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Efficient structural reliability analysis method based on advanced Kriging model

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ABSTRACT

Reliability analysis becomes increasingly complex when facing the complicated expensive-to-evaluate engineering applications, especially problems involve the implicit finite element models. In order to balance the accuracy and efficiency of implementing reliability analysis, an advanced Kriging method is proposed for efficiently analyzing the structural reliability. The method starts with an incipient Kriging model built from a very small number of samples generated by the simple random sampling method, then determines the most probable region in the probabilistic viewpoint and chooses the subsequent samples located in this region by employing the probabilistic classification function. Besides, the leave-one-out technique is used to update the current model. By locating samples in the probabilistic most probable region, only a small number of samples are used to build a precise surrogate model in the end, and only a few actual limit state function evaluations are required correspondingly. After the high quality surrogate of the implicit limit state is available by the advanced Kriging model, the Monte Carlo simulation method is employed to implement reliability analysis. Some engineering examples are introduced to demonstrate the accuracy and efficiency of the proposed method.

1. Introduction

Reliability analysis aims at evaluating the safety level of systems or structures. In the past few decades, many reliability analysis techniques have been developed. Difficulty in computing the failure probability has lead to the development of various approximation methods [1], among which the first-order reliability method (FORM) [2,3] and the second-order reliability method (SORM) [4–6] focus on searching for a single most probable point (MPP) in the failure domain, and then quantify the reliability by building a low-order approximation to the limit state function at MPP. These methods may wrongly assess the safety level in case of multiple MPPs, besides, they rely on MPP convergence and the evaluated results are affected by the precision of the limit state function approximation to a great extent.

As presented by many practitioners, engineering applications are always complex with highly nonlinear limit state models. Thus engineers resort to study the sampling methods, which do not rely on a lower-order approximation of the limit state function. The Monte Carlo simulation (MCS) technique [7–9] is a basic reference approach and is widely used. However, for the implicit limit state models where the finite element model (FEM) analysis is employed to obtain the output, the MCS

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method is infeasible due to the large computational cost. Based on MCS, the importance sampling (IS) method [10,11] is developed. Melchers [10] employed a standard normal probability density function (PDF) centered on the MPP. Au [11] used a kernel smoothing approximation of the optimal instrumental PDF built from some failed samples. IS method requires fewer evaluations of the actual limit state function comparing with MCS, however it needs large number of evaluations all the same with respect to the rare events.

In order to reduce the calls of limit state functions, especially for the FEM analysis, some approximation methods based on the meta-models are proposed including quadratic response surfaces [12–14], neural networks [15], support vector machines [16,17] and Kriging [18–21]. However, it is often difficult to decide how many samples should be selected to construct the surrogates and it is difficult to quantify the error of the surrogate model. The traditional Kriging based methods used a number of randomly selected samples to build the surrogate, and the accuracy of the approximate model depends on the information provided by the given samples. If few samples are used, the prediction capability of the approximate model would be insufficient. On the contrary, if large numbers of samples are used, the accuracy can be ensured, but the correspondingly computational cost would be expensive, especially for the computational intensive models.

Obviously, an efficient meta-model based reliability analysis method is needed to balance the accuracy and cost. Bichon [21,22] proposed an expected feasibility function based on the Kriging model and depended on it to locate the samples near the limit state, which decreased the actual limit state function evaluations. Dubourg [23,24] employed a probabilistic classification function based on the Kriging model to approximate the failure indicator in order to refine the models and build a quasi-optimal importance sampling density. This paper starts from the probabilistic viewpoint and proposes an advanced Kriging-based method to efficiently estimate the reliability of the structural system. The method begins with an initial Kriging model constructed from a very small number of samples obtained by the simple random sampling method, then employs the probabilistic classification function to determine the most probable region, and selects the subsequent samples with high level of uncertainty to enrich the experiment points for updating the model. Besides, the leave-one-out technique is used as the stopping criterion to refine the model. By choosing the subsequent samples which locate in the most probable region with the probabilistic viewpoint, only a small number of evaluations of the actual limit state function are needed to build an accurate meta-model.

This paper is organized as follows. Section 2 reviews the basic reliability analysis and the Kriging method. Section 3 presents methods to find the probabilistic most probable region and choose the subsequent experiment samples in this region for refining the model. Section 4 gives the implementation progress of reliability analysis by the proposed advanced Kriging method. Section 5 illustrates the accuracy and efficiency of the proposed method. Section 6 provides the conclusions.

2. Reliability analysis

2.1. Basic reliability methods

The goal of reliability analysis is to compute the failure probability P_f to evaluate the safety level of systems or structures. For the response function $Z = g(x)$ relating with the n-dimensional independent random input vector $x = \{x_1, x_2, \ldots, x_n\}$, the failure probability P_f is defined by

$$P_f = \int_F f_x(\boldsymbol{x})d\boldsymbol{x} = \int_{R^n} I_{g\leqslant 0}(\boldsymbol{x})f_x(\boldsymbol{x})d\boldsymbol{x}, \tag{1}$$

where $f_x(\boldsymbol{x}) = \prod_{i=1}^{n} f_{X_i}(x_i)$ is the joint probability density function (PDF) of random variable vector $\boldsymbol{x}$, $f_{X_i}(x_i)$ is the marginal PDF of x_i. The integration is performed over the failure region F, which is defined by the response function $Z = g(\boldsymbol{x})$ as $F = \{\boldsymbol{x} : g(\boldsymbol{x}) \leqslant 0\}$. $I_{g\leqslant 0}(x)$ is the failure indictor, it equals to one if $g(\boldsymbol{x}) > 0$ and zero otherwise.

The basic approximation to compute the failure probability is FORM, which linearizes the limit state surface and computes the failure probability by

$$P_f \approx \Phi(-\beta), \tag{2}$$

where β is the reliability index and represents the distance from the origin to the MPP in the standard normal space. SORM computes the failure probability in the same way by using a quadratic surface fitted at the MPP. However, for a complex engineering application, the limit state functions may be multimodal and possess multiple MPPs, thus the reliability estimated by the methods only on the information of the single MPP may be inaccurate.

