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Diagrams in ancient Egyptian geometry Survey and assessment

Gregg De Young

Department of Mathematics and Actuarial Science, The American University in Cairo, P.O. Box 74, New Cairo, Egypt

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Abstract

This article surveys and catalogs the geometric diagrams that survive from ancient Egypt. These diagrams are often overspecified and some contain inaccuracies in their construction. The diagrams accompany algorithmic texts and support the mathematical programme of their authors. The study concludes with a brief comparison with the diagram traditions of ancient Babylon, early India, and Greece.

Résumé

Cet article étudie les diagrammes géométriques qui survivent dans l'Egypte antique. Ces diagrammes sont plus spécifiques qu'il n'est nécessaire pour le problème et certains contiennent des inexactitudes dans leur construction. Les diagrammes accompagnent des textes algorithmiques et soutiennent le programme mathématique de leurs auteurs. L'étude se termine par une comparaison brève aux traditions diagrammatiques à Babylone et en Grèce.

MSC: 01A16

Keywords: Diagrams; Egyptian geometry

1. Preliminary disclaimer

I am not a specialist in the study of ancient Egypt. I do not write about ancient Egyptian diagrams and mathematical texts based on my linguistic or mathematical background. I have chosen to focus on ancient Egypt because there has come down to us a clearly defined and limited body of diagrams for which good-quality photographs are readily

E-mail address: gdeyoung@aucegypt.edu

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available in the published literature [Clagett, 1999, 313–462; Parker, 1972, Plates 1–25]. I hope that by considering a complete and clearly delimited diagram tradition I can avoid possible historical pitfalls of "selective ignorance" in which the historian may focus on "interesting" cases while ignoring the broader context and the scope of the entire tradition.

2. Introduction

The mathematical tradition of ancient Egypt has been studied for more than a century, and assessments of that tradition have varied considerably. Most of these studies have focused on the mathematical content of the surviving treatises. There have been repeated attempts to uncover the algorithms employed to create the tables of unit fractions used by the ancient Egyptians, as well as discussions of the historical influence of Egyptian fractions on the Greek arithmetical tradition.[1] Others have focused on the various mathematical problems and speculated about whether the Egyptian solution techniques prescribed in the texts reveal an (unexpressed) understanding of mathematical principles and formulae that would be used to solve similar classes of problems later in history.[2] In discussions of Egyptian mathematics, little attention has so far been given to the diagrams that accompany various geometrical problems. It is this lacuna in the scholarly and historical discussions that is addressed in this paper.

Modern editors of early mathematical works seem to feel a deep urge to "help" the ancient author or copyist by "improving" his diagrams—they straighten lines, adjust angles, and generally reformulate the diagrams to bring them into line with modern standards of abstraction and generality. This may produce a more aesthetically pleasing and perhaps more "mathematically correct" version of the diagram, but in doing so the editor imposes his own views of mathematics on the earlier text rather than providing the reader with the necessary data to make his own interpretations. One way to preserve the integrity of the ancient text is to publish a photographic record of it. Such records are of utmost importance, but they provide information only on the holistic level. They do not make any distinction among the various parts and layers of meaning in the text or diagram. A facsimile edition of an early text is important in that it makes the text more widely available for study, but it can never replace a careful critical edition of the work based on the surviving testimonia that can help the reader understand the history of the text and allow him then to formulate his own understanding of the material. In the same way, mathematical diagrams need to be edited and not merely photographed. Readers of early mathematical texts often need assistance in extracting the essential geometrical features of premodern diagrams—the points that define lines and the relationships between these lines that define angles and planes and surfaces.

Taking advantage of the limited number of examples of geometrical diagrams from ancient Egypt—there are only 34 in total represented in the scholarly literature—I provide in Appendixes I and II a catalog of all published extant diagrams in the hieratic and demotic Egyptian papyri, respectively, along with edited versions of the diagrams. To these, I add a third appendix, comparing the Egyptian diagrams to similar classes of diagrams

[1] Gillings [1972] and Knorr [1982] are classic examples of these interests. More recently, the work of Abdulaziz [2008] and Miatello [2008] has added to the discussion of mathematical techniques in ancient Egypt.

[2] See, for example, Guillemot's discussion [1998] on false-position solutions, or Lumpkin [1980] on Pythagorean triples.

that occur in two other mathematical traditions—that of ancient Mesopotamia and the early medieval Sanskrit mathematical tradition of India. Through the study of these diagrams, the paper aims to make a modest contribution to the recent discussions of geometrical diagrams in various cultures and time periods [Crozet, 1999; Netz, 1999; Chemla, 2005; De Young, 2005; Keller, 2005; Saito, 2006].

3. Sources for studying Egyptian diagrams

Present-day assessments of ancient Egyptian mathematics rest primarily on the written treatises which have come down to us. These treatises derive almost exclusively from two periods of Egyptian history. The first group of treatises, written in hieratic script, comes from the late Middle Kingdom and Second Intermediate Period (1850–1550 B.C.E.). The second, and somewhat less well attested,[3] body of evidence consists of mostly fragmentary remains of material written in demotic script dating from the later Ptolemaic period and the Greco-Roman era of Egyptian history (300 B.C.E.–300 A.D.). That even these few sources have survived is thanks in large part to the dry climate of Egypt, which has acted to preserve the papyrus on which most treatises were inscribed. Although the majority of our extant sources do not antedate the late Middle Kingdom and Second Intermediate Period, it seems probable that a mathematical tradition was well established much earlier in ancient Egypt [Imhausen, 2006, 20–21].

By far the most important of our hieratic sources is the Rhind Mathematical Papyrus (RMP). This treatise was compiled by a scribe named Ahmose in the 33rd year of the reign of Awserre (one of the Hyksos rulers of Egypt, also known to historians as Apophis, who reigned circa 1585–1542 B.C.E.). Ahmose tells us that he made his copy from an older document dating from the time of Nymaatre (traditionally identified as the throne name of Ammenemes III), who reigned during the latter half of the 19th century B.C.E. [Robins and Shute, 1987, 11]. It is often assumed that the mathematics reflected in the Rhind Papyrus is considerably older than the Middle Kingdom treatise from which it is ostensibly drawn. Reputedly found in a small structure near the Ramesseum on the Theban West Bank, the papyrus, now in two pieces, was purchased by Henry Rhind in 1858. On Rhind's death, the papyrus was sold to the British Museum, where it has resided ever since (BM 10057 and 10058). Nearly five meters long, it is the most complete and most extensive of any of the surviving mathematical documents. It contains a table of equivalent unitary fractions of $2/n$ for odd values from 5 to 101,[4] as well as some 87 problems. The problems treat various situations in which the scribe-administrator would be expected to calculate based on specific data and then take appropriate action [Imhausen, 2001, 2003b]. Among these problems, 20 deal with questions that involve geometry as we think of it today: Problems 41–46 demonstrate the calculation of volumes, 48–55 the calculation of areas, and 56–60

[3] The late material seems somewhat less well attested when considered in terms of volume. This appearance can be misleading, though, since some of the demotic material still awaits complete study and publication.

[4] The $2/n$ tables are used by scribes in both multiplication and division operations, both of which typically involve repeated doubling of fractional values. Although there is not a unique set of unitary fractions for any value of $2/n$, it is clear that the Egyptians preferred the specific sets of fractions found in the tables.

exemplify calculations related specifically to construction of a pyramid or similar structure.[5] The majority of the diagrams in the papyrus accompany the last two groups of problems.

Our second important source is the Moscow Mathematical Papyrus (MMP), originally purchased in Luxor in 1893–1894 by V. Golenishchev, and now in the Moscow Museum of Fine Arts. The papyrus is nearly as long as the Rhind Papyrus, but only eight centimeters high. On the basis of paleography and orthography, it has been suggested that it may date from early Dynasty XIII (ca. 1800–1650 B.C.E.). It is presumed to be based on material dating at least to Dynasty XII (ca. 1985–1800 B.C.E.). The treatise consists of 25 problems which appear to be arranged without any obvious organizing principle.[6] Several of these problems are now very fragmentary and their content can scarcely be guessed. Only four, Problems 4, 6, 14, and 17, contain recognizable diagrams.[7]

Several other mathematical papyri are known from ancient Egypt: the Lahun Papyri mathematical fragments, Berlin papyrus 6619 fragments, and the Mathematical Leather Roll (also acquired by the British Museum from the estate of Henry Rhind). Of these, only one of the Lahun fragments (UC 32160) contains a recognizable geometric diagram. All the documentary sources mentioned thus far are written in hieratic script—a more cursive style developed from hieroglyphics for use in personal documents. (Hieroglyphic script was typically reserved for formal settings such as the walls of temples and tombs, or other official inscriptions.)

In the late period of Egyptian history (after approximately 700 B.C.E.), hieratic script was gradually replaced by demotic script. A few fragments of demotic mathematical papyri from this late period have been identified and studied. Of these, P. Cairo J. E. 89127–30, 89137–43; P. British Museum 10520; and P. Carlsberg 30 contain recognizable diagrams and are included in our study.[8]

4. Writing on papyrus

The mathematical treatises containing these examples of diagrams are all preserved on papyrus. Use of this physical medium itself imposes certain limitations on the scribe's ability to construct diagrams and other illustrations. It is useful to begin, therefore, with a brief consideration of the physical technology of writing on papyrus in ancient Egypt.[9]

Papyrus is constructed from thin strips of the inner pith of the papyrus plant (usually identified as *Cyperus papyrus*). These strips, each roughly 30–40 cm long, after being well soaked in water, were laid down parallel to one another in two layers, the lower layer horizontal and the upper vertical, and then subjected to pressure in order to force the layers together and extract as much water as possible. When thoroughly dried, individual sheets could be glued together to make a scroll. The surface of papyrus is typically slightly

[5] Ahmose did not number his problems. Historians follow the numeration assigned by Eisenlohr [1877], who published the first comprehensive survey of the treatise.

[6] This collection of problems is a quintessential example of a "mathematical recombination text," a term coined by Friberg [2005, viii] to refer to texts that are more or less chaotic collections of material from disparate sources.

[7] The numeration of Struve [1930] is usually preferred because it includes fragmentary problems that earlier authors had ignored in their numeration.

[8] When referring to these demotic fragments, I follow the problem numeration of Parker [1972].

[9] Much of the material for this section is drawn from Parkinson and Quirke [1995, Ch. 2].

textured. If too rough, it might be smoothed by burnishing with a small stone or shell. Where two sheets are joined together in constructing a scroll, the overlap produces a thicker strip. Some scribes seem to have avoided writing on this strip, but others positioned their columns with no regard to the location of the join.

The text would typically be inscribed using a piece of a Nile rush (1.5–2.5 mm in diameter and about 20 cm in length) as a pen to apply the ink to the papyrus. These rush pens had no ink reservoir. To use them, the scribe moistened the end and rubbed it across the small ink cake or pellet in his writing chest. Probably he could write only a few signs before having to repeat the process. The main body of the text would be written in black, with titles and rubrication rendered in red ink. The scribe typically wrote the hieratic or demotic text in horizontal lines arranged in columns whose width corresponded to the amount of the scroll the scribe could easily unroll in his lap while seated cross-legged on the ground. There were neither horizontal nor vertical rulings, but the scribe could often use visible fibers in the papyrus sheet as a rough guide.

