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Two-level iteration penalty methods for the incompressible flows ☆

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ABSTRACT

In this article, we present a new iteration penalty method for incompressible flows based on the iteration of pressure with a factor of penalty parameter, which was first developed for Stokes flows by Cheng and Abdul (2006) [14]. The stability and error estimates of numerical solutions in some norms are derived for this one-level method. Then, combining the techniques of two-level method and linearization with respect to the nonlinear convective term, we propose two-level Stokes/Oseen/Newton iteration penalty methods corresponding to three different linearization method, and show the stability and error estimates of these three methods. Finally, some numerical tests are given to demonstrate the effect of penalty parameter and the efficiency of the new methods.

1. Introduction

In this paper, we consider a two-level iteration penalty method for the incompressible flows which are governed by the incompressible Navier–Stokes equations

$$\begin{cases} -\mu\Delta u + (u\cdot\nabla)u - \nabla p = f, & \text{in } \Omega, \\ \operatorname{div} u = 0, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ assumed to have a Lipschitz continuous boundary $\partial\Omega$. $\mu > 0$ represents the viscosity coefficient. $u = (u_1(x), u_2(x))$ denotes the velocity vector, $p = p(x)$ the pressure, $f = (f_1(x), f_2(x))$ the prescribed body force vector. The solenoidal condition $\operatorname{div} u = 0$ means that the flows are incompressible.

The development of appropriate mixed finite element approximations is a key component in the search for efficient techniques for solving the problem (1.1) quickly and efficiently. Roughly speaking, there exist two main difficulties. One is the nonlinear term $(u\cdot\nabla)u$, which can be processed by the linearization methods such as the Newton iteration method [1], or the two-level method [2–9]. The other is that the velocity and the pressure are coupled by the solenoidal condition. The popular technique to overcome this difficulty is to relax the solenoidal condition in an appropriate method, resulting in a

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pesudo-compressible system, such as the penalty method and the artificial compressible method [10]. Recently, using the Taylor–Hood element ($P_2 - P_1$ triangular element), Li and An [11] studied two-level penalty finite element methods for Navier–Stokes equations with nonlinear slip boundary conditions, where the main results can be extended to the problem (1.1). Denote $(u_\varepsilon^h, p_\varepsilon^h)$ the two-level penalty finite element approximation solution to $(u,p) \in (H^3(\Omega)^2, H^2(\Omega))$. The error estimate derived in [11] is

$$||u - u_\varepsilon^h||_1 + ||p - p_\varepsilon^h|| \leqslant c(\varepsilon + h^2 + H^3), \tag{1.2}$$

where $\varepsilon > 0$ is small, h and H are the fine mesh size and coarse mesh size, respectively, and satisfy $h < H < 1$. $c > 0$ is independent of ε, h and H. Thus, it suggests that ε depends on h, i.e. $\varepsilon = O(h^2)$, to yield an accurate approximation. However, the condition number of the numerical discretization of two-level penalty methods is $O(\varepsilon^{-1}h^{-2})$, which will result in a very ill-conditioned problem when mesh size $h \to 0$.

In this paper, we combine the iteration penalty method with the two-level method to solve the numerical solution to (1.1). The iterative penalty method was first introduced by Cheng [12] for the Stokes equations and further used to solve the pure Neumann problem [13] and the Navier–Stokes equations with nonlinear slip boundary conditions [14]. This iteration penalty method allows us to use a "not very small" penalty parameter ε. Our two-level iteration penalty methods can be described as follows. The first step and the second step are required to solve a small Navier–Stokes equations on the coarse mesh in terms of the iteration penalty method [12,14]. The third step is required to solve a large linearization problem on the fine mesh in terms of Stokes iteration, Oseen iteration or Newton iteration, respectively. We prove that these two-level iteration penalty finite element solutions $(u_{\varepsilon h}, p_{\varepsilon h})$ are of the following error estimate

$$||u - u_{\varepsilon h}||_1 + ||p - p_{\varepsilon h}|| \leqslant \begin{cases} c(h^2 + H^3 + \varepsilon H^2 + \varepsilon^{k+1}) & \text{Stokes/Oseen methods} \\ c(h^2 + H^4 + \varepsilon H^2 + \varepsilon^{k+2}) & \text{Newton method} \end{cases} \tag{1.3}$$

for any positive integer k. Thus, if we choose $\varepsilon = O(H) = O(h^{2/3})$, then (1.3) is of the optimal convergence rate of same order as the usual Galerkin finite element method. Therefore, compared to the two-level penalty method in [11], our iteration penalty method allows that ε is not very small. Moreover, combining with two-level methods, our method we study in this paper can save a large amount of computational time and is an efficient numerical method for solving the numerical solution to the problem (1.1).

2. Preliminary

In what follows, we employ the standard notation $H^l(\Omega)$ (or $H^l(\Omega)^2$) and $||\cdot||_l, l \geqslant 0$, for the Sobolev spaces of all functions having square integrable derivatives up to order l in Ω and the standard Sobolev norm. When $l = 0$, we shall write $L^2(\Omega)$ (or $L^2(\Omega)^2$) and $||\cdot||$ instead of $H^0(\Omega)$ (or $H^0(\Omega)^2$) and $||\cdot||_0$, respectively. Let X be a Banach space. Denote by X' the dual space of X and by $<\cdot,\cdot>_X$ the dual product between X and X'. The dual norm $||\cdot||_{X'}$ is defined by $||v||_{X'} = \sup_{w\in X} \frac{<v,w>_X}{||w||_X}$.

For the mathematical setting, we introduce the following spaces:

$$V = H_0^1(\Omega)^2, \quad M = L_0^2(\Omega) = \left\{ q \in L^2(\Omega);\ \int_\Omega q dx = 0 \right\}.$$

The space V is equipped with the norm

$$||v||_V = \left(\int_\Omega |\nabla v|^2 dx \right)^{1/2}.$$

It is well known that $||v||_V$ is equivalent to $||v||_1$. Introduce two bilinear forms

$$a(u,v) = \mu \int_\Omega \nabla u \cdot \nabla v dx, \quad \forall\ u,\ v \in V,$$
$$d(v,q) = \int_\Omega q \operatorname{div} v dx, \quad \forall\ v \in V,\ q \in M,$$

and a trilinear form

$$b(u,v,w) = \int_\Omega (u\cdot\nabla)v \cdot w dx - \frac{1}{2}\int_\Omega \operatorname{div} u\, v \cdot w dx = \frac{1}{2}\int_\Omega (u\cdot\nabla)v\cdot w dx - \frac{1}{2}\int_\Omega (u\cdot\nabla)w \cdot v dx.$$

It is easy to verify that this trilinear form satisfies the following important properties [7]:

$$b(u,v,w) = -b(u,w,v), \tag{2.1}$$
$$b(u,v,w) \leqslant N||u||_V||v||_V||w||_V, \tag{2.2}$$
$$b(u,v,w) \leqslant \frac{N}{2}||u||^{1/2}||u||_V^{1/2}(||v||_V||w||^{1/2}||w||_V^{1/2} + ||w||_V||v||^{1/2}||v||_V^{1/2}), \tag{2.3}$$

for all $u, v, w \in V$ and

$$|b(u,v,w)| + |b(v,u,w)| + |b(w,u,v)| \leqslant N||u||_V||v||_2||w||, \tag{2.4}$$

for all $u \in V$, $v \in H^2(\Omega)^2$, $w \in L^2(\Omega)^2$, where $N > 0$ depends only on Ω.

