The influence of time-dependent delay on behavior of stochastic population model with the Allee effect ☆

Miljana Jovanović *, Marija Krstić

University of Niš, Faculty of Science and Mathematics, Višegradska 33, 18000 Niš, Serbia

ABSTRACT

This paper presents the analysis of behavior of stochastic time-dependent delay population model with the Allee effect. We prove the existence-and-uniqueness of positive solution of considered model. Then, we find the sufficient conditions under which the population will become extinct. We also show that if the initial population size exceeds environmental carrying capacity and time delay is sufficiently long, considered population is non-persistent in mean. The sufficient conditions for asymptotical mean square stability and stability in probability of the positive equilibrium states of the model, in terms of Lyapunov functional method, are obtained. Finally, as an illustration, we apply our mathematical results and predict time which a population of the African wild dog *Lycaon pictus* needs to reach it is equilibrium states, and also confirm that population of brown tree snake *Boiga irregularis* is non-persistent in mean if the initial population size is greater than carrying capacity and time delay is long enough.

© 2014 Elsevier Inc. All rights reserved.

1. Introduction and motivation

One of the most interesting phenomena in the nature is that some species often collaborate among themselves in search for food, or on an occasion when they have to defend themselves. Some of the species, as fish and birds, form hunting groups (packs, prides, for instance) and in such way they become more difficult prey for their predators. Likewise, some parasitic species associate themselves so that they can overcome the defence mechanism of the host. It is also well known that ants and bees live in communities in which they have division of labor, altruism, etc. The principle reason of forming such communities is that the individuals have greater chances to survive in the nature if they belong to some group.

The possibility of a positive relationship between capability to survive in the nature and population size was studied by Allee [1]. Allee observed that many animal and plant species suffer a decrease of the per capita growth rate as their populations reach small sizes or low densities. Under such conditions, the growth rate can reach zero, or even negative values, because of a decrease in reproduction and/or survival when prominent individuals are not numerous enough. One of Allee's collaborators, Odum, firstly referred to this process as *Allee principle*, but it is now generally known as the *Allee effect*. Allee demonstrated that a negative density dependence occurs when population growth rate is reduced at low population size. The Allee effect refers to a population that has a maximal per capita growth rate at low density. This occurs when the per capita growth rate increases as density increases, and decreases after the density passes a certain value which is called Allee

threshold. The Allee effect can be caused by difficulties in finding mate, social dysfunction at small population sizes, inbreeding depression, food exploitation, predator avoidance, among the other reasons.

The Allee effect has been the subject of increased interest in the ecological literature, especially in the last two decades. Indeed, prevention of population extinction is one of the priorities in conservation biology because of the fact that populations of small sizes are often subjected to greater risk of extinction. Allee effect also plays a significant role in ecosystem invasions and, most importantly, in biological control (for more details see [2]).

At the same time, in mathematical literature, many population models with the Allee effect have been considered. Some of them are devoted to biological invasions of the alien species into the new habitat (see [3,4], for instance). More precisely, it is well-known that the Allee effect is relevant to many conservation programmes and studies of a range expansion during biological invasions because of the fact that it can result in critical population threshold under which a population becomes extinct. On the other hand, in recent years, the study of the Allee effect has attracted much attention in population dynamics. Former studies demonstrate that the Allee effect may have important dynamical effects on the local stability analysis of population models. It may have either a destabilizing or a stabilizing role in the system (see [5,6]).

All of previously mentioned models are deterministic models. However, population systems are often subjected to environmental noise. Thus, it is useful to investigate how the noise affects them. One of the papers that studies the stochastic population model with the Allee effect is [7]. Let us briefly present the results obtained there. The stochastic differential equation of the form

$$dN(t) = N(t)\left(1 - \frac{N(t)}{T}\right)\left(\frac{N(t)}{K} - 1\right)(r\,dt + \alpha\,dw(t)), \quad t \geqslant 0, \tag{1}$$

$$N(0) = n_0$$

is used to model the stochastic population model with the Allee effect. In (1) $N(t)$ represents the population size at time $t, w = \{w(t), t \geqslant 0\}$ is a one-dimensional standard Brownian motion defined on a complete probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geqslant 0}, P\}$ with a filtration $\{\mathcal{F}_t\}_{t \geqslant 0}$ satisfying the usual conditions (it is right continuous and increasing, while $\mathcal{F}_0$ contains all P-null sets), n_0 is a random variable independent of w such that $0 < n_0 < K$ a.s. Moreover, the constant T is the minimal population size which is needed for population survival and it represents the Allee threshold below which is population extinction; K is the environmental carrying capacity which represents the maximal population size, $0 < T < K$; r is the intrinsic growth rate and α^2 represents the intensity of white noise.

In [7], the authors prove existence and uniqueness of a positive solution of model (1) and investigate stability properties of equilibrium states, $E_0 = 0, E_* = T$ and $E^* = K$. By the Lyapunov function method, they obtain sufficient conditions under which the equilibrium states are asymptotically mean square stable and asymptotically stable in mean. Results of this paper are improved in [8], where authors apply the Feller test to obtain the stability conditions for equilibrium states of model (1). However, since the Feller test is just valid for one-dimensional, time-homogeneous stochastic differential equations, the Lyapunov function and Lyapunov functional methods seem more efficient in high dimensional cases and in cases when stochastic delay models are considered.

Many processes in population biology involve time delay, which represents time lag between causes and their consequences. For example, in population dynamics time delay may represent time which individuals need to arrive into the age of reproductive maturity (*reproductive time lag*), time which they need to react to environmental changes (*reaction time lag*), resource regeneration time, feeding time, etc. Time delay is source of instability in dynamic systems and it may cause population fluctuations. The destabilizing effect of time delay is often expressed by the fact that introduction of time delay which exceeds the dominant time scale of a system makes equilibrium state, which is otherwise stable, to become unstable. If the delay is too long, the population dies out.

Since all species exhibit time delay, at least due to maturation time, we can say that stochastic delay models closely approach reality. Thus, we assume that the average growth rate of Eq. (1) is a function of some specified delayed argument. In regard to the fact that time delay is not resistant to time fluctuations, in this paper we investigate the stochastic time-dependent delay population model with Allee effect

$$dN(t) = rN(t)\left(1 - \frac{N(t - \tau(t))}{T}\right)\left(\frac{N(t - \tau(t))}{K} - 1\right)dt + \alpha N(t)\left(1 - \frac{N(t)}{T}\right)\left(\frac{N(t)}{K} - 1\right)dw(t), \quad t \geqslant 0, \tag{2}$$

where $\tau(t)$ is a nonnegative, bounded, continuously differentiable function on $[0, +\infty)$ satisfying

$$\bar{\tau} = \sup_{t \geq 0} \tau(t)$$

for some constant $\bar{\tau} \geq 0$ and $\tau'(t) = \frac{d\tau(t)}{dt}$. Let $\mathcal{C} = C([-\bar{\tau}, 0]; \mathbb{R}^+)$ be the family of continuous functions φ from $[-\bar{\tau}, 0]$ to $\mathbb{R}^+$ with the norm $\|\varphi\| = \sup_{-\bar{\tau} \leqslant \theta \leqslant 0} |\varphi(\theta)|$ and $\mathcal{D}$ the space of $\mathcal{F}_0$-adapted random variables $\varphi \in \mathcal{C}$. The initial data for Eq. (2) is

$$N_0 = \{\xi(\theta), -\bar{\tau} \leqslant \theta \leq 0\} \in \mathcal{C}. \tag{3}$$

In the papers [7,8] authors suppose that the biomass satisfies $0 < n_0 < K$ a.s., where K is the environmental carrying capacity. However, in some cases number of organisms may exceed the carrying capacity of the habitat as a result of an

increase in birth rate, decline in mortality rate, increase in immigration, or depletion of resources. We use this fact to consider model (2) also for initial data (3) satisfying the condition $N_0 > K$.

In this paper we consider model (2) which describes the dynamics of a single species subjected to the strong Allee effect. This model has the same equilibrium states as the model (1): $E_0 = 0, E_* = T$ and $E^* = K$. In the next section we prove the existence-and-uniqueness of the positive solution of Eq. (2). Section 3 is dedicated to study of the asymptotic mean square stability and stability in probability of the equilibrium states of Eq. (2). We find the sufficient conditions under which the considered population becomes extinct and non-persistent in mean as well as the conditions for stability in probability of the positive equilibrium states. In Section 4 we provide numerical examples for population of African wild dog *Lycaon pictus* and brown tree snake *Boiga irregularis*, to verify our mathematical findings.