Generally, the MCS method is a basic simulation and the results are used as references. It is a simulation technique. The MCS estimator is then derived as

$$\widehat{P}_{f\ MC} = E_x[I_{g\leqslant 0}(\boldsymbol{x})] = \frac{1}{N}\sum_{i=1}^{N} I_{g\leqslant 0}(\boldsymbol{x}^{(k)}), \tag{3}$$

where $x^{(k)} (k = 1, \ldots, N)$ are a set of random input samples and $E[\cdot]$ is the expectation operator. According to the central limit theorem, this estimator is unbiased. For the events with very rare failure probability, in order to obtain a convergent result, N should be large enough (generally, $N = (10^2 \sim 10^4)/P_f$). Note that it is efficient to implement MCS estimation for the problem with explicit limit state function. However, engineering problems are often characterized by an implicit input–output relationship, usually, we only know their numerical relation constructed by FEM. And using the MCS simulation method to analyze the FEM directly is a very time-consuming progress when facing these real problems.

2.2. Meta-model based method

In order to cope with the expensive-to-evaluate problems expressed by FEM, the meta-model based method is a good choice. Among the various meta-models, Kriging model is widely used [18,25].

Different from other models, the Kriging method is a semi-parametric interpolation technique. Sacks [26] characterized the actual response function $g(\boldsymbol{x})$ by two parts: the linear regression part and the nonparametric part,

$$g(\boldsymbol{x}) = \boldsymbol{f}^T(\boldsymbol{x})\boldsymbol{\beta} + \boldsymbol{Z}(\boldsymbol{x}), \tag{4}$$

where $\boldsymbol{f}(\boldsymbol{x}) = [f_1(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x})]^T$ is the basis function, $\boldsymbol{\beta} = [\beta_1, \ldots, \beta_m]^T$ is the vector of regression coefficient which needs to be determined, and m denotes the number of the basis function. $\boldsymbol{Z}(\boldsymbol{x})$ is used to model the departure of regression model $\boldsymbol{f}^T(\boldsymbol{x})\beta$ and it is assumed to be a Gaussian stochastic process with zero mean, the covariance can be defined as

$$Cov[Z(\boldsymbol{x}_i), Z(\boldsymbol{x}_j)] = \sigma^2 \boldsymbol{R}(\boldsymbol{x}_i, \boldsymbol{x}_j), \quad i,j = 1, \ldots, N, \tag{5}$$

where N is the number of experimental points, σ^2 is the progress variance and $\boldsymbol{R}(\cdot,\cdot)$ is the correlation function which is given as

$$\boldsymbol{R}(\boldsymbol{x}_i, \boldsymbol{x}_j) = \exp\left(-\sum_{l=1}^{n} \theta_l |\boldsymbol{x}_{il} - \boldsymbol{x}_{jl}|^{pl}\right), \tag{6}$$

where n represents the dimensionality number of the input vector $\boldsymbol{x}$, pl determines the smoothness of the function in the lth coordinate direction and $pl = 2$ is widely used [26], θ_l is the correlation parameter, $\boldsymbol{x}_{il}$ and $\boldsymbol{x}_{jl}$ are the lth component of vector $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$ respectively.

Thus the unknown parameters $\boldsymbol{\beta}$ and σ^2 can be estimated as

$$\widehat{\boldsymbol{\beta}} = (\boldsymbol{F}^T \boldsymbol{R}^{-1} \boldsymbol{F})^{-1} \boldsymbol{F}^T \boldsymbol{R}^{-1} \mathbf{g}, \tag{7}$$

$$\widehat{\sigma^2} = \frac{1}{N} (\mathbf{g} - \boldsymbol{F}\widehat{\boldsymbol{\beta}})^T \boldsymbol{R}^{-1} (\mathbf{g} - \boldsymbol{F}\widehat{\boldsymbol{\beta}}), \tag{8}$$

where $\boldsymbol{F}$ is a vector of $\boldsymbol{f}(\boldsymbol{x})$ and $\mathbf{g}$ is the vector of response outputs evaluated at each of the experimental points, and $\boldsymbol{R}$ is the correlation matrix, i.e.,

$$\boldsymbol{R} = \begin{bmatrix} R(\boldsymbol{x}_1, \boldsymbol{x}_2) & \cdots & R(\boldsymbol{x}_1, \boldsymbol{x}_N) \\ \vdots & \ddots & \vdots \\ R(\boldsymbol{x}_N, \boldsymbol{x}_1) & \cdots & R(\boldsymbol{x}_N, \boldsymbol{x}_N) \end{bmatrix}. \tag{9}$$

However, before computing $\widehat{\boldsymbol{\beta}}$ and $\widehat{\sigma^2}$, we have to estimate the unknown parameters of the correlation function by using maximum likelihood:

$$\min : 1/2(N \ln \widehat{\sigma^2} + \ln |\boldsymbol{R}|). \tag{10}$$

The reason why Kriging models are set apart from other meta-models is that they provide not only a predicted value at a prediction point, but also an estimate of the prediction variance, which gives an uncertainty indication of the Kriging model. The expected value $\mu_{\widehat{G}}$ and variance $\sigma^2_{\widehat{G}}$ of the Kriging model prediction at a point $\boldsymbol{x}$ are

$$\mu_{\widehat{G}}(\boldsymbol{x}) = \boldsymbol{f}^T(\boldsymbol{x})\widehat{\boldsymbol{\beta}} + \boldsymbol{r}^T(\boldsymbol{x})\boldsymbol{R}^{-1}(\mathbf{g} - \boldsymbol{F}\widehat{\boldsymbol{\beta}}), \tag{11}$$

$$\sigma^2_{\widehat{G}}(\boldsymbol{x}) = \sigma^2 - [\boldsymbol{f}^T(\boldsymbol{x}) \quad \boldsymbol{r}^T(\boldsymbol{x})] \begin{bmatrix} \mathbf{0} & \boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{R} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{f}(\boldsymbol{x}) \\ \boldsymbol{r}(\boldsymbol{x}) \end{bmatrix}, \tag{12}$$

where $\boldsymbol{r}^T(\boldsymbol{x}) = [R(\boldsymbol{x}, \boldsymbol{x}_1), \ldots, R(\boldsymbol{x}, \boldsymbol{x}_N)]^T$ is the correlation vector between an unknown point $\boldsymbol{x}$ and all known experimental points $(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_N)$.