Given $f \in V'$, under the above notations, the variational formulation of the problem (1.1) reads as: find $(u,p) \in (V,M)$ such that for all $(v,q) \in (V,M)$:

$$\begin{cases} a(u,v) + b(u,u,v) - d(v,p) = < f,\ v >_V, \\ d(u,q) = 0. \end{cases} \tag{2.5}$$

If we define a generalized bilinear form on $(V,M) \times (V,M)$ by

$$\mathcal{B}(u,p;v,q) = a(u,v) - d(v,p) + d(u,q),$$

then the problem (2.5) also takes the following form:

$$\mathcal{B}(u,p;v,q) + b(u,u,v) = < f,\ v >_V. \tag{2.6}$$

The following existence, uniqueness and regularity results concerning the solution (u,p) to the problem (2.6) are classical [15,16,10]:

Theorem 2.1. *Assume that μ and f satisfy the following uniqueness condition:*

$$2\mu^{-2}N||f||_{V'} < 1, \tag{2.7}$$

then the problem (2.6) exists a unique solution $(u,p) \in (V,M)$ satisfying

$$||u||_V \leqslant \frac{1}{\mu}||f||_{V'} < \frac{\mu}{2N}. \tag{2.8}$$

Furthermore, assume that $\partial\Omega$ is of class $C^m, m = 2,3$. If $f \in H^{m-2}(\Omega)^2$, then the solution (u,p) to the problem (2.6) satisfies the following regularity:

$$||u||_m + ||p||_{m-1} \leqslant c||f||_{m-2}.$$

3. Iteration penalty finite element approximation

Suppose that $f \in H^1(\Omega)$ and Ω is a convex polygonal domain. Let $\mathcal{T}_h$ be a family of quasi-uniform triangular partition of Ω. The corresponding ordered triangles are denoted by $K_1, K_2, \ldots, K_n$. Let $h_i = \mathrm{diam}(K_i), i = 1, \ldots, n$, and $h = \max\{h_1, h_2, \ldots, h_n\}$. For every $K \in \mathcal{T}_h$, let $P_r(K)$ denote the space of the polynomials on K of degree at most r. For simplicity, we consider the conforming finite element spaces V_h and M_h defined by

$$V_h = \left\{ v_h \in C(\overline{\Omega})^2,\ v_h|_K \in [P_2(K)]^2,\ \forall\ K \in \mathcal{T}_h,\ v_h|_{\partial\Omega} = 0 \right\},$$

$$M_h = \left\{ q_h \in C(\overline{\Omega}),\ q_h|_K \in P_1(K),\ \forall\ K \in \mathcal{T}_h,\ \int_\Omega q_h dx = 0 \right\}.$$

It is well known that V_h and M_h satisfy the Babuška–Brezzi condition [17,18]:

$$\kappa||q_h|| \leqslant \sup_{v_h \in V_h} \frac{d(v_h, q_h)}{||v_h||_V}, \tag{3.1}$$

where $\kappa > 0$ is a constant independent of h.

Denote R_h and Q_h the L^2 orthogonal projections onto V_h and M_h, respectively, which satisfy

$$||v - R_h v|| + h||v - R_h v||_V \leqslant ch^i||v||_i, \quad \forall\ v \in H^3(\Omega)^2 \cap V,\ i = 1,2,3, \tag{3.2}$$

$$||q - Q_h q|| \leqslant ch^j||q||_j, \quad \forall\ q \in H^2(\Omega) \cap M,\ j = 1,2. \tag{3.3}$$

In [12], the authors propose the following iteration penalty finite element approximation for solving (2.5).

Step I: Find $(u^0_{\varepsilon h}, p^0_{\varepsilon h}) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\begin{cases} a(u^0_{\varepsilon h}, v_h) + b(u^0_{\varepsilon h}, u^0_{\varepsilon h}, v_h) - d(v_h, p^0_{\varepsilon h}) = (f, v_h), \\ d(u^0_{\varepsilon h}, q_h) + \varepsilon(p^0_{\varepsilon h}, q_h) = 0. \end{cases} \tag{3.4}$$

Step II: For $k = 1, 2, \ldots$, find $(u_{\varepsilon h}^k, p_{\varepsilon h}^k) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\begin{cases} a(u_{\varepsilon h}^k, v_h) + b(u_{\varepsilon h}^k, u_{\varepsilon h}^k, v_h) - d(v_h, p_{\varepsilon h}^k) = (f, v_h), \\ d(u_{\varepsilon h}^k, q_h) + \varepsilon(p_{\varepsilon h}^k, q_h) = \varepsilon(p_{\varepsilon h}^{k-1}, q_h). \end{cases} \tag{3.5}$$

Define a generalized bilinear form on $(V_h, M_h) \times (V_h, M_h)$ by

$$\mathcal{B}_{\varepsilon h}(u_h, p_h; v_h, q_h) = \mathcal{B}(u_h, p_h; v_h, q_h) + \varepsilon(p_h, q_h),$$

then the problem (3.5) also takes the following form:

$$\mathcal{B}_{\varepsilon h}(u_{\varepsilon h}^k, p_{\varepsilon h}^k; v_h, q_h) + b(u_{\varepsilon h}^k, u_{\varepsilon h}^k, v_h) = (f, v_h) + \varepsilon(p_{\varepsilon h}^{k-1}, q_h). \tag{3.6}$$

Theorem 3.1. *Suppose that $(u_{\varepsilon h}^k, p_{\varepsilon h}^k) \in (V_h, M_h)$ is the solution to the problem (3.6), then it satisfies*

$$\mu\|u_{\varepsilon h}^k\|_V^2 + \varepsilon\|p_{\varepsilon h}^k\|^2 \leqslant \frac{2k+1}{2\mu}\|f\|_1^2. \tag{3.7}$$

Proof. Setting $v_h = u_{\varepsilon h}^0, q_h = p_{\varepsilon h}^0$ in (3.4), using (2.1) and Young inequality, it yields

$$\mu\|u_{\varepsilon h}^0\|_V^2 + \varepsilon\|p_{\varepsilon h}^0\|^2 = (f, u_{\varepsilon h}^0) \leqslant \frac{\mu}{2}\|u_{\varepsilon h}^0\|_V^2 + \frac{1}{2\mu}\|f\|_1^2.$$

Thus, we obtain

$$\mu\|u_{\varepsilon h}^0\|_V^2 + 2\varepsilon\|p_{\varepsilon h}^0\|^2 \leqslant \frac{1}{\mu}\|f\|_1^2.$$

For $k = 1, 2, \ldots$, setting $v_h = u_{\varepsilon h}^k, q_h = p_{\varepsilon h}^k$ in (3.6), it yields

$$\mu\|u_{\varepsilon h}^k\|_V^2 + \varepsilon\|p_{\varepsilon h}^k\|^2 = (f, u_{\varepsilon h}^k) + \varepsilon(p_{\varepsilon h}^{k-1}, p_{\varepsilon h}^k) \leqslant \frac{\mu}{2}\|u_{\varepsilon h}^0\|_V^2 + \frac{1}{2\mu}\|f\|_1^2 + \frac{\varepsilon}{2}\|p_{\varepsilon h}^k\|^2 + \frac{\varepsilon}{2}\|p_{\varepsilon h}^{k-1}\|^2.$$

Thus, we obtain

$$\mu\|u_{\varepsilon h}^k\|_V^2 + \varepsilon\|p_{\varepsilon h}^k\|^2 \leqslant \frac{1}{\mu}\|f\|_1^2 + \varepsilon\|p_{\varepsilon h}^{k-1}\|^2 \leqslant \cdots \leqslant \frac{k}{\mu}\|f\|_1^2 + \varepsilon\|p_{\varepsilon h}^0\|^2 \leqslant \frac{2k+1}{2\mu}\|f\|_1^2.$$

The proof is complete. □

Now, we recall a theorem concerning the error estimate between the exact solution and the penalty finite element approximation solution. The proof can be found in [12,14].

Theorem 3.2. *Let $(u, p) \in H^3(\Omega)^2 \cap V \times H^2(\Omega) \cap M$ and $(u_{\varepsilon h}^k, p_{\varepsilon h}^k) \in (V_h, M_h)$ be the solutions to the problems (2.5) and (3.6), then they satisfy*

$$\|u - u_{\varepsilon h}^k\|_V + \|p - p_{\varepsilon h}^k\| \leqslant c(h^2 + \varepsilon^{k+1}), \tag{3.8}$$

for any positive integer k.

We begin to give the error estimate $\|u - u_{\varepsilon h}^k\|$ for the penalty finite element approximation (3.5). This L^2 error analysis is based on the regularity assumption that the following linearized problem (3.9) is $(H^2(\Omega)^2, H^1(\Omega))$ regular.

