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An improved regularization method for initial inverse problem in 2-D heat equation

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ABSTRACT

The main purpose of this article is to present a new method to regularize the initial inverse heat problem with inhomogeneous source. This problem is well known to be severely ill-posed. There are many regularization methods with error estimator of logarithmic order. An improved regularization method is proposed. The error estimates of Hölder type are obtained. Some numerical tests illustrate that the proposed method is feasible and effective.

1. Introduction

In this paper, we consider the non-homogeneous initial inverse heat problem (or called non-homogeneous backward heat problem) in a rectangle $I = (0,\pi) \times (0,\pi)$:

$$\begin{cases} u_t - u_{xx} - u_{yy} = f(x,y,t), (x,y,t) \in I \times (0,T), \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \quad (x,y,t) \in I \times (0,T), \\ u(x,y,T) = g(x,y), \quad (x,y) \in I, \end{cases} \tag{1}$$

where $g(x,y)$ and $f(x,y,z)$ are given. This problem is well known to be severely ill-posed and regularization methods for it are required (see [1]). As is known, the above problem is severely ill-posed, i.e. its solutions do not always exist and in the case of existence they do not depend continuously on the given data. In fact, from small noise contaminated from physical measurements, the corresponding solutions have large errors. That makes difficult to numerical calculations. Hence, a regularization is a need. Authors such as Ames and Hughes [2], Lattès and Lions [3], Showalter [4], Miller [5] and Payne [6] have approximated the Problem (1) by quasi-reversibility method and quasi-boundary value method. In [7], Schrsoter and Tautenhahn established an optimal error estimate for the homogeneous case of (1). A mollification method has been studied by Hao in [8]. Kirkup and Wadsworth used an operator-splitting method in [9]. A method of hyperbolic equation for backward heat has been considered by Masood and Zaman [10]. Very recently, the homogeneous Problem of (1) was also investigated by Hon and Takeuchi [11], Quin and Wei in [12], Rashidinia and Azarnavid in [13], Daripa et al. in [14], Wang in [15].

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Although there are many papers on the homogeneous case of the initial inverse heat problem, we only find a few results on the non-homogeneous case, especially the two-dimensional non-homogeneous case is very rare. For the case, we refer the reader to some recent works of Feng et al. [16], Li et al. [17], Nam et al. [18], Trong et al. [19–21] etc. Physically, g is measured data with an error of parameter ϵ. Let u and v^ϵ be the exact solution and the approximated solution of the backward heat problem respectively. In [20,22] , the errors are of order $\frac{1}{1+\ln\frac{T}{\epsilon}}$. The error estimates in [23] are $\epsilon^{\frac{t}{T}}$ for $t>0$ and $(\ln\frac{1}{\epsilon})^{\frac{-1}{4}}$ for $t=0$. Very recently, Trong and Tuan [22] improved the previous stability results which is of order $\epsilon^{\frac{t}{T}}\left(\frac{T}{1+\ln\left(\frac{T}{\epsilon}\right)}\right)^{1-\frac{t}{T}}$. For the literature on non-homogeneous backward heat, we refer the reader to the results in [16,24,25]. However, the error estimates in the mentioned papers are still of logarithmic order.

Very recently, Nam et al. [18] regularized the Problem (1) by truncation method and obtained the error estimate which is of order $\epsilon^q,\ 0<q<1$. Using this method, Tuan and Trong [26] considered a general version of the Problem (1) with similar results. The truncation method introduced in [18,26] is simple and effective to solve the backward heat problem with good estimates. However, in practice, the computation of the approximation solution (by the truncation method) is impossible and difficult when we consider the problem in a general two-dimensional domain (see [27]). If the spectral problem of operator $-\Delta$ in this domain is unknown then the truncation method is seems to be useless. This is may be a disadvantage point of papers [18,26]. Motivated by this reason, in the present paper, we provide another regularization method to established the Hölder estimates. Our method is similar to quasi-boundary value method (or non-local boundary value problems method, see [28,29,21]) but it seems to be in a new direction. In few words, we explain why this method is new. By a natural way, to approximate the solution of the Problem (1), in many previous methods, we usually propose a regularized solution u^ϵ for $t\in[0,T]$, then estimate the error $\|u^\epsilon(.,t)-u(.,t)\|$ (norm in L^2) for all $t\in[0,T]$. The method in this paper is first to compute the regularized solution for $t\in[0,T+(h-1)T]$ where $h\geqslant 1$, then use the resulting solution at $t+(h-1)T$ to approximate the exact solution at t. Under some suitable conditions on the exact solution, we will introduce the error which is of order $\epsilon^q(\ln(\frac{1}{\epsilon}))^{-p}$ for $p>0,\ 0<q<1$. This type of error is not introduced in many related results.

The paper is structured as follows. In Section 2, we present the solution of the 2-D initial inverse heat problem. In Section 3, motivated by the idea coming from Tuan and Trong [21], we establish stability results for our problem and propose a new strategy with Hölder estimates. The proofs of the main theoretical results will be given in Section 4. In Section 5, the numerical results of our regularized method are presented, which proved the effectiveness of our method.

2. The ill-posed initial inverse heat problem

Throughout this paper, we denote $\langle .,.\rangle, \|.\|$ by the inner product and the norm in L^2 respectively. Let us first make clear what a weak solution of the Problem (1) is. We call a function $u\in C\Big([0,T];\ L^2(I)\Big)\cap C^1((0,T);\ L^2(I))$ to be a weak solution for the (1) if

$$\frac{d}{dt}\langle u(.,.,t),W\rangle-\langle\Delta u(.,.,t),W\rangle=\langle f(.,.,t),W\rangle \tag{2}$$

for all functions $W(x,y)\in H^2(I)\cap H^1_0(I)$. In fact, it is enough to choose W in the orthogonal basis $\{\frac{2}{\pi}\sin(px)\sin(qy)\}_{p,q\geqslant 1}$ and the formula (2) is equivalent to

$$u_{pq}(t)=e^{(T-t)(p^2+q^2)}g_{pq}-\int_t^T e^{(s-t)(p^2+q^2)}f_{pq}(s)ds,\quad \forall\ p,q\geqslant 1, \tag{3}$$

which may also be written formally as

$$u(x,y,t)=\sum_{p,q=1}^{\infty}\left(e^{(T-t)(p^2+q^2)}g_{pq}-\int_t^T e^{(s-t)(p^2+q^2)}f_{pq}(s)ds\right)\sin(px)\sin(qy), \tag{4}$$

where for

$$u_{pq}(t)=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi u(x,y,t)\sin(px)\sin(qy)dxdy,$$
$$g_{pq}=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(x,y)\sin(px)\sin(qy)dxdy,$$
$$f_{pq}(t)=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y,t)\sin(px)\sin(qy)dxdy.$$

Let the function $f\in L^2((0,T);\ L^2(I))$ and $g\in L^2(I)$ be given. Note that the expression (4) is the solution of Problem (1) if it exists. In the following Theorem, we provide a condition of its existence.

Theorem 2.1. *If the Problem (1) has a solution u then we have*

$$\sum_{p,q=1}^{\infty}\left(e^{T(p^2+q^2)}g_{pq}-\int_0^T e^{s(p^2+q^2)}f_{pq}(s)ds\right)^2<\infty. \tag{5}$$

Else if (5) holds then the Problem (1) has a unique solution.