2. Positive and global solution

As $N(t)$ of Eq. (2) represents population size at time t, we are only interested in the positive solutions. Moreover, in order for a stochastic differential equation to have a unique global solution (i.e. solution that does not explode in finite time) for any given initial data, the coefficients of stochastic differential equation are generally required to satisfy the linear growth condition and local Lipschitz condition [9]. However, the coefficients of Eq. (2) satisfy local Lipschitz condition, but they do not satisfy the linear growth condition, so the solution of Eq. (2) may explode at a finite time. By the following theorem we establish some conditions under which the solution of Eq. (2) is positive and global.

Theorem 2.1. *If the delay function satisfies the condition*

$$\tau' = \sup_{t \geq 0} \tau'(t) < 1, \tag{4}$$

then there exists a unique positive global solution $N(t)$ to Eq. (2) on $t \geq -\bar{\tau}$, for any given initial data (3).

Proof. Since the coefficients of Eq. (2) are locally Lipschitz continuous, for any given initial data $N_0 \in \mathcal{C}$ there is a unique maximal local solution $N(t)$ on $t \in [-\bar{\tau}, \tau_e)$, where τ_e represents explosion time. To show this solution is global, we need to prove that $\tau_e = \infty$ a.s. Let $k_0 > 0$ be sufficiently large for

$$\frac{1}{k_0} < \min_{-\bar{\tau} \leq \theta \leq 0} |\xi(\theta)| \leq \max_{-\bar{\tau} \leq \theta \leq 0} |\xi(\theta)| < k_0.$$

For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf \left\{ t \in [0, \tau_e) : N(t) \notin \left(\frac{1}{k}, k \right) \right\},$$

where throughout this paper we set $\inf \emptyset = \infty$ (as usual $\emptyset$ denotes the empty set). Clearly, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k \to \infty} \tau_k$. If we can show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ a.s. and $N(t)$ is a positive global solution of Eq. (2). Thus, we only need to show that $\tau_\infty = \infty$ a.s or for all $\bar{T} > 0$ we have $P\{\tau_k \leq \bar{T}\} \to \infty$ when $k \to \infty$.

Let us define a C^2 function $V_1 : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$V_1(N) = \sqrt{N} - 1 - \frac{1}{2} \ln N.$$

Let $k \geq k_0$ and $\bar{T} > 0$ be arbitrary. For $0 \leq t \leq \tau_k \wedge \bar{T}$ we apply the Itô formula to $V_1(N(t))$ and obtain

$$dV_1(N(t)) = LV_1(N(t), N(t - \tau(t))) dt + \alpha \frac{\sqrt{N(t)} - 1}{2TK} (K - N(t))(N(t) - T) dw(t),$$

where

$$LV_1(N, N_1) = r \frac{\sqrt{N} - 1}{2TK} (K - N_1)(N_1 - T) - \alpha^2 \frac{\sqrt{N} - 2}{8T^2 K^2} (K - N)^2 (N - T)^2.$$

Using the elementary inequality $\pm 2uv \leq u^2 + v^2$, we obtain

$$LV_1(N, N_1) \leq \frac{|r|}{4TK} \left(\sqrt{N} - 1 \right)^2 + \frac{|r|}{4TK} (K - N_1)^2 (N_1 - T)^2 - \alpha^2 \frac{\sqrt{N}}{8T^2 K^2} (K - N)^2 (N - T)^2 + \frac{\alpha^2}{4T^2 K^2} (K - N)^2 (N - T)^2.$$

In order to eliminate the terms with delay, we introduce the non-negative functional

$$V_2(N(t)) = \frac{1}{1 - \tau'} \frac{|r|}{4TK} \int_{t - \tau(t)}^{t} (K - N(s))^2 (N(s) - T)^2 ds.$$

Note that, according to the condition $\tau' < 1$, functional V_2 is well defined. Thus,

$$V_1(N(\tau_k \wedge \overline{T})) + V_2(N(\tau_k \wedge \overline{T})) \leqslant V_1(\xi(0)) + V_2(\xi(0)) + \int_0^{\tau_k \wedge \overline{T}} F(N(s))\,ds + \alpha \int_0^{\tau_k \wedge \overline{T}} \frac{\sqrt{N(s)} - 1}{2TK}(K - N(s))(N(s) - T)\,dw(s),$$

where

$$F(N) = \frac{|r|}{4TK}\left(\sqrt{N} - 1\right)^2 + \frac{1}{4TK}\left(\frac{|r|}{1 - \tau'} + \frac{\alpha^2}{TK}\right)(K - N)^2(N - T)^2 - \alpha^2 \frac{\sqrt{N}}{8T^2K^2}(K - N)^2(N - T)^2.$$

It is straightforward to see that $F(N(t))$ is bounded, say by C in $\mathbb{R}^+$. Therefore,

$$V_1(N(\tau_k \wedge \overline{T})) + V_2(N(\tau_k \wedge \overline{T})) \leqslant V_1(\xi(0)) + V_2(\xi(0)) + C(\tau_k \wedge \overline{T}) + \alpha \int_0^{\tau_k \wedge \overline{T}} \frac{\sqrt{N(s)} - 1}{2TK}(K - N(s))(N(s) - T)\,dw(s).$$

Taking expectation in the last inequality, we have

$$EV_1(N(\tau_k \wedge \overline{T})) \leqslant EV_1(N(\tau_k \wedge \overline{T})) + EV_2(N(\tau_k \wedge \overline{T})) \leqslant V_1(\xi(0)) + V_2(\xi(0)) + C\overline{T}. \tag{5}$$

For every $\omega \in \{\tau_k \leqslant \overline{T}\}$, $N(\tau_k, \omega) \notin (k^{-1}, k)$. Thus,

$$V_1(N(\tau_k)) = \sqrt{N(\tau_k)} - 1 - \frac{1}{2}\ln N(\tau_k) = \left(\sqrt{k} - 1 - \frac{1}{2}\ln k\right) \wedge \left(\sqrt{\frac{1}{k}} - 1 + \frac{1}{2}\ln k\right)$$

and, regarding to (5), we find that

$$\infty > V_1(N(0)) + V_2(N(0)) + C\overline{T} \geqslant EV_1(N(\tau_k \wedge \overline{T})) = P\{\tau_k \leqslant \overline{T}\}V_1(N(\tau_k)) + P\{\tau_k > \overline{T}\}V_1(N(\overline{T})) \geqslant P\{\tau_k \leqslant \overline{T}\}V_1(N(\tau_k))$$

$$= P\{\tau_k \leqslant \overline{T}\}\left[\left(\sqrt{k} - 1 - \frac{1}{2}\ln k\right) \wedge \left(\sqrt{\frac{1}{k}} - 1 + \frac{1}{2}\ln k\right)\right].$$

Since $\left(\sqrt{k} - 1 - \frac{1}{2}\ln k\right) \wedge \left(\sqrt{\frac{1}{k}} - 1 + \frac{1}{2}\ln k\right)$ tends to infinity when $k \to \infty$, it follows that $\lim_{k \to \infty} P\{\tau_k \leqslant \overline{T}\} = 0$. Because of the fact that $\overline{T} > 0$ is arbitrary, it follows that $P\{\tau_\infty < \infty\} = 0$ and $P\{\tau_\infty = \infty\} = 1$ which completes the proof. $\square$

3. Stability analysis

Stability analysis is a very important research topic in many areas including population dynamics. Importance of stability analysis lies in the fact that stochastic differential equations which are used to model population's growth, in major of cases, are not effectively solvable. Thus, it is important to investigate the long time behavior of considered population model near to equilibrium states, and stability analysis enables us such investigation. In the literature, there are many papers about stability analysis of stochastic population models (see [7,8,10,11], for example).

In this section we discuss extinction and non-persistence in mean of model (2) proposed by Hallam and Ma [12,13] for some deterministic models and Liu and Wang (see [14,15], for example) for some stochastic logistic models. We also find the sufficient conditions for stability of equilibrium states E_0, E_* and E^* of Eq. (2). For this purpose, we use well known method based on construction of the appropriate Lyapunov functionals. The general method of Lyapunov functionals construction was proposed and developed by Kolmanovskii and Shaikhet (see [16–23], for instance) for stochastic functional differential equations, stochastic difference equations with discrete and continuous time and for neutral stochastic differential equations.