Given $z \in L^2(\Omega)^2$, find $(w, \pi) \in (V, M)$ such that for all $(v, q) \in (V, M)$

$$\begin{cases} a(w, v) + b(u, v, w) + b(v, u, w) - d(v, \pi) = (z, v), \\ d(w, q) = 0. \end{cases} \tag{3.9}$$

According to (2.1) and (2.8), it is easy to verify that the problem (3.9) exists a unique solution (w, π). The assumption that (3.9) is $(H^2(\Omega)^2, H^1(\Omega))$ regular means that (w, π) also belongs to $(H^2(\Omega)^2, H^1(\Omega))$ and the following inequality holds:

$$\|w\|_2 + \|\pi\|_1 \leqslant c\|z\|. \tag{3.10}$$

Theorem 3.3. *Let $(u, p) \in H^3(\Omega)^2 \cap V \times H^2(\Omega) \cap M$ and $(u_{\varepsilon h}^k, p_{\varepsilon h}^k) \in (V_h, M_h)$ be the solutions to the problems (2.5) and (3.6), then they satisfy*

$$\|u - u_{\varepsilon h}^k\| \leqslant c(h^3 + \varepsilon h^2 + \varepsilon^{k+1}), \tag{3.11}$$

for any positive integer k.

Proof. Setting $z = u - u_{\varepsilon h}^k$ and $v = u - u_{\varepsilon h}^k$ in the first equation of (3.9), we get

$$||u - u_{\varepsilon h}^k||^2 = a(w, u - u_{\varepsilon h}^k) + b(u, u - u_{\varepsilon h}^k, w) + b(u - u_{\varepsilon h}^k, u, w) - d(u - u_{\varepsilon h}^k, \pi). \tag{3.12}$$

Subtracting (2.5) from (3.5) yields

$$\begin{cases} a(u - u_{\varepsilon h}^k, v_h) + b(u, u, v_h) - b(u_{\varepsilon h}^k, u_{\varepsilon h}^k, v_h) - d(v_h, p - p_{\varepsilon h}^k) = 0, \\ d(u - u_{\varepsilon h}^k, q_h) + \varepsilon(p_{\varepsilon h}^{k-1}, q_h) - \varepsilon(p_{\varepsilon h}^k, q_h) = 0 \end{cases} \tag{3.13}$$

Taking $v_h = R_h w$ and $q_h = Q_h \pi$ in (3.13) and combining with (3.12), we obtain

$$\begin{aligned} ||u - u_{\varepsilon h}^k||^2 &= a(w - R_h w, u - u_{\varepsilon h}^k) + b(u, u - u_{\varepsilon h}^k, w) + b(u - u_{\varepsilon h}^k, u, w) + b(u_{\varepsilon h}^k, u_{\varepsilon h}^k, R_h w) - b(u, u, R_h w) \\ &\quad + d(R_h w - w, p - p_{\varepsilon h}^k) - d(u - u_{\varepsilon h}^k, \pi - Q_h \pi) + \varepsilon(p_{\varepsilon h}^{k-1}, Q_h \pi) - \varepsilon(p_{\varepsilon h}^k, Q_h \pi) = I_1 + \cdots + I_4. \end{aligned} \tag{3.14}$$

Using (3.2), (3.8) and (3.10), I_1 is estimated by

$$I_1 = a(w - R_h w, u - u_{\varepsilon h}^k) \leqslant \mu ||u - u_{\varepsilon h}^k||_V ||w - R_h w||_V \leqslant ch(h^2 + \varepsilon^{k+1})||w||_2 \leqslant ch(h^2 + \varepsilon^{k+1})||u - u_{\varepsilon h}^k||.$$

Similarly, using (3.2), (3.3), (3.8) and (3.10), I_3 is estimated by

$$\begin{aligned} I_3 &= d(R_h w - w, p - p_{\varepsilon h}^k) - d(u - u_{\varepsilon h}^k, \pi - Q_h \pi) \leqslant ||R_h w - w||_V ||p - p_{\varepsilon h}^k|| + ||u - u_{\varepsilon h}^k||_V ||\pi - Q_h \pi|| \\ &\leqslant ch(h^2 + \varepsilon^{k+1})(||w||_2 + ||\pi||_1) \leqslant ch(h^2 + \varepsilon^{k+1})||u - u_{\varepsilon h}^k||. \end{aligned}$$

Concerning I_2, we rewrite it as

$$\begin{aligned} I_2 &= b(u, u - u_{\varepsilon h}^k, w) + b(u - u_{\varepsilon h}^k, u, w) + b(u_{\varepsilon h}^k, u_{\varepsilon h}^k, R_h w) - b(u, u, R_h w) \\ &= b(u - u_{\varepsilon h}^k, u - u_{\varepsilon h}^k, w) + b(u - u_{\varepsilon h}^k, u, w - R_h w) + b(u, u - u_{\varepsilon h}^k, w - R_h w) + b(u - u_{\varepsilon h}^k, u - u_{\varepsilon h}^k, R_h w - w). \end{aligned}$$

Then, from (2.2), (2.8), (3.2), (3.7), (3.8) and (3.10), it holds that

$$\begin{aligned} I_2 &\leqslant N||u - u_{\varepsilon h}^k||_V^2 (||w||_V + ||w - R_h w||_V) + N||u||_V ||u - u_{\varepsilon h}^k||_V ||w - R_h w||_V \leqslant c(h^3 + h\varepsilon^{k+1} + \varepsilon^{2k+2})||w||_2 \\ &\leqslant c(h^3 + h\varepsilon^{k+1} + \varepsilon^{2k+2})||u - u_{\varepsilon h}^k||. \end{aligned}$$

Finally, we estimate I_4 by

$$\begin{aligned} I_4 &= \varepsilon(p_{\varepsilon h}^{k-1}, Q_h \pi) - \varepsilon(p_{\varepsilon h}^k, Q_h \pi) = \varepsilon(p_{\varepsilon h}^{k-1} - p, Q_h \pi) + \varepsilon(p - p_{\varepsilon h}^k, Q_h \pi) \leqslant \varepsilon(||p_{\varepsilon h}^{k-1} - p|| + ||p - p_{\varepsilon h}^k||)||Q_h \pi|| \\ &\leqslant \varepsilon(h^2 + \varepsilon^k)||\pi||_1 \leqslant \varepsilon(h^2 + \varepsilon^k)||u - u_{\varepsilon h}^k||. \end{aligned}$$

Combining these estimates with (3.14), we conclude that (3.11) holds. □

4. Two-level iteration penalty methods

In this section, based on the iteration penalty method described in the above section, the two-level iteration penalty finite element methods for (2.5) are proposed in terms of Stokes iteration, Oseen iteration or Newton iteration. From now on, H and h with $h < H$ are two real positive parameter. The coarse mesh triangulation $\mathcal{T}_H$ is made as like in Section 3. And a fine mesh triangulation $\mathcal{T}_h$ is generated by a mesh refinement process to $\mathcal{T}_H$. The conforming finite element space pairs (V_h, M_h) and $(V_H, M_H) \subset (V_h, M_h)$ corresponding to the triangulations $\mathcal{T}_h$ and $\mathcal{T}_H$, respectively, are constructed as like in Section 3. With the above notations, we propose the following two-level iteration finite element methods.

4.1. Two-level stokes iteration penalty method

At Step I and Step II, we solve (3.4) and (3.5) on the coarse mesh. i.e.,

Step I: Find $(u_{\varepsilon H}^0, p_{\varepsilon H}^0) \in (V_H, M_H)$ such that for all $(v_H, q_H) \in (V_H, M_H)$

$$\begin{cases} a(u_{\varepsilon H}^0, v_H) + b(u_{\varepsilon H}^0, u_{\varepsilon H}^0, v_H) - d(v_H, p_{\varepsilon H}^0) = (f, v_H), \\ d(u_{\varepsilon H}^0, q_H) + \varepsilon(p_{\varepsilon H}^0, q_H) = 0. \end{cases} \tag{4.1}$$

Step II: For $k = 1, 2, \ldots$, find $(u_{\varepsilon H}^k, p_{\varepsilon H}^k) \in (V_H, M_H)$ such that for all $(v_H, q_H) \in (V_H, M_H)$

$$\begin{cases} a(u_{\varepsilon H}^k, v_H) + b(u_{\varepsilon H}^k, u_{\varepsilon H}^k, v_H) - d(v_H, p_{\varepsilon H}^k) = (f, v_H), \\ d(u_{\varepsilon H}^k, q_H) + \varepsilon(p_{\varepsilon H}^k, q_H) = \varepsilon(p_{\varepsilon H}^{k-1}, q_H). \end{cases} \tag{4.2}$$

