_l))d\alpha = \frac{(a_m - b_m) + (a_r - b_l)}{2}.$$

Based on above,

$$\mu_P(A,B) = \frac{1}{2}\left(\frac{\int_0^1 \left((A-B)_\alpha^L + (A-B)_\alpha^U\right)d\alpha}{\|T\|} + 1\right) = \frac{1}{2}\left(\frac{\frac{(a_m-b_m)+(a_l-b_r)}{2} + \frac{(a_m-b_m)+(a_r-b_l)}{2}}{\|T\|} + 1\right)$$
$$= \frac{1}{2}\left(\frac{(a_l - b_r) + 2(a_m - b_m) + (a_r - b_l)}{2\|T\|} + 1\right).$$

Obviously,

$$\|T\| = \begin{cases} \frac{(t_l^+ - t_r^-) + 2(t_m^+ - t_m^-) + (t_r^+ - t_l^-)}{2} & \text{if } t_l^+ \geqslant t_r^-, \\ \frac{(t_l^+ - t_r^-) + 2(t_m^+ - t_m^-) + (t_r^+ - t_l^-)}{2} + 2(t_r^- - t_l^+) & \text{if } t_l^+ < t_r^-, \end{cases}$$

can be proved in the same means. The proof is omitted. □

4. The relative preference relation constructed the fuzzy preference relation

By the fuzzy preference relation P, preference degree of two fuzzy numbers is derived. However, time complexity on fuzzy operation is $O(n^2)$ for ranking n fuzzy numbers due to pair-wise comparison [30]. On the other hand, time complexity on fuzzy operation is merely $O(n)$ for ranking n fuzzy numbers by defuzzification. That is why many researches used defuzzification to rank several fuzzy numbers. To modify the fuzzy preference relation P, we propose a relative preference relation P^* based on P to rank a set of fuzzy numbers. By the relative preference relation P^*, time complexity on fuzzy operation for ranking n fuzzy numbers is $O(n)$. The related notion of the relative preference relation is presented below.

Definition 4.1. Let $S = \{X_1, X_2, \ldots, X_n\}$ denote a set composed of n fuzzy numbers. A fuzzy number $X_i = [x_{il}, x_{ir}]$ belongs to the set S , where $i = 1, 2, \ldots, n$. Assume $\overline{X} = (\sum_i X_i)/n$ derived by extension principle is average of the n fuzzy numbers in S. A relative preference relation P^* with membership function $\mu_{P^*}(X_i, \bar{X})$ represents preference degree of X_i over $\bar{X}$ in S. We define

$$\mu_{P_*}(X_i, \bar{X}) = \frac{1}{2}\left(\frac{\int_0^1 \left((X_i - \bar{X})_\alpha^L + (X_i - \bar{X})_\alpha^U\right) d\alpha}{\|T_S\|} + 1\right), \tag{10}$$

where

$$\|T_S\| = \begin{cases} \int_0^1 ((T_S^+ - T_S^-)_\alpha^L + (T_S^+ - T_S^-)_\alpha^U) d\alpha & \text{if } t_{sl}^+ \geqslant t_{sr}^-, \\ \int_0^1 ((T_S^+ - T_S^-)_\alpha^L + (T_S^+ - T_S^-)_\alpha^U + 2(t_{sr}^- - t_{sl}^+)) d\alpha & \text{if } t_{sl}^+ < t_{sr}^-, \end{cases}$$

T_S^+ is in an interval $[t_{sl}^+, t_{sr}^+]$, T_S^- is in an interval $[t_{sl}^-, t_{sr}^-]$, $t_{sl}^+ = \max_i\{x_{il}\}$, $t_{sr}^+ = \max_i\{x_{ir}\}$, $t_{sl}^- = \min_i\{x_{il}\}$, $t_{sr}^- = \min_i\{x_{ir}\}$, $i = 1, 2, \ldots, n$.

Obviously, $0 < \mu_{P^*}(X_i, \bar{X}) < 1$, where $i = 1, 2, \ldots, n$. $\mu_{P^*}(X_i, \bar{X}) < \frac{1}{2}$ expresses that $\bar{X}$ is preferred to X_i. On the other hand, $\mu_{P^*}(X_i, \overline{X}) > \frac{1}{2}$ presents that X_i is preferred to $\overline{X}$.