3. Advanced Kriging method

Generally, to implement reliability analysis, the traditional Kriging based methods used a number of randomly selected samples to construct a Kriging model, the accuracy of which depends on the information provided by the given samples. The results lack of accuracy if few samples are used. Here we take a simple structure reliability problem as an example. The limit state function reads:

$$g(\boldsymbol{x}) = 5 + 2x_1 - x_2 - x_1^2, \tag{13}$$

where x_1 and x_2 are both independent uniform distribution variables, x_1 and x_2 are characterized by an interval $[-3,5]$ and $[1,6.5]$, respectively. For this explicit example, we generate few samples of $\boldsymbol{x}$ by the simple random sampling method and use a Kriging model to substitute the limit state function $g(\boldsymbol{x}) = 0$. In Fig. 1, The blue line represents the limit state function $g(\boldsymbol{x}) = 0$, the dashed red line represents the limit state $\hat{y}(\boldsymbol{x}) = 0$ of the Kriging prediction. Fig. 1 clearly shows that for a sample x^*, it is safe according to the actual limit state function $g(\boldsymbol{x}) = 0$ but fails by the model prediction $\hat{y}(\boldsymbol{x}) = 0$.

On the contrary, if large numbers of samples are used to ensure the accuracy, then the limit state function evaluations will increase to an expensive lever correspondingly. The proposed method of this paper will avoids these problems by introducing a sample choosing technique.

3.1. Probabilistic classification function

The Kriging prediction $\hat{y}(\boldsymbol{x})$ follows a Gaussian distribution:

$$\hat{y}(\boldsymbol{x}) \sim N[\mu_{\hat{y}}(\boldsymbol{x}), \sigma_{\hat{y}}(\boldsymbol{x})], \tag{14}$$

where the mean $\mu_{\hat{y}}(\boldsymbol{x})$ and the variance $\sigma^2_{\hat{y}}(\boldsymbol{x})$ were defined in Eqs. (11) and (12), respectively.

Kriging method uses the mean $\mu_{\hat{G}}(\boldsymbol{x})$ as the response prediction. Because Kriging model can directly give the variance of the prediction at any point, Picheny [27] and Dubourg [24] proposed the probabilistic classification function $\pi(\boldsymbol{x})$ to substitute the failure indictor function $I_{g\leqslant 0}(\boldsymbol{x})$. In this paper, we employ the probabilistic classification function as a probabilistic prediction to evaluate the uncertainty of input $\boldsymbol{x}$. The probabilistic classification function is defined as

$$\pi(\boldsymbol{x}) = P[\hat{y}(\boldsymbol{x}) \leqslant 0]. \tag{15}$$

The probabilistic classification function is similar to the original definition of the failure probability P_f, which reads:

$$P_f = P[g(\boldsymbol{x}) \leqslant 0]. \tag{16}$$

Note that the probabilistic classification function $\pi(\boldsymbol{x})$ is not P_f, and it shall not be confused with the solving of P_f. Here it is simply the probability that the prediction $\hat{y}(\boldsymbol{x})$ at point $\boldsymbol{x}$ is negative.

Eq. (14) presents the Gaussian nature of the Kriging prediction, based on it, the probabilistic classification function by the Kriging model can be expressed as follows [24]:

$$\pi(\boldsymbol{x}) = \Phi\left(\frac{0 - \mu_{\hat{y}}(\boldsymbol{x})}{\sigma_{\hat{y}}(\boldsymbol{x})}\right), \quad \boldsymbol{x} \notin \chi, \tag{17}$$

where $\chi = \{\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m\}$ is a set of experiment samples which were used to build the Kriging model, m denotes the number of the experiment samples. As we know, the Kriging prediction has a zero variance at the point in the experimental samples χ, thus for the points in the experimental sample space, the probabilistic classification function is defined as

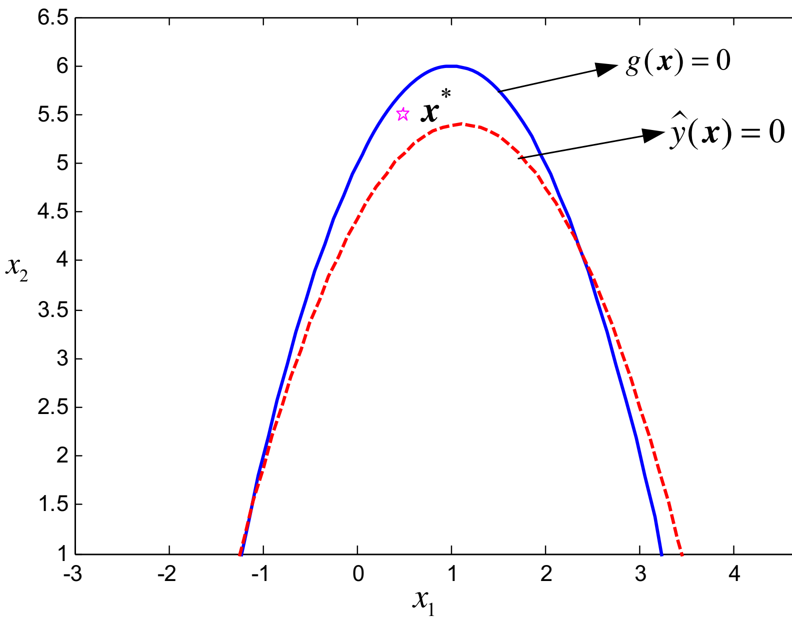

Fig. 1. Comparison of the actual limit state function and the Kriging model.

$$\pi(\boldsymbol{x}) = \begin{cases} 1 & \boldsymbol{x} \in \chi,\ g(\boldsymbol{x}) \leqslant 0 \\ 0 & x \in \chi,\ g(\boldsymbol{x}) > 0 \end{cases}. \tag{18}$$

Given the definition of the probabilistic classification function, we can classify the points in the design space in the probabilistic point of view. Fig. 2 shows the three classification strategies on the limit state function given in Eq. (14). The green line made by the circles represents the points which all satisfy $\pi(\boldsymbol{x}) = 97.5\%$, and the cyan line made by the squares represents the points which all satisfy $\pi(\boldsymbol{x}) = 2.5\%$. The three classification strategies are defined as: the region out of the green line has $\pi(\boldsymbol{x}) > 97.5\%$, the region below the cyan line has $\pi(\boldsymbol{x}) < 2.5\%$, and the points located between the above two lines have $2.5\% < \pi(\boldsymbol{x}) < 97.5\%$. In the next subsection, we will give the reason of the three classification strategies in details.