At Step III, we solve a linearized problem on the fine mesh in terms of Stokes iteration. i.e.,

Step III: Find $(u_{\varepsilon h}, p_{\varepsilon h}) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\begin{cases} a(u_{\varepsilon h}, v_h) + b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, v_h) - d(v_h, p_{\varepsilon h}) = (f, v_h), \\ d(u_{\varepsilon h}, q_h) + \varepsilon(p_{\varepsilon h}, q_h) = \varepsilon(p^k_{\varepsilon H}, q_h). \end{cases} \tag{4.3}$$

It follows from (3.7), (3.8) and (3.11) that $u^k_{\varepsilon H}$ is uniformly bounded in V and $(u^k_{\varepsilon H}, p^k_{\varepsilon H})$ satisfies

$$||u - u^k_{\varepsilon H}||_V + ||p - p^k_{\varepsilon H}|| \leqslant c(H^2 + \varepsilon^{k+1}), \tag{4.4}$$

$$||u - u^k_{\varepsilon H}|| \leqslant c(H^3 + \varepsilon H^2 + \varepsilon^{k+1}). \tag{4.5}$$

According to the definition of the generalized bilinear form $\mathcal{B}_{\varepsilon h}$, (2.6) and (4.3) also take the following forms:

$$\mathcal{B}_{\varepsilon h}(u, p; v, q) + b(u, u, v) = (f, v) + \varepsilon(p, q). \tag{4.6}$$

and

$$\mathcal{B}_{\varepsilon h}(u_{\varepsilon h}, p_{\varepsilon h}; v_h, q_h) + b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, v_h) = (f, v_h) + \varepsilon(p^k_{\varepsilon H}, q_h). \tag{4.7}$$

Then we get

$$\mathcal{B}_{\varepsilon h}(u - u_{\varepsilon h}, p - p_{\varepsilon h}; v_h, q_h) + b(u, u, v_h) - b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, v_h) = \varepsilon(p - p^k_{\varepsilon H}, q_h). \tag{4.8}$$

Next, we prove the following error estimate of the finite element approximation solution $(u_{\varepsilon h}, p_{\varepsilon h})$ to the problem (4.3).

Theorem 4.1. *Suppose that the uniqueness condition (2.7) holds. Let $(u, p) \in H^3(\Omega)^2 \cap V \times H^2(\Omega) \cap M$ and $(u_{\varepsilon h}, p_{\varepsilon h}) \in (V_h, M_h)$ be the solutions to the problems (2.5) and (4.3), then they satisfy*

$$||u - u_{\varepsilon h}||_V + ||p - p_{\varepsilon h}|| \leqslant c(h^2 + \varepsilon H^2 + H^3 + \varepsilon^{k+1}). \tag{4.9}$$

Proof. Setting $v_h = R_h u - u_{\varepsilon h}$ and $q_h = Q_h p - p_{\varepsilon h}$ in (4.8), it yields

$$\begin{aligned} \mu||R_h u - u_{\varepsilon h}||^2_V + \varepsilon||Q_h p - p_{\varepsilon h}||^2 &= \mathcal{B}_{\varepsilon h}(R_h u - u_{\varepsilon h}, Q_h p - p_{\varepsilon h}; R_h u - u_{\varepsilon h}, Q_h p - p_{\varepsilon h}) \\ &= \mathcal{B}_{\varepsilon h}(R_h u - u, Q_h p - p; R_h u - u_{\varepsilon h}, Q_h p - p_{\varepsilon h}) + \mathcal{B}_{\varepsilon h}(u - u_{\varepsilon h}, p - p_{\varepsilon h}; R_h u - u_{\varepsilon h}, Q_h p - p_{\varepsilon h}) \\ &= \mathcal{B}_{\varepsilon h}(R_h u - u, Q_h p - p; R_h u - u_{\varepsilon h}, Q_h p - p_{\varepsilon h}) + b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, R_h u - u_{\varepsilon h}) - b(u, u, R_h u - u_{\varepsilon h}) \\ &\quad + \varepsilon(p - p^k_{\varepsilon H}, Q_h p - p_{\varepsilon h}) = J_1 + J_2 + J_3. \end{aligned} \tag{4.10}$$

From Hölder inequality and Young inequality, it is easy to show that

$$\begin{aligned} J_1 &= \mathcal{B}_{\varepsilon h}(R_h u - u, Q_h p - p; R_h u - u_{\varepsilon h}, Q_h p - p_{\varepsilon h}) \\ &\leqslant \mu||R_h u - u||_V||R_h u - u_{\varepsilon h}||_V + ||r_h u - u||_V||Q_h p - p_{\varepsilon h}|| + ||R_h u - u_{\varepsilon h}||_V||Q_h p - p|| + \varepsilon||Q_h p - p||||Q_h p - p_{\varepsilon h}|| \\ &\leqslant \frac{\mu}{8}||R_h u - u_{\varepsilon h}||^2_V + c(||R_h u - u||^2_V + ||Q_h p - p||^2) + \eta||Q_h p - p_{\varepsilon h}||^2 + \varepsilon^2||Q_h p - p||^2, \end{aligned} \tag{4.11}$$

and

$$J_3 = \varepsilon(p - p^k_{\varepsilon H}, Q_h p - p_{\varepsilon h}) \leqslant \varepsilon||p - p^k_{\varepsilon H}||||Q_h p - p_{\varepsilon h}|| \leqslant \eta||Q_h p - p_{\varepsilon h}||^2 + c\varepsilon^2||p - p^k_{\varepsilon H}||^2, \tag{4.12}$$

where $\eta > 0$ is some small constant. Note that

$$b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, v_h) - b(u, u, v_h) = b(u^k_{\varepsilon H} - u, u, v_h) + (u, u^k_{\varepsilon H} - u, v_h) + b(u^k_{\varepsilon H} - u, u^k_{\varepsilon H} - u, v_h), \tag{4.13}$$

then from (2.2) and (2.4) we can estimate J_2 by

$$\begin{aligned} J_2 &= b(u^k_{\varepsilon H} - u, u, R_h u - u_{\varepsilon h}) + (u, u^k_{\varepsilon H} - u, R_h u - u_{\varepsilon h}) + b(u^k_{\varepsilon H} - u, u^k_{\varepsilon H} - u, R_h u - u_{\varepsilon h}) \\ &\leqslant N||u||_2||R_h u - u_{\varepsilon h}||_V||u^k_{\varepsilon H} - u|| + N||u^k_{\varepsilon H} - u||^2_V||R_h u - u_{\varepsilon h}||_V \leqslant \frac{\mu}{8}||R_h u - u_{\varepsilon h}||^2_V + c\Big(||u^k_{\varepsilon H} - u||^2 + ||u^k_{\varepsilon H} - u||^4_V\Big). \end{aligned}$$

Substituting these estimates into (4.10) and using triangular inequality, it yields

$$\begin{aligned} ||u - u_{\varepsilon h}||_V &\leqslant ||u - R_h u||_V + ||R_h u - u_{\varepsilon h}||_V \\ &\leqslant c(||R_h u - u||_V + ||Q_h p - p||) + \eta||Q_h p - p_{\varepsilon h}|| + c(\varepsilon||p - p^k_{\varepsilon H}|| + ||u^k_{\varepsilon H} - u|| + ||u^k_{\varepsilon H} - u||^2_V). \end{aligned} \tag{4.14}$$

Next, we estimate $||Q_h p - p_{\varepsilon h}||$. From the Babuška–Brezzi condition (3.1), we have

$$\kappa||Q_h p - p_{\varepsilon h}|| \leqslant \sup_{v_h \in V_h} \frac{d(v_h, Q_h p - p_{\varepsilon h})}{||v_h||_V} = \sup_{v_h \in V_h} \frac{d(v_h, Q_h p - p) + d(v_h, p - p_{\varepsilon h})}{||v_h||_V} \leqslant ||Q_h p - p|| + \sup_{v_h \in V_h} \frac{d(v_h, p - p_{\varepsilon h})}{||v_h||_V}. \tag{4.15}$$