Lemma 4.1. *The relative preference relation P^* is a total ordering relation.*

Lemma 4.2. *Let X_i and X_j be two fuzzy numbers in S. X_i is preferred to X_j iff $\mu_{P^*}(X_i, \bar{X}) > \mu_{P^*}(X_j, \bar{X})$.*

Undoubtedly, the n fuzzy numbers $X_1, X_2, \ldots, X_n$ can be ranked according to $\mu_{P^*}(X_1, \bar{X})$, $\mu_{P^*}(X_2, \bar{X})$,. . ., $\mu_{P^*}(X_n, \bar{X})$, i.e. relative preference degrees of the fuzzy numbers over average. Thus time complexity on fuzzy operation is $O(n)$ for ranking n fuzzy numbers. The difference between P^* and P is in comparison basis. Besides, another difference between P^* and P is in the range consisting of fuzzy numbers. The range of P^* includes a set of fuzzy numbers, whereas the range of P merely has two fuzzy numbers.

Lemma 4.3. *$X_i \succ X_j$ iff $\mu_{P^*}(X_i, \bar{X}) > \mu_{P^*}(X_j, \bar{X})$, where $\succ$ is a binary relation on fuzzy numbers presented on* Definition 2.7.

Lemma 4.4. *Assume $S = \{X_1, X_2, \ldots, X_n\}$ to indicate a set consisting of n triangular fuzzy numbers, where $X_i = (x_{il}, x_{im}, x_{ir})$, $i = 1, 2, \ldots, n$. Let $\overline{X} = (\bar{x}_l, \bar{x}_m, \bar{x}_r)$ be average of the n fuzzy numbers. The relative preference relation P^* with membership function $\mu_{P^*}(X_i, \bar{X})$ represents preference degree of X_i over $\bar{X}$ in S. Thus*

$$\mu_{P^*}(X_i, \bar{X}) = \frac{1}{2}\left(\frac{(x_{il} - \bar{x}_r) + 2(x_{im} - \bar{x}_m) + (x_{ir} - \bar{x}_l)}{2\|T_S\|} + 1\right), \tag{11}$$

where

$$\|T_S\| = \begin{cases} \frac{(t_{sl}^+ - t_{sr}^-) + 2(t_{sm}^+ - t_{sm}^-) + (t_{sr}^+ - t_{sl}^-)}{2} & \text{if } t_{sl}^+ \geqslant t_{sr}^-, \\ \frac{(t_{sl}^+ - t_{sr}^-) + 2(t_{sm}^+ - t_{sm}^-) + (t_{sr}^+ - t_{sl}^-)}{2} + 2(t_{sr}^- - t_{sl}^+) & \text{if } t_{sl}^+ < t_{sr}^-, \end{cases}$$

$t_{sl}^+ = \max_i\{x_{il}\}$, $t_{sm}^+ = \max_i\{x_{im}\}$, $t_{sr}^+ = \max_i\{x_{ir}\}$, $t_{sl}^- = \min_i\{x_{il}\}$, $t_{sm}^- = \min_i\{x_{im}\}$, $t_{sr}^- = \min_i\{x_{ir}\}$, $i = 1, 2, \ldots, n$.

5. Numerical examples

To demonstrate the fuzzy preference relation and relative preference relation clearly, some numerical examples are illustrated to present fuzzy numbers are ranked by P or P^*.

In the first example,

$$A_1 = (4, 5, 6),$$

and

$$A_2 = (1, 2, 3),$$

are two triangular fuzzy numbers.

Intuitively, A_1 is preferred to A_2.

By the fuzzy preference relation P, $\|T\|$ of A_1 and A_2 is

$$= \frac{(6-1) + 2(5-2) + (4-3)}{2} = 6,$$

where

$$t_l^+ = \max\{4, 1\} = 4,$$

$$t_m^+ = \max\{5, 2\} = 5,$$

$$t_r^+ = \max\{6, 3\} = 6,$$

$$t_l^- = \min\{4, 1\} = 1,$$

$$t_m^- = \min\{5, 2\} = 2,$$

$$t_r^- = \min\{6, 3\} = 3,$$

$$t_l^+ \geqslant t_r^-$$

Then

$$\mu_P(A_1, A_2) = \frac{1}{2}\left[\frac{(6-1) + 2(5-2) + (4-3)}{2 \times 6} + 1\right] = 1.$$

Similarly,

$$\mu_P(A_2, A_1) = \frac{1}{2}\left[\frac{(1-6) + 2(2-5) + (3-4)}{2 \times 6} + 1\right] = 0.$$

Thus A_1 is preferred to A_2 by the preference relation P. Further, preference degree of A_1 over A_2 is 1, and preference degree of A_2 over A_1 is 0. The ranking order is consistent with the intuition. Additionally, we use the preference relation P to compare A_1 with A_2 because there are merely two fuzzy numbers in this example. Further, the sorting results yielded by P and P^* are the same. This is due to $\bar{A}$ is between A_1 and A_2 if $\bar{A}$ is the average of A_1 and A_2. Obviously, $\mu_{P^*}(A_1, \bar{A}) > \mu_{P^*}(A_2, \bar{A})$ based on Lemma 4.2.

