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A Bessel collocation method for solving fractional optimal control problems

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ABSTRACT

In the present paper, we apply the truncated Bessel series approximation by using collocation scheme, for solving linear and nonlinear fractional optimal control problems (OCPs) indirectly. Therefore, the necessary (and also sufficient in most cases) optimality conditions are stated in a form of nonlinear (or linear) fractional two-point boundary value problem (TPBVP). For solving this mentioned TPBVP, we generalize a new numerical method (which is called the Bessel collocation method). One of the best advantages of this generalization is that, there is no need to use operational matrices of differentiation and also the new generalized idea can be implemented in any mathematical software. Some numerical examples are provided to confirm the accuracy of the proposed method. All of the numerical computations have been performed on a PC using several programs written in MAPLE 13.

1. Introduction

Control problems for systems governed by ordinary (or partial) differential equations arise in many applications, e.g., in astronautics, aeronautics, robotics, and economics [1,2]. Experimental studies of such problems go back recent years and computational approaches have been applied since the advent of computer age. The solution of practical control systems usually has special difficulties. Moreover, in classical theory of control, just input–output signals are considered and the basic deficiency of this theory is that it is only applicable for time invariant linear systems. Therefore, presentation of an applicable numerical approach for solving practical control problems has considerable importance.

We recall that, the approaches for numerical solutions of optimal control problems (OCPs) may be divided into two major classes: indirect methods and the direct methods. The indirect methods are based on the Pontryagin maximum principle (PMP) and require the numerical solution of boundary value problems that result from the necessary conditions of optimal control [3]. Direct optimization methods transcribe the (infinite-dimensional) continuous problem to a finite-dimensional nonlinear programming problem (NLP) through some parametrization of the state and/or control variables. In the direct methods, initial guesses have to be provided only for physically intuitive quantities such as the states and possibly controls. The indirect schemes are based on optimizing then discretizing the main OCPs, meanwhile the direct methods are based on discretizing then optimizing the main OCPs. One of the best properties of the indirect schemes is the high credit of the obtained approximate solution of the main OCPs. This specific property is based on satisfying the first order of necessary conditions that originated from the calculus of variation and the PMP. In this paper, after imposing PMP to the considered fractional OCPs, we obtain a fractional two-point boundary value problem (TPBVP) such that for solving this equation we

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generalize a new collocation method which is based on the truncated Bessel series [4]. It should be mentioned that, the Bessel collocation method (BCM) which was used in the works [5–8] just can be applicable for solving linear and some special nonlinear cases (in polynomial forms), because in these works the operational matrix of differentiation has a basic role. Therefore, a generalization of this idea should be constructed, since in most case studies of real world, the problems appear in a complex nonlinear form and operational matrices of differentiation (and also integration ones) are not efficient and applicable. This is our basic motivation for presenting such this generalization.

For solving TPBVPs, we can use many ideas. One of the well-known methods is integrating from the mentioned problems in an appropriate interval such that the boundary conditions may be imposed. After this procedure, one can use high accurate Gauss quadrature rules or operational matrices of integration. However Gauss quadrature rules [9] have a high order of accuracy for smooth data, but using them may give rise to ill-conditioned algebraic systems. On the other hand, by using operational matrices of integration (specially for orthogonal functions and polynomials) we may reach to sparse algebraic systems, but the order of accuracy is decreased usually. In practice, if we deal with a complex nonlinear problem, it is better to use Gauss quadrature rules, because operational matrices of integration have low accuracy in these cases. Otherwise, if we deal with simple (for instance polynomial forms) nonlinear problems, one may use operational matrices of integration. Using operational matrices of integration goes back to last four decades. Typical examples of such matrices are related to the Walsh functions [10], block-pulse functions [11], Laguerre polynomials [12], Chebyshev polynomials [13], Legendre polynomials [14], Hermite polynomials [15], Fourier series [16], Bernstein polynomials [17] and Bessel functions [18].

Another popular way to solve a TPBVP is direct solving. In other words, we do not integrate the mentioned TPBVP and solve it directly by any idea in the field of approximation theory such as collocation, Galerkin, etc. Some other new direct solvers do not use the classical collocation or Galerkin schemes and are based upon completeness of bases (for instance, Fourier [19]) and operational matrices of differentiation. These new approaches are very applicable and fast for solving high order linear delay (in both cases of neutral and difference) Fredholm integro-differential equations. Collocation and Galerkin techniques have a wide range of application for solving linear and nonlinear differential (including hyperbolic partial differential equations) Fredholm Volterra integro-differential difference delay equations and their systems and one can refer to the works [20–23] for collocation approaches.

It should be noted that, any direct solver has more efficiency with regard to the methods that deal with the integral forms (specially oprational matrices of integration). For instance, one can refer to [24]. In this paper, the authors used direct collocation scheme for solving high-order linear complex differential equations and transform the basic equations to the linear algebraic systems directly. However, if we want to use operational matrices of integration or Gauss quadrature rules, a two dimensional Volterra integral equation should be solved and this subject may be leads to the popularity decreasing of such methods. Moreover, collocation methods are very faster than the Galerkin methods. For clarifying this subject, we can consider a two dimensional Volterra integral equation. If we want to solve it by Galerkin schemes, four-dimensional integral expression will be appeared in computations and this may increased the computational time. Accuracy of direct collocation methods with respect to the operational matrices of integration (even in the case of Legendre polynomials) for solving Euler–Lagrange equations was provided in numerical experiments of [25].