On the other hand, it follows from (2.2), (2.4) and (4.13) that

$$\begin{aligned} d(v_h, p - p_{\varepsilon h}) &= a(u - u_{\varepsilon h}, v_h) + b(u, u, v_h) - b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, v_h) \\ &= a(u - u_{\varepsilon h}, v_h) + b(u, u - u^k_{\varepsilon H}, v_h) + b(u - u^k_{\varepsilon H}, u, v_h) - b(u - u^k_{\varepsilon H}, u - u^k_{\varepsilon H}, v_h) \\ &\leqslant (\mu||u - u_{\varepsilon h}||_V + N||u||_2||u - u^k_{\varepsilon H}|| + N||u - u^k_{\varepsilon H}||^2_V)||v_h||_V. \end{aligned}$$

Thus, we obtain

$$||Q_h p - p_{\varepsilon h}|| \leqslant c\Big(||p - Q_h p|| + ||u - u_{\varepsilon h}||_V + ||u - u^k_{\varepsilon H}|| + ||u - u^k_{\varepsilon H}||^2_V\Big). \tag{4.16}$$

Substituting (4.16) into (4.14), for sufficiently small $\eta > 0$, we get from (3.2), (3.3), (4.4) and (4.5)

$$\begin{aligned} ||u - u_{\varepsilon h}||_V &\leqslant c(||R_h u - u||_V + ||Q_h p - p||) + c\Big(\varepsilon||p - p^k_{\varepsilon H}|| + ||u^k_{\varepsilon H} - u|| + ||u^k_{\varepsilon H} - u||^2_V\Big) \\ &\leqslant c(h^2 + \varepsilon H^2 + \varepsilon^{k+2} + H^3 + \varepsilon H^2 + \varepsilon^{k+1} + H^4 + \varepsilon^{2k+2}) \leqslant c\Big(h^2 + \varepsilon H^2 + H^3 + \varepsilon^{k+1}\Big). \end{aligned}$$

From (4.16), again, we conclude

$$||p - p_{\varepsilon h}|| \leqslant c(h^2 + \varepsilon H^2 + H^3 + \varepsilon^{k+1}).$$

The proof is complete. □

4.2. Two-level Oseen iteration penalty method

At Step I and Step II, we solve (4.1) and (4.2) on the coarse mesh. i.e.,

Step I: Find $(u^0_{\varepsilon H}, p^0_{\varepsilon H}) \in (V_H, M_H)$ by (4.1).
Step II: For $k = 1, 2, \ldots$, find $(u^k_{\varepsilon H}, p^k_{\varepsilon H}) \in (V_H, M_H)$ by (4.2).
At Step III, we solve a linearized problem on the fine mesh in terms of Oseen iteration, i.e.,
Step III: Find $(u_{\varepsilon h}, p_{\varepsilon h}) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\begin{cases} a(u_{\varepsilon h}, v_h) + b(u^k_{\varepsilon H}, u_{\varepsilon h}, v_h) - d(v_h, p_{\varepsilon h}) = (f, v_h), \\ d(u_{\varepsilon h}, q_h) + \varepsilon(p_{\varepsilon h}, q_h) = \varepsilon(p^k_{\varepsilon H}, q_h). \end{cases} \tag{4.17}$$

Next, we prove the following error estimate of the finite element approximation solution $(u_{\varepsilon h}, p_{\varepsilon h})$ to the problem (4.17).

Theorem 4.2. *Suppose that the uniqueness condition (2.7) holds. Let $(u, p) \in (H^3(\Omega)^2 \cap V, H^2(\Omega) \cap M)$ and $(u_{\varepsilon h}, p_{\varepsilon h}) \in (V_h, M_h)$ be the solutions to the problems (2.5) and (4.17), then they satisfy*

$$||u - u_{\varepsilon h}||_V + ||p - p_{\varepsilon h}|| \leqslant c\Big(h^2 + \varepsilon H^2 + H^3 + \varepsilon^{k+1}\Big). \tag{4.18}$$

Proof. Proceeding as the proof in Theorem 4.1, we have

$$\begin{aligned} \mu||R_h u - u_{\varepsilon h}||^2_V + \varepsilon||Q_h p - p_{\varepsilon h}||^2 &= \mathcal{B}_{\varepsilon h}(R_h u - u, Q_h p - p; R_h u - u_{\varepsilon h}, Q_h p - p_{\varepsilon h}) + b(u^k_{\varepsilon H}, u_{\varepsilon h}, R_h u - u_{\varepsilon h}) \\ &\quad - b(u, u, R_h u - u_{\varepsilon h}) + \varepsilon(p - p^k_{\varepsilon H}, Q_h p - p_{\varepsilon h}). \end{aligned} \tag{4.19}$$

Here, we only estimate $b(u^k_{\varepsilon H}, u_{\varepsilon h}, R_h u - u_{\varepsilon h}) - b(u, u, R_h u - u_{\varepsilon h})$ by

$$\begin{aligned} b\big(u^k_{\varepsilon H}, u_{\varepsilon h}, R_h u - u_{\varepsilon h}\big) - b(u, u, R_h u - u_{\varepsilon h}) &= b\big(u^k_{\varepsilon H}, u_{\varepsilon h} - R_h u, R_h u - u_{\varepsilon h}\big) + b\big(u^k_{\varepsilon H}, R_h u - u, R_h u - u_{\varepsilon h}\big) \\ &\quad + b\big(u^k_{\varepsilon H} - u, u, R_h u - u_{\varepsilon h}\big) \\ &= b\big(u^k_{\varepsilon H}, R_h u - u, R_h u - u_{\varepsilon h}\big) + b\big(u^k_{\varepsilon H} - u, u, R_h u - u_{\varepsilon h}\big) \\ &\leqslant N||u^k_{\varepsilon H}||_V||R_h u - u||_V||R_h u - u_{\varepsilon h}||_V + N||u||_2||R_h u - u_{\varepsilon h}||_V||u^k_{\varepsilon H} - u|| \\ &\leqslant \frac{\mu}{8}||R_h u - u_{\varepsilon h}||^2_V + c\Big(||u^k_{\varepsilon H} - u||^2 + ||R_h u - u||^2_V\Big), \end{aligned} \tag{4.20}$$

where we use (2.1), (2.2) and (2.4). Substituting (4.11), (4.12) and (4.20) into (4.19), it yields

$$\begin{aligned} ||u - u_{\varepsilon h}||_V &\leqslant ||u - R_h u||_V + ||R_h u - u_{\varepsilon h}||_V \\ &\leqslant c(||R_h u - u||_V + ||Q_h p - p||) + \eta||Q_h p - p_{\varepsilon h}|| + c(\varepsilon||p - p^k_{\varepsilon H}|| + ||u^k_{\varepsilon H} - u||), \end{aligned} \tag{4.21}$$

where $\eta > 0$ is some small constant. Note that

$$d(v_h, p - p_{\varepsilon h}) = a(u - u_{\varepsilon h}, v_h) + b(u, u, v_h) - b(u^k_{\varepsilon H}, u_{\varepsilon h}, v_h) = a(u - u_{\varepsilon h}, v_h) + b(u - u^k_{\varepsilon H}, u, v_h) + b(u^k_{\varepsilon H}, u - u_{\varepsilon h}, v_h)$$
$$\leqslant (\mu||u - u_{\varepsilon h}||_V + N||u||_2||u - u^k_{\varepsilon H}|| + N||u^k_{\varepsilon H}||_V||u - u_{\varepsilon h}||_V)||v_h||_V.$$

Then, from (3.7) and (4.15) we obtain

$$||Q_h p - p_{\varepsilon h}|| \leqslant c(||p - Q_h p|| + ||u - u_{\varepsilon h}||_V + \|u - u^k_{\varepsilon H}\|). \tag{4.22}$$

Substituting (4.22) into (4.21), for sufficiently small $\eta > 0$, we get from (3.2), (3.3), (4.4) and (4.5)

$$||u - u_{\varepsilon h}||_V \leqslant c(||R_h u - u||_V + ||Q_h p - p|| + \varepsilon\|p - p^k_{\varepsilon H}\| + \|u^k_{\varepsilon H} - u\|) \leqslant c(h^2 + \varepsilon H^2 + \varepsilon^{k+2} + H^3 + \varepsilon H^2 + \varepsilon^{k+1})$$
$$\leqslant c(h^2 + \varepsilon H^2 + H^3 + \varepsilon^{k+1}).$$