In the second example,

$$B_1 = (0.22, 0.3, 0.51),$$

$$B_2 = (0.16, 0.32, 0.58),$$

and

$$B_3 = (0.25, 0.4, 0.71),$$

are three triangular fuzzy numbers.

By the preference relation P, $\|T\|$ of B_1 and B_2 is

$$\frac{(0.22 - 0.51) + 2(0.32 - 0.3) + (0.58 - 0.16)}{2} + 2(0.51 - 0.22) = 0.665,$$

where

$$t_l^+ = \max\{0.22, 0.16\} = 0.22,$$

$$t_m^+ = \max\{0.3, 0.32\} = 0.32,$$

$$t_r^+ = \max\{0.51, 0.58\} = 0.58,$$

$$t_l^- = \min\{0.22, 0.16\} = 0.16,$$

$$t_m^- = \min\{0.3, 0.32\} = 0.3,$$

$$t_r^- = \min\{0.51, 0.58\} = 0.51,$$

$$t_l^+ < t_r^-.$$

Then

$$\mu_P(B_1, B_2) = \frac{1}{2}\left[\frac{(0.22 - 0.58) + 2(0.3 - 0.32) + (0.51 - 0.16)}{2 \times 0.665} + 1\right] = 0.481 < \frac{1}{2}.$$

Thus B_2 is preferred to B_1.

Similarly,

$$\mu_P(B_1, B_3) = \frac{1}{2}\left[\frac{(0.22 - 0.71) + 2(0.3 - 0.4) + (0.51 - 0.25)}{2 \times 0.735} + 1\right] = 0.354 < \frac{1}{2}.$$

B_3 is preferred to B_1.

Additionally,

$$\mu_P(B_2, B_3) = \frac{1}{2}\left[\frac{(0.22 - 0.58) + 2(0.3 - 0.32) + (0.51 - 0.16)}{2 \times 0.85} + 1\right] = 0.388 < \frac{1}{2}.$$

Hence B_3 is preferred to B_2.

Based on above, $B_3 \succ B_2 \succ (B_1$.

By the relative preference relation P^*, average $\bar{B}$ of B_1, B_2 and B_3 is derived according to extension principle, i.e.

$$\bar{B} = (0.21, 0.34, 0.6).$$

In addition, $\|T_S\|$ of B_1, B_2 and B_3 is

$$\frac{(0.25 - 0.51) + 2(0.4 - 0.3) + (0.71 - 0.16)}{2} + 2(0.51 - 0.25) = 0.765,$$

where

$$t_{sl}^+ = \max\{0.22, 0.16, 0.25\} = 0.25,$$

$$t_{sm}^+ = \max\{0.3, 0.32, 0.4\} = 0.4,$$

$$t_{sr}^+ = \max\{0.51, 0.58, 0.71\} = 0.71,$$

$$t_{sl}^- = \min\{0.22, 0.16, 0.25\} = 0.16,$$

$$t_{sm}^- = \min\{0.3, 0.32, 0.4\} = 0.3,$$

$$t_{sr}^- = \min\{0.51, 0.58, 0.71\} = 0.51,$$

$$t_{sl}^+ < t_{sr}^-.$$

Thus

$$\mu_{P^*}(B_1, \bar{B}) = \frac{1}{2}\left[\frac{(0.22 - 0.6) + 2(0.3 - 0.34) + (0.51 - 0.21)}{2 \times 0.765} + 1\right] = 0.448,$$

$$\mu_{P^*}(B_2, \bar{B}) = \frac{1}{2}\left[\frac{(0.16 - 0.6) + 2(0.32 - 0.34) + (0.58 - 0.21)}{2 \times 0.765} + 1\right] = 0.464,$$

and

$$\mu_{P^*}(B_3,\bar{B}) = \frac{1}{2}\left[\frac{(0.25-0.6)+2(0.4-0.34)+(0.71-0.21)}{2\times 0.765}+1\right] = 0.588.$$

According to $\mu_{P^*}(B_1,\bar{B})$, $\mu_{P^*}(B_2,\bar{B})$ and $\mu_{P^*}(B_3,\bar{B})$, we know that $B_3 \succ B_2 \succ B_1$. The ranking result is consistent with the computation by P.