From the last paragraph, one can conclude that direct collocation methods are suitable for solving TPBVPs which arise from OCPs. On the other hand, operational matrices of integration and Gauss quadrature rules have been used in a huge size of research works, meanwhile direct collocation methods for solving the mentioned TPBVPs have had few results. This motivate us to present a new idea. Therefore, in this paper we generalize a new collocation method, that was applied for solving several classes of applied mathematics models [5–8] (in the form of differential or integral equations), for approximating the solution of fractional ordinary differential equations (ODEs) systems. These systems of fractional ODEs, which are the necessary (and also are sufficient in several special cases) conditions for optimal solutions of fractional OCPs, originate from the PMP and have considerable importance in optimal control and calculus of variation. For saving computational time and clarity of presentation, we modify and then generalize the basic idea of BCM in two aspects. The modification is based on time saving. In all of the works [5–8], the authors collocate the basic equation at the whole of computational interval and then remove last constraints (in algebraic system) with the number of boundary conditions and then replace the conditions which appear from the boundary conditions. But, in this paper, we collocate the basic equations at the nodes with the number of unknown coefficients which the boundary conditions are subtracted. This really decreased computational time. The generalization is related to the representation of the truncated Bessel series and then removing the operational matrix of differentiation writing. It should be mentioned that, the main reason of presenting the operational matrices of differentiation is for obtaining the associated algebraic systems more clearly and no computational applicability may not be observed. Moreover, instead of representing the truncated Bessel series $J_n(t),\ n = 0, 1, \ldots, n$ in terms of monomials, we shall use $TaylorPolynomial(BesselJ(n,t), t, order = N)$ command in MAPLE software, where N is the order of approximation.

It should be mentioned that, the structure of this research work is different from [26] in two important aspects. The problem of under investigation in this paper is a fractional OCP, meanwhile in [26] a typical OCP with integer order derivatives has been considered. Moreover, in [26], we have used operational matrices, meanwhile in the present study a general collocation method by using truncated Bessel series has been used for saving computational time and clarity of presentation.

The paper is organized as follows. Optimality conditions for fractional OCPs are introduced in the next section. In Section 3, the basic idea for solving the obtained fractional TPBVPs are provided. Numerical experiments are given for illustrating applicability of the presented technique in Section 4. Section 5 contains some conclusions and future works.

2. Fractional optimal control problems

Before of introducing fractional OCPs, we recall two important definitions from fractional calculus [27]. Assume that β be a real positive constant and $n = [\beta] + 1$, where $[\beta]$ denotes the integer part of β. Right Reimann–Liouville fractional derivative for $\lambda(t)$ can be defined as follows

$$ {}_tD^{\beta}_{t_f}\lambda(t) = \frac{(-1)^n}{\Gamma(n-\beta)}\frac{d^n}{dt^n}\int_t^{t_f}(\eta - t)^{n-\beta-1}\lambda(\eta)d\eta. $$

Also, left Caputo fractional derivative for $x(t)$ can be defined in the following form

$$ {}^{C}_{to}D^{\beta}_{t}x(t) = \frac{1}{\Gamma(n-\beta)}\int_{t_0}^{t}(t-\eta)^{n-\beta-1}x^{(n)}(\eta)d\eta. $$

We now suppose that α be a real number in (0,1), and $F, G : [t_0, t_f] \times \Re^2 \to \Re$ be two continuously differentiable functions. A general form of fractional OCPs can be introduced as

$$ Minimize \quad J(x,u) = \int_{t_0}^{t_f} F(t, x(t), u(t))dt, \tag{1} $$

subject to the fractional dynamic control system

$$ A\dot{x}(t) + B\,{}^{C}_{to}D^{\alpha}_{t}x(t) = G(t, x(t), u(t)), \tag{2} $$

and the initial condition

$$ x(t_0) = x_0, \tag{3} $$

where $(A, B) \neq (0, 0)$ and x_0 is a given constant. According to discussions in [27], if (x, u) be a minimum solution of (1)–(3), then there exists a $\lambda(t)$ which (x, u, λ) satisfies

$$ \begin{aligned} &A\dot{\lambda}(t) - B\,{}_tD^{\alpha}_{t_f}\lambda(t) = -\frac{\partial H}{\partial x}(t, x, u, \lambda), \\ &A\dot{x}(t) + B\,{}^{C}_{to}D^{\alpha}_{t}x(t) = \frac{\partial H}{\partial \lambda}(t, x, u, \lambda), \\ &\frac{\partial H}{\partial u}(t, x, u, \lambda) = 0, \quad t \in [t_0, t_f], \\ &x(t_0) = x_0, \quad \lambda(t_f) = 0, \end{aligned} \tag{4} $$

where H denotes the Hamiltonian and is defined in the form of $H(t, x, u, \lambda) = F(t, x, u) + \lambda G(t, x, u)$. It should be mentioned that in practice, we obtain u in terms of λ and x from the condition $\frac{\partial H}{\partial u}(t, x, u, \lambda) = 0$. Therefore, the above-mentioned system can be rewritten in the following form

$$ \begin{aligned} &A\dot{\lambda}(t) - B\,{}_tD^{\alpha}_{t_f}\lambda(t) = M(t, x(t), \lambda(t)), \\ &A\dot{x}(t) + B\,{}^{C}_{to}D^{\alpha}_{t}x(t) = N(t, x(t), \lambda(t)), \\ &x(t_0) = x_0, \quad \lambda(t_f) = 0, \end{aligned} \tag{5} $$

where $M(t, x(t), \lambda(t))$ and $N(t, x(t), \lambda(t))$ are known functions in terms of x and λ.