From (4.22), again, we conclude

$$||p - p_{\varepsilon h}|| \leqslant c(h^2 + \varepsilon H^2 + H^3 + \varepsilon^{k+1}).$$

The proof is complete. □

4.3. Two-level newton iteration penalty method

At Step I and Step II, we solve (4.1) and (4.2) on the coarse mesh. i.e.,

Step I: Find $(u^0_{\varepsilon H}, p^0_{\varepsilon H}) \in (V_H, M_H)$ by (4.1).
Step II: For $k = 1, 2, \ldots$, find $(u^k_{\varepsilon H}, p^k_{\varepsilon H}) \in (V_H, M_H)$ by (4.2).
At Step III, we solve a linearized problem on the fine mesh in terms of Newton iteration. i.e.,
Step III: Find $(u_{\varepsilon h}, p_{\varepsilon h}) \in (V_h, M_h)$ such that for all $(v_h, q_h) \in (V_h, M_h)$

$$\begin{cases} a(u_{\varepsilon h}, v_h) + b(u^k_{\varepsilon H}, u_{\varepsilon h}, v_h) + b(u_{\varepsilon h}, u^k_{\varepsilon H}, v_h) - d(v_h, p_{\varepsilon h}) = (f, v_h) + b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, v_h), \\ d(u_{\varepsilon h}, q_h) + \varepsilon(p_{\varepsilon h}, q_h) = \varepsilon(p^k_{\varepsilon H}, q_h). \end{cases} \tag{4.23}$$

Next, we prove the following error estimate of the finite element approximation solution $(u_{\varepsilon h}, p_{\varepsilon h})$ to the problem (4.23).

Theorem 4.3. *Suppose that the uniqueness condition (2.7) holds. Let $(u, p) \in (H^3(\Omega)^2 \cap V, H^2(\Omega) \cap M)$ and $(u_{\varepsilon h}, p_{\varepsilon h}) \in (V_h, M_h)$ be the solutions to the problems (2.5) and (4.23), then for sufficiently small h, H, ε they satisfy*

$$||u - u_{\varepsilon h}||_V + ||p - p_{\varepsilon h}|| \leqslant c(h^2 + \varepsilon H^2 + H^4 + \varepsilon^{k+2}). \tag{4.24}$$

Proof. Proceeding as the proof in Theorem 4.1, we have

$$\begin{aligned} \mu||R_h u - u_{\varepsilon h}||^2_V + \varepsilon||Q_h p - p_{\varepsilon h}||^2 &= \mathcal{B}_{\varepsilon h}(R_h u - u, Q_h p - p; R_h u - u_{\varepsilon h}, Q_h p - p_{\varepsilon h}) \\ &\quad + b(u^k_{\varepsilon H}, u_{\varepsilon h}, R_h u - u_{\varepsilon h}) + b(u_{\varepsilon h}, u^k_{\varepsilon H}, R_h u - u_{\varepsilon h}) - b(u, u, R_h u - u_{\varepsilon h}) \\ &\quad - b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, R_h u - u_{\varepsilon h}) + \varepsilon(p - p^k_{\varepsilon H}, Q_h p - p_{\varepsilon h}) = J_1 + J_3 + J_4, \end{aligned} \tag{4.25}$$

where J_1 and J_3 have been estimated by (4.11) and (4.12), respectively. We rewrite J_4 as

$$\begin{aligned} J_4 &= b(u^k_{\varepsilon H}, u_{\varepsilon h}, R_h u - u_{\varepsilon h}) + b(u_{\varepsilon h}, u^k_{\varepsilon H}, R_h u - u_{\varepsilon h}) - b(u, u, R_h u - u_{\varepsilon h}) - b(u^k_{\varepsilon H}, u^k_{\varepsilon H}, R_h u - u_{\varepsilon h}) \\ &= b(u_{\varepsilon h} - u, u, R_h u - u_{\varepsilon h}) + b(u_{\varepsilon h}, u_{\varepsilon h} - u, R_h u - u_{\varepsilon h}) - b(u_{\varepsilon h} - u^k_{\varepsilon H}, u_{\varepsilon h} - u^k_{\varepsilon H}, R_h u - u_{\varepsilon h}) \\ &= b(u_{\varepsilon h} - R_h u, u, R_h u - u_{\varepsilon h}) + b(R_h u - u, u, R_h u - u_{\varepsilon h}) + b(u_{\varepsilon h}, R_h u - u, R_h u - u_{\varepsilon h}) \\ &\quad + b(R_h u - u_{\varepsilon h}, R_h u - u^k_{\varepsilon H}, R_h u - u_{\varepsilon h}) + b(u^k_{\varepsilon H} - R_h u, R_h u - u^k_{\varepsilon H}, R_h u - u_{\varepsilon h}) \\ &= J_5 + \cdots + J_9. \end{aligned} \tag{4.26}$$

From (2.2) and (2.8), J_5 and J_6 are estimated by

$$J_5 = b(u_{\varepsilon h} - R_h u, u, R_h u - u_{\varepsilon h}) \leqslant N||u||_V||R_h u - u_{\varepsilon h}||^2_V \leqslant \frac{\mu}{2}||R_h u - u_{\varepsilon h}||^2_V,$$

and

$$J_6 = b(R_h u - u, u, R_h u - u_{\varepsilon h}) \leqslant N||u||_V||R_h u - u_{\varepsilon h}||_V||R_h u - u||_V \leqslant \frac{\mu}{2}||R_h u - u_{\varepsilon h}||_V||R_h u - u||_V$$
$$\leqslant \frac{\mu}{8}||R_h u - u_{\varepsilon h}||_V^2 + \frac{\mu}{2}||R_h u - u||_V^2.$$

Using (2.2) and (3.2), we estimate J_7 by

$$J_7 = b(u_{\varepsilon h}, R_h u - u, R_h u - u_{\varepsilon h}) = b(u_{\varepsilon h} - R_h u, R_h u - u, R_h u - u_{\varepsilon h}) + b(R_h u, R_h u - u, R_h u - u_{\varepsilon h})$$
$$\leqslant N||R_h u - u||_V||R_h u - u_{\varepsilon h}||_V^2 + N||R_h u||_V||R_h u - u||_V||R_h u - u_{\varepsilon h}||_V \leqslant ch^2||R_h u - u_{\varepsilon h}||_V^2 + \frac{\mu}{8}||R_h u - u_{\varepsilon h}||_V^2 + c||R_h u - u||_V^2.$$

It follows from (2.2), (3.2) and (4.4) that J_8 is estimated by

$$J_8 = b(R_h u - u_{\varepsilon h}, R_h u - u_{\varepsilon H}^k, R_h u - u_{\varepsilon h}) \leqslant N||R_h u - u_{\varepsilon H}^k||_V||R_h u - u_{\varepsilon h}||_V^2 \leqslant N\big(||R_h u - u||_V + ||u - u_{\varepsilon H}^k||_V\big)||R_h u - u_{\varepsilon h}||_V^2$$
$$\leqslant c\Big(h^2 + H^2 + \varepsilon^{k+1}\Big)||R_h u - u_{\varepsilon h}||_V^2.$$

Finally, J_9 is estimated by

$$J_9 = b\big(u_{\varepsilon H}^k - R_h u, R_h u - u_{\varepsilon H}^k, R_h u - u_{\varepsilon h}\big) \leqslant N||R_h u - u_{\varepsilon H}^k||_V^2||R_h u - u_{\varepsilon h}||_V \leqslant \frac{\mu}{8}||R_h u - u_{\varepsilon h}||_V^2 + c\Big(||R_h u - u||_V^4 + ||u - u_{\varepsilon H}^k||_V^4\Big).$$