Given the rather fragile nature of the rush pen and the inherent physical texture of the papyrus surface, it is remarkable that the ancient scribes were able to attain the results we see in the diagrams of the mathematical texts. Through long training, they became adept at living with the limitations of their technology. Despite the statement by Parkinson and Quirke [1995, 32] that while using a rush pen the scribe would have found it difficult if not impossible to draw a neat line from the lower left to the upper right of the papyrus sheet, we find a variety of diagrams with lines slanting either toward the left or the right, and they appear to have been constructed reasonably professionally. We can only conclude that the scribes who wrote the mathematical treatises were not novices but were well versed in their profession.

5. Egyptian geometrical diagrams as artifacts

5.1. General characteristics of Egyptian diagrams

Diagrams accompany specific classes of problems in Egyptian geometry and are omitted from other classes of problems that we today might consider to lie within the domain of geometry. These diagrams differ in certain respects from the diagrams we are accustomed to see in our modern geometry textbooks. One of the most obvious differences is that the ancient Egyptian diagrams are without any labels. That is to say, none of the geometrical components of the diagram—points, lines, angles, planes, surfaces—are given names or designated by letters. What might initially appear to be labels are actually specific numerical metrical values assigned to the length of lines or the area of plane figures or the volume of solid objects. These numerical values represent either (a) the data that are to be operated upon (using various known numerical constants and the assigned calculation procedures) in solving the problem or (b) the desired results produced from the prescribed calculation processes [Ritter, 1995]. These "labels" emphasize the fact that the geometrical diagrams found in ancient Egyptian mathematical treatises accompany problems that are both numerical and algorithmic in character [Imhausen, 2002b, 2003a]. The diagrams are produced with sufficient accuracy to enable the scribe or reader to conceptualize the problem, but there is no evidence of effort expended to obtain careful metrical precision.

Diagrams appearing in Egyptian mathematical treatises are typically "overspecified"—that is, they are produced with greater specificity than is actually required for the solution of the problem. Modern geometry texts strive for the greatest generality possible, and this affects also the style of diagram construction favored by the editor. A modern editor, wishing to create a general triangle, would probably choose to construct it as scalene. In the Egyptian diagrams, we find that triangles are more likely to be portrayed as isosceles or right-angled.[10]

Another general feature of geometrical diagrams, at least in the hieratic period, is the preference for a horizontal axis rather than a vertical alignment. For example, in a modern geometry text, triangles will typically be oriented with a base parallel to the bottom of the page. Thus, the altitude line will be vertical. In Egyptian triangles, the shortest side will most often be arranged more or less vertically, with the longer sides meeting either toward the left or toward the right. Moreover, the Egyptian scribe seems to have had a strong preference for making the largest angle in the triangle no more than a right angle, so obtuse angles are relatively rare. Thus to the modern reader, the Egyptian triangle typically appears to be lying on its side, pointing either toward the left or toward the right margin of the column.

5.2. *Placement of diagrams*

There are two basic types of diagrammatic structure in ancient Egyptian treatises and the type of structure influences placement of diagrams within the text. Some diagrams are very small, so that they fit into a line of text without distorting the flow of symbols. Such diagrams were termed "in-line drawings" by Ritter [Imhausen, 2008, 30]. Other diagrams are of larger dimensions and are set off from the surrounding text by open space and, in some other premodern traditions, by actual boundary lines.[11] When larger diagrams occur in the hieratic papyri, they are often placed near the end of the problem they accompany. In the RMP, which is read from right to left, they are typically at the left of the column; see Figs. 1 and 2. In a few cases, however (Problems 49 and 51), the diagram appears in the middle of the text of the problem, with a sub-column of text to the right and another to the left of the diagram. The diagrams in the RMP, Problems 41, 48, and 50, are "in-line drawings." Problem 48 is the only example in this treatise in which the diagram occurs at the beginning of the problem and at the head of a column.[12] In the MMP, two diagrams (Problem 4 and Problem 14) appear at the left side of the column in which the problem is given and two diagrams (Problem 6 and Problem 17) appear at the bottom of the column. In either case, placement of the diagram is at the end of the problem.

Because the demotic papyri are often in quite fragmented condition, it is not always easy to generalize concerning the placement of diagrams. The examples in P. Cairo J.E. 89127–30 and 89137–43 occur in columns 137, 140, 141, 143. In each case, they appear to be aligned at the right margin of the text column. If the diagram is relatively large, it occupies the entire width of the column. There does not seem to be a marked preference for any specific placement within the proposition itself. In Problems 36–38, we find multiple diagrams.

[10] A similar situation occurs, as we shall see, in the Mesopotamian tradition. Thus it is not too surprising to find similar practices being followed in ancient Greek mathematical treatises also. For example, the right triangle used in the "Pythagorean Theorem" (*Elements*, I, 47) is almost always shown as isosceles in Byzantine Greek and medieval Arabic manuscripts [Saito, 2006].

[11] While diagram boundary lines are unknown in ancient Egypt, they appear frequently in Mesopotamian mathematical tablets and in some palm-leaf Sanskrit codices.

[12] The "diagram" of Problem 41 is an unusual case, since it occurs in the statement of the problem, not the working out of the problem, as is more typical.

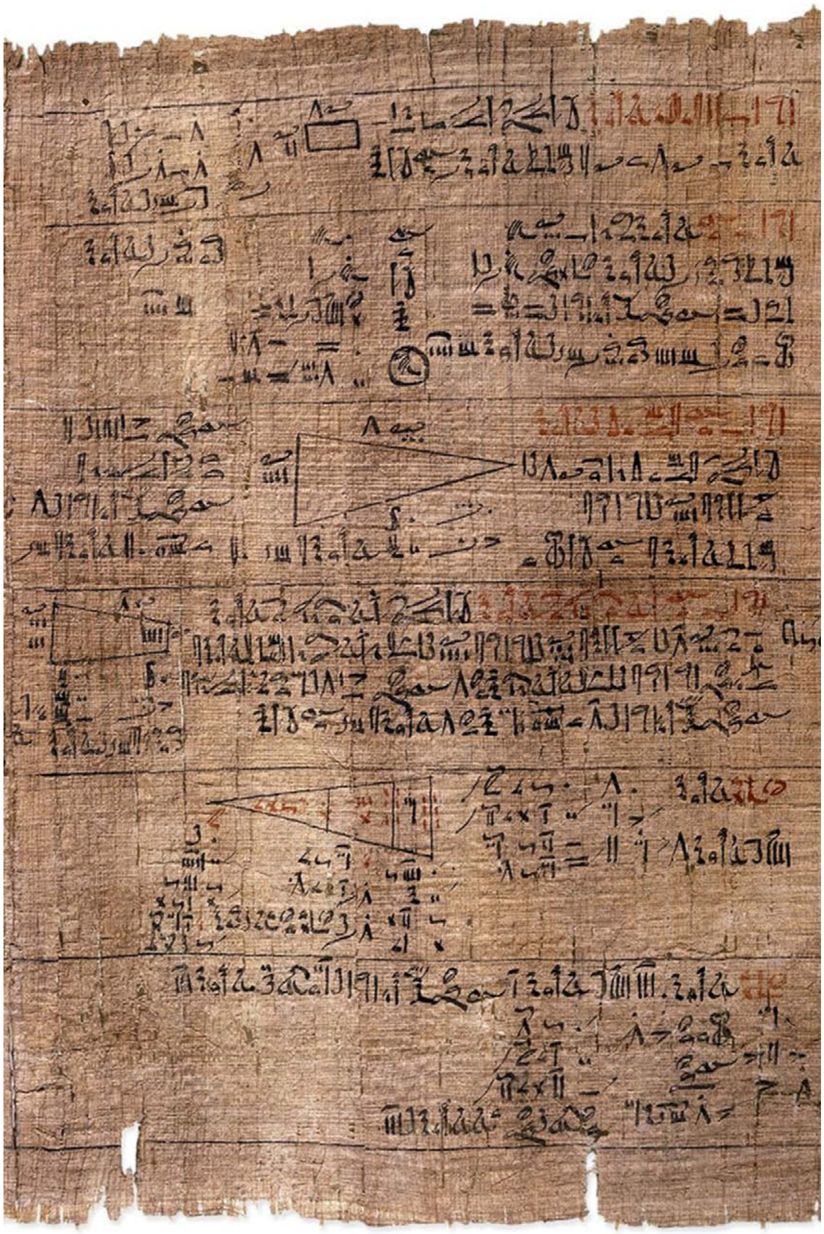

Fig. 1. Rhind Mathematical Papyrus, Problems 49–55 with their diagrams. ©Trustees of the British Museum. Used with the permission of the British Museum.

In each case, the problem begins with a circular area in which is inscribed a regular rectilinear figure (either equilateral triangle or square) and the scribe is asked to calculate the area of each of the components of the circular area. In Problems 36 and 38, the diagram illustrating the specific situation appears near the beginning of the problem. The diagrams for the subunits of the figure occur where the text begins to explain the calculation process to find their areas. In the case of Problem 37, however, both diagrams appear near the end of the problem. Problems 39 and 40 concern calculations relating to pyramids. In both cases, the diagram occurs near the beginning of the problem. (The text of the latter problem

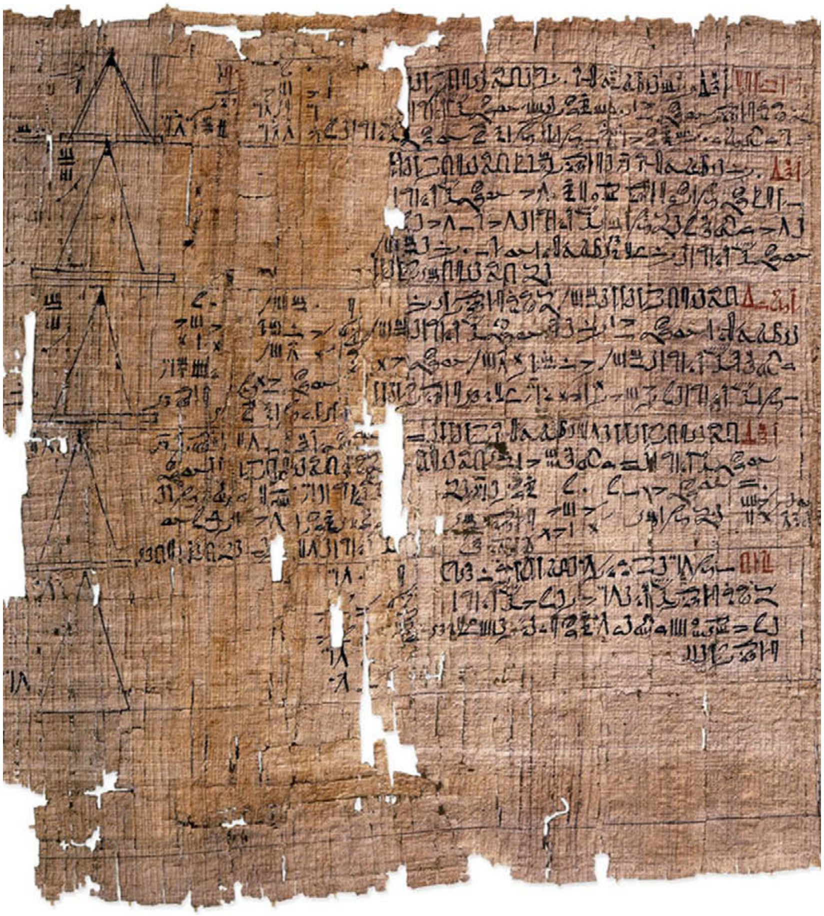

Fig. 2. Rhind Mathematical Papyrus, Problems 57–60 with their diagrams. ©Trustees of the British Museum. Used with the permission of the British Museum.

occupies six lines, two of which precede the diagram and two of which follow it. Thus, the diagram literally occurs in the middle of the problem, with two lines of text to the left of the diagram.)