As it was pointed out in [27], the above-mentioned fractional system contains necessary conditions for optimality of solutions of (1)–(3). If $F(t, x, u)$ and $G(t, x, u)$ be two convex functions in terms of x and u, then (5) contains necessary and sufficient condition for optimal solutions x^* and u^*. It should be recalled that, we should approximate $x(t)$ and $\lambda(t)$ by $x_N(t)$ and $\lambda_N(t)$, respectively in (5). Therefore, one may obtain an approximate optimal solution of (1)–(3). In the next section, we provide our basic idea, in which the numerical solutions $x_N(t)$ and $\lambda_N(t)$ will be obtained.

3. Method of the solution

In this part of paper, we will focus on the numerical solution of fractional TPBVM (5) by proposing the generalized Bessel collocation method (BCM) [4]. It should be noted that, in all of the research works [4–8], the authors used operational matrix of differentiation even for nonlinear problems. Moreover, all of the considered nonlinear problems have polynomial forms and no attempts was done for a general complex nonlinear problem by BCM. On the other hand, the selected Bessel basis in the mentioned references is the truncated Bessel series, so that one can comput it by some suitable commands in mathematical softwares such as MAPLE and MATLAB. In the sequel, we will illustrate this fact clearly. We now assume that, the solutions of (5) can be approximated by the truncated Bessel series as follows

$$
\begin{aligned}
x(t) &\approx x_N(t) = \sum_{n=0}^{N} X_n J_n(t), \\
\lambda(t) &\approx \lambda_N(t) = \sum_{n=0}^{N} \Lambda_n J_n(t),
\end{aligned} \tag{6}
$$

where X_n and $\Lambda_n,\ n = 0,1,2,\ldots,N$ are the unknown Bessel coefficients to be determined and $J_n(t),\ n = 0,1,2,\ldots,N$ are the Bessel polynomials of first kind defined by [4]

$$
J_n(t) = \sum_{k=0}^{\left[\frac{N-n}{2}\right]} \frac{(-1)^k}{k!(k+n)!}\left(\frac{t}{2}\right)^{2k+n}, \quad 0 \leqslant t < \infty.
$$

The Bessel polynomials $J_n(t)$ can be written in the MAPLE software in the form

$$
J_n(t) = TaylorPolynomial(BesselJ(n,t), t, order = N),
$$

where N is the order of approximation. This representation is more clear than the matrix forms that were used in [4–8]. Now, we assume that $t_s = t_0 + \frac{t_f - t_0}{N}s,\ s = 0,1,\ldots,N$. Since there exists $2N+2$ unknown coefficients X_n and Λ_n $(n = 0,1,\ldots,N)$, we should construct the associated system of $2N+2$ algebraic equations in a manner which needs less computational time. For this purpose, we collocate the first equation of (5) at the nodes t_k, where $k = 0,1,\ldots,N-1$ and collocate the second equation of (5) at the nodes t_j, where $j = 1,2,\ldots,N$ as follows

$$
\begin{aligned}
&A\dot{\lambda}_N(t_k) - B\,{}_{t_k}D_{t_f}^{\alpha}\lambda_N(t_k) = M(t_k, x_N(t_k), \lambda_N(t_k)), \quad k = 0,1,\ldots,N-1, \\
&A\dot{x}_N(t_j) + B\,{}_{t0}^{C}D_t^{\alpha}xN(t_j) = N(t_j, x_N(t_j), \lambda_N(t_j)), \quad j = 1,2,\ldots,N,
\end{aligned} \tag{7}
$$

where $t_s = t_0 + \frac{t_f - t_0}{N}s,\ s = 0,1,\ldots,N$. Therefore, the above system consists of $2N$ algebraic equations. These equations together with the two boundary conditions $x_N(t_0) = x_0$ and $\lambda_N(t_f) = 0$ form a system that has $2N+2$ algebraic equations and $2N+2$ unknowns. For solving this system one can apply the Newton algorithm in many softwares such as MAPLE. This procedure may be done by *f solve* command in MAPLE software. After solving the above-mentioned system of nonlinear algebraic equations, the unknown Bessel coefficients X_n and Λ_n $(n = 0,1,\ldots,N)$ will be extracted and replaced in (6).

Remark 1. As it was noted in [28], since in our computations we have the expression ${}_tD_{t_f}^{\alpha}\lambda_N(t)$, it is desirable to use its equivalent form as follows

$$
{}_tD_{t_f}^{\alpha}\lambda_N(t) = \frac{\lambda_N(t_f)}{\Gamma(1-\alpha)}(t_f - t)^{-\alpha} - \frac{1}{\Gamma(1-\alpha)}\int_t^{t_f}(\eta - t)^{-\alpha}\lambda'(\eta)d\eta.
$$

Evidently, we may check the accuracy of our collocation approach. Since $x_N(t)$ and $\lambda_N(t)$ are the numerical solutions of (5), the expressions $x_N(t), \lambda_N(t), \dot{x}_N(t), \dot{\lambda}_N(t), {}_{t_0}^{C}D_t^{\alpha}x(t)$ and ${}_tD_{t_f}^{\alpha}\lambda(t)$ can be replaced in Eq. (5). Therefore, (5) may be satisfied approximately. In other words for $t = t_q \in [t_0, t_f],\ q = 0,1,2,\ldots$

$$
\begin{aligned}
E_1(t_q) &= |A\dot{\lambda}_N(t_q) - B\,{}_{t_q}D_{t_f}^{\alpha}\lambda_N(t_q) - M(t_q, x_N(t_q), \lambda_N(t_q))| \cong 0, \\
E_2(t_q) &= |A\dot{x}_N(t_q) + B\,{}_{to}^{C}D_t^{\alpha}xN(t_q) - N(t_q, x_N(t_q), \lambda_N(t_q))| \cong 0,
\end{aligned}
$$

and $E_{i,N}(t_q) \leq 10^{-k_q},\ i = 1,2$ (k_q positive integer).