3.1. Extinction and non-persistence in mean

A major problem in population biology is to understand what determines the risk of extinction of population. It is well known that the risk of extinction is greater for populations consisting of a few individuals than for those having many. Species subjected to a strong Allee effect might be more susceptible to catastrophic population collapses with only a slight increase in mortality, resulting either from harvesting or natural causes. In fisheries, for example, the existence of multiple equilibria has been recognized, and the existence of a critical threshold for harvested populations has been advanced as a highly plausible explanation for the collapse of fisheries in several parts of the world [2]. The higher the critical threshold, the greater is the Allee effect and the less effort might be needed to extinct population.

The dynamics of model (2) is unusual in a sense that the convergence to the equilibrium states depends on the size of a delay. As it might be expected, if the delay is too long, the population may not become extinct, but the size of that population can be close to zero so that the population can be endangered. This means that there exists a threshold between extinction and survival of population denoted as non-persistence in mean.

Before presenting stability conditions of the equilibrium state $E_0 = 0$ of model (2), we need some definitions of stability theory (see [24]).

Consider the d-dimensional stochastic functional differential equation

$$dy(t) = f(t, y_t)dt + g(t, y_t)dw(t), \geqslant 0,$$
$$y_0 = \varphi = \{\varphi(\theta) : -\tau \leqslant \theta \leqslant 0\}, \tag{6}$$

where $y_t = \{y(t + \theta) : -\tau \leqslant \theta \leqslant 0\}$ is a $\mathcal{C}$-valued stochastic process and $y_0 \in \mathcal{D}$, such that $E\|\varphi\|^2 < \infty$, while $f(t, \varphi)$ is a d-dimensional vector and $g(t, \varphi)$ is a $d \times m$-dimensional matrix, both defined for $t \geqslant 0$. We assume that Eq. (6) has a unique global solution $y(t; \varphi)$, as well as that $f(t, 0) = g(t, 0) \equiv 0$. So, Eq. (6) has the trivial solution $y(t) \equiv 0$ corresponding to the initial condition $y_0 = 0$.

Definition 3.1. The trivial solution of Eq. (6) is said to be stochastically stable (stable in probability) if for every $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, 0) > 0$ such that

$$P\{|y(t; \varphi)| > r, t \geqslant 0\} \leqslant \varepsilon$$

for any initial condition $\varphi \in \mathcal{D}$ satisfying $P\{\|\varphi\| \leqslant \delta\} = 1$.

Definition 3.2. The trivial solution of Eq. (6) is said to be mean square stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $E|y(t; \varphi)|^2 < \varepsilon$ for any $t \geqslant 0$ provided that $\sup_{-\tau \leqslant \theta \leqslant 0} E|\varphi(\theta)|^2 < \delta$.

Definition 3.3. The trivial solution of Eq. (6) is said to be asymptotically mean square stable if it is mean square stable and $\lim_{t \to \infty} E|y(t; \varphi)|^2 = 0$.

Let us consider stability of equilibrium state $E_0 = 0$. The linear part of Eq. (2) is given by

$$d\widetilde{N}(t) = -\widetilde{N}(t)(r\, dt + \alpha\, dw(t)). \tag{7}$$

It is well known that the condition $r > \frac{\alpha^2}{2}$ is the necessary and sufficient condition for asymptotic mean square stability of the trivial solution of Eq. (7) (see Example 7.1 in [21]). Thus, we have just proved the following.

Lemma 3.1. *Let the parameters of model* (2) *satisfy conditions* (4) *and* $r > \frac{\alpha^2}{2}$ *for any given initial data* (3), *such that* $N_0 < T$. *Then, the trivial solution of Eq.* (7) *is asymptotically mean square stable.*

Note that Eq. (2) has the order of nonlinearity more than one. From [18,19] it follows that if the order of nonlinearity of the equation under consideration is more than one then the conditions which are sufficient for asymptotic mean square stability of the trivial solution of the linear part of this equation, are sufficient for stability in probability of the trivial solution of the original equation. Thus, by the following theorem we obtain stability in probability of trivial equilibrium state E_0 under the same conditions as in the previous lemma.

Corollary 3.1. *If the conditions of Lemma* 3.1 *are satisfied, then, the trivial solution of Eq.* (2) *is stable in probability.*

We need to introduce a condition when the population is bare, i.e. we define the critical number between extinction and persistence of population.

Definition 3.4. The solution of Eq. (6) is said to be non-persistent in mean if

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t y(s)\, ds = 0 \quad \text{a.s.}$$

If a delay in a population is too long and the initial data exceeds carrying capacity, then the considered population declines because its environment can not support the excess number any longer. In many situations this can happen very rapidly because of excessive demand degrades or even devastates the environment and there is a sudden catastrophic feedback effect. Such a feedback effect can only eradicate those numbers of population in excess of the carrying capacity of an environment but under certain circumstances it can cause the near extinction of an entire species [25].

In the sequel of this section we will use the exponential martingale inequality.

Lemma 3.2 [9]. *Let* $g : [0, +\infty) \to \mathbb{R}^{1 \times m}$ *be* $\mathcal{F}_t$-adapted process such that for any $T > 0, \int_0^T |g(t)|^2 dt < \infty$ a.s. Then for any constants $\alpha, \beta > 0$

$$P\left\{\sup_{t \in [0, T]}\left[\int_0^t g(s)dw(s) - \frac{\alpha}{2}\int_0^t |g(s)|^2 ds\right] > \beta\right\} \leqslant e^{-\alpha\beta}. \tag{8}$$

In the following theorem we use notation

$$M = \inf_{-\tau \leqslant \theta \leqslant 0} \xi(\theta).$$

Theorem 3.1.

(a) *Let any of the conditions below be satisfied:*
  (i) *let the parameters of model* (2) *satisfy conditions* (4), $r > 0$ *and*

$$\bar{\tau} > \frac{(1 - \tau')TK}{r\left[(1 - \tau')M^2 - (T + K)\|\xi\|\right]} \ln \frac{T + K}{TK} \frac{\xi(0)}{1 - \tau'} \tag{9}$$

for any given initial data (3), *such that* $\frac{T+K}{1-\tau'} < N_0 < (1 - \tau')\frac{M^2}{T+K}$;
  (ii) *let the parameters of model* (2) *satisfy conditions* (4), $T > \frac{1}{1-\tau'}$, $r > 0$ *and*

$$\bar{\tau} < \frac{(1 - \tau')TK}{r\left[(T + K)\|\xi\| - (1 - \tau')\left[\|\xi\|^2 - (T + K)(\|\xi\| - M)\right]\right]} \ln \frac{TK}{T + K} \frac{1 - \tau'}{\xi(0)} \tag{10}$$

for any given initial data (3), *such that* $0 < N_0 < T$ *and* $\xi(0) < (1 - \tau')\frac{TK}{T+K}$.
Then, the solution $N(t)$ *of Eq.* (2) *is non-persistent in mean.*
  (b) *In particular, let* $\tau(t) \equiv \tau$, *for all* $t \geqslant 0$, *and let any of the following conditions be satisfied:*
  (i) *let the parameters of model* (2) *satisfy conditions* $r > 0$ *and*

$$\tau > \frac{TK}{rM(M - T - K)} \ln \frac{T + K}{TK} \xi(0) \tag{11}$$

for any given initial data (3), *such that* $N_0 > T + K$;
  (ii) *let the parameters of model* (2) *satisfy conditions* $r > 0$ *and*

$$\tau < \frac{TK}{r\|\xi\|(T + K - \|\xi\|)} \ln \frac{TK}{\xi(0)(T + K)} \tag{12}$$

for any given initial data (3), *such that* $0 < N_0 < T$ *and* $\xi(0) < \frac{TK}{T+K}$.

Then, the solution $N(t)$ *of Eq.* (2) *is non-persistent in mean.*

Proof.