P. British Museum 10520 contains only one recognizable diagram. This diagram, illustrating an area calculation, is inserted into the column following the first line of the problem. The diagram is aligned on the right margin of the column and occupies approximately one-third of an empty horizontal band across the width of the column.

P. Carlsberg 30 is too fragmentary to make many generalizations. Three rectilinear figures, which seem to share a common measure of length and so may belong to a single problem, occupy the width of an entire column. Only one of these three diagrams is reasonably complete. In another fragment from the same papyrus, a small equilateral triangle appears as an "in-line diagram" within the truncated statement of a now undecipherable problem.

6. Assessing ancient Egyptian diagrams

What features of the geometric diagram did the ancient Egyptian scribe consider important or essential? To approach this question, I have extracted the basic geometric content from all available published Egyptian geometrical diagrams based on photographs of the manuscript sources.[13] The extraction of the geometrical data uses DRaFT, a software tool produced under the leadership of Ken Saito (Osaka Prefecture University) as part of the Scriptorium project. The primary goal of the Scriptorium project is to develop software tools for the creation of electronic critical editions of premodern mathematical sources. Recognizing that diagrams are an essential part of early mathematical treatises, Saito has incorporated a function (DRaFT) in order to edit the diagrammatic features of mathematical manuscripts.[14] Using this newly developed software tool, the modern editor can extract very precisely the relations among the points, lines, and arcs that make up the textual diagrams and can represent them accurately in the editing process. This ability to present the essential geometrical features of the diagrams is especially important in cases when the manuscript diagram is incorrectly drawn or when the various manuscript testimonia differ in their diagrams.[15] In addition to preserving the diagram data themselves, the editing process also offers a window into the drafting skills with which ancient scribes constructed their diagrams. And, by allowing a more precise analysis of relations among the salient geometrical features of ancient Egyptian diagrams, this software tool also offers insight into the relation between text and diagram in ancient Egyptian mathematics.

DRaFT registers the points that the user specifies in the diagram and links those points as instructed by the user to produce diagram components (lines and arcs). The program is currently capable of producing three types of lines. The specific characteristics of these lines can be defined by the user within the parameters of the software. For my editing purposes, I have chosen them to be solid, dashed, and dotted. The solid line is the default setting and I use it to indicate the actual lines as they exist in the diagrams in the papyri. When sections of diagrams are missing and it is possible to reconstruct the missing portion without excessive ambiguity, I signify that reconstruction by using a dashed line. For example, if the central portion of a side of a rectangle is missing but the end portions and vertices are present, I presume that the missing piece must have been a straight line joining the remaining end portions and use dashed lines to indicate these reconstructed line segments. If part of the diagram is missing because of damage to the papyrus itself, I use the dotted line to mark the existing edge of the papyrus. When lines or arcs within the diagram are distorted from their correct shape, either rectilinear or circular, as described in the text, I sometimes

[13] I have relied primarily on the photographs published in Clagett [1999] and Parker [1972]. Full color photographic plates from the RMP are available in Robins and Shute [1987], and digital images are available on the British Museum Web site. Use of the color plates does not affect diagram studies so much as textual studies, however. The black and white photographs accompanying the first edition of Struve [1930] may reveal more fine detail than Clagett's reproductions, but the basic geometric data of the diagrams are not affected.

[14] DRaFT and its associated software toolkit may be downloaded gratis from Saito's Web site (http://www.greekmath.org/diagram/).

[15] Unfortunately, such problems are not rare or isolated occurrences. Crozet [1999, 146–162] and Saito [2006, 82–90] have discussed these issues in regard to Arabic and Greek manuscripts, respectively. Crozet concluded that, when possible, a facsimile of the manuscript diagrams should be included in the critical apparatus of the edition in such cases and that a correctly constructed diagram should be included in the text of the edition. Although the principle is well taken, facsimile reproductions can be difficult to obtain and costly to produce. Saito, in his discussion of these issues, has used DRaFT software to collect and present the data. De Young [2008] has used DRaFT software in preparing an edition of a unique Arabic manuscript and its diagrams, some of which were damaged and some of which were drawn/copied incorrectly.

include a true straight line or circular arc in dotted line format to emphasize the distortion in the diagram. Thus, I use the dotted line for two purposes, but I do not believe any serious ambiguity arises from that double usage.

Even using this software, though, it is impossible to indicate completely the minute variations in line structure and width found in these hand-drawn diagrams. The results produced from DRaFT are not the equivalent of a photographic record. Although the program registers very precisely the points selected by the editor, there are still editorial decisions involved in selecting the specific points. When a line has a very noticeable thickness, for example, should one choose points on the outer edge of the line, the inner edge, or somewhere in the middle? When the scribe produces a distinctly rounded vertex to a quadrilateral, should one choose the point of the vertex to be somewhere in the rounded portion of the line or should the editor choose a point outside the line where the two sides would intersect if continued rectilinearly? In cases of such ambiguity, I find it is often necessary to describe the editing decisions taken, in addition to simply giving my edited rendition of the original diagram. The final result of such choices is an edited version of the diagram.[16] Just as the choices made when editing a text have consequences for the reading and understanding of the treatise, so the choices made when selecting the points and defining the lines of the diagram have consequences as well. Even though the editor endeavors to make wise and informed choices when editing the diagrams, the results cannot in any direct way be said to literally recreate the diagram precisely as drawn by the ancient scribe. (Only where there are significant and visible variations from straight lines, for example, do I attempt to approximate those divergences from rectilinearity.)

Another limitation of DRaFT, at the moment, is that it is able to assign labels using only Roman or Greek script. In my edited diagrams, therefore, I am using Roman rather than hieroglyphic or hieratic fonts. The editor can, however, specify with considerable precision where he wishes these labels to be placed. Because the Roman labels occupy a different space within the edited diagram than did the original numerical values, I have not attempted to place the labels precisely where the ancient scribe placed his labels. Instead, I have tried to achieve a placement that reflects as clearly as possible my understanding of the intent of the scribe and yet allows the label to be easily read within the diagram. This leaves open the possibility that modern sensibilities of aesthetics may sometimes unconsciously come into play. I can only hope that in the process I have not betrayed the cultural identity of the scribes who originally produced the diagrams.

7. Diagram figures in Egyptian mathematics

For the purposes of this paper, I have divided the surviving diagrams into several categories[17]: squares and rectangles, triangles, pyramids, trapezia, and circles; I will discuss each category in turn in this section. From an initial inspection, it appears that the diagrams in the papyri have not been constructed using drawing aids such as a straightedge

[16] But one should not conclude that therefore a photographic record is a better way to preserve information. In a revealing series of photographs of the Mesopotamian tablet YBC 7289, it is clear that differences in lighting and photographic angle can dramatically change the visual record of the tablet, emphasizing some lines and making others almost invisible. These photos may be viewed on the Web site of W. Casselman: http://www.math.ubc.ca/~cass/Euclid/ybc/ybc.html.

[17] These are, of course, modern categories and they do not necessarily represent the way an ancient Egyptian mathematician might have divided his subject matter.

or compass. Straight horizontal and vertical lines were probably not too difficult to render with considerable accuracy because the visible fibers in the papyrus could provide a useful guide for the scribe. Even straight lines slanting across the papyrus fibers were typically done quite neatly. Most scribes, through considerable training, would probably have had extensive practice in producing straight lines. Nevertheless, in rectilinear figures, it is not uncommon that lines fail to meet neatly at vertices but extend slightly beyond them. Circles, especially when they are larger than "in-line drawings," are more problematic, since the papyrus fibers provided no guidance and circles were not used nearly as often as straight lines in ordinary scribal practice.

7.1. Squares and rectangles

These squares and rectangles (there are no examples of parallelograms without right angles) appear most often in conjunction with area problems—especially areas of fields. Even when a problem explicitly calls for a square, the scribe draws it as an elongated rectangle. When any rectangular figure is drawn, the longer sides are always horizontal. Although no straightedge seems to be used, the lines are often reasonably straight and the angles are reasonably close to being right angles. In one late example (Fig. 35), we find a diagonal drawn from upper right to lower left vertex in a rectangle.

The actual metric of the scribe's diagram is at variance with the dimensions required by the text of the problem in several cases. For example, RMP, Problem 49 states that we are to find the area of a rectangular plot measuring 2 *khet* by 10 *khet*.[18] The rectangle in the papyrus labels the shorter side "2" and the longer side "10" (Fig. 10), but the dimensions drawn are inconsistent with these values, having a length-to-width ratio of 2:1. In another diagram, accompanying MMP, Problem 6, the text requires that the shorter side of the rectangle should be 3/4 the length of the longer side (Fig. 21). Again, the diagram metric does not correspond to what is required in the problem, even if one ignores the fact that one vertical side is significantly shorter than the other (left side is 3.8 and right side is 4.8 units). If we take the longer base (9.2 units) and compare it to an average of the two shorter sides (4.3 units), the length-to-width ratio is slightly less than 2:1. It sometimes seems that an implicit aesthetic norm may override any desire for representational accuracy in establishing the metric of the diagram.

7.2. Triangles

We have examples of equilateral, isosceles, and scalene triangles. Equilateral triangles (seen only in demotic papyri) typically have one side as a horizontal base and have their axis of symmetry placed vertically. Isosceles triangles, if representing pyramids, are also positioned with a horizontal baseline. When not representing pyramids, they have their axis of symmetry placed horizontally. Scalene triangles occur only in the case of right triangles. These are always oriented with the longer side of the right angle toward the top of the roll or column. In some demotic examples, a vertical altitude line is included in the diagram of the triangle.

[18] Despite this explicit statement of the problem, the scribe actually calculates the area of a plot 1 *khet* by 10 *khet* [Chace, 1927, 92].