Proof. Suppose the Problem (1) has a solution $u\in C([0,T];\ H_0^1(I))\cap C^1((0,T);\ L^2(I))$, then u is defined by

$$u(x,y,t)=\sum_{p,q=1}^{\infty}\left(e^{-(t-T)(p^2+q^2)}g_{pq}-\int_t^T e^{-(t-s)(p^2+q^2)}f_{pq}(s)ds\right)\sin(px)\sin(qy). \tag{6}$$

This implies

$$u_{pq}(0)=e^{T(p^2+q^2)}g_{pq}-\int_0^T e^{s(p^2+q^2)}f_{pq}(s)ds. \tag{7}$$

Then

$$\|u(.,.,0)\|^2=\sum_{p,q=1}^{\infty}\left(e^{T(p^2+q^2)}g_{pq}-\int_0^T e^{s(p^2+q^2)}f_{pq}(s)ds\right)^2<\infty. \tag{8}$$

If (5) holds, then we define

$$v(x,y)=\frac{\pi^2}{4}\sum_{p,q=1}^{\infty}\left(e^{T(p^2+q^2)}g_{pq}-\int_0^T e^{s(p^2+q^2)}f_{pq}(s)ds\right)\sin(px)\sin(qy).$$

Since (8), we see that $v\in L^2(I)$.
Consider the problem

$$\begin{cases} u_t-u_{xx}-u_{yy}=f(x,y,t),\\ u(0,y,t)=u(\pi,y,t)=u(x,0,t)=u(x,\pi,t)=0, \quad t\in(0,T),\\ u(x,y,0)=v(x,y), \quad (x,y)\in(0,\pi)\times(0,\pi). \end{cases} \tag{9}$$

It is clear that (9) is the direct problem so it has a unique solution u (see [30]). We have

$$u(x,y,t)=\sum_{p,q=1}^{\infty}\left(e^{-t(p^2+q^2)}\langle v(x,y),\sin(px)\sin(qy)\rangle+\int_0^t e^{(s-t)(p^2+q^2)}f_{pq}(s)ds\right)\sin(px)\sin(qy). \tag{10}$$

By letting $t=T$ in (10), we obtain

$$\begin{aligned} u(x,y,T)&=\sum_{p,q=1}^{\infty}\left[e^{-T(p^2+q^2)}\left(e^{T(p^2+q^2)}g_{pq}-\int_0^T e^{s(p^2+q^2)}f_{pq}(s)ds\right)+\int_0^T e^{(s-T)(p^2+q^2)}f_{pq}(s)ds\right]\sin(px)\sin(qy)\\ &=\sum_{p,q=1}^{\infty}g_{pq}\sin(px)\sin(qy)=g(x,y). \end{aligned}$$

Hence, u is a solution of (1). To prove the uniqueness of the solution to the Problem (1), we refer the readers to [21]. □

3. The main theoretical results

Let us recall $(g,f)\in L^2(I)\times L^2(0,T:L^2(I))$ be the exact data. Assume that the noisy data $(g^\epsilon,f^\epsilon)\in L^2(I)\times L^2(0,T:L^2(I))$ satisfies

$$\|g^\epsilon-g\|\leqslant\epsilon;\quad \|f^\epsilon(.,t)-f(.,t)\|_{L^2(0,T:L^2(I))}=\left(\int_0^T\|f^\epsilon(.,t)-f(.,t)\|^2dt\right)^{\frac{1}{2}}\leqslant\epsilon.$$

In this paper, we establish a method to regularize the Problem (1). In fact, letting $h\geqslant 1$ be a fixed number, we denote $T_h=hT,\ T_{h-1}=(h-1)T$. Let β be a positive constant (is called parameter regularization) which depends on ϵ such that $\lim_{\epsilon\to0}\beta=0$. We consider the following well-posed problem

$$\begin{cases} u_t^\epsilon-u_{xx}^\epsilon-u_{yy}^\epsilon=\sum_{p,q=1}^{\infty}\frac{e^{-T_h(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}}f_{pq}(t)\sin(px)\sin(qy), \quad (x,y,t)\in I\times(0,T_h),\\ u^\epsilon(0,y,t)=u^\epsilon(\pi,y,t)=u^\epsilon(x,0,t)=u^\epsilon(x,\pi,t)=0, \quad (x,y,t)\in I\times[0,T_h],\\ u^\epsilon(x,y,T_h)=\sum_{p,q=1}^{\infty}\frac{e^{-T_h(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}}g_{pq}\sin(px)\sin(qy), \quad (x,y)\in I, \end{cases} \tag{11}$$

and the regularized problem

$$\begin{cases} v^\epsilon_t - v^\epsilon_{xx} - v^\epsilon_{yy} = \sum_{p,q=1}^{\infty} \frac{e^{-T_h(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}} f^\epsilon_{pq}(t)\sin(px)\sin(qy), \quad (x,y,t)\in I\times(0,T_h), \\ v^\epsilon(0,y,t) = v^\epsilon(\pi,y,t) = v^\epsilon(x,0,t) = v^\epsilon(x,\pi,t) = 0, \quad (x,y,t)\in I\times[0,T_h], \\ v^\epsilon(x,y,T_h) = \sum_{p,q=1}^{\infty} \frac{e^{-T_h(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}} g^\epsilon_{pq}\sin(px)\sin(qy), \quad (x,y)\in I, \end{cases} \tag{12}$$

where $g_{pq},\ g^\epsilon_{pq},\ f_{pq}(t)$ are defined by

$$g_{pq} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g(x,y)\sin(px)\sin(qy)dxdy,$$

$$g^\epsilon_{pq} = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi g^\epsilon(x,y)\sin(px)\sin(qy)dxdy,$$

$$f_{pq}(t) = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f(x,y,t)\sin(px)\sin(qy)dxdy,$$

$$f^\epsilon_{pq}(t) = \frac{4}{\pi^2}\int_0^\pi\int_0^\pi f^\epsilon(x,y,t)\sin(px)\sin(qy)dxdy,$$

and then we take $v^\epsilon(.,.,t+T_{h-1})$ of (12) as an approximation to $u(.,.,t)$. The main purpose of our method is of considering the error

$$\|v^\epsilon(.,.,t+T_{h-1}) - u(.,.,t)\|.$$

Notice readers that when $h=1$, Problem (11) is considered in [21] (see p. 874). However, the error estimates in [21] is only the logarithmic form (see Theorem 2.3, p. 878, [21]). Thus, our present work significantly improves related results on the backward problem (we mention this in Remark 1).