(a) Applying the Itô formula to $\ln N(t) + \frac{r}{1-\tau'} \frac{T+K}{TK} \int_{t-\tau(t)}^{t} N(s)\,ds$, we have

$$d\left(\ln N(t) + \frac{r}{1 - \tau'} \frac{T + K}{TK} \int_{t-\tau(t)}^{t} N(s)ds \right) = \left[r\left(1 - \frac{N(t - \tau(t))}{T}\right)\left(\frac{N(t - \tau(t))}{K} - 1\right) - \frac{\alpha^2}{2}\left(1 - \frac{N(t)}{T}\right)^2\left(\frac{N(t)}{K} - 1\right)^2 \right.$$
$$\left. + \frac{r}{1 - \tau'} \frac{T + K}{TK} N(t) - r\frac{1 - \tau'(t)}{1 - \tau'} \frac{T + K}{TK} N(t - \tau(t)) \right] dt$$
$$+ \alpha\left(1 - \frac{N(t)}{T}\right)\left(\frac{N(t)}{K} - 1\right) dw(t).$$

Then

$$\ln N(t) + \frac{r}{1 - \tau'} \frac{T + K}{TK} \int_{t-\tau(t)}^{t} N(s)ds \leqslant \ln \xi(0) + \frac{r}{1 - \tau'} \frac{T + K}{TK} \int_{-\bar{\tau}}^{0} \xi(\theta)d\theta + M(t)$$
$$+ \int_{0}^{t} \left[r\left(1 - \frac{N(s - \tau(s))}{T}\right)\left(\frac{N(s - \tau(s))}{K} - 1\right) - \frac{\alpha^2}{2}\left(1 - \frac{N(s)}{T}\right)^2\left(\frac{N(s)}{K} - 1\right)^2 \right] ds$$
$$+ \frac{r}{1 - \tau'} \frac{T + K}{TK} \int_{0}^{t} N(s)ds - r\frac{T + K}{TK} \int_{0}^{t} N(s - \tau(s))ds, \tag{13}$$

where $M(t) = \alpha \int_{0}^{t} \left(1 - \frac{N(s)}{T}\right)\left(\frac{N(s)}{K} - 1\right) dw(s)$ is a real-valued continuous local martingale vanishing at $t = 0$. For every integer $n \geqslant 1$, by exponential martingale inequality (8), we get

$$P\left\{ \sup_{0 \leqslant t \leqslant n} \left[M(t) - \frac{\alpha^2}{2} \int_{0}^{t} \left(1 - \frac{N(s)}{T}\right)^2 \left(\frac{N(s)}{K} - 1\right)^2 ds \right] > 2\ln n \right\} \leqslant \frac{1}{n^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, using Borel–Cantelli lemma yields that there exists an $\Omega_0 \subset \Omega$ such that $P(\Omega_0) = 1$ and for any $\omega \in \Omega_0$ there exists an integer $n_0(\omega)$, when $n \geqslant n_0(\omega)$ and $t \in [n - 1, n]$. Then

$$M(t) \leqslant 2\ln n + \frac{\alpha^2}{2} \int_0^t \left(1 - \frac{N(s)}{T}\right)^2 \left(\frac{N(s)}{K} - 1\right)^2 ds.$$

That is

$$\ln N(t) \leqslant \ln N(t) + \frac{r}{1-\tau'} \frac{T+K}{TK} \int_{t-\tau(t)}^t N(s)ds$$

$$\leqslant \ln \xi(0) + \frac{r}{1-\tau'} \frac{T+K}{TK} \|\xi\|\bar{\tau} + r \int_0^t \left(1 - \frac{N(s-\tau(s))}{T}\right)\left(\frac{N(s-\tau(s))}{K} - 1\right)ds + 2\ln n + r$$

$$\times \frac{T+K}{TK} \left(\frac{1}{1-\tau'} \int_0^t N(s)ds - \int_0^t N(s-\tau(s))ds\right).$$

Denote by $\psi(t) = t - \tau(t)$, for every $t \geqslant 0$. From condition (4) we conclude that $\psi(t)$ is monotonically increasing function. Then there exists $t_1 > 0$ such that $t_1 - \tau(t_1) = 0$. It follows that

$$\ln N(t) \leqslant \ln \xi(0) + \frac{r}{1-\tau'} \frac{T+K}{TK} \|\xi\|\bar{\tau} + r \int_{-\bar{\tau}}^0 \left(1 - \frac{\xi(\theta)}{T}\right)\left(\frac{\xi(\theta)}{K} - 1\right)d\theta + r \int_{t_1}^t \left(1 - \frac{N(s-\tau(s))}{T}\right)\left(\frac{N(s-\tau(s))}{K} - 1\right)ds + 2$$

$$\times \ln n + r\frac{T+K}{TK} \left(\frac{1}{1-\tau'} \int_0^t N(s)\,ds - \int_{-\bar{\tau}}^0 \xi(\theta)d\theta - \int_{t_1}^t N(s-\tau(s))ds\right)$$

for $t \in [n-1, n]$ and $n \geqslant n_0(\omega)$. For sufficiently large $t \in (n-1, n)$ we have $\frac{2\ln n}{n-1} \leqslant \epsilon$, which implies that $2\ln n \leqslant \epsilon t$.

(i) For any given initial data (3), such that $M = \inf_{-\tau \leqslant \theta \leqslant 0} \xi(\theta) > \frac{T+K}{1-\tau'}$ and $\|\xi\| < (1-\tau')\frac{M^2}{T+K}$, we obtain that $\left(1 - \frac{\xi(\theta)}{T}\right)\left(\frac{\xi(\theta)}{K} - 1\right) \leqslant -\frac{(M-T)(M-K)}{TK}$. Then

$$\ln N(t) \leqslant \ln \xi(0) + r\left(\frac{1}{1-\tau'} \frac{T+K}{TK} \|\xi\| - \frac{(M-T)(M-K)}{TK} - \frac{T+K}{TK}M\right)\bar{\tau} + r\frac{T+K}{TK} \int_{t_1}^t N(s-\tau(s))\,ds - r(t-\tau(t_1)) + \epsilon t$$

$$+ \frac{r}{1-\tau'} \frac{T+K}{TK} \int_0^t N(s)\,ds - r\frac{T+K}{TK} \int_{t_1}^t N(s-\tau(s))\,ds$$

$$\leqslant \ln \xi(0) + r\left(\frac{1}{1-\tau'} \frac{T+K}{TK} \|\xi\| - \frac{M^2+TK}{TK} + 1\right)\bar{\tau} + \frac{r}{1-\tau'} \frac{T+K}{TK} \int_0^t N(s)\,ds - (r-\epsilon)t.$$

Let $f(t) = \int_0^t N(s)ds$. Then, the previous inequality becomes

$$\ln \frac{df(t)}{dt} - \frac{r}{1-\tau'} \frac{T+K}{TK} f(t) \leqslant \ln \xi(0) + r\left(\frac{1}{1-\tau'} \frac{T+K}{TK} \|\xi\| - \frac{M^2}{TK}\right)\bar{\tau} - (r-\epsilon)t.$$

Denote that $A = \left(\frac{1}{1-\tau'} \frac{T+K}{TK} \|\xi\| - \frac{M^2}{TK}\right)r\bar{\tau}$. Thus, we have

$$e^{-\frac{r}{1-\tau'} \frac{T+K}{TK} f(t)}df(t) \leqslant \xi(0)e^{A-(r-\epsilon)t}dt.$$

Integrating both sides of the last inequality from 0 to t, we obtain

$$\int_0^t e^{-\frac{r}{1-\tau'} \frac{T+K}{TK} f(s)}df(s) \leqslant \xi(0)e^A \int_0^t e^{-(r-\epsilon)s}ds.$$

Hence,

$$\frac{1-\tau'}{r} \frac{TK}{T+K} \left(1 - e^{-\frac{r}{1-\tau'} \frac{T+K}{TK} f(t)}\right) \leqslant \xi(0)e^A \frac{1}{r-\epsilon} \left(1 - e^{-(r-\epsilon)t}\right),$$

that is,

$$e^{-\frac{r}{1-\tau'} \frac{T+K}{TK} f(t)} \geqslant 1 - \frac{r}{r-\epsilon} \frac{T+K}{TK} \frac{\xi(0)}{1-\tau'} e^A \left(1 - e^{-(r-\epsilon)t}\right).$$

Then, we deduce that

$$f(t) \leqslant -\frac{1-\tau'}{r} \frac{TK}{T+K} \ln\left[1 - \frac{r}{r-\epsilon} \frac{T+K}{TK} \frac{\xi(0)}{1-\tau'} e^A \left(1 - e^{-(r-\epsilon)t}\right)\right] \leqslant -\frac{1-\tau'}{r} \frac{TK}{T+K} \ln\left[1 - \frac{T+K}{TK} \frac{\xi(0)}{1-\tau'} e^A \left(1 - e^{-(r-\epsilon)t}\right)\right].$$