In the above example, B_1, B_2 and B_3 are pair-wise compared by P. That is to say, we have to compare B_1 with B_2, B_1 with B_3, and B_2 with B_3. The fuzzy operation number is three (i.e. $C_2^3 = 3$). On the other hand, the fuzzy operation number by P^* is four(i.e. 3 + 1). These fuzzy computations include $\bar{B}$, $\mu_{P^*}(B_1,\bar{B})$, $\mu_{P^*}(B_2,\bar{B})$ and $\mu_{P^*}(B_3,\bar{B})$. Although P^* is bigger on fuzzy operation number than P in this example, the situation will be reversed as lots of fuzzy numbers are ranked. Further, $C_2^n < n+1$ for $n \leqslant 3$, whereas $C_2^n > n+1$ for $n > 3$. Obviously, ranking fuzzy numbers by the relative preference relation is similar to defuzzification on fuzzy operation. Practically, fuzzy pair-wise comparison number is $C_2^n = \frac{n(n-1)}{2}$ for ranking n fuzzy numbers. The work is easy as n is small, whereas fuzzy pair-wise comparison is complex and difficult once n is big. For instance, fuzzy pair-wise comparison number is $C_2^{10} = \frac{10(10-1)}{2} = 45$ for ranking ten fuzzy numbers. This is a hard work. Oppositely, fuzzy operation number is merely eleven for ranking ten fuzzy numbers by P^*. To describe the computation of P^*, an example of ranking ten fuzzy numbers is expressed below. In the example,

$$C_1 = (0.637, 0.752, 0.916),$$

$$C_2 = (0.519, 0.704, 0.822),$$

$$C_3 = (0.314, 0.573, 0.657),$$

$$C_4 = (0.413, 0.567, 0.675),$$

$$C_5 = (0.358, 0.591, 0.764),$$

$$C_6 = (0.236, 0.768, 0.811),$$

$$C_7 = (0.335, 0.416, 0.559),$$

$$C_8 = ((0.163, 0.431, 0.801),$$

$$C_9 = (0.362, 0.535, 0.715),$$

and

$$C_{10} = (0.461, 0.719, 0.923),$$

are ten triangular fuzzy numbers that will be ranked.

Firstly, average $\bar{C}$ of $C_1, C_2, \ldots, C_{10}$ is obtained based on extension principle, i.e.

$$\bar{C} = (0.3798, 0.6056, 0.7643).$$

By P^*, $\|T_S\|$ of $C_1, C_2, \ldots, C_{10}$ is

$$\frac{(0.637-0.559)+2(0.768-0.416)+(0.923-0.163)}{2} = 0.771,$$

where

$$t_{sl}^+ = 0.637,$$

$$t_{sm}^+ = 0.768,$$

$$t_{sr}^+ = 0.923,$$

$$t_{sl}^- = 0.163,$$

$$t_{sm}^- = 0.416,$$

$$t_{sr}^- = 0.559,$$

$$t_{sl}^+ > t_{sr}^-.$$

Then

$$\mu_{P^*}(C_1,\bar{C}) = \frac{1}{2}\left[\frac{(0.637-0.7643)+2(0.752-0.6056)+(0.916-0.3798)}{2\times 0.771}+1\right] = 0.728,$$

$$\mu_{P_*}(C_2,\bar{C}) = \frac{1}{2}\left[\frac{(0.519-0.7643)+2(0.704-0.6056)+(0.822-0.3798)}{2\times 0.771}+1\right] = 0.628,$$

$$\mu_{P^*}(C_3,\bar{C}) = \frac{1}{2}\left[\frac{(0.314-0.7643)+2(0.573-0.6056)+(0.657-0.3798)}{2\times 0.771}+1\right] = 0.423,$$

$$\mu_{P^*}(C_4,\bar{C}) = \frac{1}{2}\left[\frac{(0.413-0.7643)+2(0.567-0.6056)+(0.675-0.3798)}{2\times 0.771}+1\right] = 0.457,$$

$$\mu_{P^*}(C_5,\bar{C}) = \frac{1}{2}\left[\frac{(0.358-0.7643)+2(0.591-0.6056)+(0.764-0.3798)}{2\times 0.771}+1\right] = 0.483,$$

$$\mu_{P^*}(C_6,\bar{C}) = \frac{1}{2}\left[\frac{(0.236-0.7643)+2(0.768-0.6056)+(0.811-0.3798)}{2\times 0.771}+1\right] = 0.574,$$

$$\mu_{P^*}(C_7,\bar{C}) = \frac{1}{2}\left[\frac{(0.335-0.7643)+2(0.416-0.6056)+(0.559-0.3798)}{2\times 0.771}+1\right] = 0.296,$$

$$\mu_{P^*}(C_8,\bar{C}) = \frac{1}{2}\left[\frac{(0.163-0.7643)+2(0.431-0.6056)+(0.805-0.3798)}{2\times 0.771}+1\right] = 0.330,$$

$$\mu_{P^*}(C_9,\bar{C}) = \frac{1}{2}\left[\frac{(0.362-0.7643)+2(0.535-0.6056)+(0.711-0.3798)}{2\times 0.771}+1\right] = 0.431,$$

and

$$\mu_{P^*}(C_{10},\bar{C}) = \frac{1}{2}\left[\frac{(0.461-0.7643)+2(0.719-0.6056)+(0.923-0.3798)}{2\times 0.771}+1\right] = 0.651.$$