4. Numerical examples

In this section, we conduct three numerical examples to illustrate the efficiency and applicability of the presented idea. It should be mentioned that, in the first example, we select an OCP from [29] and extend its integer order derivative to the fractional order derivative. Since we assume that $0 < \alpha < 1$ and the case $\alpha = 1$ are easy to handel, we approximate the solution of this example for several values of α such as =0.6, 0.7, 0.8, 0.9 and 1. On the other hand, since we have the analytical solution for the case of $\alpha = 1$, we can show the credit of the proposed idea by depicting numerical solutions associated to these values of α in Figs. 1 and 2. As we have seen numerical solutions $x_8(t)$ and $u_8(t)$ tend to the exact solutions $x(t)$ and $u(t)$ as α tends to 1. Moreover, in Figs. 3–10, we provide numerical results of the first and second example associated to the trajectory and control variables in the case of $\alpha = 1$. In addition, in Tables 1–5, numerical results of all examples in the case of $\alpha = 1$ are given to show the effectiveness of our presented technique.

Example 4.1 [29]. As the first example, we consider a single-input scalar system as follows

$$
\begin{aligned}
&{}_0^CD_t^{\alpha}x(t) = -x(t) + u(t), \\
&J = \frac{1}{2}\int_0^1 (x^2(t) + u^2(t))dt,
\end{aligned}
$$

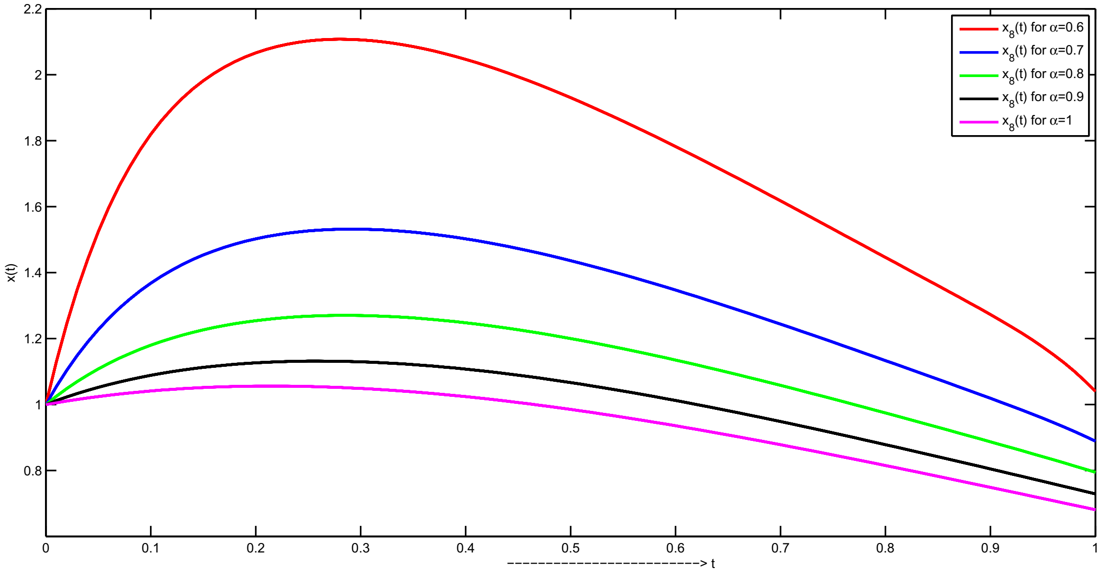

Fig. 1. History of approximate solution $x_8(t)$ by assuming different values of α for the first example.