Theorem 3.1. *Let $g\in L^2(I)$ be the function such that $\|g_{xx}+g_{yy}\|<\infty$.*

(i) *If the regularized solution $u^\epsilon(x,y,T_{h-1})$ converges in $L^2(I)$, then the Problem (1) has a unique solution u. Furthermore, $u^\epsilon(x,y,t+T_{h-1})$ converges to $u(x,y,t)$ uniformly in t as ϵ tends to zero.*

(ii) *If there is a positive constant A_1 such that*

$$\|u_{xx}(.,.,0)+u_{yy}(.,.,0)\|\leqslant A_1,$$

then with $\beta=\epsilon$ one has

$$\|v^\epsilon(.,.,T_{h-1}) - u(.,.,0)\| \leqslant \sqrt{2+T}\,\epsilon^{1-\frac{1}{h}}\left(\frac{hT}{1+\ln\left(\frac{hT}{\epsilon}\right)}\right)^{\frac{1}{h}} + A_1\frac{hT}{\ln\left(\frac{hT}{\epsilon}\right)}. \tag{13}$$

(iii) *If there are positive constants $\gamma\in(0,hT]$ and A_2 such that*

$$\frac{\pi^2}{4}\sum_{p,q=1}^{\infty}|(p^2+q^2)e^{\gamma(p^2+q^2)}\langle u(x,y,t),\sin(px)\sin(qy)\rangle|^2\leqslant A_2^2 \tag{14}$$

for all $t\in[0,T]$, then with $\beta=\epsilon^{\frac{hT}{T+\gamma}}$, we have

$$\|v^\epsilon(.,.,t+T_{h-1}) - u(.,.,t)\| \leqslant \begin{cases} B_2\epsilon^{\frac{\gamma}{T+\gamma}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{\gamma}{hT}-1}, & 0<\gamma\leqslant(h-1)T \\ B_2\epsilon^{\frac{\gamma}{T+\gamma}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{-1}{h}}, & (h-1)T<\gamma\leqslant hT \end{cases} \tag{15}$$

for all $t\in[0,T]$, where the function $H(m,n)$ is defined by $H(m,n)=n^{1-\frac{m}{n}}$ for all $0\leqslant m\leqslant n$ and

$$B_2 = H(\gamma,T_h)A_2 + \sqrt{2+T}\,\epsilon^{\frac{t}{T+\gamma}}(hT)^{\frac{1}{h}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{t}{hT}}. \tag{16}$$

Remark 1.

1. If $h = 1$ then the error (13) is similar to the results in [21](Theorem 2.3, p. 878). In many results concerning the backward heat, the optimal error is of order $(\ln\frac{T}{\epsilon})^{-m}$ where $m > 0$. The error order of logarithmic form has been investigated in many recent papers, such as [31,28,24,29,16,25,19,23,20,32,22,21]. The logarithmic type estimate is, in general, much worse than any Hölder type estimate, i.e. ϵ^q for some $q > 0$. To retain the Hölder order in $[0, T]$, we should introduce a different priori assumption on u such as (14).
2. As we know, the convergence rate of $\epsilon^{\frac{\gamma}{T+\gamma}}\left[\ln\left(\frac{T}{\frac{hT}{\epsilon^{T+\gamma}}}\right)\right]^{\frac{\gamma}{hT}-1}$ or $\epsilon^{\frac{\gamma}{T+\gamma}}\left[\ln\left(\frac{T}{\frac{hT}{\epsilon^{T+\gamma}}}\right)\right]^{\frac{-1}{h}}$ is faster than the Hölder type estimate ϵ^k, for any $0 < k < \frac{\gamma}{T+\gamma}$ when $\epsilon \to 0$. This implies that the error (15) is effective and useful.
3. The method in this paper is inherently restricted to the square domain, and does not apply to more general domain due to its reliance on the Fourier method. We hope that in the future, we can derive similar estimates without resorting to the Fourier method. The method can be applied to fairly general domains is very difficult and will be presented in next reports.

4. Proof of the main theoretical results

Lemma 1. *For $0 < \alpha < en,\ 0 \leqslant m \leqslant n$, the following inequalities are true*

$$(a)\ \frac{1}{\alpha x + e^{-xn}} \leqslant \frac{n}{\alpha\left(1 + \ln\left(\frac{n}{\alpha}\right)\right)}, \tag{17}$$

$$(b)\ \frac{e^{-xm}}{\alpha x + e^{-xn}} \leqslant H(m,n)\alpha^{\frac{m}{n}-1}\left[1 + \ln\left(\frac{n}{\alpha}\right)\right]^{\frac{m}{n}-1}, \tag{18}$$

where the function $H(m,n)$ is defined by $H(m,n) = n^{1-\frac{m}{n}}$.

Proof. The proof of part (a) of this lemma can be found in [20].

Proof of (b). We have

$$\frac{e^{-xm}}{\alpha x + e^{-xn}} = \frac{e^{-xm}}{(\alpha x + e^{-xn})^{\frac{m}{n}}(\alpha x + e^{-xn})^{1-\frac{m}{n}}} \leqslant \frac{1}{(\alpha x + e^{-xn})^{1-\frac{m}{n}}} \leqslant \left(\frac{n}{\alpha\left(1 + \ln\left(\frac{n}{\alpha}\right)\right)}\right)^{1-\frac{m}{n}} \leqslant H(m,n)\alpha^{\frac{m}{n}-1}\left[\ln\left(\frac{n}{\alpha}\right)\right]^{\frac{m}{n}-1}.$$

□

Lemma 2. *The Problem (12) has a unique solution $v^\epsilon \in C\left([0,T];\ H_0^1(I)\right) \cap C^1((0,T);\ L^2(I))$ satisfying*

$$v^\epsilon(x,y,t) = \sum_{p,q=1}^{\infty}\left(\frac{e^{-t(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)}}g^\epsilon_{pq} - \int_t^{T_h}\frac{e^{(s-t-T_h)(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)}}f_{pq}(s)ds\right)\cdot\sin(px)\sin(qy) \quad 0 \leqslant t \leqslant T_h. \tag{19}$$

The solution also depends continuously on $g^\epsilon \in L^2(I)$ and we have

$$\|u^\epsilon(.,.,t+T_{h-1}) - v^\epsilon(.,.,t+T_{h-1})\| \leqslant \sqrt{2+T}\epsilon\beta^{\frac{t-T}{hT}}\left(\frac{hT}{1 + \ln\left(\frac{hT}{\beta}\right)}\right)^{\frac{T-t}{hT}}, \quad 0 \leqslant t \leqslant T. \tag{20}$$

Proof. The proof of this lemma is divided into two steps.

Step 1. The existence, the uniqueness and the stability of a solution of (12) can be found in the paper [21] (Theorem 2.1, p. 875).

Step 2. We shall prove (20).

In fact, from u^ϵ, v^ϵ are solutions of Problem (1) corresponding to the exact data g and noisy data g^ϵ respectively, we have

$$u^\epsilon(x,y,t) = \sum_{p,q=1}^{\infty}\left(\frac{e^{-t(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)}}g_{pq} - \int_t^{T_h}\frac{e^{(s-t-T_h)(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)}}f_{pq}(s)ds\right)\cdot\sin(px)\sin(qy) \quad 0 \leqslant t \leqslant T_h, \tag{21}$$

$$v^\epsilon(x,y,t) = \sum_{p,q=1}^{\infty}\left(\frac{e^{-t(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)}}g^\epsilon_{pq} - \int_t^{T_h}\frac{e^{(s-t-T_h)(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)}}f^\epsilon_{pq}(s)ds\right)\cdot\sin(px)\sin(qy) \quad 0 \leqslant t \leqslant T_h. \tag{22}$$

Thus using Lemma 1b, we get

$$\|u^\epsilon(.,.,t)-v^\epsilon(.,.,t)\|^2=\frac{\pi^2}{2}\sum_{p,q=1}^{\infty}\left(\frac{e^{-t(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}}(g^\epsilon_{pq}-g_{pq})\right)^2$$
$$+\frac{\pi^2}{2}\sum_{p,q=1}^{\infty}\left(\int_t^{T_h}\frac{e^{(s-t-T_h)(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}}(f^\epsilon_{pq}(s)-f_{pq}(s)ds)\right)^2$$
$$\leqslant \beta^{\frac{2t}{T_h}-2}\left(\frac{T_h}{1+\ln\left(\frac{T_h}{\beta}\right)}\right)^{2-\frac{2t}{T_h}}\left[2\|g^\epsilon-g\|^2+\left(\int_t^{T_h}(f^\epsilon_{pq}(s)-f_{pq}(s)ds)\right)^2\right].$$