Triangles that are explicitly specified as equilateral occur only in the later demotic material. P. Cairo J.E. 89140, Problem 36 involves an equilateral triangle inscribed within a circle (Fig. 26), and the scribe is asked to calculate the areas of the various parts of the figure. It is followed in the problem by a diagram of the independent triangle (Fig. 27), which now contains an altitude line that was not shown in the initial composite diagram. Perhaps due to the present condition of the papyrus, this triangle is not recognizable as equilateral. Later, in Problem 38, which also involves a triangle, the figure does appear more nearly equilateral (Fig. 29), although measurements of its angles and sides indicate small variations from the specified form.

Isosceles triangles are most often used to represent pyramids, which will be discussed in the next section. The triangular figure accompanying the much-debated RMP, Problem 53 appears very much like an isosceles triangle (Fig. 14). Like most hieratic triangles, it is arranged on a horizontal, rather than a vertical, axis. The problem and its diagram have generated considerable discussion because the numerical values inscribed in the triangle do not seem to correspond to the values used in the actual calculations of the problem. Nor are they totally consistent with a true triangular figure. (In a triangle, for example, the value for the length of the vertical (base) side and for the length of the vertical line cutting the triangle parallel to the base cannot both be "6", although that value is clearly written in the diagram.) Even this apparent isosceles triangle is not constructed with close precision, as immediately appears from the measurements of the sides and angles of the triangle.

Most triangles, at least in the hieratic sources, are right- or nearly right-angled. Most examples of right triangles are not oriented to take maximum advantage of the fibers of the papyrus. This is somewhat surprising because one would have expected that following the fibers in the papyrus would have made the construction task easier for the scribe. Instead, they are always oriented with the right angle toward the top of the column and the shorter side of the right angle vertical or more often leaning slightly from the vertical. Because the shorter side usually has a definite deviation from the vertical, the longer side of the right angle is not parallel to the horizontal lines of text. The hypotenuse, then, typically lies at the bottom of the figure but not parallel to the base of the column. Two examples of right triangles from the MMP, Problems 4 (Fig. 20) and 17 (Fig. 23), are each oriented with the shortest side upright on the left side of the figure and leaning slightly toward the left. The right angle is at the top of this shortest side. Thus, the longer side of the right angle is not horizontal, but rises from left to right at a small angle. Each triangle as a whole, thus, seems to be pointing slightly upward toward the right. It seems quite likely that the triangle in RMP, Problem 51 (Fig. 12) was also intended as a right triangle, since it shares similar structural characteristics, except that in this case the shorter side leans toward the right and so the triangle as a whole points to the right and slightly downward.

7.3. Pyramids

Triangles representing pyramids are found in RMP, Problems 56–59, forming a subclass of triangle diagrams, each of which appears isosceles. They differ from other triangular diagrams in that they are typically shown resting on a solid base parallel to the bottom of the scroll. This base is shaped like an elongated brick and extends beyond the pyramid to the right and the left. Most probably this structure is intended to represent the stone foundation on which pyramids were usually constructed.[19] The pyramids are shown as isosceles triangles resting on these bases. Interestingly, each triangle has slightly different angle measures and side lengths, indicating that they were not constructed using a common pattern. Moreover, Gillings [1972, 187] has calculated the slope of the face of each pyramid based on the data given in these problems. He found each of them to be close to 53°, although in the diagrams the angle is typically somewhat more than 60°. At the apex of each triangle is a distinctive darkened area, perhaps intended to represent the pyramidion or the capstone of the pyramid. The lower limit of this darkened area is indicated in the edited diagrams with a dashed line. In RMP, Problem 56 (Fig. 15), the pyramid itself is shown as though having a thickness or an outer skin, the only example of this technique.[20]

Two examples of triangular figures representing pyramids occur in the demotic literature. The first, P. Cairo 89143, Problem 39 (Fig. 30), has been somewhat damaged, but it does not differ too much from an isosceles triangle. In this case, the baseline appears not quite level. Whether this is a feature of the scribe's carelessness or an artifact introduced by the rather clumsy repair of the papyrus is not something I can decide easily. The diagram for Problem 40 (Fig. 31) contains the numeral "10" on each of the two sides. It is true that in a pyramid these sides should be equal, but the value "10" is not consistent with the text of the calculation or with the solution that is found. Perhaps the scribe was misled by the fact that each side of the base of the pyramid is specified to be 10 cubits?[21] The demotic pyramids are not drawn on any sort of base, nor do they have the darkening of the apex as in the hieratic examples.

Both the hieratic and demotic pyramid diagrams provide a purely two-dimensional representation of a pyramid. There is no attempt to draw a pyramid with anything like perspective in order to represent a three-dimensional reality. In that sense, these diagrams have been created at a fairly high level of abstraction. They are schematic two-dimensional representations, perhaps a kind of vertical cross section through the pyramid, not miniaturized but visually accurate reproductions of three-dimensional structures.

7.4. Trapezia

In this category we place especially the truncated triangle of RMP, Problem 52 (Fig. 13) and the truncated pyramid of MMP, Problem 14 (Fig. 22). In many respects the two diagrams are very similar in structure, although the former involves calculation of an area and the latter accompanies a calculation of volume. The truncated triangle lies horizontally with

[19] The best-known exception is the "Bent Pyramid" of King Snefru at Dahshur. Built directly on the sand, it developed marked subsidence during construction, resulting in interior cracking. In an effort to reduce the weight of the pyramid, the builders tried altering the angle of its sides from 55° to 43°, giving the pyramid its distinctive "bent" shape. Ultimately, however, the interior cracking forced them to abandon the structure [Lehner, 1997, 102–103].

[20] Whether this was intended to represent the limestone casing with which the pyramids were typically finished is impossible to guess. The RMP was copied nearly a thousand years after the great age of stone pyramid construction in Dynasties IV and V, but the treatise from which it is copied dates from the time of Middle Kingdom mud-brick pyramid construction. These later pyramids, although their core was mud brick, were still faced with limestone. It is difficult to decide how much technical information to read into this single example.

[21] Parker [1972, 52] believes that the text of the solution is more likely to be correct and that the values on the sides in this diagram are probably not correct.

the smaller cutoff side toward the right. The truncated pyramid, since it represents a volume, is placed vertically. We might understand it to be vertical cross sections through the pyramid. In the case of truncated pyramids, as in the case of complete pyramids, we find that there is no attempt to portray the structure three-dimensionally. It is only from the text of the problem that we know the diagrams are intended to represent a three-dimensional, rather than a two-dimensional, entity.

7.5. Circular figures

Circular figures are, as a rule, considerably more distorted than rectilinear figures. Typically they appear as if flattened at the poles and bulging irregularly at the equator, producing a figure resembling an ellipsoid rather than a circle. When used as a small "inline drawing," the circle is often fairly well drawn, with only a small amount of distortion, as in RMP, Problems 41 (Fig. 6) and 50 (Fig. 11). By contrast, the larger circles used in the diagrams of demotic fragment, P. Cairo J.E. 89137, Problems 32 (Fig. 24) and 33 (Fig. 25) appear remarkably crude.

P. Cairo J.E. 89140–89143, Problems 36–38 (Figs. 27–29), involve calculating the area of segments of circles. The diagrams of these segments are typically as distorted as the diagrams of complete circles in other problems. In each case, we must use one or more Bézier curves to approximate the rounded shape and include a true circle or circular arc in dotted lines for reference purposes.

Only in RMP, Problem 48 (Fig. 8) do we find a polygonal figure used to approximate a circle.[22] Such approximating polygonal figures are used much more commonly in the Greek mathematical tradition, but the approximating figure will usually have more vertices than the eight that are commonly identified in this diagram and they will also be regular polygons, rather than the irregular form shown in this diagram. Whether the technique of approximating circles using polygons can be said to have originated in ancient Egypt cannot be adequately answered on the basis of the surviving evidence.

7.6. Nonrepresentational line diagrams

Among the demotic mathematical fragments there are a few "diagrams" unlike the two-dimensional geometrical diagrams we have discussed so far. Apparently intended primarily for examples of land measurement, these "diagrams" present the geometrical information at an even more abstract level. For example, P. British Museum 10520 includes two problems involving calculation of the area of a rectangular field. The first of these (Fig. 32) is provided with a typical rectangular diagram. (This diagram provides another example of a scribe using a metric that contradicts the explicit length values noted in the diagram.) The second problem does not contain a diagram in the literal sense of the word, but has only a line on either side of which are noted the length of the field and at whose ends there is given the width of the field (Fig. 3).[23]

[22] At least, this is one of the more common interpretations of this diagram. Imhausen [2008, 30] is among the few who have taken the diagram literally to represent the situation given in the text, a circle inscribed in a square.

[23] Parker [1972, 72] implies that use of a line instead of a full rectangle is common, although I have encountered few examples.

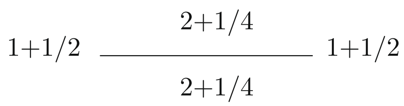

Fig. 3. P. British Museum 10520, an example of the nonrepresentational line diagrams occasionally found in demotic sources.

When dimensions on parallel sides are equal, one often finds the value given only on one side of the line and a dot representing the same value placed on the opposite side of the line. Another example of a line replacing a rectangle in area measurements is found on Theban Ostracon D 12.[24]

Two further examples of line diagrams occur in P. British Museum 10399 (Fig. 4). Both problems involve calculation of the volume of a "mast" in terms of the number of *hin* of water it would contain.[25] In both these problems, the mast is illustrated as a vertical line with the diameter of its ends noted at top and bottom of the line, respectively, and its length noted along the side of the line. In both these cases, the line diagram is placed at the end of the problem.

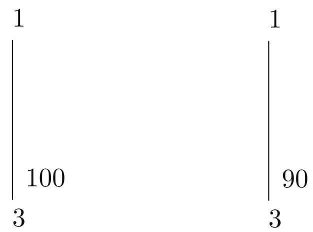

Fig. 4. P. British Museum 10399r, Problem 42 (left) and Problem 43 (right).

The line diagrams, although they present geometrical knowledge in a diagrammatic form, are not included in our study, since they represent a different level of abstraction than the more customary two-dimensional geometrical figures.

7.7. Geometric problems without diagrams

There are a number of problems in ancient Egyptian treatises that we would today class as geometric questions, but they are not given diagrams to support them. These problems occur in both the hieratic and the demotic materials. Why they have not been provided with diagrams is not known. These problems include the following:

- RMP, Problems 42–43 and 45–46—calculating the volume of a granary[26]
- MMP, Problem 7—calculating the area of a triangle.

[24] This is reported by Parker [1972, 72]. I have not seen the image of the Ostracon.
[25] The practical value of such calculation is not evident to me. For a survey of ancient Egyptian metrology, see Chace [1927, 30–34], Gillings [1972, 207–213], or Clagett [1999, 7–15]. Serious researchers must also consult the recent comprehensive study of measures of capacity in ancient Egypt by Pommerening [2005].
[26] Perhaps the scribe intended the diagram of Problem 41 and Problem 44 to serve for all three problems of each respective series.