Let us note that ln function in the last inequality is well defined and it has negative value. Really, from condition (9), we have that

$$A = \left(\frac{1}{1-\tau'} \frac{T+K}{TK} \|\xi\| - \frac{M^2}{TK}\right)r\bar{\tau} < \ln\left(\frac{TK}{T+K} \frac{1-\tau'}{\xi(0)}\right),$$

so that $\frac{\xi(0)}{1-\tau'}\frac{T+K}{TK}e^A < 1$, and therefore $\frac{T+K}{TK}\frac{\xi(0)}{1-\tau'}e^A\left(1 - e^{-(r-\epsilon)t}\right) < 1$. The proof of this part of theorem follows straightforwardly because

$$\limsup_{t\to\infty}\frac{1}{t}f(t) = \limsup_{t\to\infty}\frac{1}{t}\int_0^t N(s)ds \leqslant \limsup_{t\to\infty}\frac{1}{t}\left\{-\frac{1-\tau'}{r}\frac{TK}{T+K}\ln\left[1 - \frac{\xi(0)}{1-\tau'}\frac{T+K}{TK}e^A\left(1 - e^{-(r-\epsilon)t}\right)\right]\right\} = 0.$$

(ii) For any given initial data (3), such that $0 < N_0 < T$ and $\xi(0) < (1-\tau')\frac{TK}{T+K}$, we get $\left(1 - \frac{\xi(0)}{T}\right)\left(\frac{\xi(0)}{K} - 1\right) \leqslant -\frac{(T-\|\xi\|)(K-\|\xi\|)}{TK}$. Then we have

$$\ln N(t) \leqslant \ln\xi(0) + r\left(\frac{1}{1-\tau'}\frac{T+K}{TK}\|\xi\| - \frac{\|\xi\|^2 - (T+K)(\|\xi\| - M)}{TK}\right)\bar\tau + \frac{r}{1-\tau'}\frac{T+K}{TK}\int_0^t N(s)\,ds - (r-\epsilon)t.$$

Denote that $f(t) = \int_0^t N(s)\,ds$. By repeating completely the previous procedure, we find that

$$f(t) \leqslant -\frac{1-\tau'}{r}\frac{TK}{T+K}\ln\left[1 - \frac{T+K}{TK}\frac{\xi(0)}{1-\tau'}e^B\left(1 - e^{-(r-\epsilon)t}\right)\right],$$

where $B = \left(\frac{1}{1-\tau'}\frac{T+K}{TK}\|\xi\| - \frac{\|\xi\|^2 - (T+K)(\|\xi\|-M)}{TK}\right)r\bar\tau$. By using condition (10), we obtain $B < \ln\left(\frac{TK}{T+K}\frac{1-\tau'}{\xi(0)}\right)$, so that $\frac{TK}{T+K}\frac{1-\tau'}{\xi(0)}e^B\left(1 - e^{-(r-\epsilon)t}\right) < 1$. Similarly to the proof of part (i), we easily get $\limsup_{t\to\infty}\frac{1}{t}\int_0^t N(s)ds = 0$, which proves part (ii) of the assertion (a).

(b) Specially, if $\tau(t) \equiv \tau$, for all $t \geqslant 0$, applying the Itô formula to $\ln N(t)$ and repeating the previous procedure, we prove the theorem. □

Remark 3.1. From the conditions (9)–(12) of Theorem 3.1, we can conclude that the stochastic noise α has no impact on the non-persistence in mean of the population.

3.2. Stability of the positive equilibrium states

In the sequel we regard Eq. (2) as a neutral stochastic differential equation. Before presenting stability conditions of the positive equilibrium states E_* and E^* of Eq. (2), let us formulate some assertions for Itô neutral stochastic differential equations (see [24], for instance).

Let us consider the following d-dimensional neutral type stochastic functional differential equation

$$d(x(t) - G(t,x_t)) = f(t,x_t)dt + g(t,x_t)dw(t), t \geqslant 0,$$
$$x_0 = \varphi = \{\varphi(\theta) : -\tau \leqslant \theta \leqslant 0\},$$
(14)

where $x_t = \{x(t+\theta) : -\tau \leqslant \theta \leqslant 0\}$ is regarded as a $\mathcal{C}$-valued stochastic process, $w(t)$ be the m-dimensional Brownian motion, the d-dimensional vector $f(t,\varphi)$ and $d \times m$-dimensional matrix $g(t,\varphi)$ are defined for $t \geqslant 0, \varphi \in \mathcal{D}, f(t,0) = 0, g(t,0) = 0$, and

$$|G(t,\varphi)| \leqslant \int_0^\tau |\varphi(-s)|dK(s), \quad \int_0^\tau dK(s) < 1.$$
(15)

Let us note that the Definitions 3.1, 3.2, 3.3 can be extended to Eq. (14).

The differential operator associated to Eq. (14) is defined by the formula

$$LV(t,\varphi) = \limsup_{\Delta\to 0}\frac{\mathbf{E}_{t,\varphi}V(t+\Delta,x_{t+\Delta}) - V(t,\varphi)}{\Delta},$$

where $x(s), s \geqslant t$ is the solution of Eq. (14) satisfying the initial condition $x_t = \varphi$, and $V(t,\varphi)$ is a functional defined for $t \geqslant 0$ and for functions $\varphi \in \mathcal{D}$.

Let us reduce a class of functionals $V(t,\varphi)$ so that the operator L can be calculated. First, for $t \geqslant 0$ and function $\varphi \in \mathcal{D}$, let $V(t,\varphi) = V(t,\varphi(0),\varphi(\theta))$, $-\tau \leqslant \theta \leqslant 0$. Then, we define the function

$$V_\varphi(t,x) = V(t,\varphi) = V(t,x_t) = V(t,x,x(t+\theta)), \quad -\tau \leqslant \theta \leqslant 0,$$

where $\varphi = x_t, x = \varphi(0) = x(t)$.

Let us denote that $C_{1,2}$ is a class of functionals $V(t,\varphi)$ so that, for almost all $t \geqslant 0$, the first and second derivatives with respect to x of $V_\varphi(t,x)$ are continuous, and the first derivative with respect to t is continuous and bounded. Then, the application of the generating operator L of Eq. (14) yields

$$LV(t,x_t) = \frac{\partial V_\varphi(t,x)}{\partial t} + f^T(t,x_t)\frac{\partial V_\varphi(t,x)}{\partial x} + \frac{1}{2}trace\left[g^T(t,x_t)\frac{\partial^2 V_\varphi(t,x)}{\partial x^2}g(t,x_t)\right].$$

In the sequel we state the theorem that contains sufficient conditions for asymptotic mean square stability of the trivial solution of Eq. (14) in terms of Lyapunov functionals [24].

Theorem 3.2. *Let condition* (15) *hold and there exist the functional*

$$V(t, \varphi) = W(t, \varphi) + |\varphi(0) - G(t, \varphi)|^2,$$

such that $V(t, \varphi) \in C_{1,2}$,

$$0 \leqslant EW(t, x_t) \leqslant K_1 \sup_{-\tau \leqslant \theta \leqslant 0} E|x(t + \theta)|^2,$$

$$ELV(t, x_t) \leqslant -K_2 E|x(t)|^2$$

for $K_i > 0$, $i = 1, 2$. *Then the trivial solution of Eq.* (14) *is asymptotically mean square stable.*

In [26] the author considers stability of different types of stochastic differential equations with time dependent delay via Lyapunov functional method, but not especially for linear stochastic differential equation with time-dependent delay of the form

$$dx(t) = -bx(t - \tau(t)) + \sigma x(t) dw_t \tag{16}$$

for $b > 0$, $\sigma \in \mathbb{R}$, and with initial data

$$x_0 = \{\xi(\theta), -\bar{\tau} \leqslant \theta \leqslant 0\}. \tag{17}$$

In this section, our aim is to give sufficient conditions for stability in probability of positive equilibrium states $E_* = T$ and $E^* = K$ to Eq. (2), by applying a sufficient conditions that provide asymptotic mean square stability of the linear part of this nonlinear system, which is of the form (16).