By replace t with $t=t+T_{h-1}$, we obtain

$$\|u^\epsilon(.,.,t+T_{h-1})-v^\epsilon(.,.,t+T_{h-1})\|^2\leqslant \beta^{\frac{2t+2T_{h-1}}{T_h}-2}\left(\frac{T_h}{1+\ln\left(\frac{T_h}{\beta}\right)}\right)^{2-\frac{2t+2T_{h-1}}{T_h}}\left[2\|g^\epsilon-g\|^2+\left(\int_{t+T_{h-1}}^{T_h}(f^\epsilon_{pq}(s)-f_{pq}(s)ds)\right)^2\right]$$
$$=\beta^{\frac{2t-2T}{hT}}\left(\frac{hT}{1+\ln\left(\frac{hT}{\beta}\right)}\right)^{\frac{2T-2t}{hT}}\left(2\|g^\epsilon-g\|^2+T\|f^\epsilon(.,t)-f(,t)\|^2_{L^2(0,T;L^2(I))}\right),$$
$$\leqslant (2+T)\epsilon^2\beta^{\frac{2t-2T}{hT}}\left(\frac{hT}{1+\ln\left(\frac{hT}{\beta}\right)}\right)^{\frac{2T-2t}{hT}}.$$

The lemma is proved. □

Now we are in a position to prove the main result.

4.1. Proof of Theorem 3.1

4.1.1. Proof of Part (i)

Assume that $\lim_{\epsilon\to 0}u^\epsilon(x,y,T_{h-1})=w(x,y)\in L^2(I)$ exists. We put

$$u(x,y,t)=\sum_{p,q=1}^{\infty}\left(e^{-t(p^2+q^2)}w_{pq}+\int_0^t e^{(s-t)(p^2+q^2)}f_{pq}(s)ds\right)\sin(px)\sin(qy),\quad 0\leqslant t\leqslant T, \tag{23}$$

where

$$w_{pq}=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi w(x,y)\sin(px)\sin(qy)dxdy.$$

We shall prove that $u(x,y,t)$ is the unique solution of the Problem (1). First, it is clear to see that u satisfies the system

$$\begin{cases} u_t-u_{xx}-u_{yy}=f(x,y,t),(x,y,t)\in I\times(0,T),\\ u(0,y,t)=u(\pi,y,t)=u(x,0,t)=u(x,\pi,t)=0,\quad (x,y,t)\in I\times(0,T).\end{cases} \tag{24}$$

Next, we show that $u(x,y,T)=g(x,y)$. In fact, we have the formula of $u^\epsilon(x,y,t+T_{h-1})$

$$u^\epsilon(x,y,t+T_{h-1})=\sum_{p,q=1}^{\infty}\left(e^{-(t+T_{h-1})(p^2+q^2)}u^\epsilon_{pq}(T_{h-1})-\int_{T_{h-1}}^{t+T_{h-1}}\frac{e^{(s-t-T_{h-1}-T_h)(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}}f_{pq}(s)ds\right)\sin(px)\sin(qy), \tag{25}$$

where

$$u^\epsilon_{pq}(T_{h-1})=\frac{4}{\pi^2}\int_0^\pi\int_0^\pi u^\epsilon(x,y,T_{h-1})\sin(px)\sin(qy)dxdy.$$

Combining (23) and (25), we obtain

$$\langle u^\epsilon(x,y,t+T_{h-1})-u(x,y,t),\sin(px)\sin(qy)\rangle=e^{-t(p^2+q^2)}\left(u^\epsilon_{pq}(T_{h-1})-w_{pq}(t)\right)+\int_{T_{h-1}}^{t+T_{h-1}}$$
$$\times\frac{e^{(s-t-T_{h-1}-T_h)(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}}f_{pq}(s)ds-\int_0^t e^{(s-t)(p^2+q^2)}f_{pq}(s)ds. \tag{26}$$

By change variables $s_1 = s - T_{h-1}$, the second term of the right hand side of (26) can be rewritten as follows

$$\int_{T_{h-1}}^{t+T_{h-1}} \frac{e^{(s-t-T_{h-1}-T_h)(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}} f_{pq}(s)ds = \int_0^t \frac{e^{(s_1+T_{h-1}-t-T_{h-1}-T_h)(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}} f_{pq}(s_1)ds_1 = \int_0^t \frac{e^{(s-t-T_h)(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}} f_{pq}(s)ds. \quad (27)$$

Combining (26) and (27), we have

$$\langle u^\epsilon(x,y,t+T_{h-1}) - u(x,y,t), \sin(px)\sin(qy)\rangle = e^{-t(p^2+q^2)}\left(u^\epsilon_{pq}(T_{h-1}) - w_{pq}(t)\right) - \int_0^t \frac{\beta e^{(s-t)(p^2+q^2)}(p^2+q^2)}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}} f_{pq}(s)ds.$$

By using the inequality (17), we get

$$|\langle u^\epsilon(x,y,t+T_{h-1}) - u(x,y,t), \sin(px)\sin(qy)\rangle| \leqslant |u^\epsilon_{pq}(T_{h-1}) - w_{pq}(t)| + \int_0^t \frac{T_h}{\ln\left(\frac{T_h}{\beta}\right)}(p^2+q^2)|f_{pq}|(s)ds.$$

By using the inequality $(a+b)^2 \leqslant 2(a^2+b^2)$, we arrive at

$$\begin{aligned}\|u^\epsilon(x,y,t+T_{h-1}) - u(x,y,t)\|^2 &\leqslant \frac{\pi^2}{2}\sum_{p,q=1}^{\infty}\left(u^\epsilon_{pq}(T_{h-1}) - w_{pq}\right)^2 + \frac{\pi^2}{2}T\sum_{p,q=1}^{\infty}\int_0^t\left(\frac{T_h}{\ln\left(\frac{T_h}{\beta}\right)}(p^2+q^2)|f_{pq}|(s)\right)^2 ds \\ &\leqslant 2\|u^\epsilon(.,.,T_{h-1}) - w(.,.)\|^2 + \frac{T\pi^2}{2}\left(\frac{T_h}{\ln\left(\frac{T_h}{\beta}\right)}\right)^2\int_0^t\sum_{p,q=1}^{\infty}(p^2+q^2)^2 f_{pq}^2(s)ds \\ &\leqslant 2\|u^\epsilon(.,.,T_{h-1}) - w(.,.)\|^2 + 2T\left(\frac{T_h}{\ln\left(\frac{T_h}{\beta}\right)}\right)^2\int_0^T\|f_{xx}(.,.,s) + f_{yy}(.,.,s)\|^2 ds.\end{aligned}$$