- MMP, Problem 10—calculating the area of a curved surface (?).
- MMP, Problem 18—calculating the area of a piece of cloth.
- P. Cairo 89130–89138, Problems 7–18—calculating how to change the dimensions of a piece of cloth while keeping its total area the same.
- P. Cairo 89129–89139, Problems 24–31—calculations relating to a pole leaning against a wall [Melville, 2004, 155–160].
- P. Cairo 89137–89140, Problems 34–35—calculating the dimensions of a rectangle.
- P. BM 10399 recto—two more problems calculating the volume of a "mast".

8. Concluding thoughts

Saito [2006, 82], when describing the tendency of Greek diagrams toward "overspecification," ponders whether this characteristic of geometrical diagrams is an artifact of the transmission process, perhaps the result of incompetent medieval copyists, or whether it had its origins in antiquity with the development of mathematics itself. He concludes in favor of an ancient origin, arguing that (1) overspecification is so widespread that it is unlikely to be the result of a conscious decision to modify diagram construction methods, and (2) the existence of "incorrect" diagram figures in which the diagram as constructed does not reflect the construction techniques described in the problem yet still functions to represent the geometric situation under discussion implies that diagrams were never conceived as miniaturized photographs of fields or granaries, but rather were schematic representations of relations. Our survey of ancient Egyptian diagrams has confirmed the validity of Saito's conclusion. The origins of the overspecification in geometrical diagrams do indeed spring from antiquity, since the phenomenon appears in nearly all the diagrams at the very beginning of the recorded history of geometry. And the existence of "wrong" diagrams in ancient Egypt (for example, Fig. 10—a diagram constructed with a metric that contradicts the explicit statement of the problem) implies that from very early times the diagram had schematic, rather than literal value. The existence of abstract and nonrepresentational diagrams (for example, Figs. 3 and 4) lends further credence to the hypothesis that diagrams were intended to be schematic, not fully representational.

Friberg [2005] has discussed at length the question of possible intellectual exchanges between Old Babylon and Egypt, but with a focus on the conceptual formulations and computational techniques internal to the mathematics.[27] My focus here, however, is on the use and structure of geometrical diagrams. In this regard, there seems to be a remarkable degree of interest in truncated triangles and pyramids in both hieratic and Mesopotamian traditions (see Appendix III). This parallel interest finds expression in similar diagram structures. In both Egyptian and Mesopotamian traditions, diagrams of such figures are "overspecified"—usually appearing as isosceles. Moreover, in both traditions, the diagram

[27] Friberg's interpretations remain controversial. While not denying that some intellectual interchanges probably occurred, I would suggest that it may be more useful to contextualize what we know of ancient mathematics within the culture that developed it rather than focus on questions of priority of discovery.

figures are arrayed with their baselines vertical and the major axis of symmetry placed horizontally.[28] In addition, neither the Mesopotamian nor the Egyptian geometrical tradition produced diagrams that attempt to present three dimensions. The scribes must have understood something of three dimensions from daily experience, but in their diagrams they made no effort to produce anything other than two-dimensional schematic representations.[29] Such similarities in problems posed and their accompanying diagram construction techniques would lead quite naturally to the supposition that there occurred some cross-fertilization of mathematical concepts and usage. But beyond that broad generalization, it appears to me to be impossible to argue questions of cultural priority or superiority.

Despite such intriguing parallels between the two mathematical traditions, both in the content of the problems and in their diagrams, there are also some obvious differences. One quickly notices, for example, that Mesopotamian diagrams seem in general to be more precisely constructed than their Egyptian counterparts. With no need for concern that use of tools might lead ink to smudge on the papyrus, it seems probable that the ancient Mesopotamians may have used simple tools to help them construct circles and straight lines. At any rate, the circles in our small sample of Mesopotamian diagrams are much closer to true circles (Figs. 39 and 45) than are those in the Egyptian diagram tradition, where scribes typically produced ellipsoidal rather than circular shapes (Figs. 5, 24, 25, and 26).[30] Lines, too, are, on the whole, much more likely to be straight and true in the Mesopotamian tradition. In part this may derive from use of a different medium in which to record their diagrams.

The Mesopotamian mathematical tradition addresses a much wider range of problems than that found in the extant papyri from ancient Egypt [Robson, 2008b]. This extended range of problems is represented, of course, in the diagrams they included in their mathematical texts. Many of these problems seem, like those of ancient Egypt, to be algorithmic in nature and are deeply rooted in the everyday life of the scribes and their professional duties. Ritter [1995, 2000] and Imhausen [2001, 2003b, 2006, 2008] have taken important steps toward placing the subject matter of Egyptian mathematical problems within the broader cultural tradition of practical concerns facing the scribe-administrator. Høyrup [1996], Høyrup and Damerow [2001], and Robson [1999, 2008b] are among those who have advocated a somewhat similar approach to Mesopotamian mathematics.

[28] Not all Mesopotamian trapezoid figures fall into this category. The diagrams drawn by Yuste [2006] seem to indicate that in other contexts trapezia are not overspecified and may be arranged with the longest side at the bottom of the diagram, rather like the diagram of MMP, Problem 14 (Fig. 22). Unfortunately, as is often the case with redrawn diagrams, the actual characteristics of the diagrams found on the tablet are left obscure.

[29] Based on the selection of diagrams presented here, this conclusion seems to hold true. A wider exploration of the range of Mesopotamian diagrams is necessary, though, in order to assess the limits of applicability of the generalization. Hoyrup [2001, 89], when discussing a typical "sub-scientific" problem (Number 23 from BM 13901) called "four fronts and the area," includes a redrawn diagram of the problem in which the "sides" of the figure are shown as rectangles projecting from the square base. Whether the redrawn diagram is constructed on the basis of an original diagram is not clear. The only pictures of BM 13901 that I have been able to see, since Neugebauer's studies are unavailable to me, do not seem to show any diagrams.

[30] The exception to this generalization is the "rough work" tablets, where circles can also appear somewhat crudely or carelessly drawn (Fig. 45). Robson [1999, 9–15] introduced the term "rough work" to describe tablets, often reused, that have hasty sketches rather than finished diagrams and rarely contain a verbal statement of a problem or question.

In a fascinating and sometimes controversial discussion of the origins of Greek deductive thought and mathematics, Netz [1999] has argued that, at its inception, Greek abstract mathematical thought developed from discussions about diagrams. The deductive proofs were developed primarily to explicate the relationships that existed in the diagrams. In that sense, he argued, diagrams were primary. In fact, the diagram could be said to stand as a metonym for an entire proposition [Netz, 1998, 37–38]. I am not convinced that the same can be said of the diagrams of ancient Egypt.[31] For if it is indeed possible to speak of the ancient Egyptian diagram as metonym, then the diagram used in ancient Egypt is most immediately a metonym for a very specific problem. It did not stand for an abstract presentation of a mathematical relationship such as was the case of the Euclidean or Archimedean propositions as developed in ancient Greece.

A part of this argument for the primacy of diagrams in early Greek mathematics involves a semiotic analysis of diagrams and diagram labels [Netz, 1999, Ch. 1]. In the course of this analysis, Netz [1999, 41–43] noted the ubiquity of diagrams in Greek mathematics. Diagrams are used even in situations where they may appear redundant or irrelevant, such as Euclid's use of line segments to represent odd and even numbers in Books VII–IX of the *Elements*. Diagrams are not, however, ubiquitous in ancient Egyptian mathematics. Rather, they appear to be reserved for propositions or problems dealing with specific subjects, and I have mentioned several problems where we might well expect to find diagrams but the ancient scribes did not include them. Not only are diagrams less frequent in ancient Egypt, but also the system of labeling them is very different. This observation, too, would seem to militate against the existence of a simple linear case of Greek borrowing from the Egyptian tradition.

The Egyptian numerical labels tie the diagram to its specific problem context, but both Ritter [1995, 2000] and Imhausen [2002b, 2003a] have argued forcefully that the Egyptian algorithmic process is not inherently antithetical to a search for generality in mathematics. If one accepts that algorithmic argument can lead to mathematical generality, then perhaps the disjunction between ancient Egypt and the abstract mathematics of the Greeks is not so great as it at first appears. And then we might well wonder whether "intellectual tourists" from ancient Greece, such as Herodotus and Plato, both of whom visited Egypt during the late period of its history, may not have returned home with some new understandings of mathematics that helped to propel Greece toward greater abstraction. Such "tourists" would have been visiting Egypt more than a thousand years after the flowering of the earlier hieratic tradition, but there might still have been important mathematical remains surviving from that era that have now fallen below our historical horizon. At the moment, we can do little more than speculate.

Although the changes in mathematical approach make it difficult to argue for a simple linear transmission between Egypt and ancient Greece, the remarkable similarity in mathematical problems, calculation techniques, and even diagrams between the mathematics of ancient Mesopotamia and ancient Egypt more strongly suggests some sort of common scholarly tradition. Whether that common thread included direct exchanges between scholars of the two traditions or some less direct transmission, I leave to others more expert in the languages and traditions of the period. Friberg [2005, 2007] has argued at length for the

[31] Although not precisely the same as Netz's metonym, Guillemot [1992, 128–129] has noted the close connection between the form of Egyptian diagrams of pyramids and other three-dimensional figures and the hieroglyphic symbols for these structures. On the other hand, there might be a sense in which the "rough work" tablets of Mesopotamia, which usually contain only a diagram and calculation, could be said to represent something like metonyms for classes of problems.

primacy of the Mesopotamian mathematical tradition and has advanced the claim that both the Egyptian and the Greek traditions borrowed numerous mathematical techniques from ancient Mesopotamia. At least at the level of diagram construction, my results from this initial overview of Egyptian geometry and its diagrams seem not to contradict Friberg's claims. For the moment, however, I find it impossible to determine the direction of any presumed historical causality or even the existence of a historical priority. All that can be said with any certainty is that there are intriguing parallels in diagram construction between the Mesopotamian and Egyptian (especially hieratic) mathematical documents. Høyrup [1990] has argued for a more nuanced interpretation of early mathematical traditions, suggesting that we must distinguish between craft or practical mathematics and scholastic mathematics. He designated the former as "sub-scientific" mathematics and proposed that it was an important conduit for transmitting mathematical knowledge in the premodern period. This often unwritten transmission may account for the appearance of nearly identical problems in a variety mathematical traditions, crossing linguistic and temporal boundaries.