Theorem 3.3. *Let the parameters of Eq.* (16) *satisfy the conditions* (4),

$$-\infty < \underline{\tau}' = \inf_{t \geqslant 0} \tau'(t) \tag{18}$$

and

$$b > \max\left\{0, \frac{1 - \tau'}{2(1 - \tau') - T'(2 - \tau')} \sigma^2\right\} \tag{19}$$

$$\bar{\tau} < \frac{2b(1 - \tau') - bT'(2 - \tau') - \sigma^2(1 - \tau')}{b^2(2 - \tau' + 2T')} \tag{20}$$

for any initial data (17), *where* $\max\{|\underline{\tau}'|, \tau'\} = T'$. *Then, the trivial solution of Eq.* (16) *is asymptotically mean square stable.*

Proof. Let us consider the neutral form of Eq. (16), that is

$$\frac{d}{dt}\left[x(t) - b \int_{t-\tau(t)}^{t} x(s) ds\right] = -bx(t) - b\tau'(t)x(t - \tau(t)) + \sigma x(t)\dot{w}(t). \tag{21}$$

Define the Lyapunov functional $V = V_1 + V_2$, where

$$V_1(x(t)) = \left(x(t) - b \int_{t-\tau(t)}^{t} x(s) ds\right)^2$$

and V_2 will be chosen later. If we apply the generating operator L on V_1, use the elementary inequality $\pm 2uv \leqslant u^2 + v^2$ and $|\tau'(t)| \leqslant T'$, we find that

$$LV_1(x(t), x(t - \tau(t))) = -2bx^2(t) - 2b\tau'(t)x(t)x(t - \tau(t)) + 2b^2 \int_{t-\tau(t)}^{t} x(t)x(s) ds + 2b^2\tau'(t) \int_{t-\tau(t)}^{t} x(t - \tau(t))x(s) ds + \sigma^2 x^2(t)$$

$$\leqslant -2bx^2(t) + b|\tau'(t)|(x^2(t) + x^2(t - \tau(t))) + b^2\left(\tau(t)x^2(t) + \int_{t-\tau(t)}^{t} x^2(s) ds\right)$$

$$+ b^2|\tau'(t)|\left(\tau(t)x^2(t - \tau(t)) + \int_{t-\tau(t)}^{t} x^2(s) ds\right) + \sigma^2 x^2(t)$$

$$= b\left[-2 + |\tau'(t)| + b\tau(t) + \frac{\sigma^2}{b}\right]x^2(t) + b|\tau'(t)|(1 + b\tau(t))x^2(t - \tau(t)) + b^2(1 + |\tau'(t)|)\int_{t-\tau(t)}^{t} x^2(s) ds$$

$$\leqslant b\left[-2 + T' + b\bar{\tau} + \frac{\sigma^2}{b}\right]x^2(t) + bT'(1 + b\bar{\tau})x^2(t - \tau(t)) + b^2(1 + T')\int_{t-\tau(t)}^{t} x^2(s) ds.$$

We choose the functional V_2 to eliminate the terms with delay

$$V_2(x(t)) = \frac{bT'(1+b\bar{\tau})}{1-\tau'} \int_{t-\tau(t)}^t x^2(s)ds + \frac{b^2(1+T')}{1-\tau'} \int_{t-\tau(t)}^t (s-t+\tau(t))x^2(s)ds$$

and for Lyapunov functional V we obtain

$$LV(x(t), x(t-\tau(t))) \leqslant -b\left[2 - \frac{2-\tau'}{1-\tau'}T' - \frac{\sigma^2}{b} - \bar{\tau}\frac{b(2-\tau'+2T')}{1-\tau'}\right]x^2(t).$$

Providing that conditions (19) and (20) hold, the quantity in the bracket is positive. Hence, the trivial solution of Eq. (16) is asymptotically mean square stable by virtue of Theorem 3.2. $\square$

As it is shown in [18,19], if the order of nonlinearity of the system under consideration is more than one then a sufficient condition for asymptotic mean square stability of the linear part of the initial nonlinear system is also a sufficient condition for stability in probability of the initial system. Thus, we can prove the following corollaries.

Corollary 3.2. *Let the parameters of model* (2) *satisfy the conditions* (4), (18) *and*

$$r > \max\left\{0, \frac{K-T}{T}\frac{1-\tau'}{2(1-\tau')-T'(2-\tau')}\alpha^2\right\} \tag{22}$$

$$\bar{\tau} < \frac{2rT(1-\tau')-rTT'(2-\tau')-\alpha^2(K-T)(1-\tau')}{r^2(K-T)(2-\tau'+2T')}, \tag{23}$$

for any initial data (3), *such that* $T < N_0 < K$ *or* $N_0 > K$. *Then, the positive equilibrium state* $E^* = K$ *of Eq.* (2) *is stable in probability.*

Proof. In order to investigate stability properties of the equilibrium state $E^* = K$, let us make the change of variable $y = K - N$. Thus, Eq. (2) becomes

$$dy(t) = -ry(t-\tau(t))\frac{(K-y(t))(K-T-y(t-\tau(t)))}{TK}dt - \alpha y(t)\frac{(K-y(t))(K-T-y(t))}{TK}dw(t), \tag{24}$$

with initial data

$$y_0 = \{K - \xi(\theta), -\bar{\tau} \leqslant \theta \leqslant 0\}. \tag{25}$$

Obviously, the stability of equilibrium state $E^* = K$ of Eq. (2) is equivalent to the stability of trivial solution of Eq. (24). The linear part of Eq. (24) is given by

$$d\tilde{y}(t) = -\frac{K-T}{T}r\tilde{y}(t-\tau(t))dt - \frac{K-T}{T}\alpha\tilde{y}(t)dw(t). \tag{26}$$

Eq. (26) has the form (16). Hence, from Theorem 3.3 we conclude that trivial solution of this equation is asymptotically mean square stable under the conditions (4), (18), (22) and (23). Since the order of nonlinearity of Eq. (24) is more than one, from the previous discussion we have that the trivial solution of Eq. (24) is stable in probability. $\square$

The following corollary gives sufficient conditions under which the positive equilibrium state $E_* = T$ of Eq. (2) is stable in probability.

Corollary 3.3. *Let the parameters of model* (2) *satisfy the conditions* (4), (18) *and*

$$r < \min\left\{0, -\frac{K-T}{K}\frac{1-\tau'}{2(1-\tau')-T'(2-\tau')}\alpha^2\right\} \tag{27}$$

$$\bar{\tau} < \frac{-2rK(1-\tau')+rKT'(2-\tau')-\alpha^2(K-T)(1-\tau')}{r^2(K-T)(2-\tau'+2T')} \tag{28}$$

for any given initial data (3), *such that* $N_0 < T$ *or* $T < N_0 < K$. *Then, the positive equilibrium state* $E_* = T$ *of Eq.* (2) *is stable in probability.*

Proof. Similarly to the proof of Theorem 3.2, we introduce the new variable $x = N - T$ and transform Eq. (2) into

$$dx(t) = rx(t-\tau(t))\frac{(T+x(t))(K-T-x(t-\tau(t)))}{TK}dt + \alpha x(t)\frac{(T+x(t))(K-T-x(t))}{TK}dw(t), \tag{29}$$

with initial data

$$x_0 = \{\xi(\theta) - T, -\bar{\tau} \leqslant \theta \leqslant 0\}. \tag{30}$$

Since the stability of equilibrium state $E_* = T$ of Eq. (2) is equivalent to the stability of trivial solution of Eq. (29), we consider the linear part of Eq. (29), given by

$$d\tilde{x}(t) = \frac{K-T}{K} r\tilde{x}(t - \tau(t)) dt + \frac{K-T}{K} \alpha \tilde{x}(t) dw(t). \tag{31}$$

This equation is equation of type (16) for $b = -\frac{K-T}{K}r$ and $\sigma = \frac{K-T}{K}\alpha$. Thus the proof of stability in probability of equilibrium state $E_* = T$ of Eq. (2) follows straightforwardly from Theorem 3.3 and the discussion after it. □

4. Real world examples and numerical simulation

In order to verify mathematical results obtained in previous sections, we try to put our theoretical research into the context of real-world settings. With this aim, we consider the discretization of Eqs. (1), (2) their deterministic analogues, according to the Euler–Maruyama approximate method (see [27]), in the form

$$x_i = \xi(i\Delta), i = -m, \dots, -1, 0,$$

$$x_i = x_{i-1} + x_{i-1}\left(r\left(1 - \frac{x_{i-1-\left[\frac{\tau((i-1)\Delta)}{\Delta}\right]}}{T}\right)\left(\frac{x_{i-1-\left[\frac{\tau((i-1)\Delta)}{\Delta}\right]}}{K} - 1\right)\Delta + \alpha\left(1 - \frac{x_{i-1}}{T}\right)\left(\frac{x_{i-1}}{K} - 1\right)\sqrt{\Delta}\,\phi_{i-1}\right), \quad i = 1, \dots, n,$$

where $\Delta = 1/365$ years; $m = \bar{\tau}/\Delta$; $\phi_i, i = 1, \dots, n$ are the Gaussian random variables $\mathcal{N}(0,1)$; $[x]$ gives the greatest integer less than or equal to x.[1]