It gives $\lim_{\epsilon\to 0} u^\epsilon(x,y,t+T_{h-1}) = u(x,y,t)$. Thus $\lim_{\epsilon\to 0} u^\epsilon(x,y,T_h) = u(x,y,T)$. On the other hand, we have

$$\begin{aligned}\|u^\epsilon(.,.,T_h) - g(.,.)\|^2 &= \frac{\pi^2}{4}\sum_{p,q=1}^{\infty}\left[\frac{e^{-T_h(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)})} - 1\right]^2 g_{pq}^2 = \frac{\pi^2}{4}\sum_{p,q=1}^{\infty}\left[\frac{\beta(p^2+q^2)}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)})}\right]^2 g_{pq}^2 \\ &\leqslant \frac{\pi^2}{4}\left(\frac{T_h}{\ln\left(\frac{T_h}{\beta}\right)}\right)^2\sum_{p,q=1}^{\infty}(p^2+q^2)^2 g_{pq}^2.\end{aligned}$$

Thus

$$\|u^\epsilon(.,.,T_h) - g(.,.)\| \leqslant \frac{\pi^2}{4}\left(\frac{T_h}{\ln\left(\frac{T_h}{\beta}\right)}\right)\sqrt{\sum_{p,q=1}^{\infty}(p^2+q^2)^2 g_{pq}^2} = \left(\frac{T_h}{\ln\left(\frac{T_h}{\beta}\right)}\right)\|g_{xx} + g_{yy}\|.$$

Therefore $\lim_{\epsilon\to 0} u^\epsilon(x,y,T_h) = g(x,y)$. This implies $u(x,y,T) = g(x,y)$. Hence, $u(x,t)$ is the unique solution of the Problem (1). We also see that $u^\epsilon(x,y,t+T_{h-1})$ converges to $u(x,y,t)$ uniformly in t.

4.1.2. Proof of part (ii)

Using the triangle inequality, we get

$$\|u(.,.,0) - v^\epsilon(.,.,T_{h-1})\| \leqslant \|u(.,0) - u^\epsilon(.,.,T_{h-1})\| + \|u^\epsilon(.,.,T_{h-1}) - v^\epsilon(.,.,T_{h-1})\|. \quad (28)$$

First, from (20) in Theorem 2, we estimate

$$\|u^\epsilon(.,.,T_{h-1}) - v^\epsilon(.,.,T_{h-1})\| \leqslant \sqrt{2+T}\epsilon\beta^{\frac{-1}{h}}\left(\frac{hT}{1+\ln\left(\frac{hT}{\beta}\right)}\right)^{\frac{1}{h}}. \quad (29)$$

Next, we continue to estimate the error $\|u(.,.,0) - u^\epsilon(.,.,T_{h-1})\|$. From (21), we have

$$\begin{aligned}u^\epsilon(x,y,T_{h-1}) &= \sum_{p,q=1}^{\infty}\left(\frac{e^{-T_{h-1}(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)})}g_{pq} - \int_{T_{h-1}}^{T_h}\frac{e^{(s-T_{h-1}-T_h)(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)}}f_{pq}(s)ds\right)\cdot\sin(px)\sin(qy) \\ &= \sum_{p,q=1}^{\infty}\frac{e^{-T_{h-1}(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)})}\left(g_{pq} - \int_{T_{h-1}}^{T_h}e^{(s-T_h)(p^2+q^2)}f_{pq}(s)ds\right)\sin(px)\sin(qy). \quad (30)\end{aligned}$$

On the other hand, by direct calculation, we get

$$\int_{T_{h-1}}^{T_h} e^{(s-T_h)(p^2+q^2)} f_{pq}(s)ds = \int_0^T e^{(s-T)(p^2+q^2)} f_{pq}(s)ds,$$

and since

$$u(x,y,0) = \sum_{p,q=1}^{\infty} e^{T(p^2+q^2)} \left(g_{pq} - \int_0^T e^{(s-T)(p^2+q^2)} f_{pq}(s)ds \right) \sin(px)\sin(qy), \tag{31}$$

gives us

$$g_{pq} - \int_0^T e^{(s-T)(p^2+q^2)} f_{pq}(s)ds = e^{-T(p^2+q^2)} \langle u(x,y,0), \sin(px)\sin(qy)\rangle. \tag{32}$$

Combining (30) and (32), we obtain

$$\begin{aligned}\langle u^\epsilon(x,y,T_{h-1}), \sin(px)\sin(qy)\rangle &= \frac{e^{-T_{h-1}(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)})} e^{-T(p^2+q^2)} \langle u(x,y,0), \sin(px)\sin(qy)\rangle \\ &= \frac{e^{-T_h(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)})} \langle u(x,y,0), \sin(px)\sin(qy)\rangle.\end{aligned} \tag{33}$$

From (31) and (33), we have

$$\langle u^\epsilon(x,y,T_{h-1}) - u(x,y,0), \sin(px)\sin(qy)\rangle = \frac{-\beta(p^2+q^2)}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)})} \langle u(x,y,0), \sin(px)\sin(qy)\rangle.$$

It follows from (17) that

$$\begin{aligned}\|u^\epsilon(.,.,T_{h-1}) - u(.,.,0)\|^2 &= \frac{\pi^2}{4} \sum_{p,q=1}^{\infty} |\langle u^\epsilon(x,y,T_{h-1}) - u(x,y,0), \sin(px)\sin(qy)\rangle|^2 \\ &= \frac{\pi^2}{4} \sum_{p,q=1}^{\infty} \left(\frac{\beta(p^2+q^2)}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)})} \right)^2 |\langle u(x,y,0), \sin(px)\sin(qy)\rangle|^2 \\ &\leqslant \left(\frac{T_h}{\ln\left(\frac{T_h}{\beta}\right)} \right)^2 \|u_{xx}(.,.,0) + u_{yy}(.,.,0)\|^2.\end{aligned}$$

Hence

$$\|u^\epsilon(.,.,T_{h-1}) - u(.,.,0)\| \leqslant \frac{hT}{\ln\left(\frac{hT}{\beta}\right)} A_1. \tag{34}$$

From $\beta = \epsilon$ and combining (28), (29) and (34), we conclude that

$$\|u(.,.,0) - v^\epsilon(.,.,T_{h-1})\| \leqslant \sqrt{2+T}\epsilon\beta^{\frac{-1}{h}} \left(\frac{hT}{1+\ln\left(\frac{hT}{\beta}\right)} \right)^{\frac{1}{h}} + \frac{hT}{\ln\left(\frac{hT}{\beta}\right)} A_1 \leqslant \sqrt{2+T}\epsilon^{1-\frac{1}{h}} \left(\frac{hT}{1+\ln\left(\frac{hT}{\epsilon}\right)} \right)^{\frac{1}{h}} + \frac{hT}{\ln\left(\frac{hT}{\epsilon}\right)} A_1.$$

4.1.3. Proof of Part (iii)

It follows from (21) that

$$\begin{aligned}u^\epsilon(x,y,t+T_{h-1}) &= \sum_{p,q=1}^{\infty} \left(\frac{e^{-(t+T_{h-1})(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)})} g_{pq} - \int_{t+T_{h-1}}^{T_h} \frac{e^{(s-t-T_{h-1}-T_h)(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)})} f_{pq}(s)ds \right) \sin(px)\sin(qy) \\ &= \sum_{p,q=1}^{\infty} \frac{e^{-(t+T_{h-1})(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)})} \left(g_{pq} - \int_{t+T_{h-1}}^{T_h} e^{(s-T_h)(p^2+q^2)} f_{pq}(s)ds \right) \sin(px)\sin(qy) \\ &= \sum_{p,q=1}^{\infty} \frac{e^{-(t+T_{h-1})(p^2+q^2)}}{\beta(p^2+q^2) + e^{-T_h(p^2+q^2)})} \left(g_{pq} - \int_t^T e^{(s-T)(p^2+q^2)} f_{pq}(s)ds \right) \sin(px)\sin(qy).\end{aligned}$$