Acknowledgments

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Appendix I. Catalog of diagrams in hieratic papyri

In this appendix, I investigate all the diagrams available in the extant hieratic mathematical texts, based on the published photographs of these documents. For each diagram, I provide measures of angles and lengths of lines. These measurements were taken from the EPS figures produced by the DRaFT software using a simple school protractor for angle measures and a straightedge for linear measures. One problem arises when the lines used to produce the diagram are relatively thick or reflect the wavering of the scribe's pen as he drew the lines. Does one use the outer edge of the lines, the inner edge, or perhaps a median position? Such ambiguities often confront the editor of early diagrams. The decisions made by the modern editor thus have a distinct impact on the measurements recorded. In my analysis, I have tried to strike a balanced interpretation based on what I believe the scribe intended with the diagram and what we know about the limitations of the ancient writing technology. Therefore, I record angle measurements only to the nearest degree. Because I have rounded off the angle measures to the nearest degree, the sum of the angle measures might not always total the precise number of degrees required for a particular plane figure. Length measures are recorded in centimeters and length values refer only to the EPS diagrams constructed by the DRaFT software tools. I have not made any attempt to restore the metric of these EPS diagrams to what is actually found in the original papyri from which they are edited. Although the values derived from the EPS figures do not represent the literal metric of the original diagram, these edited diagrams retain the basic relations found in the original. Thus they can be used to assess the essential geometric features of the original diagrams. These data can also help to reveal specific trends in the practice of drawing diagrams, such as preferred ratios between sides of rectilinear figures and the degree of accuracy of angle constructions.

For each diagram entry, I begin with a statement of the problem to which it is attached, followed by references to the location of reproductions of the original diagram. This is followed by the edited diagram. Where appropriate, metrical features of the edited diagram are indicated in the diagram caption. Each entry concludes with any additional general discussion of the diagram and its editing.

Lahun Papyrus UC 32160

Among the Middle Kingdom cache of papyrus fragments excavated at Lahun, one contains a small "in-line" circular diagram [Clagett, 1999, 415; Imhausen, 2002a]. At the moment, this appears to be one of the earliest examples of a geometrical diagram preserved from ancient Egypt. The problem statement itself is not preserved in the extant fragment. From the surviving calculation, it appears to be similar to RMP, Problem 43—calculation of the volume of a circular granary [Imhausen, 2008, 36–37; Imhausen and Ritter, 2004, 84–87].

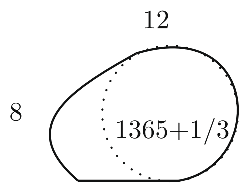

Fig. 5. Major axis = 4.7; minor axis = 3.3.

The diagram illustrates the problem of varying thicknesses of lines, since the line bounding the figure changes dramatically in width during the circumscription of the area—a feature that cannot be accommodated in the editing process. Almost certainly the scribe drew the figure in one bold sweep of his brush/pen, beginning from the lower right and proceeding upward and around to the left, as indicated in the changing thicknesses of the line. I have drawn the edited diagram following the outer edge of the scribe's line.

Because the circle diverges considerably from a true circle, I have approximated it using two Bézier curves and a straight line. The joins between the straight line and the curves are slightly noticeable, but well within the ambiguity of the scribe's line with its varying thickness. I have measured the axes (if it is fair to call them axes) at the widest distances horizontally and vertically. For comparison purposes a true circle, shown with a dotted line, has been included. Given the degree of distortion in the figure, one can only guess at the scribe's intended center and the size of the radius. I have chosen center and radius (which is very close to half the minor axis of the oval) to obtain the most complete fit to what the scribe has drawn on the papyrus.

RMP, Problem 41

To find the volume of a cylindrical granary whose diameter is 9 and whose height is 10 [Chace, 1927, Plate 63; Clagett, 1999, 357; Robins and Shute, 1987, Plate 14].

This diagram is an example of an "in-line drawing," as can be seen in the detail from the RMP. It is an unusual diagram because it is given in red, indicating that it is part of the statement of the problem.[32]

Fig. 6. Edited diagram (left) and detail (right) from the RMP showing the diagram in its context at the end of the statement of the problem (reading from right to left). Major axis = 4.3; minor axis = 3.1. ©Trustees of the British Museum. Used with permission of the British Museum.

The diagram diverges considerably from a true circle. It is more like a flattened oval, with its major axis oriented horizontally. Such distortion is typical of circles produced in Egyptian mathematical papyri. The figure has been approximated using two Bézier curves adjusted to follow the outer edge of the scribe's pen/brush stroke. A comparison circle, shown as a dotted line, emphasizes the degree of distortion in the figure. I have adjusted the diameter of this comparison circle to fit along the minor axis of the scribe's oval figure.

The granary is obviously a three-dimensional object, but it has been symbolized or diagrammed as only a circle. The reduction of three dimensions to two appears to be a typical convention of the ancient Egyptian scribe-mathematician.

RMP, Problem 44

To find the volume of a rectangular granary whose length, width, and height are 10 [Chace, 1927, Plate 66; Clagett, 1999, 358; Robins and Shute, 1987, Plate 15].

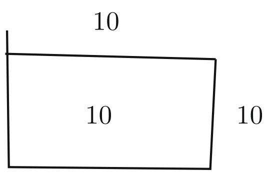

Fig. 7. Angle measures, beginning from the lower left and moving clockwise: 92°, 88°, 90°, 93°. Perimeter measures, beginning from the base and proceeding clockwise: 4.1, 2.3, 4.3, 2.2.

The diagram appears to have been drawn with one stroke, probably beginning at the top left corner and proceeding horizontally. Each of the corners is slightly rounded, and the process of changing directions at the two lower corners has produced a small ink blot at

[32] Chace has pointed out, in his transliteration of the text opposite plate 63, that the circle is not really a necessary feature of the problem, but was probably added by the scribe as a kind of illustration. I consider it a kind of diagram, since it has written within it one of the pieces of data (the value 9) on which the scribe is called to operate in order to solve the problem.

each corner. The final upward stroke extends slightly beyond the corner, but is much thinner than the remainder of the line. The edited diagram does not preserve the fine detail of the scribe's brush stroke. In producing the edited diagram, I have tended to follow a line approximately at the middle of the scribe's brush stroke. In measuring the sides, I have ignored the extension of the left side beyond the upper vertex.

The value 10 is written above the longer, horizontal side and beside the right-hand vertical side. Another 10 is written inside the diagram. Perhaps the scribe intended this inner value to represent the height of the granary. The diagram is drawn as a rectangle, the ratio of whose length to width is approximately 2:1, whereas the problem calls for a square base. It is not clear whether this deviation from the stated values is the reflection of some unexpressed aesthetic ideal or whether it derives from the sign for a four-sided structure (shown in this problem in ideographic form [Guillemot, 1992, 128]). Whatever the case, we find that squares are typically diagrammed as rectangles in the Egyptian mathematical tradition. Also, as in the case of Problem 41, the granary is a three-dimensional structure, but it is diagrammed as two-dimensional.

This problem and Problem 45 are the correlates of Problems 41 and 42, respectively, but this time with a square instead of a circular base. Many of the mathematical problems, such as these, arise from everyday problems that scribes might expect to encounter in their work.

RMP, Problem 48

The statement of the problem is missing. From the calculation, it appears to be a comparison of the area of a circle with the area of its circumscribing square [Clagett, 1999, 360; Chace, 1927, Plate 70; Robins and Shute, 1987, Plate 15].[33]

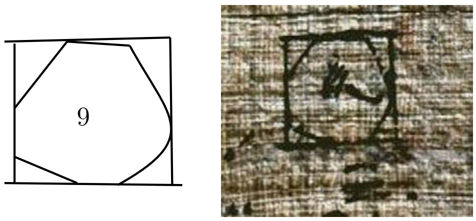

Fig. 8. Edited diagram (left) with detail from RMP (right). Angle measurements of the square, beginning from lower left and moving clockwise: 90°, 92°, 88°, 90°. Perimeter measures of the square, beginning from the base and proceeding clockwise: 5.5, 4.9, 5.4, 5.1. ©The Trustees of the British Museum. Used with the permission of the British Museum.

The diagram is another example of an "in-line drawing." The sides of the square do not meet neatly at the corners. The horizontal sides extend slightly beyond the vertical sides in both directions. I have preserved these irregularities in the edited diagram. The use of the polygonal figure, usually described as an irregular octagon, to approximate a circle is unusual. If this is indeed the intention of the scribe, it indicates a relatively sophisticated level of mathematical abstraction. Chace [1927, Plate 70] redrew the diagram as a somewhat irregular octagon within a square (Fig. 9) when producing his hieroglyphic transcription of the hieratic text. Gillings [1972, 140–146] explicitly argued that the figure was intended to be an

[33] Note that in his statement of the problem in the plate caption, Chace has incorrectly stated that the radius of the circle is 9, rather than having its diameter as 9. The problem is stated correctly on p. 91, where he outlines the calculation.

octagon and attempted to explain how this approximating figure could generate the calculation recorded by the scribe (Fig. 9). Clagett [1999, 162], in his translation, indicated the possibility that the circumscribed figure might be intended to represent a circle, but seems to favor the octagon hypothesis, as does Guillemot [1992, 138–140]. Only Imhausen [2008, 30] has interpreted the inscribed figure as a literal circle.

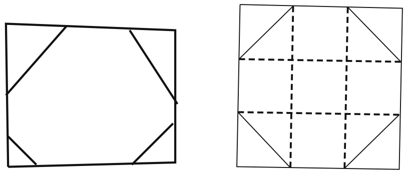

Fig. 9. Problem 48 diagram as redrawn by Chace (left) and Gillings (right).

The varying interpretations arise because the copyist has not given a statement of the problem, but only a diagram and a calculation. If we accept the octagon hypothesis, we have in this diagram the only example of the approximating polygon technique that I have encountered in documents from ancient Egypt. Given the irregularity of other circular figures in the papyri (Figs. 5, 6, 24, and 25, for example) and the apparent intent of the associated calculation, I think the interpretation of Imhausen is more likely to be correct. The figure is so irregular that I have approximated the portion on the right side of the diagram by a curve while using straight lines on the left side elements. The value 9 *setet* is noted in the diagram and is used in the calculation.

RMP, Problem 49

Calculate area of a rectangular piece of land 10 *khet* by 2 *khet* [Chace, 1927, Plate 71; Clagett, 1999, 361; Robins and Shute, 1987, Plate 16].

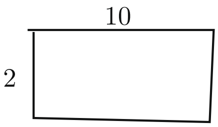

Fig. 10. Angle measures of the rectangle, beginning from lower left and proceeding clockwise: 91°, 90°, 88°, 92°. Perimeter measures, beginning from the base and moving clockwise: 6.2, 3, 6.3, 3.2.

This diagram is another example of a small "in-line drawing." The logic of the calculation requires that the vertical side should be one *khet*, rather than two *khet* as written in the diagram and as stated in the papyrus.[34] The diagram is drawn with a length-to-width ratio of approximately 2:1, rather than the 5:1 ratio derived from the values included in the diagram or the 10:1 ratio that the statement of the problem indicates. It often seems that a kind of aesthetic imperative overrides literalness when it comes to constructing these mathematical diagrams.

RMP, Problem 50

Calculate the area of a round field 9 *khet* in diameter [Clagett, 1999, 361; Robins and Shute, 1987, Plate 16].