Example 1. The African wild dog (*Lycaon pictus*) is one of the most endangered large carnivores in Africa. At the beginning of the 20th century, this species was observed in large numbers and was distributed over most of the African continent. Despite current legal protection, its survival is not guaranteed. In the literature many causes have been proposed to explain wild dogs current decline, such as human persecution, diseases, habitat fragmentation, competition with other predators, etc. However, this does not explain why sympatric species that suffer from similar anthropogenic pressures are not as endangered as wild dogs. There is another factor which would render this species more sensitive to other mortality factor, the existence of an Allee effect. More preciously, the hunting strategy of the group usually requires a threshold group size to be successful because of kleptoparasitism by hyenas, which can be very costly to small groups of wild dogs. In addition, helpers are required by the breeding female: litters are very large (up to 20 pups), and the breeding female, then the pups, need to be feed by other members of the group. Group members also help by chasing predators from the den area, and by staying at the den to protect the pups while the pack is hunting. Consequently, a critical number of helpers might be needed for wild dog groups to survive. Results suggest that habitat fragmentation and destruction, as well as increased human pressure, increase the effects of the Allee effect. The Allee effect at the pack level (with a critical number of individuals), generates on Allee effect at the population level (with a critical number of packs) ([2,28]).

African wild dog are always found at lower population densities, compared to sympatric large carnivores. Consequently, most populations of wild dogs are small and only a handful exceed 500 individuals. They live in packs which have from three to 20 adults. If we take into account their yearlings and pups, pack size may fluctuate from three to 44. Three of the largest remaining wild dog populations are found in Kruger National Park (South Africa), the Selous Game Reserve (Tansania) and Northern Botswana. In Kruger and Botswana growth rates were surprisingly similar, both very close to zero. The Selous population showed slightly positive growth $r = 0.038$ [29]. We use those growth rates to preform the simulations of number of African wild dogs in Kruger, Botswana and Selous. Since African wild dogs attain sexual maturity between 12 and 18 months ([30]), we choose that $\tau(t) = \frac{1}{4}(5 + \sin\frac{t}{3})$, $t \geqslant 0$, for example. Then $\bar{\tau} = 1.5$, $\tau' = \frac{1}{12}$, $\underline{\tau}' = -\frac{1}{12}$ and $T' = \frac{1}{12}$.

Using these results, we get the following parameters of the studied model (2) for determining the number of African wild dogs in Selous

$$T = 3, \quad K = 20, \tag{32}$$
$$r = 0.038, \quad \alpha = 0.07, \tag{33}$$
$$\xi(\theta) = 8e^{-0.2\theta}, \quad -1.5 \leqslant \theta \leqslant 0. \tag{34}$$

The conditions of the Corollary 3.2 are satisfied and the equilibrium $E^* = 20$ of model (2) is stable in probability (Fig. 1).

The simulation in Fig. 1 shows that the population size in Selous will reach the carrying capacity in 22 years, approximately.

When we change parameters

$$r = 0.3, \quad \alpha = 0.07, \tag{35}$$

the condition (23) of Corollary 3.2 is not satisfied and the equilibrium $E^* = 20$ of model (2) is unstable (Fig. 2, left).

If initial data drop below Allee threshold $T = 3$, and

$$\xi(\theta) = 2e^{-0.2\theta}, \quad -1.5 \leqslant \theta \leqslant 0, \tag{36}$$

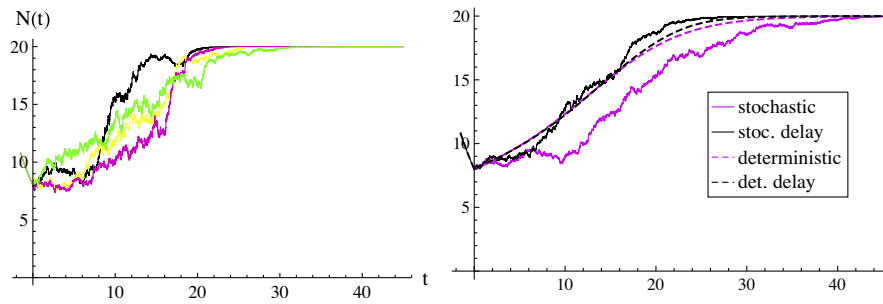[1] All calculations are made by using *MATHEMATICA* programme.

Fig. 1. (The Selous population) The left figure shows four trajectories of model (2) with parameters (32)–(34); the right one shows the corresponding mean of solutions to models (1) and (2) and the trajectories of corresponding deterministic models (time increment $\Delta t = 0.003$, or one day).
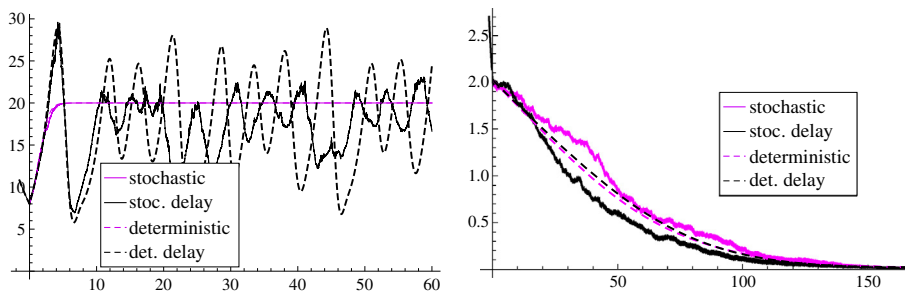

Fig. 2. (The Selous population) The figure shows the mean of four solutions to models (1) and (2) with parameters: (32), (34), (35) (left) and (32), (33), (36) (right) and the trajectories of corresponding deterministic models (time increment $\Delta t = 0.003$).
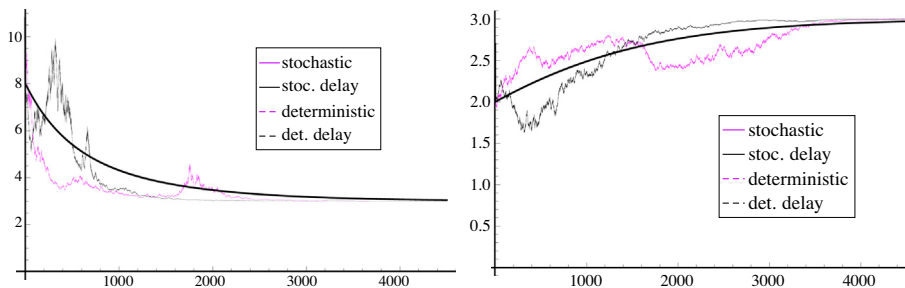

Fig. 3. (The Kruger and Botswana populations) The figure show the mean of solutions to models (1) and (2) with parameters (32), (37) and: (34) (left), (36) (right) and the trajectories of corresponding deterministic models (time increment $\Delta t = 0.03$, or ten days).

then parameters (32) and (33) satisfy the conditions of the Corollary 3.1 and population will be extinct in 150 years, approximately (see Fig. 2, right).

Since growth rates in Kruger and Botswana are very close to zero, we consider the case where they drop bellow zero and get negative values

$$r = -0.001, \quad \alpha = 0.05. \tag{37}$$

Then the parameters (32), (37) and (34) or (36) of model (2) satisfy conditions of Corollary 3.3 and equilibrium $T = 3$ is stable (Fig. 3).

When initial data satisfy (36), the number of wild dogs in Kruger or Botswana will rich the Allee threshold $E_* = 3$ in about 2600 years (Fig. 3, right), and in the case when initial data is (34), in 1600 years (Fig. 3, left).

Moreover, the introduction of time delay in model (1) makes model (2) to reach all equilibrium states faster than model (1) and their deterministic analogs.