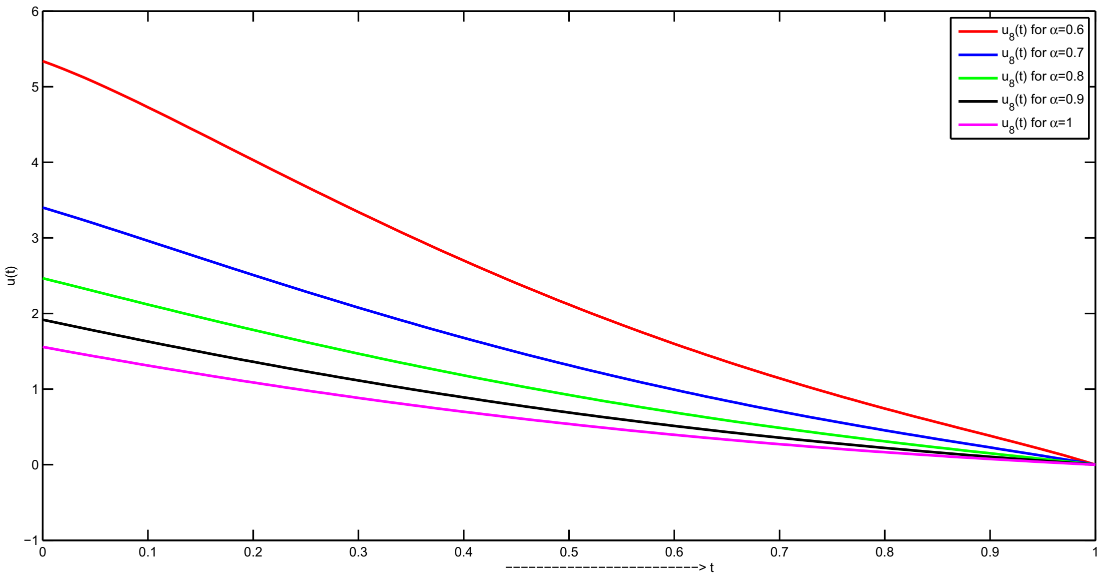

Fig. 2. History of approximate solution $u_8(t)$ by assuming different values of α for the first example.

with free terminal condition and the initial condition

$$x(0) = 1.$$

According to [29], the analytical solution of the above-mentioned problem (in the case of $\alpha = 1$) is

$$x(t) = \cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t),$$
$$u(t) = (1 + \sqrt{2}\beta)\cosh(\sqrt{2}t) + (\sqrt{2} + \beta)\sinh(\sqrt{2}t),$$

where

$$\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2}\sinh(\sqrt{2})}{\sqrt{2}\cosh(\sqrt{2}) + \sinh(\sqrt{2})}.$$

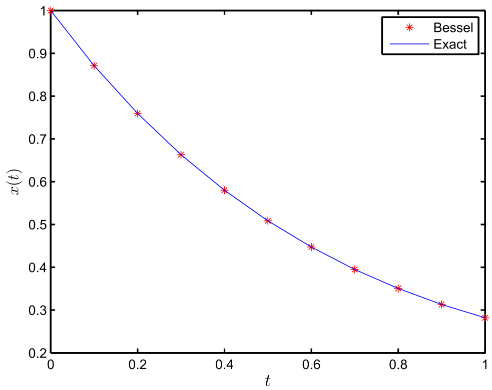

Fig. 3. History of $x(t)$ and $x_N(t)$ for $N = 6$.

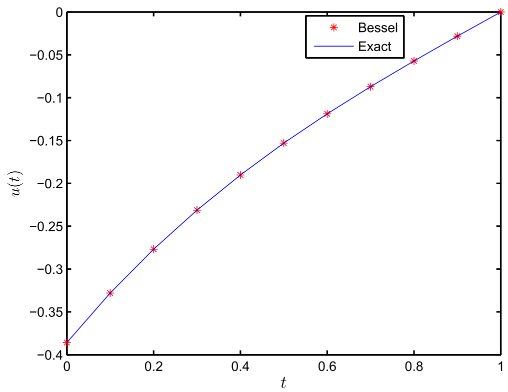

Fig. 4. History of $u(t)$ and $u_N(t)$ for $N = 6$.

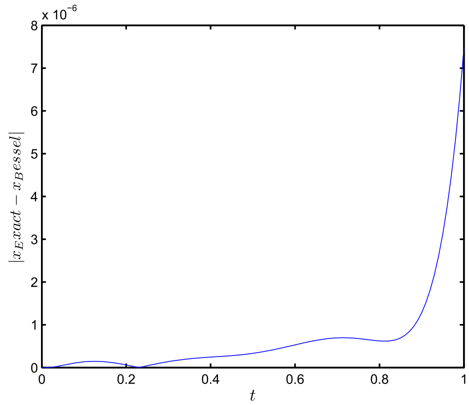

Fig. 5. History of $|x^*(t) - x_N(t)|$ for $N = 6$.

According to (5) we should have

$$ {}_0^C D_t^\alpha x(t) = -x(t) - \lambda(t), $$
$$ {}_t D_1^\alpha \lambda(t) = x(t) - \lambda(t), $$
$$ x(0) = 1, \quad \lambda(1) = 0. $$

Also, the following optimal control law may be computed by using $\frac{\partial H}{\partial u} = 0$

$$ u^*(t) = -\lambda(t). $$

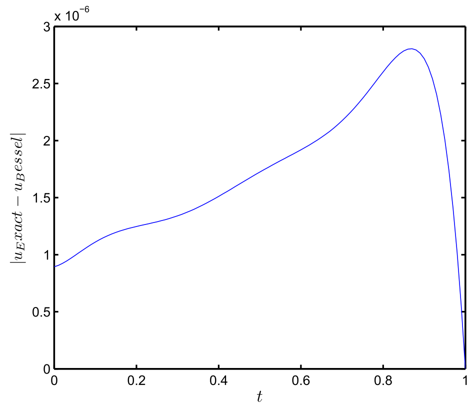

Fig. 6. History of $|u^*(t) - u_N(t)|$ for $N = 6$.

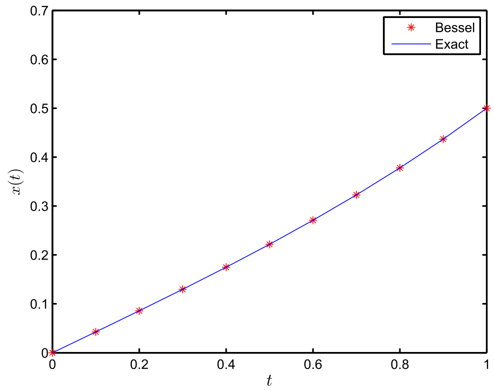

Fig. 7. History of $x(t)$ and $x_N(t)$ for $N = 5$.