[34] The actual calculation is carried out using cubits, rather than *khet* [Chace, 1927, 92].

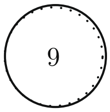

Fig. 11.

This diagram is another small "in-line drawing." I have not included measurements along the major and minor axes since the figure is so nearly circular. Unlike the circle in Problem 41, there is no doubt that the circle here is intended as a kind of diagram. The circle, although executed fairly accurately, bulges slightly toward the right. A true circle, shown as a dotted line, has been inserted for comparison purposes.

RMP, Problem 51

Calculate the area of a triangular field whose side is 10 *khet* and whose base is 4 *khet* [Chace, 1927, Plate 73; Clagett, 1999, 362; Robins and Shute, 1987, Plate 16].

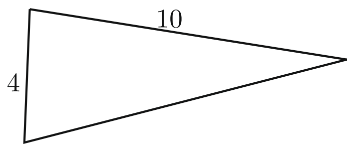

Fig. 12. Angle measures, beginning at the lower left and proceeding clockwise: 73°, 84°, 23°. Perimeter measures, beginning with the shortest side and proceeding clockwise: 3.5, 8.9, 9.5.

Although the diagram appears almost as an isosceles triangle, the calculation indicates that the side labeled "10" must have been intended to represent the altitude of the triangle and hence the diagram must represent a right triangle with the right angle at the top of the shorter vertical side. This problem is almost exactly like MMP, Problem 4.[35] The construction of the diagram by the two scribes is also remarkably similar, having the longer side of the right angle slanting slightly toward the right and the shorter side of the right angle leaning slightly from the vertical.

RMP, Problem 52

Calculate the area of a truncated triangular parcel of land whose base is 6 *khet*, whose summit is 4 *khet*, and whose side is 20 *khet* [Chace, 1927, Plate 74; Clagett, 1999, 362; Robins and Shute, 1987, Plate 16].

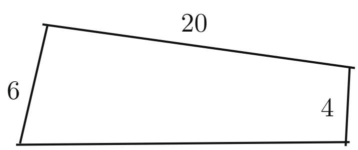

Fig. 13. Angle measures, beginning at the lower left and moving clockwise: 75°, 96°, 94°, 95°. Perimeter measures, beginning from the bottom and proceeding clockwise: 7.8, 2.9, 7.3, 1.8.

[35] In the present problem, as in RMP, Problem 49, the actual calculation of the area is performed in terms of cubits, rather than *khet* [Chace, 1927, 92–93].

In the diagram, the lines conspicuously do not meet neatly at the corners. This feature has been preserved in the edited diagram. When inserting the dimensions of the trapezium, the scribe has placed the word *khet* outside the right face of the figure but the value 4 inside. This seems to have been purely a scribal decision, since at the base (left face), the word *khet* is written above the value 6, both outside the baseline.[36] The diagram has some similarities to the diagram of the previous problem. It appears that the scribe intended the figure to be a truncated right triangle, with the right angle again at the upper left. In this diagram also the longer side of the right angle (if my assumption is correct) slants slightly downward.

RMP, Problem 53

As in Problem 48, the statement of the problem is omitted, leaving only a diagram and calculations [Chace, 1927, Plate 75; Clagett, 1999, 363; Robins and Shute, 1987, Plate 16].

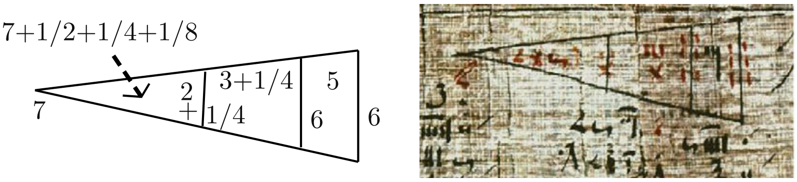

Fig. 14. Edited diagram (left) with detail from the RMP (right). Because the area measure that appears in the apex area of the figure takes so much space when converted to Roman characters, I have moved the label outside the figure and inserted a dashed arrow pointing to its original position. Angle measures of large triangle, beginning with the apex and moving clockwise: 20°, 82°, 78°. Angle measures of interior lines with lower and upper sides, respectively: longer line = 78°, 82°; shorter line = 81°, 79°. Perimeter measures, beginning from the lower side and proceeding clockwise: 9.8, 9.6, 3.2. Measure of the two interior dividing lines: 1.6 and 2.6, respectively. The altitude (measured on the perpendicular to the base) = 9.5, giving an altitude-to-base ratio of approximately 3:1. ©Trustees of the British Museum. Used with permission of the British Museum.

The diagram is unusual in that it is the only diagram in the RMP that has its value labels added in red ink. All other diagrams, with the exception of the "diagram" in Problem 41, are labeled in black.

Chace [1927, 50] considers the triangle to be isosceles. This is very close to what the scribe has actually constructed. The interpretation of the problem itself is difficult because the numbers used in the calculation do not completely agree with the numbers indicated in the diagram. Chace (1927, 94) interprets this problem to be an exercise in calculating the area of various parts of the triangle in the accompanying diagram. To do so, he must assume that some of the numbers in the diagram are incorrect and probable mistakes. Clagett [1999, 165] does not differ substantially in his interpretation. I have given the numbers in the diagram as they appear to me to be written.[37] In so doing, I have disagreed in

[36] As in Problem 49, the actual calculation of the area was performed in terms of cubits, rather than *khet* [Chace, 1927, 93].

[37] Gillings [1972, Appendix 12] provides a basic and accessible guide to interpreting common hieratic numeral forms that was generally adequate for the purposes of this paper. Serious investigation of the hieratic mathematical writings should, of course, rely on a standard palaeographic resource such as Möller [1909, 59–63] for a more scientific introduction to hieratic numeral forms.

several places with the readings of earlier scholars, who have been convinced that the numbers in the text are correct and those in the diagram are in error.

RMP, Problem 56

Calculate the *seked* of a pyramid given that the side of its base is 360 and its altitude is 250 cubits [Chace, 1927, Plate 78; Clagett, 1999, 364; Robins and Shute, 1987, Plate 17].[38]

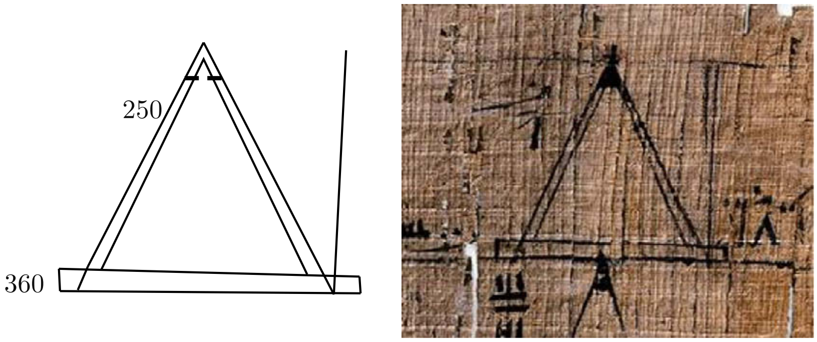

Fig. 15. Edited diagram (left) with detail from the RMP (right). Angle measures of the outer face, beginning with the lower left and proceeding clockwise: 64°, 56°, 60°. Angle measures of the inner face, beginning with the lower left and proceeding clockwise: 66°, 52°, 62°. Base angle measures, beginning with lower left and proceeding clockwise: 92°, 89°, 93°, 90°. Angle of standing line: 92° as measured from the baseline of the base. Pyramid sides (measured on the outer surfaces), beginning from the base and proceeding clockwise: 7, 7.5, 7.7. Base perimeter measures, starting from the bottom and proceeding clockwise: 8.6, 0.5, 8.6, 0.4. Measure of standing line: 7. Base extension beyond pyramid: left = 0.5; right = 0.9. Altitude of the pyramid (measured as the perpendicular from the apex to the top of the base or foundation) = 6.8, giving an altitude-to-base ratio slightly less than 1:1. ©Trustees of the British Museum. Used with permission of the British Museum.

In this diagram, the triangle representing the pyramid is shown bounded by a double wall on both slopes. This is the only example of a pyramid diagram with such a feature. Possibly it was drawn thus to indicate the outer shell of white limestone that was characteristic of pyramids.

There appears to be a vertical line erected to the right side of the pyramid, although its function is not known. Was it intended to represent the altitude of the pyramid? But if that was the scribe's intent, its placement exterior to the pyramid seems an unusual technique. Was this placement chosen because a pyramid is actually a solid object and hence any measure of its altitude must be done exteriorly? We can only speculate. In a later demotic example, P. Cairo 89143, Problem 40 (Fig. 31), the altitude line is placed inside the triangle representing the pyramid.

Using the value of the *seked* found in this problem, Gillings [1972, 187] has calculated the slope of the face of the pyramid to be 54°14′. Thus the actual figure is drawn somewhat

[38] The *seked* of a pyramid is the slope of the face of the pyramid. It is important that a pyramid rise uniformly from a level base in order to maintain its symmetric shape. It has been suggested that the ancient pyramid builders often worked with rise and run of 11 units in for every 14 units up [Lehner, 1997, 218–221].

incorrectly in regard to the stated values of the problem. This phenomenon is common to each of the diagrams of pyramids that accompany Problems 57–59. It is difficult to know how we should interpret this divergence between the characteristics of the actual pyramids and their visual representation by the scribe. Does it arise from a disconnect between the stonemasonry practices of the pyramid builders and the construction of working diagrams in mathematical papyri, a divide between theory and practice? Or does this discrepancy represent the operation of an underlying aesthetic value that overrode other concerns such as accuracy in visual representation?

RMP, Problem 57

Calculate the altitude of a pyramid whose *seked* is 5 palms 1 finger per cubit and the side of whose base is 140 cubits [Clagett, 1999, 365; Robins and Shute, 1987, Plate 17].

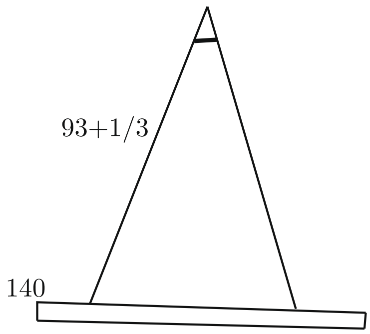

Fig. 16. Pyramid angle measures, beginning with the lower left and proceeding clockwise: 71°, 38°, 71°. Pyramid perimeter measures, beginning from the base and proceeding clockwise: 5.7, 8.8, 8.8. Base angle measures, beginning from lower left and proceeding clockwise: 93°, 90°, 88°, 94°. Base perimeter measures, beginning from the base and proceeding clockwise: 9.1, 0.5, 9.2, 0.5. Base extension beyond pyramid: left = 1.5; right = 2.0. Altitude (measured on the perpendicular from the apex to the base) = 8.9, giving an altitude-to-base ratio slightly greater than 3:2.

Using the values stated in the problem, Gillings [1972, 187] calculates the slope of the face of the pyramid as 53°8′. The measured values in the diagram are considerably greater, suggesting that the scribe was guided by aesthetics more than mathematics or the visual appearance of the pyramids.