Based on the above computations, the ranking result is $C_1 \succ C_{10} \succ C_2 \succ C_6 \succ C_5 \succ C_4 \succ C_9 \succ C_3 \succ C_8 \succ C_7$. Through the example, we know that time complexity on fuzzy operation is $O(n)$ to rank n fuzzy numbers by P^*. On the other hand, time complexity on fuzzy operation is $O(n^2)$ (i.e. fuzzy operation number is $C_2^n = \frac{n(n-1)}{2}$) when n fuzzy numbers are ranked by fuzzy pair-wise comparison. For fuzzy operation on ranking ten fuzzy numbers, computation number by P^* (i.e. eleven) is smaller than calculation number by these pair-wise comparison methods(i.e. forty-five) including P. This will be strength of the relative preference relation P^* on fuzzy operation. Additionally, the previous examples are expressed through triangular fuzzy numbers because the fuzzy numbers are widely applied in fuzzy field.

To demonstrate the proposed method rationality, we utilize two similar methods to derive the illustrated examples. The two methods are presented below: one is the method of Wang and Lee [31], and the other is the method of Chen and Hsieh [32]. The method of Wang and Lee was revised from that of Chu and Tsao [14]. They defined the centroid point $(\bar{x}(A), \bar{y}(A))$ for a fuzzy number $A = (a_l, a_m, a_r)$ to be

$$\bar{y}(A) = \frac{\int_0^1 \alpha A_\alpha^L d\alpha + \int_0^1 \alpha A_\alpha^U d\alpha}{\int_0^1 A_\alpha^L d\alpha + \int_0^1 A_\alpha^U d\alpha}, \tag{12}$$

and

$$\bar{x}(A) = \frac{\int_{a_l}^{a_m} \beta (A^{-1})_\beta^L d\beta + \int_{a_m}^{a_r} \beta (A^{-1})_\beta^U d\beta}{\int_{a_l}^{a_m} (A^{-1})_\beta^L d\beta + \int_{a_m}^{a_r} (A^{-1})_\beta^U d\beta}, \tag{13}$$

where A^{-1} was the inverse function of A.

Wang and Lee supposed that fuzzy number A was compared with fuzzy number B according to the following situations.

(i) If $\bar{x}(A) > \bar{x}(B)$, then $A > B$.
(ii) If $\bar{x}(A) < \bar{x}(B)$, then $A < B$.
(iii) If $\bar{x}(A) = \bar{x}(B)$, then
 if $\bar{y}(A) > \bar{y}(B)$, then $A > B$;
  else $\bar{y}(A) < \bar{y}(B)$, then $A < B$;

else $\bar{y}(A) = \bar{y}(B)$, then $A = B$.

In short, they first compared A with B through $\bar{x}$'s values, then they compared A with B through $\bar{y}$'s values as $\bar{x}(A) = \bar{x}(B)$. On the other hand, Chen and Hsieh defined graded mean integration representation $G(A)$ of $A = (a_l, a_m, a_r)$ to be

$$G(A) = \frac{\int_0^1 \alpha(A_\alpha^L + A_\alpha^U)d\alpha}{2\int_0^1 \alpha d\alpha} = \frac{1}{6}(a_l + 4a_m + a_r). \tag{14}$$

The above three examples are yielded by the methods of Wang and Lee, as well as Chen and Hsieh. Then the computation results compared with those derived by the fuzzy preference relation or relative preference relation are respectively shown in the following tables (see Tables 1–3).

In the first example, sorts yielded by the fuzzy preference relation are consistent with those of the other methods. In the second example, sorts yielded by the relative preference relation are consistent with those of the other methods. In the third example, sorts yielded by the relative preference relation are consistent with those derived by the method of Wang and Lee. In addition, sorts yielded by the relative preference relation are as similar as those derived by the method of Chen and Hsieh. The sort variety is merely in fuzzy numbers C_3 and C_9 for the two methods. This is due to computation variety of the two methods and characteristic of fuzzy numbers. In fact, it is said that C_3 and C_9 are very similar. Furthermore, sorts derived by the method of Wang and Lee coincide with those of the proposed method. Obviously, the proposed method can rank fuzzy numbers effectively and efficiently.