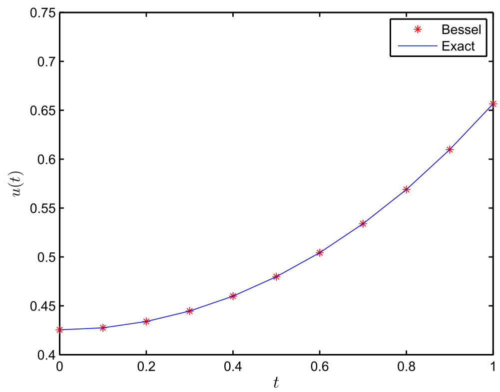

Fig. 8. History of $u(t)$ and $u_N(t)$ for $N = 5$.

As it is declared before, we obtain the numerical solutions $x_8(t)$ and $u_8(t)$ for several values of α and depict them in Figs. 1 and 2, respectively. Moreover, we obtain the approximate solutions by applying the presented method for $N = 6$ in the case of $\alpha = 1$. In Figs. 3 and 4, the approximate solutions $x_N(t)$ and $u_N(t)$ of the suggested idea for $N = 6$ are compared with the exact solutions. For the approximate solutions $x_N(t)$ and $u_N(t)$ which are gained by the proposed technique for $N = 6$, we depict the error functions associated with trajectory and control functions in Figs. 5 and 6. According to [29], the optimal value of the performance index is $J^* = 0.1929092981$. The numerical values of performance index which obtained by the presented algorithm, are given in Table 1 for different values of N. Moreover in Table 2, absolute error histories of the state and control functions at the selected points for $N = 6$ are provided. As seen from this Table, our method converges more rapidly than [30]. These results confirm the accuracy of the presented technique.

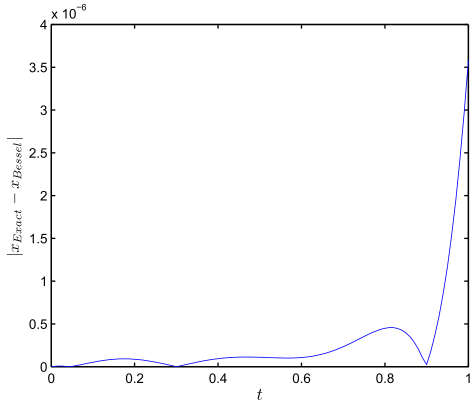

Fig. 9. History of $|x^*(t) - x_N(t)|$ for $N = 5$.

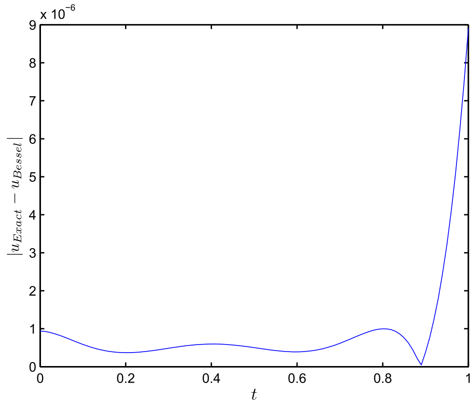

Fig. 10. History of $|u^*(t) - u_N(t)|$ for $N = 5$.

Table 1
The optimal cost functional J_N of Example 4.1 for different values of N.

N	J_N
3	0.1923867316
4	0.1929041515
5	0.1929065847
6	0.1929092756

Table 2
Absolute error histories of the state and control functions at the selected points for $N = 6$.

t_i	$\lvert u^*(t_i) - u_N(t_i)\rvert$	$\lvert x^*(t_i) - x_N(t_i)\rvert$
0	8.9399e−007	0
0.2	1.2469e−006	6.2315e−008
0.4	1.5108e−006	2.4565e−007
0.6	1.9190e−006	5.3182e−007
0.8	2.6017e−006	6.2441e−007
1.0	8.3000e−011	7.4770e−006

Example 4.2 [29]. As the second example we consider the following linear OCP

$$\dot{x} = u(t),$$
$$J = \int_0^1 (x^2(t) + u^2(t))dt,$$

Table 3
The optimal cost functional J_N of Example 4.2 for different values of N

N	Performance index value
2	0.32872544026
3	0.32776465094
4	0.32826209799
5	0.32825878417

Table 4
Absolute error histories of the state and control functions at the selected points for $N = 5$.

t_i	$\lvert u^*(t_i) - u_N(t_i)\rvert$	$\lvert x^*(t_i) - x_N(t_i)\rvert$
0	9.3588e−007	0
0.2	3.6901e−007	8.8267e−008
0.4	5.9620e−007	9.4765e−008
0.6	3.9054e−007	1.0804e−007
0.8	9.9867e−007	4.5014e−007
1.0	8.9136e−006	3.5843e−006

Table 5
Numerical results of Example 4.3.