3.2. The most probable region

Combing the definition of the probabilistic classification function and the three classification strategies, we can see that the points located in the region out of the green line which is indicated by "f" in Fig. 3 has $P[\hat{y}(\boldsymbol{x}) \leqslant 0] > 97.5\%$, and the points located in the region below the cyan line which is indicated by "S" in Fig. 3 has $P[\hat{y}(\boldsymbol{x}) > 0] > 97.5\%$, thus the current Kriging model can make a certain prediction of the sign of the points in these two regions with a probability greater than 97.5%. Then we can approximately say the points located in both of the two region have a certain sign, which means the points located in these regions absolutely fail or be safe, respectively.

What we concern is the region between the green and cyan lines. We are sure the sign of the points in "f" and "S", they fail or be safe with a confidence level greater than 97.5% as above presented, in other words, this means we are not really exactly know the sign of the points located between the two regions. $2.5\% < \pi(\boldsymbol{x}) < 97.5\%$ indicates a 95% confidence interval of a uncertain region, so the green line and the cyan line are the 95% confidence margin. Similar to the meaning of the MPP, here we define this region as the "probabilistic most probable region (P_MPR)", which means the uncertain region, and we are not sure the sign of the points in it (see Fig. 3).

The most important part of the proposed method is to find the P_MPR. With the P_MPR available, we can make improvement in building the Kriging model. So the next step is to choose some points in the P_MPR. From the large number of points generated by the simple random sampling method, we choose some points that satisfy $2.5\% < \pi(\boldsymbol{x}) < 97.5\%$ as the subsequent experiment points to update the model efficiently. Thus in this paper, from the viewpoint of probability, consequently, more complex and nonlinear limit state surfaces can be modeled accurately by adding the experiment points in P_MPR, and we can directly say that the more samples are located in the P_MPR, the more precise model we will get. Actually, by concentrating the experiment points in the important area for constructing a surrogate model, only a small number of actual function evaluations are required to construct a high quality meta-model. It is both accurate for an arbitrarily shaped limit state surface and computationally efficient for the expensive-to-evaluate response functions.

Note that there are not many points in the 95% confidence interval of the uncertain region, for the large number of samples, the corresponding realization of probabilistic classification functions are close to zero or one mostly. And the more

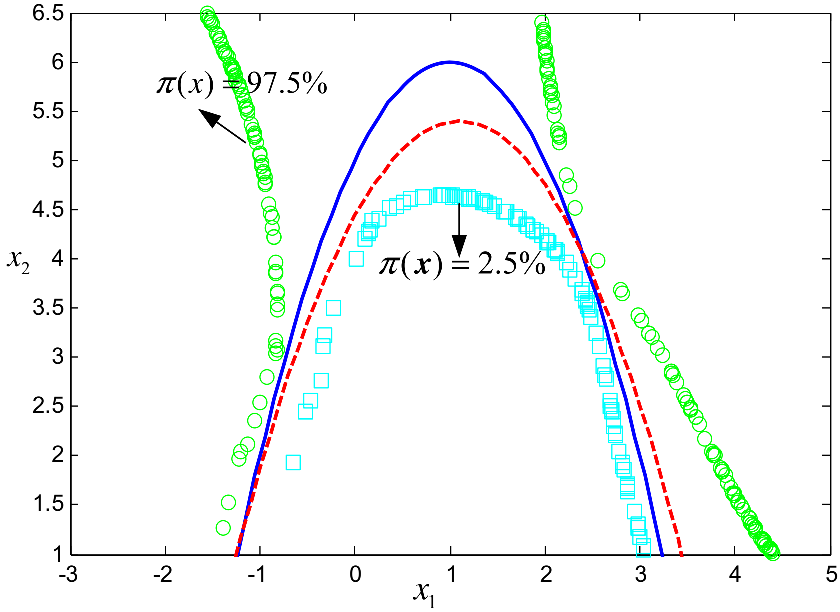

Fig. 2. Three classification strategies of the numerical example.

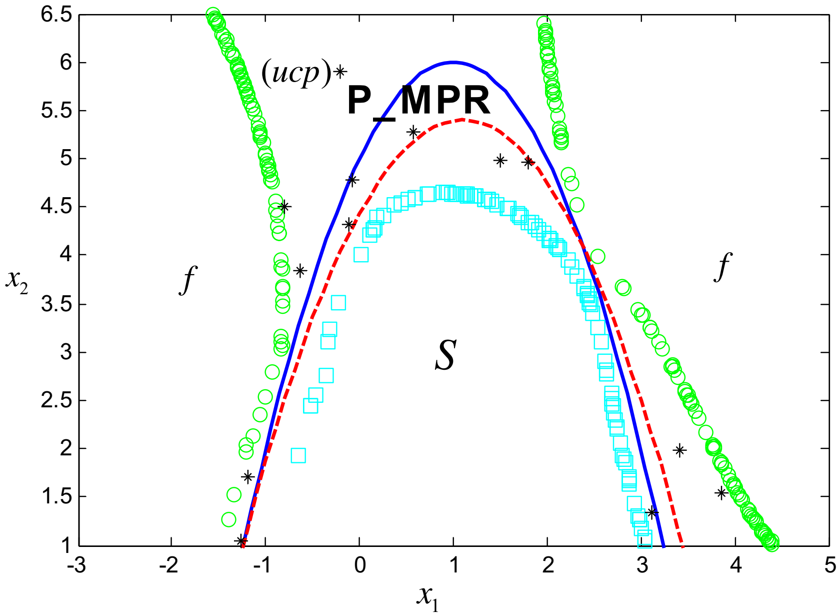

Fig. 3. The P_MPR and *ucp* of the numerical problem.

subsequent points are added to enrich the experiment points, the more accurate the surrogate will be, besides, the fewer points there will be in the P_MPR. Analogously, we also can set $\pi(\boldsymbol{x}) \geqslant 98\%$ and $\pi(\boldsymbol{x}) \leqslant 2\%$ to define the certain region "f" and "S", respectively, or other margins, and the number of points in P_MPR will change related with the margins. Here we use the "uncertain point (*ucp*)" to denote these points located in the P_MPR, see Fig. 3. These *ucp* in P_MPR are all very important, here we choose some points having large values of product of marginal PDFs among them to enrich the experiment points, because they have larger contribution to the evaluation of p_f than others.