6. The extension in general fuzzy numbers

In this section, the relative preference relation P^* is used to rank general fuzzy numbers because P^* does not only rank triangular fuzzy numbers but also other forms of fuzzy numbers. Since P^* is from the fuzzy preference relation P, we first propose the lemma of comparing two trapezoidal fuzzy numbers by P as similar as Lemma 3.5.

Lemma 6.1. *Let A and B be two trapezoidal fuzzy numbers, where* $A = (a_l, a_h, a_m, a_r)$ *and* $B = (b_l, b_h, b_m, b_r)$. *Thus*

$$\mu_P(A,B) = \frac{1}{2}\left(\frac{(a_l - b_r) + (a_h - b_m) + (a_m - b_h) + (a_r - b_l)}{2\|T\|} + 1\right), \tag{15}$$

where

$$\|T\| = \begin{cases} \frac{(t_l^+ - t_r^-)+(t_h^+ - t_m^-)+(t_m^+ - t_h^-)+(t_r^+ - t_l^-)}{2} & \text{if } t_l^+ \geqslant t_r^-, \\ \frac{(t_l^+ - t_r^-)+(t_h^+ - t_m^-)+(t_m^+ - t_h^-)+(t_r^+ - t_l^-)}{2} + 2(t_r^- - t_l^+) & \text{if } t_l^+ < t_r^-, \end{cases}$$

$t_l^+ = \max\{a_l, b_l\}$, $t_h^+ = \max\{a_h, b_h\}$, $t_m^+ = \max\{a_m, b_m\}$, $t_r^+ = \max\{a_r, b_r\}$, $t_l^- = \min\{a_l, b_l\}$, $t_h^- = \min\{a_h, b_h\}$, $t_m^- = \min\{a_m, b_m\}$, $t_r^- = \min\{a_r, b_r\}$.

Moreover, it is easily extended to utilize the relative preference relation P^* for ranking several trapezoidal fuzzy numbers as similar as Lemma 4.4.

Lemma 6.2. *Assume* $S = \{X_1, X_2, \ldots, X_n\}$ *to indicate a set composed of n trapezoidal fuzzy numbers, where* $X_i = (x_{il}, x_{ih}, x_{im}, x_{ir})$, $i = 1, 2, \ldots, n$. *Let* $\overline{X} = (\bar{x}_l, \bar{x}_h, \bar{x}_m, \bar{x}_r)$ *be average of the n fuzzy numbers. The relative preference relation P^* with membership function* $\mu_{P^*}(X_i, \bar{X})$ *represents preference degree of X_i over $\bar{X}$ in S. Thus*

$$\mu_{P^*}(X_i, \bar{X}) = \frac{1}{2}\left(\frac{(x_{il} - \bar{x}_r) + (x_{ih} - \bar{x}_m) + (x_{im} - \bar{x}_h) + (x_{ir} - \bar{x}_l)}{2\|T_S\|} + 1\right), \tag{16}$$

where

$$\|T_S\| = \begin{cases} \frac{(t_{sl}^+ - t_{sr}^-)+(t_{sh}^+ - t_{sm}^-)+(t_{sm}^+ - t_{sh}^-)+(t_{sr}^+ - t_{sl}^-)}{2} & \text{if } t_{sl}^+ \geqslant t_{sr}^-, \\ \frac{(t_{sl}^+ - t_{sr}^-)+(t_{sh}^+ - t_{sm}^-)+(t_{sm}^+ - t_{sh}^-)+(t_{sr}^+ - t_{sl}^-)}{2} + 2(t_{sr}^- - t_{sl}^+) & \text{if } t_{sl}^+ < t_{sr}^-, \end{cases}$$

Table 1
Results of the first example yielded by the fuzzy preference relation and the methods of Wang and Lee, as well as Chen and Hsieh.

Fuzzy numbers	The fuzzy preference relation		The method of Wang and Lee		The method of Chen and Hsieh	
	Values	Sorts	Values ($\bar{x}$, $\bar{y}$)	Sorts	Values	Sorts
$A_1 = (4, 5, 6)$	1	1	(5,0.5)	1	5	1
$A_2 = (1, 2, 3)$	0	2	(2,0.5)	2	2	2

Table 2
Results of the second example yielded by the relative preference relation and the methods of Wang and Lee, as well as Chen and Hsieh.