Substituting these estimates into (4.26) and combining with (4.11), (4.12), for sufficiently small h, H, ε we get

$$||u - u_{\varepsilon h}||_V \leqslant ||u - R_h u||_V + ||R_h u - u_{\varepsilon h}||_V \leqslant c(||R_h u - u||_V + ||Q_h p - p||) + \eta||Q_h p - p_{\varepsilon h}||$$
$$+ c\Big(\varepsilon||p - p_{\varepsilon H}^k|| + ||R_h u - u||_V^2 + ||u - u_{\varepsilon H}^k||_V^2\Big), \tag{4.27}$$

where $\eta > 0$ is some small constant. Note that

$$d(v_h, p - p_{\varepsilon h}) = a(u - u_{\varepsilon h}, v_h) + b(u, u, v_h) - b\big(u_{\varepsilon H}^k, u_{\varepsilon h}, v_h\big) - b\big(u_{\varepsilon h}, u_{\varepsilon H}^k, v_h\big) + b\big(u_{\varepsilon H}^k, u_{\varepsilon H}^k, v_h\big). \tag{4.28}$$

Since

$$\begin{aligned}
b(u,u,v_h) - b\big(u_{\varepsilon H}^k, u_{\varepsilon h}, v_h\big) - b\big(u_{\varepsilon h}, u_{\varepsilon H}^k, v_h\big) + b\big(u_{\varepsilon H}^k, u_{\varepsilon H}^k, v_h\big) &= b(u - u_{\varepsilon h}, u, v_h) + b(u, u - R_h u, v_h) - b(u - u_{\varepsilon h}, u - R_h u, v_h) \\
&\quad + b\big(u_{\varepsilon h} - u_{\varepsilon H}^k, R_h u - u_{\varepsilon H}^k, v_h\big) - b\big(u_{\varepsilon H}^k, u_{\varepsilon h} - R_h u, v_h\big) \\
&\leqslant (N||u||_V||u - u_{\varepsilon h}||_V + N||u||_V||u - R_h u||_V \\
&\quad + ||u - R_h u||_V||u - u_{\varepsilon h}||_V)||v_h||_V + (||u - u_{\varepsilon h}||_V \\
&\quad + ||u - u_{\varepsilon H}^k||_V)\big(||u - R_h u||_V + ||u - u_{\varepsilon H}^k||_V\big)||v_h||_V \\
&\quad + N||u_{\varepsilon H}^k||_V(||u - u_{\varepsilon h}||_V + ||u - R_h u||_V)||v_h||_V \\
&\leqslant c\Big(1 + h^2 + H^2 + \varepsilon^{k+1}\Big)||u - u_{\varepsilon h}||_V||v_h||_V \\
&\quad + c\Big(||u - R_h u||_V + ||u - R_h u||_V^2 + ||u - u_{\varepsilon H}^k||_V^2\Big)||v_h||_V,
\end{aligned}$$

then combining (4.28) and (4.15), we obtain

$$||Q_h p - p_{\varepsilon h}|| \leqslant c\Big(1 + h^2 + H^2 + \varepsilon^{k+1}\Big)||u - u_{\varepsilon h}||_V + c\Big(||u - R_h u||_V + ||p - Q_h p|| + ||u - R_h u||_V^2 + ||u - u_{\varepsilon H}^k||_V^2\Big). \tag{4.29}$$

Substituting (4.29) into (4.27), we obtain

$$||u - u_{\varepsilon h}||_V \leqslant c(||R_h u - u||_V + ||Q_h p - p||) + c(\varepsilon||p - p_{\varepsilon H}^k|| + ||R_h u - u||_V^2 + ||u - u_{\varepsilon H}^k||_V^2)$$
$$\leqslant c(h^2 + \varepsilon H^2 + \varepsilon^{k+2} + h^4 + H^4 + \varepsilon^{2k+2}) \leqslant c(h^2 + \varepsilon H^2 + H^4 + \varepsilon^{k+2}), \tag{4.30}$$

which proves (4.24). □

5. Numerical results

In this section, we take some numerical tests to illustrate the performance of the present iteration penalty methods. The testing example is quoted from [12], namely the exact solution is chosen as

$$u = (x^2(x-1)^2 y(y-1)(2y-1), -x(x-1)(2x-1)y^2(y-1)^2),$$

Table 1
Effect of penalty parameter $\varepsilon = \varepsilon_0 H$ in the full error $\frac{\|u-u_h\|_V+\|p-p_h\|}{\|u\|_V+\|p\|}$ for the three methods with $H = 1/36$ and $h = 1/216$.

ε_0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
Stokes	6.86784e−06	6.84687e−06	6.84666e−06	6.84665e−06
Oseen	6.86749e−06	6.84652e−06	6.84631e−06	6.84631e−06
Newton	6.86738e−06	6.84641e−06	6.84620e−06	6.84620e−06

Table 2
Convergence of the one-level penalty method.

h	$\frac{\|u-u_h\|}{\|u\|}$	$\frac{\|u-u_h\|_V}{\|u\|_V}$	$\frac{\|p-p_h\|}{\|p\|}$	Iteration	Time (s)
$1/2^3$	4.90246e−3	4.46192e−2	3.90625e−3	3	0.51
$1/3^3$	1.25834e−4	4.03434e−3	3.42936e−4	2	4.15
$1/4^3$	9.40634e−6	7.20093e−4	6.10352e−5	2	26.19
$1/5^3$	1.26250e−6	1.88860e−4	1.60000e−5	2	131.37
$1/6^3$	2.44682e−7	6.32562e−5	5.35837e−6	2	475.93
Order	3.00370	1.99288	2		

Table 3
Convergence of the two-level Stokes iteration penalty method with penalty parameter $\varepsilon = \varepsilon_0 H$ ($\varepsilon_0 = 0.01$).

H	h	$\frac{\|u-u_h\|}{\|u\|}$	$\frac{\|u-u_h\|_V}{\|u\|_V}$	$\frac{\|p-p_h\|}{\|p\|}$	Iteration	Time (s)
$1/2^2$	$1/2^3$	4.90877e−3	4.46272e−2	3.90625e−3	3	0.25
$1/3^2$	$1/3^3$	1.25834e−4	4.03542e−3	3.42936e−4	3	1.33
$1/4^2$	$1/4^3$	9.47324e−6	7.20277e−4	6.10352e−5	3	4.86
$1/5^2$	$1/5^3$	1.27665e−6	1.88907e−4	1.60000e−5	3	19
$1/6^2$	$1/6^3$	2.48640e−7	6.32720e−5	5.35837e−6	3	83.21
	Order	2.99848	1.99286	2		

and

$$p = x^2 - y^2,$$

in the unit square $\Omega = [0, 1]^2$, and the force f is determined by the original system (1.1).

In all the experiments, we choose $\mu = 0.01$ and implement all algorithms by the finite element software FreeFem++ [19]. Firstly, we check the effect of the penalty parameter on the approximation error $\frac{\|u-u_h\|_V+\|p-p_h\|}{\|u\|_V+\|p\|}$ for the three methods. With the fixed two uniform meshes with $H = 1/36$ and $h = 1/216$, the computational results are shown in Table 1. Here, the symbols "Stokes", "Oseen" and "Newton" mean using two-level Stokes/Oseen/Newton iteration penalty method, respectively. We can see that, for our present testing case, it suffices to set $\varepsilon = 0.01H$ if it is hoped to be as large as possible.

Then, to verify the theoretical analysis for two-level Stokes/Oseen/Newton iteration penalty methods given in Theorems 4.1,4.2,4.3, we use several mesh pairs $H = 1/2^2, 1/3^2, \ldots, 1/6^2$ and $h = H^{3/2}$. Also, for one-level iteration method (3.4) and (3.5), we only use the corresponding fine meshes. The numerical results are displayed in Tables 3–5, and Table 2. In all these tables, the symbol "Iteration" denotes the number of Newton iteration in Step II of corresponding method. the From these tables, the observations and conclusions are presented as follows:

- From Table 2, the numerical convergence orders of one-level iteration penalty method (3.4) and (3.5) coincide with the ones predicted by theoretical analysis in Theorems 3.2 and 3.3 very well, namely, $O(h^2)$ for velocity in H^1-norm and pressure in L^2-norm, and $O(h^3)$ for velocity in L^2-norm.
- Based on Tables 3–5, all three two-level Stokes/Oseen/Newton iteration penalty methods can reach the optimal convergence orders of $O(h^2)$ for both velocity and pressure, in H^1- and L^2-norms, respectively, as proven in Theorems 4.1,4.2,4.3. Besides, we find that these three iteration methods can achieve the numerical convergence orders of $O(h^3)$ for velocity in the sense of L^2-norm as expected.
- As for the comparisons between these three iteration penalty methods, we can find from Tables 3–5 that, the two-level Newton iteration penalty method obtains a little better approximations results than the Oseen one, and both methods exceed the Stokes one.
- From the view of computational cost, we can obviously observe by comparing Tables 3–5 and Table 2 that these three two-level iteration penalty methods significantly save CPU time than the one-level iteration penalty method, meanwhile, obtain nearly the same approximation results.