3.3. Leave-one-out technique

The accuracy and efficiency of the reliability analysis mostly rely on the quality of the Kriging model, and in the above subsections, we have found method of selecting points to enrich the experiment points, which are important to the model accuracy. However, it is difficult to arbitrarily determine how many subsequent points should be added and how accurate model we need, thus it is proposed here to adaptively refine the Kriging model. Consequently, a metric is needed to refine the model as a stopping criterion.

The direct way to measure the quality of the Kriging model is comparing the predicted values $\hat{y}(\boldsymbol{x})$ and the real values $g(\boldsymbol{x})$ on some new points. However, in the early steps of the refinement procedure, the Kriging model is with low quality and the prediction is not accurate to implement reliability analysis, so this direct way will cause many extra evaluations of the actual limit state function in every refinement loop with unnecessary cost. As a result the efficiency of the proposed method may not be notable and it would even be ineffective to waste such costly evaluations. Allen [28] proposed to use a leave-one-out estimate of the mean squared error as the predicted residual sum of squares in a regression context

$$\text{PRESS} = \frac{1}{m}\sum_{i=1}^{m}\left(\hat{y}_{\chi\backslash \boldsymbol{x}_i}(\boldsymbol{x}_i) - g(\boldsymbol{x}_i)\right)^2. \tag{19}$$

As we know, the relative error is more meaningful than the absolute error, and it avoids the effect of the dimension and can better indicate the accuracy of the prediction. Thus, here we employ the leave-one-out technique with respect to the relative error as a stopping criterion to refine the Kriging model, namely

$$\text{RPRESS} = \frac{1}{m}\sum_{i=1}^{m}\left(\frac{\hat{y}_{\chi\backslash x_i}(\boldsymbol{x}_i) - g(\boldsymbol{x}_i)}{g(\boldsymbol{x}_i)}\right)^2, \tag{20}$$

where $\hat{y}_{\chi\backslash \boldsymbol{x}_i}(\boldsymbol{x}_i)$ is the ith leave-one-out Kriging prediction of the limit state function constructed from the experiment points without the ith sample $\boldsymbol{x}_i$.

The leave-one-out error is an important estimator of the performance of a learning algorithm. And it is a method which does not require any extra evaluations of the limit state function and only uses the available observations in the experiment design obtained to build the Kriging model. For different problems and different requirement, we can give an upper limit of the leave-one-out error, denoted *e_given*, to stop the refine strategy. In the first few steps, the surrogate model has a high level of error, and generally, we can obtain a surrogate with an acceptable level of accuracy when *e_given* is set to 0.1.

4. Implementation of reliability analysis

In this section, we will make a complete summary of the implementation of the reliability analysis. In the above sections, we have finished building a high quality Kriging model, thus we can use it next for reliability analysis directly. For an implicit problem, such as the complicated expensive-to-evaluate FEM, it is a time-consuming progress for the direct use of sampling method, however, for a given limit state model, we can use the MCS method for reliability analysis, it is efficient all the same and the results generally can be made as reference. Thus the accuracy of the results mainly relies on the accuracy of the model. In this paper we make use of the MCS method for subsequent reliability analysis, and the failure probability is evaluated by Eq. (3). Besides, enough samples will be used to make sure a convergent result.

The flowchart of the proposed method is given in Fig. 4. It can be simply divided into five steps. The method proceeds as follows:

1. Sample N_0 samples using the simple random sampling method and evaluate the corresponding limit state function. In the initial step, we do not need too many points, so less than ten points are sampled. Note that one can also use the Latin hypercube sampling method to generate the initial points, which has low discrepancy property.
2. Construct the Kriging model, compute the leave-one-out estimate of the mean squared error RPRESS and judge whether it is smaller than the given *e_given*. If so, turn to step 5 directly.
3. Generate N_R points ($N_R = 3 \times 10^4$ in this paper) by the simple random sampling method and compute the probabilistic classification function correspondingly so as to determine the P_MPR and select the *ucp*.
4. Add N_{ucp} *ucp* with large values of product of marginal PDFs to the experiment points and loop back to step 2. In this paper, we aim at using the least points to accurately construct the surrogate, so we only choose 2 points to enrich the experiment points in every loop.
5. Use the MCS method with N random points to implement reliability analysis.

5. Application examples

The proposed method in this paper mainly focuses on the engineering problems, including the implicit input–output models. In this section, three engineering examples including two explicit problems and one implicit problem are used to

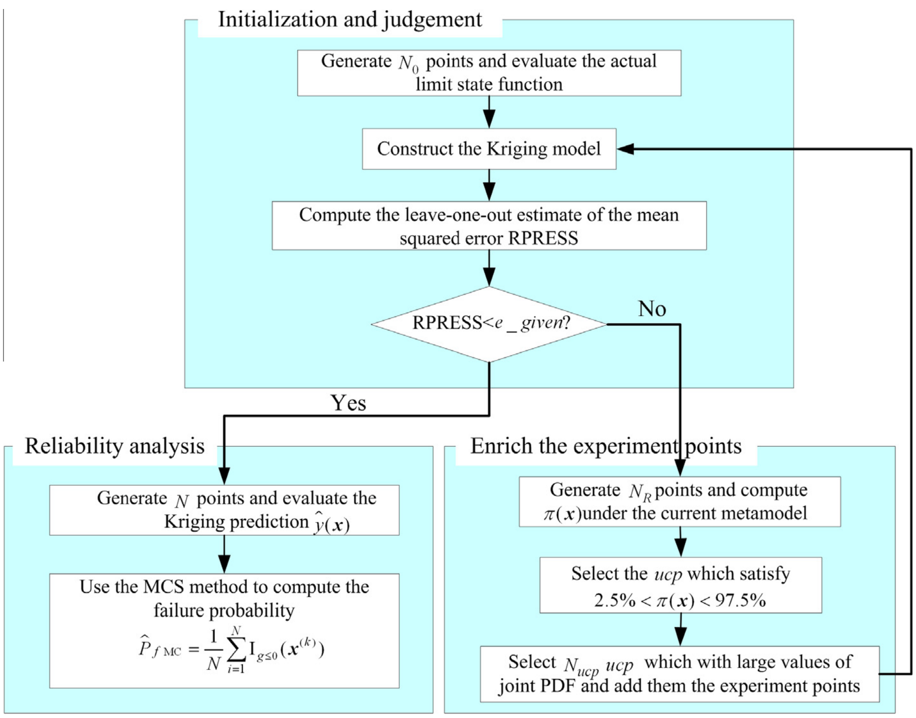

Fig. 4. Flowchart of the proposed method based on the Kriging model.

demonstrate the efficiency and accuracy of the proposed method. N_{call} denotes the number of the actual limit state function evaluations. In the first two explicit examples, Ori-Kriging represents the traditional use of the Kriging method with experimental samples generated by the simple random sampling method, and then combines MCS method to estimate the failure probability. While in the last implicit example, Ori-Kriging represents the traditional use of the Kriging model constructed by Latin hypercube sampling method. Adv-Kriging denotes the proposed method of this paper.