Fuzzy numbers	The relative preference relation		The method of Wang and Lee		The method of Chen and Hsieh	
	Values	Sorts	Values $(\bar{x}, \bar{y})$	Sorts	Values	Sorts
$B_1 = (0.22, 0.3, 0.51)$	0.448	3	(0.343,0.484)	3	0.322	3
$B_2 = (0.16, 0.32, 0.58)$	0.464	2	(0.353,0.488)	2	0.337	2
$B_3 = (0.25, 0.4, 0.71)$	0.588	1	(0.453,0.485)	1	0.427	1

Table 3
Results of the third example yielded by the relative preference relation and the methods of Wang and Lee, as well as Chen and Hsieh.

Fuzzy numbers	The relative preference relation		The method of Wang and Lee		The method of Chen and Hsieh	
	Values	Sorts	Values $(\bar{x}, \bar{y})$	Sorts	Values	Sorts
$C_1 = (0.637, 0.752, 0.916)$	0.728	1	(0.768,0.497)	1	0.760	1
$C_2 = (0.519, 0.704, 0.822)$	0.628	3	(0.682,0.504)	3	0.693	3
$C_3 = (0.314, 0.573, 0.657)$	0.423	8	(0.515,0.514)	8	0.544	7
$C_4 = (0.413, 0.567, 0.675)$	0.457	6	(0.552,0.503)	6	0.559	6
$C_5 = (0.358, 0.591, 0.764)$	0.483	5	(0.571,0.504)	5	0.581	5
$C_6 = (0.236, 0.768, 0.811)$	0.574	4	(0.605,0.532)	4	0.687	4
$C_7 = (0.335, 0.416, 0.559)$	0.296	10	(0.437,0.494)	10	0.426	10
$C_8 = (0.163, 0.431, 0.801)$	0.330	9	(0.465,0.491)	9	0.448	9
$C_9 = (0.362, 0.535, 0.715)$	0.431	7	(0.537,0.499)	7	0.536	8
$C_{10} = (0.461, 0.719, 0.923)$	0.651	2	(0.701,0.503)	2	0.710	2

$t^+_{sl} = \max_i\{x_{il}\}$, $t^+_{sh} = \max_i\{x_{ih}\}$, $t^+_{sm} = \max_i\{x_{im}\}$, $t^+_{sr} = \max_i\{x_{ir}\}$, $t^-_{sl} = \min_i\{x_{il}\}$, $t^-_{sh} = \min_i\{x_{ih}\}$, $t^-_{sm} = \min_i\{x_{im}\}$, $t^-_{sr} = \min_i\{x_{ir}\}$, $i = 1, 2, \ldots, n$.

Especially for triangle and trapezoidal fuzzy numbers, we can use P and P^* to compare varied forms of fuzzy numbers. Based on Definition 3.1,

$$\int_0^1 ((A-B)^L_\alpha + (A-B)^U_\alpha)d\alpha$$

$$= \int_0^1 (A^L_\alpha - B^U_\alpha + A^U_\alpha - B^L_\alpha)d\alpha$$

$$= \int_0^1 (A^L_\alpha + A^U_\alpha - B^L_\alpha - B^U_\alpha)d\alpha$$

$$= \int_0^1 (A^L_\alpha + A^U_\alpha)d\alpha - \int_0^1 (B^L_\alpha + B^U_\alpha)d\alpha. \quad (17)$$

According to the above computation, $\int_0^1((A-B)^L_\alpha + (A-B)^U_\alpha)d\alpha$ are transferred into two different calculation components, i.e. $\int_0^1(A^L_\alpha + A^U_\alpha)d\alpha$ and $-\int_0^1(B^L_\alpha + B^U_\alpha)d\alpha$. No matter what fuzzy numbers A and B are, we can respectively calculate the part of A and the part of B.

Lemma 6.3. *For two varying forms of fuzzy numbers A and B, where $A = [a_l, a_r]$ and $B = [b_l, b_r]$. The fuzzy preference relation P is a fuzzy subset of $\Re \times \Re$ with membership function $\mu_P(A, B)$ representing preference degree of A over B. Thus*

$$\mu_P(A,B) = \frac{1}{2}\left(\frac{\int_0^1(A^L_\alpha + A^U_\alpha)d\alpha - \int_0^1(B^L_\alpha + B^U_\alpha)d\alpha}{\|T\|} + 1\right), \quad (18)$$

where

$$\|T\| = \begin{cases} \int_0^1((T^+)^L_\alpha + (T^+)^U_\alpha)d\alpha - \int_0^1((T^-)^L_\alpha + (T^-)^U_\alpha)d\alpha & \text{if } t^+_l \geqslant t^-_r, \\ \int_0^1((T^+)^L_\alpha + (T^+)^U_\alpha)d\alpha - \int_0^1((T^-)^L_\alpha + (T^-)^U_\alpha)d\alpha + 2(t^-_r - t^+_l) & \text{if } t^+_l < t^-_r, \end{cases}$$

$T^+ = [t^+_l, t^+_r]$, $T^- = [t^-_l, t^-_r]$, $t^+_l = \max\{a_l, b_l\}$, $t^+_r = \max\{a_r, b_r\}$, $t^-_l = \min\{a_l, b_l\}$, $t^-_r = \min\{a_r, b_r\}$.
Obviously, $\int_0^1(A^L_\alpha + A^U_\alpha)d\alpha - \int_0^1(B^L_\alpha + B^U_\alpha)d\alpha > 0$ iff A is preferred to B (i.e. $\mu_P(A,B) > \frac{1}{2}$).