Example 2. The population of invasive specie Brown Tree Snake, (*Boiga irregularis*) of Australia and New Guinea was introduced to the previously snake-free island Guam in the 1950s, but may not have become conspicuous away from the port area until the early 1960s. By the mid 1960s, the snake had colonized over half of the island. Since trade between Guam and Hawaii is extensive, the snakes also spread on the island of Oahu in Hawaii. There are approximately 150,000 ha of potential
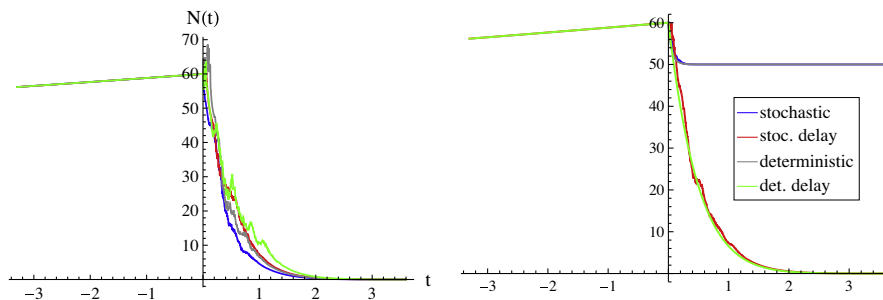
Fig. 4. The left figure shows four trajectories of model (2) with parameters (38)–(40); the right one shows the corresponding mean of the solutions to models (1) and (2) and the trajectories of corresponding deterministic models (time increment $\Delta t = 0.003$).

snake habitat on Oahu. This species is responsible for the extinction of 11 of 18 bird species in Hawaii and for power outages which last 1.5 h up from one every three to four days in 1997. Assuming a maximum population density of 50 snakes/ha, carrying capacity for the island of Oahu is 7,500,000. In the paper [31] authors simulated number of brown tree snakes in local area. They incorporated an strong Allee effect in model and assumed that minimum population level under which growth of the snake is not possible is two. However, the minimum reproducible population size could be much higher than two, given the size of the island and possible gender distributions. The intrinsic growth rate is 0.6, based on estimated population densities at different time periods on Guam. Sexual maturity is estimated to occur during a snake's third or fourth year (for details see [32]) and we choose $\tau = 3.3$ years. Since the brown tree snake is invasive species, in some cases the number of snakes may exceed the carrying capacity. We consider model (2), for determining the number of brown tree snake, with parameters

$$T = 2, \quad K = 50, \tag{38}$$
$$r = 0.6, \quad \alpha = 0.08, \tag{39}$$
$$\xi(\theta) = 60e^{0.02t}, \quad -3.3 \leqslant \theta \leqslant 0. \tag{40}$$

Since $M = \inf_{-3.3 \leqslant \theta \leqslant 0} \xi(\theta) = 56.17 > 52$ and $\tau > 2.446$, conditions of Theorem 3.1(b(i)) are satisfied and solution of model (2) is non-persistent in mean (Fig. 4). In Fig. 4 we can also observe that population of Brown Tree Snake exceed the carrying capacity of its environment for a short while by using up the stored resources or natural capital of its environment. Once the capital is exhausted, snake number inevitably fall because there are no longer enough resources available to support the number of individuals.

Acknowledgment

References

[1] W.C. Allee, Animal Aggregations, A Study in General Sociology, University of Chicago Press, Chicago, 1931.
[2] F. Courchamp, T. Clutton-Brock, B. Grenfell, Inverse density dependence and the Allee effect, TREE 14 (1999) 405–410.
[3] A. Liebhold, J. Bascompte, The Allee effect, stochastic dynamics and the eradication of alien species, Ecol. Lett. 6 (2003) 133–140.
[4] A. Ackleh, L. Allen, J. Carter, Establishing a beachhead: a stochastic population model with an Allee effect applied to species invasion, Theor. Popul. Biol. 71 (2007) 290–300.
[5] O. Duman, H. Merdan, Stability analysis of continuous population model involving predation and Allee effect, Chaos Solitons Fract. 41 (2009) 1218–1222.
[6] H. Merdan, O. Duman, O. Akin, C. Celik, Allee effects on population dynamics in continuous (overlapping) case, Chaos Solitons Fract. 39 (2009) 1994–2001.
[7] M. Krstić, M. Jovanović, On stochastic population model with the Allee effect, Math. Comput. Model. 52 (2010) 370–379.
[8] Q. Yang, D. Jiang, A note on asymptotic behaviors of stochastic population model with Allee effect, Appl. Math. Model. 35 (2011) 4611–4619.
[9] X. Mao, Stochastic Differential Equations and Applications, second ed., Horwood, Chichester, UK, 2007.
[10] C. Ji, D. Jiang, N. Shi, D. O'Regan, Existence, uniqueness, stochastic persistence and global stability of positive solutions of the logistic equation with random perturbation, Math. Methods Appl. Sci. 30 (2007) 77–89.
[11] D. Jiang, N. Shi, Y. Zhao, Existence, uniqueness and global stability of positive solutions to the food-limited population model with random perturbation, Math. Comput. Model. 42 (2005) 651–658.
[12] T.G. Hallam, Z.E. Ma, Persistence in population models with demographic fluctuations, J. Math. Biol. 24 (1986) 327–399.
[13] Z.E. Ma, T.G. Hallam, Effects of parameter fluctuations on community survival, Math. Biosci. 86 (1987) 35–49.
[14] M. Liu, K. Wang, Survival analysis of stochastic single-species population models in polluted environments, Ecol. Model. 220 (2009) 1347–1357.
[15] M. Liu, K. Wang, Persistence and extinction in stochastic non-autonomous logistic systems, J. Math. Anal. Appl. 375 (2011) 42–57.
[16] E. Beretta, V. Kolmanovskii, L. Shaikhet, Stability of epidemic model with time delays influenced by stochastic perturbations, Math. Comput. Simul. 45 (1998) 269–277.

[17] L. Shaikhet, Stability of a positive point of equilibrium of one nonlinear system with aftereffect and stochastic perturbations, Dyn. Syst. Appl. 17 (2008) 235–253.
[18] L. Shaikhet, Stability in probability of nonlinear stochastic systems with delay, Math. Notes 57 (1–2) (1995) 103–106.
[19] L. Shaikhet, Stability in probability of nonlinear stochastic hereditary systems, Dyn. Syst. Appl. 4 (2) (1995) 199–204.
[20] P. Borne, V. Kolmanovskii, L. Shaikhet, Stabilization of inverted pendulum by control with delay, Dyn. Syst. Appl. 9 (2000) 501–514.
[21] V. Kolmanovskii, L. Shaikhet, Construction of Lyapunov functionals for stochastic hereditary systems: a survey of some recent results, Math. Comput. Model. 36 (2002) 691–716.
[22] L. Shaikhet, Some new aspects of Lyapunov type theorems for stochastic differential equations of neutral type, SIAM J. Control Optim. 48 (7) (2010) 4481–4499.
[23] L. Shaikhet, Stability of predator–prey model with aftereffect by stochastic perturbation, SACTA 1 (1) (1998) 3–13.
[24] V.B. Kolmanovskii, V.R. Nosov, Stability of Functional Differential Equations, Academic press, New York, 1986.
[25] <www.populationmatters.orgb0130>.
[26] L. Shaikhet, Lyapunov Functionals and Stability of Stochastic Functional Differential Equations, Springer, Dordrecht, Heidelberg, New York, London, 2013.
[27] P.E. Kloeden, E. Platen, Numerical Solution of Stochastic Differential Equations, Springer, Berlin, 1995.
[28] F. Courchamp, T. Clutton-Brock, B. Grenfell, Multipac dynamic and the Allee effect in the African wild dog, Lycaon pictus, Anim. Behav. 3 (2000) 277–285.
[29] S. Creel, M.G.L. Mills, J.W. McNutt, Demography and Population Dynamics of African Wild Dogs in Three Critical Populations in Biology and Conservations of Wild Canids, Oxford University Press, 2004. pp. 337–350.
[30] D. Macdonald, The Encyclopedia of Mammals, Facts on File Publications, New York, 1984.
[31] K.M. Burnett, S. D'Evelyn, B.A. Kaiser, P. Nantamanasikarn, J.A. Roumasset, Beyond the lamppost: optimal prevention and control of the brown tree snake in Hawaii, Ecol. Econ. 67 (2008) 6674.
[32] J.A. Savidge, F.J. Qualls, G. Rodda, Reproductive biology of the Brown Tree Snake, Boiga irregularis (Reptilia: Colubridae), during colonization of Guam and comparation with that in their native range, Pac. Sci. 61 (2) (2007) 191–199.