5.1. Automobile front axle

In the automobile engineering, the front axle beam is used to carry the weight of the front part of the vehicle [19]. As the complete front part of the body rests on the body front axle beam, it must be robust in construction. Note that the I-beam structures are widely used in the design of front axle due to its high bend strength and light weight. As shown in Fig. 5, the dangerous cross-section happens in the I-beam part. The maximum normal stress and shear stress are $\sigma = M/W_x$ and $\tau = T/W_\rho$ respectively, where M and T are the bending moment and torque, W_x and W_ρ are section factor and polar section factor which can be written as:

$$W_x = \frac{a(h-2t)^3}{6h} + \frac{b}{6h}[h^3 - (h-2t)^3], \tag{21}$$

$$W_\rho = 0.8bt^2 + 0.4\frac{a^3(h-2t)}{t}. \tag{22}$$

To check the static strength of front axle, the performance function can be given as:

$$g = \sigma_s - \sqrt{\sigma^2 + 3\tau^2}, \tag{23}$$

where σ_s is the limit stress of yielding. According to the material property of the front axle, the limit stress of yielding σ_s is 460 Mpa. The geometry variables of I-beam a, b, t, h and the loads M and T are independent normal variables with distribution parameters listed in Table 1. Results of the failure probability are given in Table 2.

It can be seen from Table 2 that according to different number of the actual limit state function evaluations, the results of all the other methods agree with the result of MCS very well. The Subsim method, which was proposed by Au [29], is an efficient simulation method for the high dimensional rare events. Taking 10^6 evaluations of the actual limit state function,

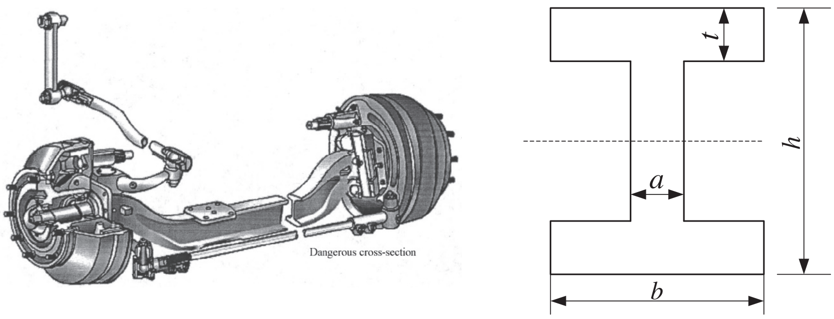

Fig. 5. The schematic diagram of automobile front axle.

Table 1
Distribution parameters of the input variables of front axle.

Random variables	a/mm	b/mm	t/mm	h/mm	$M/(\mathrm{N \cdot mm})$	$T/(\mathrm{N\,mm})$
Mean	12	65	14	85	3.5×10^6	3.1×10^6
Standard deviation	0.060	0.325	0.070	0.425	1.75×10^5	1.55×10^5

Table 2
Results of the reliability analysis for front axle.

Method	P_f	N_{call}	Error
MCS	0.0196	10^6	/
Subsim [29]	0.0202	6×10^4	3.1%
Ori-Kriging	0.0195	65	0.5%
Adv-Kriging	0.0199	13	1.5%

the coefficient of variation for the result obtained by MCS method is 0.0051 and it is convergent, thus the result can be made as a reference. In order to get a convergent result, the traditional Kriging method and the proposed advanced method of this paper both sample enough points to compute the failure probability after the Kriging model is built. We can see that Subsim method takes 6×10^4 times evaluations and it is also expensive, the traditional Kriging method require 65 calls to get a close result, and it has decreased the computational burden to a great extent. However our proposed method only needs 13 evaluations of the actual limit state function and it is efficient enough.

To demonstrate the efficiency of our proposed method, for different *e_givens*, different N_{call} are required to obtain a right result. Then for each of the N_{call}, using the Ori-Kriging method to compute the failure probability, all the results are shown in Fig. 6. The dashed green line is the convergent result of MCS method with 10^6 calls of the actual limit state function and is a reference. We can see that our proposed method converges in a fast rate with a few actual function evaluations.

5.2. Roof truss

A roof truss is shown in Fig. 7, the top boom and the compression bars are reinforced by concrete, and the bottom boom and the tension bars are steel. Assume the uniformly distributed load q is applied on the roof truss, and the uniformly distributed load can be transformed into the nodal load $P = ql/4$. Taking the safety and applicability into account, the perpendicular deflection Δ_C of the peak of structure node C not exceeding 3.2 cm is taken as the constraint condition, the

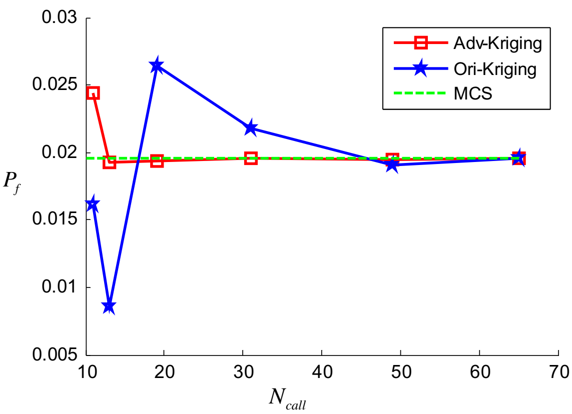

Fig. 6. The results of different N_{call} for different *e_givens* of front axle.