N	Max $Res_1(t)$ $(0 \leqslant t \leqslant 1)$	Max $Res_2(t)$ $(0 \leqslant t \leqslant 1)$	J_N obtained by the presented method
2	2.147e−02	8.390e−01	0.0318147
3	5.793e−03	2.864e−01	0.0280687
4	4.373e−04	1.523e−01	0.0287585
5	2.385e−04	4.779e−02	0.0289898
6	7.013e−05	2.119e−02	0.0290147
7	1.498e−05	6.378e−03	0.0290266
8	4.862e−06	2.557e−03	0.0290278
9	1.071e−06	7.625e−04	0.0290275
10	4.326e−07	2.853e−04	0.0290273

with the initial condition

$$x(0) = 0.$$

The analytical solution of this linear OCP is [29]

$$x(t) = \frac{e(e^t - e^{-t})}{2(e^2 - 1)}, \quad u(t) = \frac{e(e^t + e^{-t})}{2(e^2 - 1)}.$$

Therefore, we should have

$$\dot{x} = u(t),$$
$$\dot{\lambda} = -2x,$$
$$x(0) = 0, \quad \lambda(1) = 0.$$

Also, we can obtain the following optimal control law

$$u^*(t) = -\frac{\lambda}{2}.$$

Therefore, we reach to the following Hamiltonian system

$$\dot{x}(t) = -\frac{1}{2}\lambda(t), \tag{8}$$
$$\dot{\lambda}(t) = -2x(t), \tag{9}$$
$$x(0) = 0, \quad \lambda(1) = 0. \tag{10}$$

Similar to the previous example, we obtain the approximate solutions by applying the presented method for $N = 5$. In Figs. 7 and 8, the approximate solutions $x_N(t)$ and $u_N(t)$ of the suggested idea for $N = 5$ are compared with the exact solutions. For the approximate solutions $x_N(t)$ and $u_N(t)$ which are gained by the proposed technique for $N = 5$, we depict the error functions associated with trajectory and control functions in Figs. 9 and 10. According to [29], the optimal value of the performance index is $J^* = 0.3282588215$. The numerical values of performance index which are obtained by the presented algorithm, are provided in Table 3 for different values of N. Moreover in Table 4, absolute error histories of the state and control functions at the selected points for $N = 5$ are given. Again, robustness of our presented method can be observed from these Tables and Figures.

Example 4.3. As a complex test problem, we now consider the following nonlinear OCP

$$\dot{x} = 2\sqrt{x(t)u(t)}, \quad t \in [0.1, \tfrac{\pi}{2}], \quad x(0.1) = 0.009967, \quad u(0.1) = 0.009769,$$
$$J = \int_{0.1}^{\frac{\pi}{2}} (x(t) + u(t))dt.$$

In this problem, the control function must have the condition $u(\frac{\pi}{2}) \neq 0$ at the end point of the computational interval. It should be mentioned that, this nonlinear OCP is more general than the forms that introduced in Section 2. After considering the Hamiltonian function in the form $H(x, \lambda, t) = (x + u) + \lambda(2\sqrt{xu})$, one can use the PMP and reach to the following results

$$\dot{\lambda}(t) = -H_x = -\left(1 + \frac{\lambda u}{\sqrt{xu}}\right),$$
$$H_u = 1 + \frac{\lambda x}{\sqrt{xu}} = 0.$$

A simple computation yields to the following nonlinear Hamiltonian system

$$\dot{\lambda}(t) = -(1 + \lambda(t)^2), \quad x(0.1) = 0.009967,$$
$$\dot{x}(t) = 2x(t)\lambda(t), \quad \lambda\left(\frac{\pi}{2}\right) = 0.$$

Trivially, the optimal control law is given by

$$u^*(t) = x(t)\lambda(t)^2.$$

Since we assume that $u(\frac{\pi}{2}) \neq 0$, then $\lambda(\frac{\pi}{2}) \neq 0$. Therefore, the above-mentioned Hamiltonian system should be modified in the form

$$\begin{aligned} &\dot{\lambda}(t) = -(1 + \lambda(t)^2), \quad x(0.1) = 0.009967, \\ &\dot{x}(t) = 2x(t)\lambda(t), \quad \lambda(0.1) = 0.99017, \end{aligned} \tag{11}$$

where $\lambda(0.1) = \sqrt{\frac{u(0.1)}{x(0.1)}} = 0.99017$. For solving this system of nonlinear differential equations, similar to the linear cases, we suppose that the state and co-state variables be written in terms of linear combination of Bessel polynomials which are defined in Section 3 with the unknown Bessel coefficients. These coefficients will be determined after imposing the above system of differential equations at the uniform mesh in the interval (0,1). In other words, applying these collocation points to the main system together with the considered boundary conditions on $x(t)$ and $\lambda(t)$ transform the basic problem to a corresponding system of nonlinear algebraic equations. By assuming different values of N such as 2,3,4,5,6,7,8,9 and 10 we solve the above-mentioned system. In Table 5 we provide the approximated performance index J_N which is obtained by our proposed method and also the maximum residual for the first and second equations of (11) for the considered values of N.

5. Conclusions

This paper contributes to present an indirect method for solving optimal control problems (OCPs) using truncated Bessel series together with the collocation method on a uniform mesh. In this new idea, the role of operational matrices of differentiation has been removed, since this matter has not any efficiency in computations. Needing to less computational time and being easy to handle are two important properties that encourage us to apply such method for solving OCPs. Efficiency of the proposed method is confirmed from the obtained results in the section of numerical examples. It should be recall that, the structure of this research work is different from [26]. For instance, the problem of under investigation in this paper is a fractional OCP, meanwhile in [26] a typical OCP with integer order derivatives has been considered. Our idea can be extended for solving OCP governed by fractional partial differential equations, but some modifications are need.

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References

[1] J.T. Betts, Practical Methods for Optimal Control Using Nonlinear Programming, SIAM, Philadelphia, 2001.
[2] J.T. Betts, S.O. Erb, Optimal low thrust trajectories to the moon, SIAM J. Appl. Dyn. Syst. 2 (2003) 144–170.
[3] L.S. Pontryagin, V. Boltyanskii, R. Gamkrelidze, E. Mischenko, The Mathematical Theory of Optimal Processes, Wiley Interscience, 1962.