Table 4
Convergence of the two-level Oseen iteration penalty method with penalty parameter $\varepsilon = \varepsilon_0 H$ ($\varepsilon_0 = 0.01$).

H	h	$\frac{\|u-u_h\|}{\|u\|}$	$\frac{\|u-u_h\|_V}{\|u\|_V}$	$\frac{\|p-p_h\|}{\|p\|}$	Iteration	Time (s)
$1/2^2$	$1/2^3$	4.90789e−3	4.46209e−2	3.90625e−3	3	0.29
$1/3^2$	$1/3^3$	1.25772e−4	4.03484e−3	3.42936e−4	3	1.79
$1/4^2$	$1/4^3$	9.46497e−6	7.20181e−4	6.10352e−5	3	6.66
$1/5^2$	$1/5^3$	1.27493e−6	1.88883e−4	1.60000e−5	3	27.65
$1/6^2$	$1/6^3$	2.48162e−7	6.32638e−5	5.35837e−6	3	93.42
	Order	2.99910	1.99286	2		

Table 5
Convergence of the two-level Newton iteration penalty method with penalty parameter $\varepsilon = \varepsilon_0 H$ ($\varepsilon_0 = 0.01$).

H	h	$\frac{\|u-u_h\|}{\|u\|}$	$\frac{\|u-u_h\|_V}{\|u\|_V}$	$\frac{\|p-p_h\|}{\|p\|}$	Iteration	Time (s)
$1/2^2$	$1/2^3$	4.90459e−3	4.46188e−2	3.90625e−3	3	0.3
$1/3^2$	$1/3^3$	1.25511e−4	4.03467e−3	3.42936e−4	3	1.71
$1/4^2$	$1/4^3$	9.42821e−6	7.20151e−4	6.10352e−5	3	6.95
$1/5^2$	$1/5^3$	1.26706e−6	1.88875e−4	1.60000e−5	3	32
$1/6^2$	$1/6^3$	2.45952e−7	6.32613e−5	5.35837e−6	3	110.86
	Order	3.00203	1.99286	2		

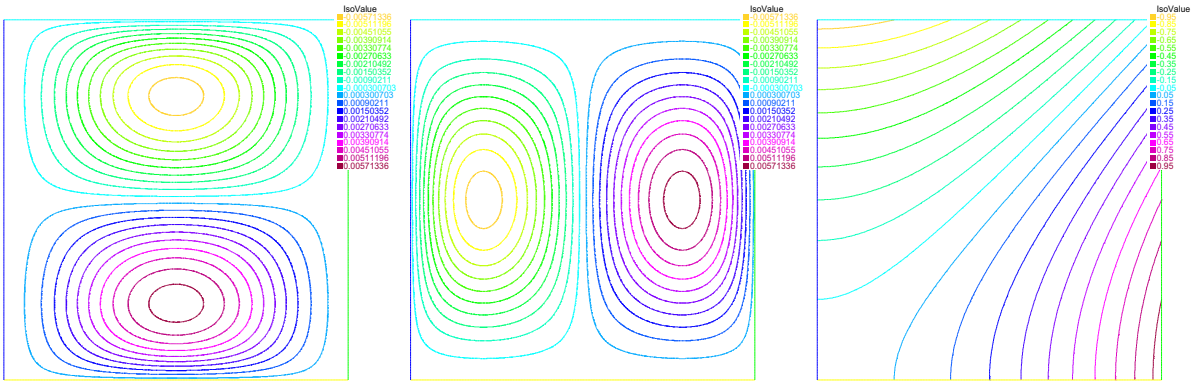

Fig. 1. Contour plots of exact solution. From left to right: two components of velocity and pressure.

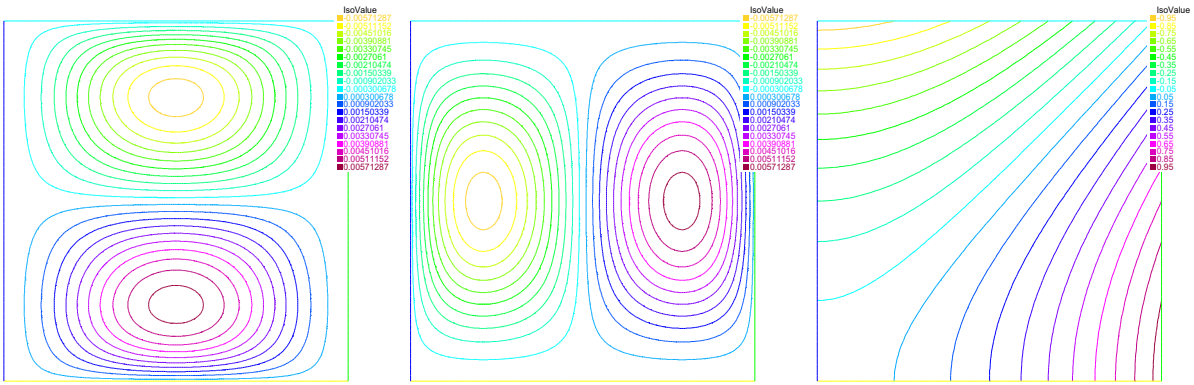

Fig. 2. Contour plots of numerical solution by one-level iteration penalty method. From left to right: two components of velocity and pressure.

Moreover, we show the contour plots of exact and numerical velocity and pressure to exhibit the approximation profiles in details. Figs. 1 and 2 present the exact solution and the numerical one by one-level iteration penalty method. Besides, as to the two-level method, here only the numerical solution by Newton iteration penalty one is displayed in Fig. 3. From these three groups of contour plots, we can observe the good coincidence with each other to illustrate the stability of the present methods.

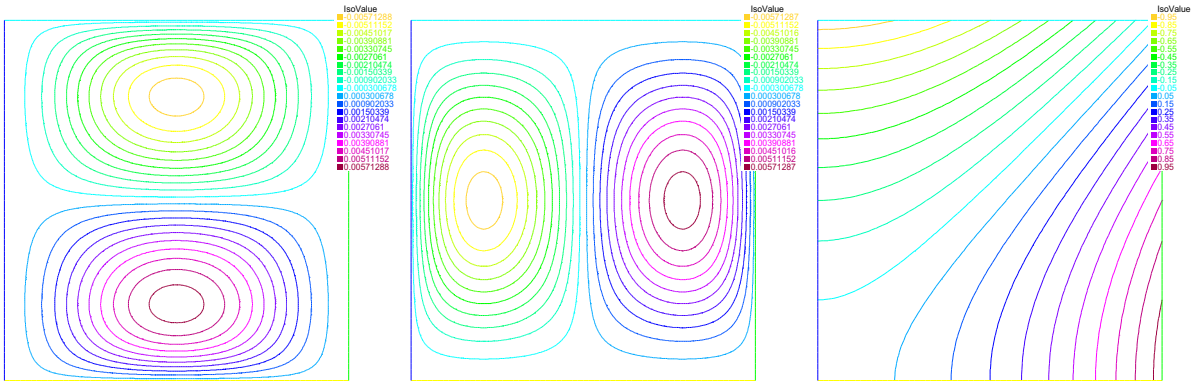

Fig. 3. Contour plots of numerical solution by two-level Newton iteration penalty method. From left to right: two components of velocity and pressure.

6. Conclusion

In this paper, we present a one-level iteration penalty method for solving Navier–Stokes equations and derived its stability and error estimates. Then we propose three two-level iteration penalty methods based on the linearization techniques of Stokes, Oseen and Netwon types on fine mesh. Also we obtain the related error estimates of these three methods. Some numerical experiments are made to show the stability and efficiency of the present methods. Combining the present methods with some stabilization techniques like the Streamline-Upwind/Petrov–Galerkin (SUPG) method or variational multiscale (VMS) method, by using lowest equal order element for example, and solving Navier–Stokes equations with large Reynolds number will be our further work.

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