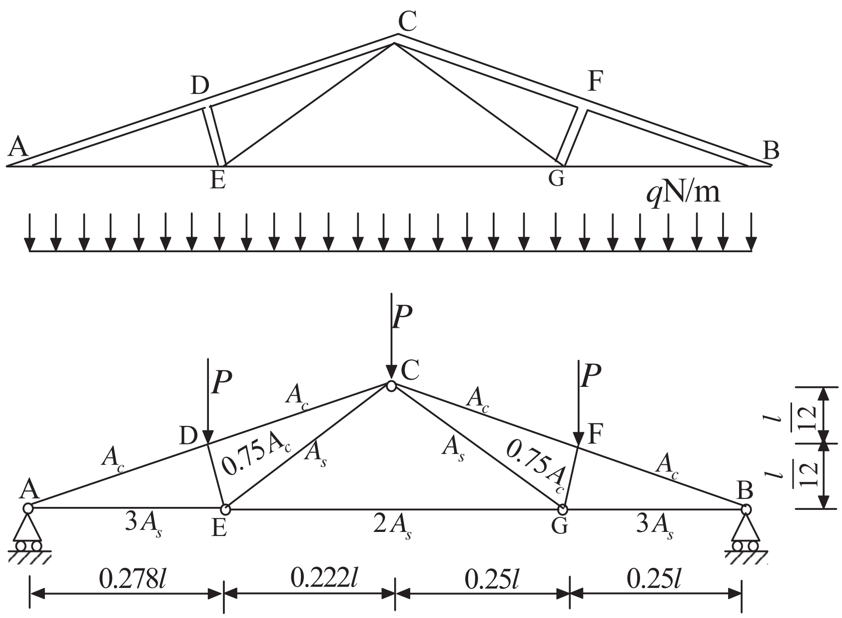

Fig. 7. The schematic diagram of roof truss.

performance response function can be constructed by $g(x) = 0.032 - \Delta_C$, where Δ_C is the function of the basic random variables, and $\Delta_C = \frac{ql^2}{2}\left(\frac{3.81}{A_C E_C} + \frac{1.13}{A_S E_S}\right)$, A_C, A_S, E_C, E_S, l respectively are sectional area, elastic modulus, length of the concrete and steel bars, the distribution parameters of these independent normal basic random variables are listed in Table 3. Results of the failure probability are given in Table 4.

The coefficient of variation for the result obtained by MCS method is 0.0042 and the result can be a reference. We can see from Table 4 that the results of Subsim method, Ori-Kriging method and our proposed Adv-Kriging method are all close to the result of MCS. The Subsim method also takes 2×10^4 times of evaluations, and Ori-Kriging need 10^2 calls of the actual function to get a close result, note that our proposed method is really an efficient method and only needs 26 evaluations of the actual function to get an accurate result.

Given different *e_givens*, our proposed method can get different N_{call} and the corresponding failure probability, for comparison, results of the Ori-Kriging method using the different N_{call} are obtained and presented in Fig. 8. It can be seen from Fig. 8 that the proposed method only needs a few samples to approximate the actual limit state surface very well and to get an accuracy result efficiently.

5.3. A planar ten-bar structure

For better illustrating the engineering application of our proposed method, we introduce this planar ten-bar structure [30] with an implicit input–output relationship, the schematic diagram of the structure is shown in Fig. 9. The length and elastic modulus of all the horizontal and vertical bars are L and E respectively. The section area of each bar is $A_i (i = 1, 2, \ldots, 10)$ and $P_i (i = 1, 2, 3)$ are the point loads. The fifteen input variables, i.e., L, E, $A_i (i = 1, 2, \ldots, 10)$ and $P_i (i = 1, 2, 3)$ are all normally distributed, and the distribution parameters are shown in Table 5. Fig. 10 is the finite element model of the structure constructed in Ansys 11.0. According to the analysis of the finite model, we assume the limit state function to be $g = 0.0035 - \Delta_y$, where Δ_y is the displacement of node 3 in vertical direction. Results of the failure probability are given in Table 6.

It can be seen from Table 6 that the efficiency of our proposed method is more noticeable for an engineering problem with an expensive-to-evaluate FEM. MCS method takes 3×10^5 evaluations of the FEM and the coefficient of variation of the result

Table 3
Distribution parameters of the input variables of roof truss.

Random variables	q(N/m)	l(m)	A_S(m^2)	A_C(m^2)	E_S(N/m^2)	E_C(N/m^2)
Mean	20000	12	9.82×10^{-4}	0.04	1×10^{11}	2×10^{10}
Coefficient of variation	0.07	0.01	0.06	0.12	0.06	0.06

Table 4
Results of the reliability analysis for roof truss.

Method	P_f	N_{call}	Error
MCS	0.0546	10^6	/
Subsim [29]	0.0541	2×10^4	0.92%
Ori-Kriging	0.0545	10^2	0.18%
Adv-Kriging	0.0549	26	0.55%

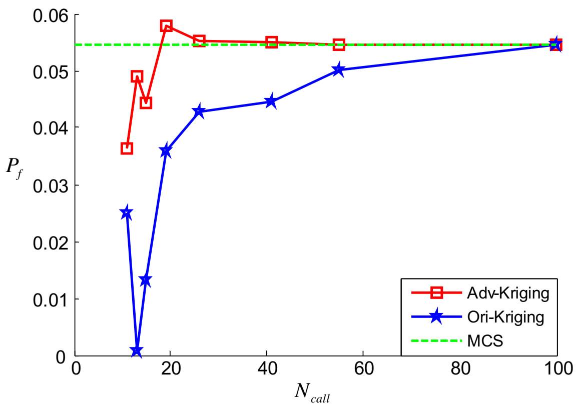

Fig. 8. The results of different N_{call} for different *e_givens* of roof truss.

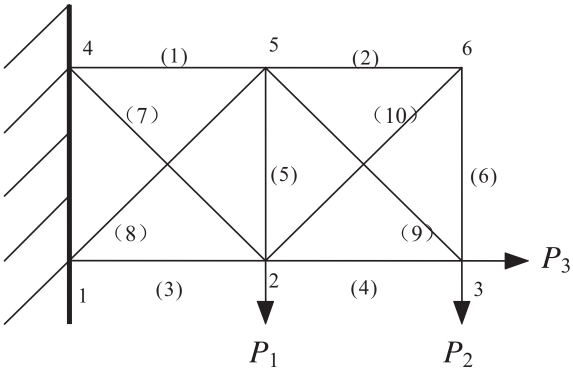

Fig. 9. The schematic diagram of planar ten-bar structure.