Based on Lemma 6.3, *utilizing the relative preference relation P^* to rank several forms of fuzzy numbers is presented in* Lemma 6.4.

Lemma 6.4. *Let $S = \{X_1, X_2, \ldots, X_n\}$ be a set consisting of n fuzzy numbers belonging to several forms of fuzzy numbers. A fuzzy number $X_i = [x_{il}, x_{ir}]$ belongs to S, where $i = 1, 2, \ldots, n$. Let $\bar{X}'$ be estimated average of $X_1, X_2, \ldots, X_n$. $\bar{X}'$ is a triangular fuzzy number $(\bar{x}'_l, \bar{x}'_m, \bar{x}'_r)$ or a trapezoidal fuzzy number $(\bar{x}'_l, \bar{x}'_h, \bar{x}'_m, \bar{x}'_r)$, because any a fuzzy number normally has lower boundary, upper boundary, and membership function of a point or a range being 1. The relative preference relation P^* with membership function $\mu_{P^*}(X_i, \bar{X}')$ represents preference degree of X_i over $\bar{X}'$ in S. Thus*

$$\mu_{P^*}(X_i, \bar{X}') = \frac{1}{2}\left(\frac{\int_0^1 \left((X_i)_\alpha^L + (X_i)_\alpha^U\right)d\alpha - \int_0^1 \left((\bar{X}')_\alpha^L + (\bar{X}')_\alpha^U\right)d\alpha}{\|T_S\|} + 1\right), \tag{19}$$

where

$$\|T_S\| = \begin{cases} \int_0^1((T_S^+)_\alpha^L + (T_S^+)_\alpha^U)d\alpha - \int_0^1((T_S^-)_\alpha^L + (T_S^-)_\alpha^U)d\alpha & \text{if } t_{sl}^+ \geqslant t_{sr}^-, \\ \int_0^1((T_S^+)_\alpha^L + (T_S^+)_\alpha^U)d\alpha - \int_0^1((T_S^-)_\alpha^L + (T_S^-)_\alpha^U)d\alpha + 2(t_{sr}^- - t_{sl}^+) & \text{if } t_{sl}^+ < t_{sr}^-, \end{cases}$$

$T_S^+ = [t_{sl}^+, t_{sr}^+]$, $T_S^- = [t_{sl}^-, t_{sr}^-]$, $t_{sl}^+ = \max_k\{x_{kl}\}$, $t_{sr}^+ = \max_k\{x_{kr}\}$, $t_{sl}^- = \min_k\{x_{kl}\}$, $t_{sr}^- = \min_k\{x_{kr}\}$, $k = 1, 2, \ldots, n$.

To sum up, the fuzzy preference relation between varied forms of fuzzy numbers is derived by Lemma 6.3, *and the relative preference relations for several forms of fuzzy numbers are obtained by* Lemma 6.4. *Since the computations are similar to the operations of* Section 5, *numerical examples are omitted.*

7. Conclusions

In this paper, we first propose the fuzzy preference relation P to compare fuzzy numbers. The fuzzy preference relation P, satisfying reciprocity and transitivity, is a total ordering relation on fuzzy numbers. However, time complexity on fuzzy operation is $O(n^2)$ for ranking n fuzzy numbers by P because P belongs to fuzzy pair-wise comparisons. Thus we propose the relative preference relation P^* revised from P to resolve the tie of fuzzy pair-wise comparisons. Time complexity on fuzzy operation is $O(n)$ for ranking n fuzzy numbers by P^*. Since P^* with membership function represents preference degrees of n fuzzy numbers over average (i.e. a specify comparison basis), the fuzzy numbers are easily ranked according to the relative preference degrees. Further, the main differences of P and P^* are in range composed of fuzzy numbers, and comparison basis. Obviously, P^* has strength which fuzzy pair-wise comparisons have, but no weakness of pair-wise comparisons. Thus ranking a set of fuzzy numbers is easy and fast by the relative preference relation P^*. Besides, we compare the proposed method with other similar methods through the illustrated examples to demonstrate our method rationality and sorting results consistency.

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