Moreover, from

$$g_{pq} - \int_t^T e^{(s-T)(p^2+q^2)} f_{pq}(s)ds = e^{(t-T)(p^2+q^2)} \langle u(x,y,t), \sin(px)\sin(qy)\rangle,$$

leads to

$$|\langle u^\epsilon(x,y,t+T_{h-1}) - u(x,y,t), \sin(px)\sin(qy)\rangle| = \frac{\beta(p^2+q^2)e^{-\gamma(p^2+q^2)}}{\beta(p^2+q^2)+e^{-T_h(p^2+q^2)})}\left|e^{\gamma(p^2+q^2)}\langle u(x,y,t), \sin(px)\sin(qy)\rangle\right|$$
$$\leqslant H(\gamma,T_h)\beta^{\frac{\gamma}{T_h}}\left(\ln\left(\frac{T_h}{\beta}\right)\right)^{\frac{\gamma}{T_h}-1}\left|(p^2+q^2)e^{\gamma(p^2+q^2)}\langle u(x,y,t), \sin(px)\sin(qy)\rangle\right|.$$

Therefore, we have

$$\|u^\epsilon(.,.,t+T_{h-1}) - u(.,.,t)\|^2 = \frac{\pi^2}{4}\sum_{p,q=1}^{\infty}|\langle u^\epsilon(x,y,t+T_{h-1}) - u(x,y,t), \sin(px)\sin(qy)\rangle|^2$$
$$\leqslant \frac{\pi^2}{4}H(\gamma,T_h)^2\beta^{2\frac{\gamma}{T_h}}\left(\ln\left(\frac{T_h}{\beta}\right)\right)^{\frac{2\gamma}{T_h}-2}\sum_{p,q=1}^{\infty}|(p^2+q^2)e^{\gamma(p^2+q^2)}\langle u(x,y,t), \sin(px)\sin(qy)\rangle|^2$$
$$\leqslant H(\gamma,T_h)^2\beta^{2\frac{\gamma}{T_h}}\left(\ln\left(\frac{T_h}{\beta}\right)\right)^{\frac{2\gamma}{T_h}-2}\frac{\pi^2}{4}\sum_{p,q=1}^{\infty}|(p^2+q^2)e^{\gamma(p^2+q^2)}\langle u(x,y,t), \sin(px)\sin(qy)\rangle|^2$$
$$\leqslant H(\gamma,T_h)^2\beta^{2\frac{\gamma}{T_h}}\left(\ln\left(\frac{T_h}{\beta}\right)\right)^{\frac{2\gamma}{T_h}-2}A_2^2.$$

Thus

$$\|u^\epsilon(.,.,t+T_{h-1}) - u(.,.,t)\| \leqslant H(\gamma,T_h)\beta^{\frac{\gamma}{T_h}}\left(\ln\left(\frac{T_h}{\beta}\right)\right)^{\frac{\gamma}{T_h}-1}A_2. \quad (35)$$

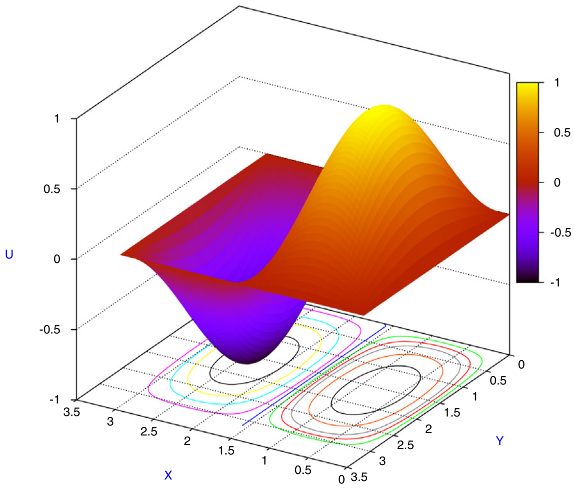
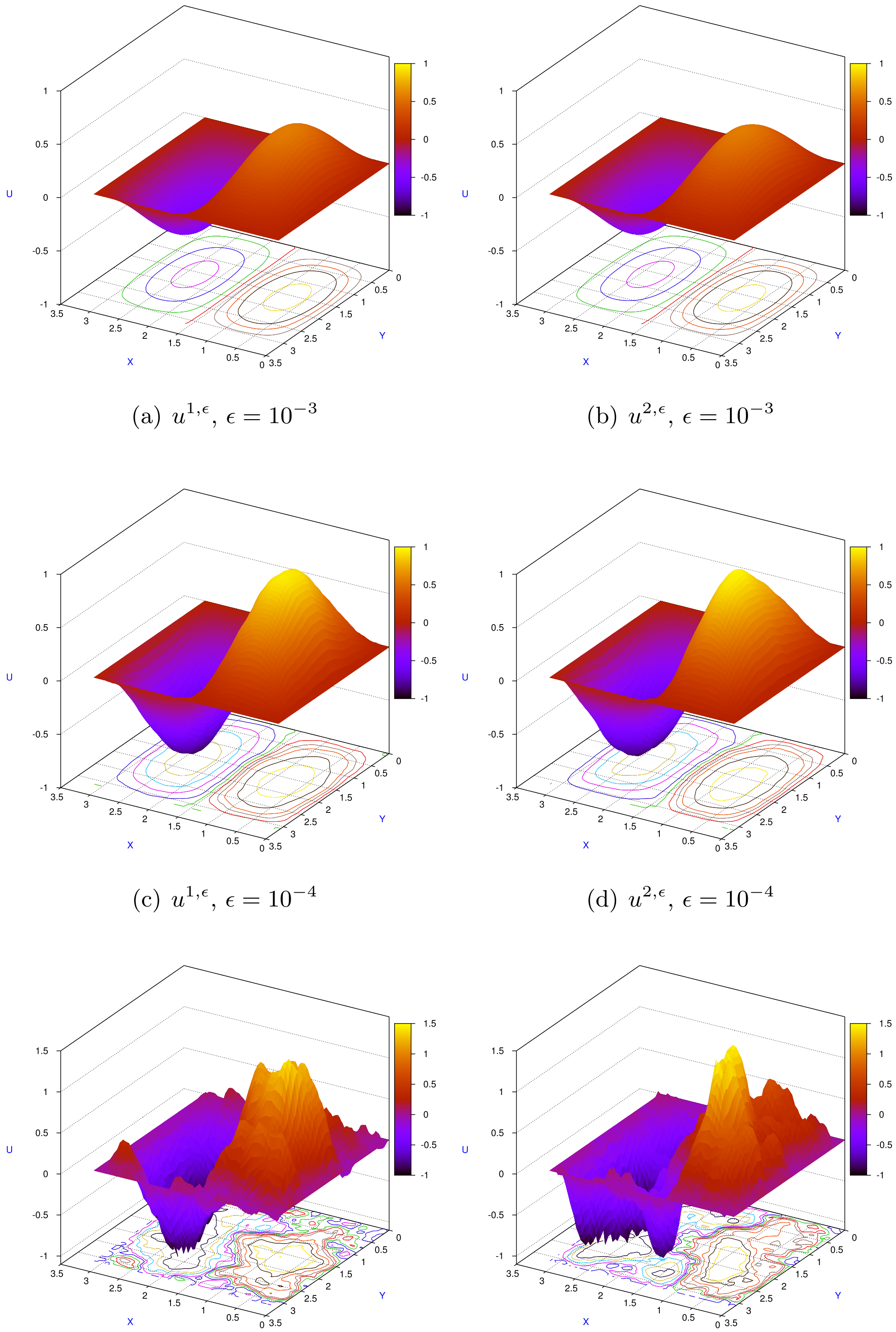


Fig. 1. Example 1, exact solutions at $t = 0$.