Table 5
Distribution parameters of the input variables of ten-bar structure.

Random variables	$A_i(\mathrm{m}^2)$	$L(\mathrm{m})$	$E(\mathrm{GPa})$	$P_1(\mathrm{kN})$	$P_2(\mathrm{kN})$	$P_3(\mathrm{kN})$
Mean	0.001	1	100	80	10	10
Coefficient of variation	0.15	0.05	0.05	0.05	0.05	0.05

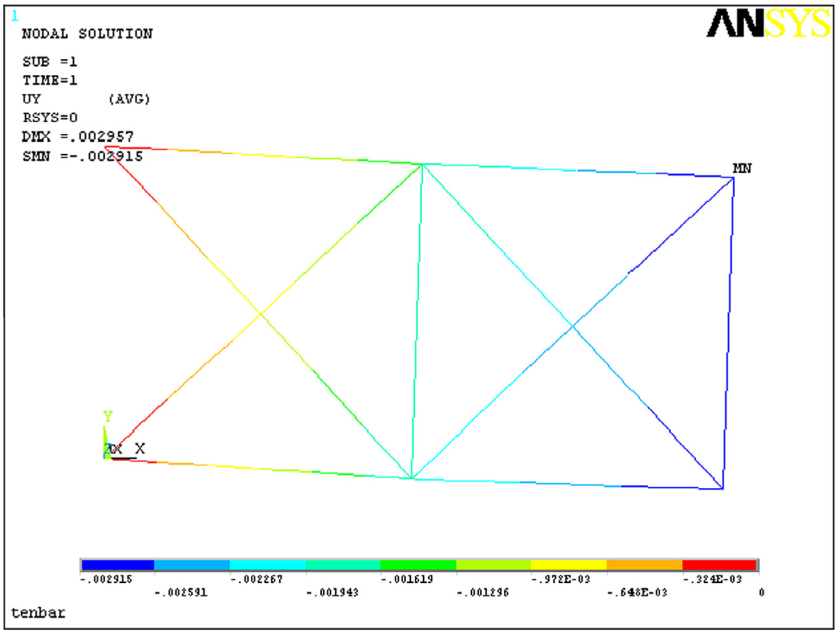

Fig. 10. The finite element model of the ten-bar structure.

is 0.0069, thus it is a very time-consuming progress. The computational cost is also unaffordable for the Subsim method. The RVM-C method, which was proposed by Zhou [17], is an advanced meta-model based method using the relevance vector machine to approximate the actual limit state function to perform reliability analysis. We can see that our proposed method and the RVM-C method are efficient equivalently, which demonstrates that our proposed method is an available efficient method for implement reliability analysis on the complicated structures.

Since the Latin hypercube sampling method is also a commonly used sampling method and is used to sample the experimental points to construct the meta-model [22]. In this example, the Ori-Kriging method uses the Latin hypercube sampling method to generate the experimental points and then constructs the Kriging model. We can see from Table 6 that the Ori-Kriging method obtains close result with 329 calls of the FEM, however our proposed method only needs 83 times of actual function evaluations to get a reasonable result.

Table 6
Results of the reliability analysis for roof truss.

Method	P_f	N_{call}	Error
MCS	0.0678	3×10^5	/
Subsim [29]	0.0671	10^5	1.01%
Ori-Kriging	0.0683	329	0.74%
RVM-C [17]	0.0680	98	0.29%
Adv-Kriging	0.0672	83	0.88%

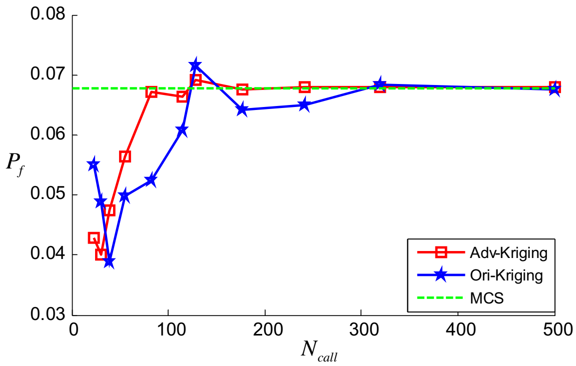

Fig. 11. The results of different N_{call} for different *e_givens* of ten bar.

Similarly, for different N_{call} obtained on the condition of different *e_givens*, we compute the failure probability using the Ori-Kriging method with the experimental points generated by Latin hypercube sampling method for comparison. We can see from Fig. 11 that when the obtained N_{call} is 83, the result of our proposed method gets a close result with MCS, and the result has already converged to the MCS result from this N_{call}. However, for these obtained N_{call}, the result of the Ori-Kriging method is not convergent until $N_{call} = 329$. Thus, our proposed advanced Kriging method is an available method for problems with implicit input–output relationship.

6. Conclusions

Reliability analysis of the engineering problems is becoming increasingly complex, because the problems are often involved in complex expensive-to-evaluate FEM. The approximate methods, such as FORM, rely on the accuracy of MPP, and the computational cost of sampling methods to analyze the FEM is unacceptable. Note that the meta-model based methods, such as Kriging model, which can balance the contradictions occurred in the above methods. However, the traditional use of Kriging cannot be efficient enough when facing the complex engineering problems, especially for those with implicit limit state functions.

Thus this paper proposes an efficient reliability analysis method based on an advanced Kriging model. Starting from the probabilistic point of view, we employ the probabilistic classification function to determine the highly uncertain region P_MPR and points, namely *ucp*, among these *ucp* we select a few ones with large values of product of marginal PDFs to enrich the experiment points. Besides, the leave-one-out technique is used to refine the Kriging model. Finally, MCS method is employed to implement reliability analysis when the high quality Kriging model is available.

Both explicit and implicit engineering applications are introduced to demonstrate the accuracy and efficiency of our proposed method. MCS and Subsim method have high accuracy but needs too many evaluations of the actual function, and the traditional method using Kriging also requires many evaluations and it is not efficient enough. Thus the proposed method is a good method to balance efficiency and accuracy and provides an available efficient method for reliability analysis.

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