[4] S. Yuzbasi, Bessel polynomial solutions of linear differential, integral and integro-differential equations, Master Thesis, Graduate School of Natural and Applied Sciences, Mugla University, 2009.
[5] S. Yuzbasi, N. Sahin, M. Sezer, A Bessel collocation method for numerical solution of generalized pantograph equations, Numer. Methods Partial Differ. Equ. (2011), http://dx.doi.org/10.1002/num.20660.
[6] S. Yuzbasi, A numerical approximation based on the Bessel functions of first kind for solutions of Riccati type differential-difference equations, Comput. Math. Appl. (2012), http://dx.doi.org/10.1016/j.camwa.2012.01.026.
[7] S. Yuzbasi, A numerical approach for solving the high-order linear singular differential-difference equations, Comput. Math. Appl. 62 (2011) 2289–2303.
[8] S. Yuzbasi, N. Sahin, M. Sezer, A collocation approach for solving linear complex differential equations in rectangular domains, Math. Methods Appl. Sci. 35 (2012) 1126–1139.
[9] O.R.N. Samadi, E. Tohidi, The spectral method for solving systems of Volterra integral equations, J. Appl. Math. Comput. (2012), http://dx.doi.org/10.1007/s12190-012-0582-8.
[10] M.S. Corrington, Solution of differential and integral equations with Walsh functions, IEEE Trans. 20 (1973) 470–476.
[11] N.S. Hsu, B. Chang, Analysis and optimal control of time-varying linear systems via block-pulse functions, Int. J. Control 33 (1989) 1107–1122.
[12] C. Hwang, Y.P. Shih, Parameter identification via Laguerre polynomials, J. Syst. Sci. 13 (1982) 209–217.
[13] G.N. Elnagar, State-control spectral Chebyshev parameterization for linearly constrained quadratic optimal control problems, J. Comput. Appl. Math. 79 (1997) 19–40.
[14] R.Y. Chang, M.L. Wang, Shifted Legendre series direct method for variational problems, J. Optim. Theory Appl. 39 (1983) 299–307.
[15] G.Th. Kekkeris, P.N. Paraskevopoulos, Hermite series approach to optimal control, Int. J. Control 47 (1988) 557–567.
[16] P.N. Paraskevopoulos, P.D. Sparis, S.G. Mouroutsos, The Fourier series operational matrix of integration, Int. J. Syst. Sci. 16 (1985) 171–176.
[17] E.H. Doha, A.H. Bhrawy, M.A. Saker, Integrals of Bernstein polynomials: an application for the solution of high even-order differential equations, Appl. Math. Lett. 24 (2011) 559–565.
[18] P.N. Paraskevopoulos, P.G. Sklavounos, G.C.H. Georgiou, The operational matrix of integration for Bessel functions, J. Franklin Inst. 327 (1990) 329–341.
[19] F. Toutounian, E. Tohidi, A. Kilicman, Fourier operational matrices of differentiation and transmission: introduction and applications, Abstr. Appl. Anal. (2013) (Article ID 198926).
[20] A.A. Dascioglu, Chebyshev polynomial approximation for high-order partial differential equations with complicated conditions, Numer. Methods Partial Differ. Eq. 25 (2009) 610–621.
[21] M. Gulsu, B. Gurbuz, Y. Ozturk, M. Sezer, Laguerre polynomial approach for solving linear delay difference equations, Appl. Math. Comput. 217 (2011) 6765–6776.
[22] M. Sezer, S. Yalcinbas, N. Sahin, Approximate solution of multi-pantograph equation with variable coefficients, J. Comput. Appl. Math. 214 (2008) 406–416.
[23] S. Yalcinbas, M. Aynigul, M. Sezer, A collocation method using Hermite polynomials for approximate solution of pantograph equations, J. Franklin Inst. 384 (2011) 1128–1139.
[24] F. Toutounian, E. Tohidi, S. Shateyi, A collocation method based on Bernoulli operational matrix for solving high order linear complex differential equations in a rectangular domain, Abstr. Appl. Anal. (2013) (Article ID 823098).
[25] E. Tohidi, A. Kilicman, A collocation method based on the Bernoulli operational matrix for solving nonlinear BVPs which arise from the problems in calculus of variation, Math. Prob. Eng. (2013) (Article ID 757206).
[26] E. Tohidi, F. Soleymani, A. Kilicman, Robustness of operational matrices of differentiation for solving state-space analysis and optimal control problems, Abstr. Appl. Anal. 2013 (2013) (Article ID 535979).
[27] S. Pooseh, R. Almeida, D.F.M. Torres, A numerical scheme to solve fractional optimal control problems, Conference Papers in Mathematics, 2013 (Article ID 165298) 10pp.
[28] N.H. Sweilam, T.M. Al-Ajami, H.W. Hoppe, Numerical solution of some types of fractional optimal control problems, Sci. World J. 2013 (Article ID 306237) 9pp.
[29] H. Saberi Nik, S. Effati, A. Yildirim, Solution of linear optimal control systems by differential transform method, Neural Comput. Appl. (2012), http://dx.doi.org/10.1007/s00521-012-1073-4.
[30] M. ElKady, E.M.E. Elbarbary, A Chebyshev expansion method for solving optimal control problems, Appl. Math. Comput. 129 (2002) 171–182.