Table 1
Relative error for regularized solutions at $t = 0$; noise amplitude $\rho = 0$ (exact data), $\rho = 10^{-r}, r = 1,3,5$ (disturbed data) and $\epsilon = 10^{-k}, k = 1,\ldots,6$.

ϵ	$\rho = 0$		$\rho = 10^{-1}$		$\rho = 10^{-3}$		$\rho = 10^{-5}$	
	$\delta^{1,\epsilon}(\rho)$	$\delta^{2,\epsilon}(\rho)$	$\delta^{1,\epsilon}(\rho)$	$\delta^{2,\epsilon}(\rho)$	$\delta^{1,\epsilon}(\rho)$	$\delta^{2,\epsilon}(\rho)$	$\delta^{1,\epsilon}(\rho)$	$\delta^{2,\epsilon}(\rho)$
10^{-1}	9.867E−01	9.870E−01	9.807E−01	9.831E−01	9.866E−01	9.869E−01	9.867E−01	9.870E−01
10^{-2}	8.812E−01	8.859E−01	8.306E−01	8.540E−01	8.807E−01	8.852E−01	8.812E−01	8.859E−01
10^{-3}	4.258E−01	4.426E−01	7.145E−01	6.652E−01	4.233E−01	4.395E−01	4.258E−01	4.426E−01
10^{-4}	6.884E−02	7.493E−02	6.735E+00	5.675E+00	8.963E−02	8.511E−02	6.879E−02	7.491E−02
10^{-5}	1.061E−02	1.069E−02	6.020E+01	4.699E+01	4.754E−01	4.371E−01	1.041E−02	1.297E−02
10^{-6}	6.526E−02	5.770E−02	diverged	diverged	3.947E+00	3.863E+00	6.572E−02	8.443E−02

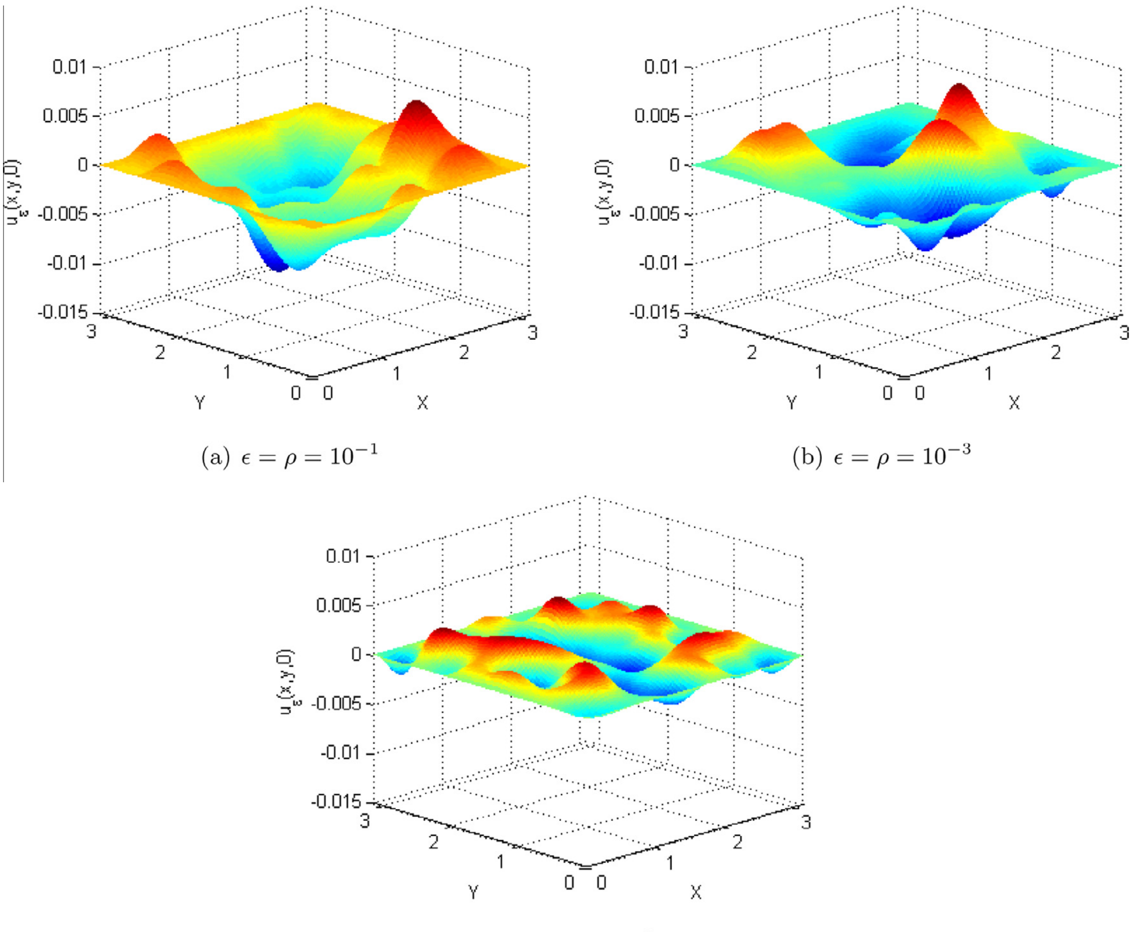


(a) $u^{1,\epsilon}$, $\epsilon = 10^{-3}$ (b) $u^{2,\epsilon}$, $\epsilon = 10^{-3}$

(c) $u^{1,\epsilon}$, $\epsilon = 10^{-4}$ (d) $u^{2,\epsilon}$, $\epsilon = 10^{-4}$

(e) $u^{1,\epsilon}$, $\epsilon = 10^{-5}$ (f) $u^{2,\epsilon}$, $\epsilon = 10^{-5}$

Fig. 2. Example 1, convergence and divergence of regularized solutions at $t = 0$, noise amplitude $\rho = 10^{-3}$.



(a) $\epsilon = \rho = 10^{-1}$ (b) $\epsilon = \rho = 10^{-3}$ (c) $\epsilon = \rho = 10^{-5}$

Fig. 3. Example 2, convergence of regularized solutions at $t = 0$.

Using the triangle inequality and (20), (35), we get

$$\begin{aligned}\|u(.,.,t) - v^\epsilon(.,.,t+T_{h-1})\| &\leqslant \|u(.,t) - u^\epsilon(.,.,t+T_{h-1})\| + \|u^\epsilon(.,.,t+T_{h-1}) - v^\epsilon(.,t+T_{h-1})\| \\ &\leqslant H(\gamma,T_h)\beta^{\frac{\gamma}{hT}}\left(\ln\left(\frac{hT}{\beta}\right)\right)^{\frac{\gamma}{hT}-1}A_2 + \sqrt{2+T}\,\beta^{\frac{t-T}{hT}}\left(\frac{hT}{1+\ln\left(\frac{hT}{\beta}\right)}\right)^{\frac{T-t}{hT}}\epsilon \\ &\leqslant H(\gamma,T_h)\epsilon^{\frac{\gamma}{T+\gamma}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{\gamma}{hT}-1}A_2 + \sqrt{2+T}\,\epsilon^{\frac{t+\gamma}{T+\gamma}}\left(\frac{hT}{1+\ln\left(\frac{hT}{\epsilon^{\frac{hT}{T+\gamma}}}\right)}\right)^{\frac{T-t}{hT}} \\ &\leqslant H(\gamma,T_h)\epsilon^{\frac{\gamma}{T+\gamma}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{\gamma}{hT}-1}A_2 + \sqrt{2+T}\,\epsilon^{\frac{t+\gamma}{T+\gamma}}(hT)^{\frac{1}{h}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{t-T}{hT}}. \end{aligned} \tag{36}$$

If $0 < \gamma \leqslant (h-1)T$, then $\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{\gamma}{hT}-1} \geqslant \left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{-1}{h}}$ and (36) can be written as follows

$$\|u(.,.,t) - v^\epsilon(.,.,t+T_{h-1})\| \leqslant \epsilon^{\frac{\gamma}{T+\gamma}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{\gamma}{hT}-1}\left[H(\gamma,T_h)A_2 + \sqrt{2+T}\,\epsilon^{\frac{t}{T+\gamma}}(hT)^{\frac{1}{h}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{t}{hT}}\right].$$

If $(h-1)T < \gamma \leqslant hT$ then (36) becomes

$$\|u(.,.,t) - v^\epsilon(.,.,t+T_{h-1})\| \leqslant \epsilon^{\frac{\gamma}{T+\gamma}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{-1}{h}}\left[H(\gamma,T_h)A_2 + \sqrt{2+T}\,\epsilon^{\frac{t}{T+\gamma}}(hT)^{\frac{1}{h}}\left[\ln\left(\frac{T}{\epsilon^{\frac{hT}{T+\gamma}}}\right)\right]^{\frac{t}{hT}}\right].$$

5. Numerical experiments

In reality, we do not know exact solution of the problem, hence, the priori assumptions (ii) and (iii) become implicit and may be skipped in practice. Thus, the regularized solution is expected that it is a reasonable solution of the problem. At the time, its certificate of use depends on its convergence.

We consider two examples of Problem (1) in the region $I = (0,\pi) \times (0,\pi)$:

$$\begin{cases} u_t - u_{xx} - u_{yy} = f(x,y,t), \quad (x,y,t) \in I \times (0,1), \\ u(0,y,t) = u(\pi,y,t) = u(x,0,t) = u(x,\pi,t) = 0, \quad (x,y,t) \in I \times (0,1), \\ u(x,y,1) = g(x,y), \quad (x,y) \in I. \end{cases} \tag{37}$$

Let $g^\rho(x,y)$ be the disturbed measure data such that $||g(x,y) - g^\rho(x,y)|| \leqslant \rho$.

(a) In the first example, we take:

$$g(x,y) = e^{-xy}\sin(2x)\sin(y),$$
$$g^\rho(x,y) = g(x,y) + \frac{\rho}{\pi}\cdot \texttt{rand()}\ ,$$
$$f(x,y,t) = -e^{-txy}\left((xy + t^2y^2 + t^2x^2 - 5)\sin y \sin 2x - 2tx\cos y \sin 2x - 4ty \cos 2x \sin y\right),$$
$$f^\rho(x,y,t) = -e^{-txy}\left((xy + t^2y^2 + t^2x^2 - 5)\sin y \sin 2x - 2tx\cos y \sin 2x - 4ty \cos 2x \sin y\right) + \frac{\rho}{\pi}\cdot \texttt{rand()}\ .$$

The problem has exact solution $u(x,y,t) = e^{-txy}\sin(2x)\sin(y)$. Select $h = 1.0 + 10^{-5}$, $\gamma = h/100$. For each regularization parameter $\beta_1 = \epsilon$ (from (ii)) and $\beta_2 = \epsilon^{\frac{h}{1+\gamma}}$ (from (iii)) we have the relative error for regularized solution is

$$\delta^{k,\epsilon}(\rho) = \frac{||u^{k,\epsilon} - u||}{||u||},$$

where $u^{k,\epsilon}$ is the regularized solution by parameter β_k. Let's observe the convergent behavior of the regularized solution regarding the changes of parameter ϵ and magnitude ρ of data error. Using our proposed method with two types of regularization parameter β_k, we set up the computation to find regularized solution at time $t = 0$, $\rho = 10^{-r}$, $r = 0,1,3,5$ and parameter $\epsilon = 10^{-k}$, $k = 1,\ldots,6$. Fig. 1 visualizes the exact solution at $t = 0$. The computational result is showed in type of relative error in Table 1 and in 3D graphs in Fig. 2. We see that the second method yields more accurate result though not very significant. Regularized solution converges as ϵ decreases but diverges when ϵ becomes smaller than ρ. Therefore, we make an assumption that the best regularized solution for problem with perturbed data is obtained when $\epsilon \approx \rho$.

(b) Consider the second example with given functions:

$$g(x,y) = x(\pi - x)y(\pi - y),$$
$$g^\rho(x,y) = g(x,y) + \frac{\rho}{\pi}\cdot \texttt{rand()}\ ,$$
$$f(x,y,t) = x(\pi - x)\left[\frac{1}{2}y(\pi - y) - 2t\right] + y(\pi - y)\left[\frac{1}{2}x(\pi - x) - 2t\right],$$
$$f(x,y,t) = x(\pi - x)\left[\frac{1}{2}y(\pi - y) - 2t\right] + y(\pi - y)\left[\frac{1}{2}x(\pi - x) - 2t\right] + \frac{\rho}{\pi}\cdot \texttt{rand()}\ .$$

It is easy to check that the exact solution $u(x,y,t) = tx(\pi - x)y(\pi - y)$. Hence, we have $u(x,y,0) = 0$. We will find the regularized solution at $t = 0$ as the approximation to plane $z = 0$. Using presumption in the first example, we will set $\epsilon = \rho$ and choose $\beta = \beta_1 = \epsilon$. With noise amplitude $\rho = 10^{-r}$, $r = 1,3,5$ the calculated relative errors for regularized solution are 0.349, 0.2172 and 0.1154, respectively. We also visualize 3D graphs of these regularized solutions in Fig. 3. As expected, they converge smoothly to the exact solution $z = 0$.

6. Conclusion

In this paper, we considered an improved regularization method for initial inverse heat problem in 2D case. In the theoretical results, we obtained the error estimation of Hölder type $\epsilon^m \ln^n(\frac{1}{\epsilon^k})$ for $m,n,k > 0$ based on the smooth assumptions of the exact solution. These estimates improve the results in many earlier works. Finally, in the numerical experiments, two numerical examples are presented to show that our proposed method is effective. In future work, we will consider the regularized problem for the following problem

$$\begin{cases} \frac{\partial u}{\partial t} = (b(x,y,t)u_x)_x + \left(c(x,y,t)u_y\right)_y + f(x,y,t), \quad (x,y,t) \in I \times (0,T), \\ u|_{\partial\Omega} = 0, \ t \in (0,T), \\ u(x,y,T) = g(x,y), \quad (x,y,t) \in I \times (0,T), \end{cases} \tag{38}$$

where $b(x,y,t)$ and $c(x,y,t)$ are functions depending on variables x,y,t.

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