RMP, Problem 58

A pyramid is 93 1/3 cubits high and the side of its base is 140 cubits. What is its *seked*? [Clagett, 1999, 365; Robins and Shute, 1987, Plate 17].

The papyrus has sustained some damage and the base of the pyramid is now incomplete. The dotted line indicates the edge of the break in the papyrus. Based on the evidence of the other RMP pyramid diagrams, the base does not extend uniformly on either side of the pyramid, so it is not possible to reconstruct the diagram based on a prediction of how far the base would have extended.

Using the values stated in the problem, Gillings [1972, 187] has calculated the slope of the side of the pyramid as 53°8′. His calculated value does not correspond with the actual diagram constructed by the scribe.

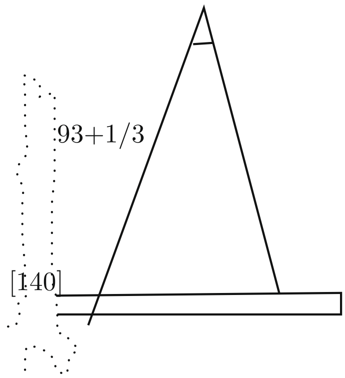

Fig. 17. Pyramid angle measures, beginning from the lower left and proceeding clockwise: 69°, 35°, 75°. Pyramid perimeter measure, beginning from the base and proceeding clockwise: 4.9, 7.3, 7.0. (The extension of the left-side line through the base has been ignored in these measurements.) Base angle measures (right), bottom to top: 88°, 95°. Base extension beyond pyramid on the right: 1.5. The altitude of the triangular figure (measured on the perpendicular from the apex to the base) = 7.1, giving an altitude-to-base ratio of somewhat less than 3:2.

RMP, Problem 59

Calculate the *seked* of a pyramid whose height is 8 cubits and the side of whose base is 12 cubits [Clagett, 1999, 366; Robins and Shute, 1987, Plate 17].

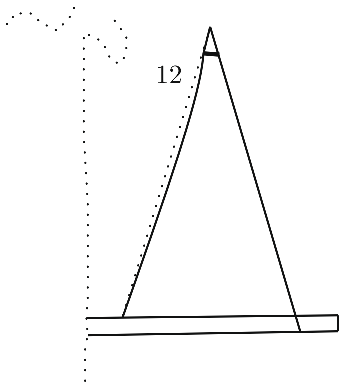

Fig. 18. Pyramid angle measures, beginning from the lower left and proceeding clockwise: 71°, 32°, 73°. Pyramid perimeter measures, beginning from the base and proceeding clockwise: 4.2, 7.2, 7.1. (The extension of the right-side line through the base has been ignored in these measurements.) Base angle measures (right), bottom to top: 88°, 88°. Base extension beyond pyramid, right: 0.9. Altitude of the triangular figure (measured on the perpendicular from the apex to the base) = 7.2, giving an altitude-to-base ratio of nearly 7:4.

There is a distinct inward bowing of the left side of the pyramid in the diagram. The dotted line indicates the true straight line between the two vertices. Measures are taken assuming the side is rectilinear. The dashed line at the apex marks the limit of the blackening applied to the tip of the triangle. The dotted line to the left of the pyramid indicates the line of damage to the papyrus. In this diagram, also, a portion of the base on the left side is missing.

Using the values given in the problem, Gillings [1972, 187] has calculated the slope of the side of the pyramid to be 53°8′. Once again, the actual diagram constructed by the scribe shows a somewhat steeper slope. Perhaps we see a kind of aesthetic drive overruling true representation.

RMP, Problem 60

Calculate the *seked* of a cone (?) whose base diameter is 15 cubits and whose height is 30 cubits [Clagett, 1999, 366; Robins and Shute, 1987, Plate 17].

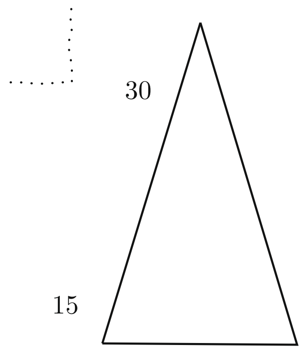

Fig. 19. Angle measures, beginning from the lower left and proceeding clockwise: 73°, 34°, 72°. Perimeter measures, beginning from the base and proceeding clockwise: 4.8, 8.2, 8.2. Altitude of the triangular figure (measured on the perpendicular from the apex to the base) = 7.9, giving an altitude-to-base ratio of a little greater than 3:2.

The meaning of the problem is obscure. Chace [1927, 28] translates it as referring to a pillar (?) rather than to a pyramid. He also notes [1927, 99] that the scribe has not expressed the *seked* in terms of palms as he had in the previous pyramid problems. The absence of a base below the triangular figure may lend credence to the interpretation that it does not represent a pyramid but some other object. Clagett [1999, 200, note 88] suggests that a cone might have been interesting to the ancient Egyptians through its potential use in a water clock. If this were the motivation, though, the diagram is inverted from what we might expect for a water clock or clepsydra mechanism.

Using the values stated in the problem, Gillings [1972, 187] has calculated the value of the slope of the side of the cone (?) represented by this diagram to be 75°58′. This calculated value differs only slightly from the metric of the diagram.

MMP, Problem 4

Calculate the area of a triangle whose side is 10 *khet* and whose base is 4 *khet* [Clagett, 1999, 387].

The diagram has suffered from damage to the papyrus. The dotted lines indicate the breaks in the papyrus itself. The dashed lines indicate the proposed reconstruction of the diagram.

This problem is parallel to RMP, Problem 51. Much of the calculation is missing here, so it is difficult to tell whether the same calculation procedures were followed.

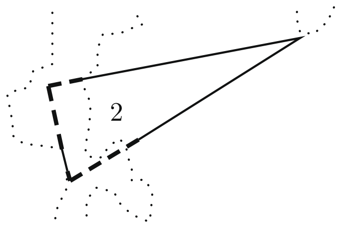

Fig. 20. Angle measures, beginning from lower left and proceeding clockwise: 71°, 89°, 22°. The values in parentheses are based on my reconstructions of the diagram. Perimeter measures, beginning from the left side and proceeding clockwise: 2.3, 6.1, 6.5. These measurements are based on my reconstruction of the diagram. Assuming that a right triangle is intended, these measures will give an altitude-to-base ratio of a little more than 5:2.

MMP, Problem 6

Calculate the area of a rectangle whose total area is 12 *setat* and whose breadth is $1/2 + 1/4$ its length [Clagett, 1999, 388].

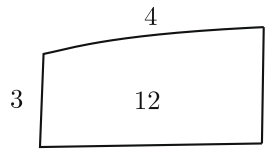

Fig. 21. Angle measures, beginning from the lower left and proceeding clockwise: 86°, 100° (?), 83° (?), 93°. The upper angle measurements are approximations necessitated by the distinct bowing of the upper horizontal line. If we assume that the side was intended to be straight, and measure the angles on that hypothesis, they are much closer to right angles. Perimeter measures: the base is 9.2; the two vertical sides are 3.8 (left) and 4.8 (right). The top line is not measurable because of the distinct bowing in its construction. But if we assume a hypothetical straight line, it is clear that the figure would be very close to the measure of the lower side.

This diagram is another example of a small "in-line drawing." The left side is distinctly shorter, which is somewhat puzzling. The scribe must have perceived the difference in side lengths. Why did he not simply extend the left side to make it closer to the right side in length? Was he not particularly interested in accuracy in the drawing? Or was there perhaps an implicit scribal practice that militated against correcting any sort of errors unless they made an essential difference in the text? At the moment, it is difficult to do more than speculate.

The number in the interior of the diagram is quite easily recognizable as 12 and seems intended to represent the area. The numerical values on the sides are much more difficult to decipher. In part, one is guided by the internal logic of the problem to conclude that they must be three and four, respectively.

MMP, Problem 14

Calculate the volume of a square pyramid whose base is four units on a side and whose height is truncated at 6 units, with an upper surface of 2 units on a side [Clagett, 1999, 396].

There is observable inward bowing of the lateral lines in this diagram. They have been approximated by Bézier curves in the edited diagram. True straight lines between the

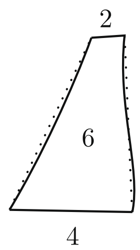

Fig. 22. Angle measures, beginning from lower left and proceeding clockwise: 66°, 117°, 91°, 85°. Perimeter measures, beginning from the base and proceeding clockwise: 2.9, 4.5, 0.8, 4.2.

vertices, shown as dotted lines, are included for comparison purposes. For purposes of measurement, I have used the hypothetical straight lines, rather than the bowed lines.

The usual interpretation of this problem is that it represents a truncated pyramid [Clagett, 1999, 221]. The term pyramid is not actually used in the hieratic text, however. Instead, the figure is named by using a small "in-line drawing" of the figure [Imhausen, 2008, 33–34]. Perhaps it would be more appropriate to use the term frustum? Imhausen also points out that this problem is unusual in that it includes in the diagram labeling some of the actual calculation results in addition to the basic data values [Clagett, 1999, 412]. To avoid cluttering the edited diagram, I have omitted these calculation values.

Using the values given in the problem, Gillings [1972, 187] has calculated that the slope of the side of this pyramid, if it is indeed intended to be a pyramid, would be 80°34′. No surviving pyramids show such steep sides. Even the late Nubian pyramids, which typically have steeper sides than Old and Middle Kingdom Egyptian pyramids, rarely exceeded 70° inclination [Lehner, 1997, 195].

MMP, Problem 17

Find the length (or height) and width of a triangle whose total area is 20 *setat* and whose width is 1/3 + 1/15 its length [Clagett, 1999, 398].

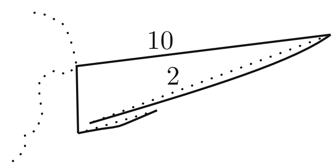

Fig. 23. Angle measures, beginning with lower left and proceeding clockwise: 74°, 96°, 13°. Perimeter measures, beginning from the left side and proceeding clockwise: 1.6, 6.1, 6.5. Assuming a right triangle is intended, these measures will give an altitude-to-base ratio of nearly 4:1.

Although not stated in the problem, the calculation indicates that the figure should be considered a right triangle. The numeral "10" is clearly written above the upper line, the longer side of the right angle, which is called the length (the equivalent of the altitude) of the triangle in this problem. This value probably represents the outcome of the calculation. The situation is confused, however, because there seems to be a numeral "1" written immediately to the right of the 10, which would typically signal a multiplication. The shorter side of the right angle is called here the width of the triangle. Its value, we are told, is 1/3 + 1/15 (or 2/5) of the triangle's length. The number "2" placed inside the triangle is more difficult. It is not the value of the area, since we are given that it is "20". Probably it is a part of the operation of the problem, since the calculation part of the problem seems to begin by doubling twenty to obtain 40.

In this diagram, the longest line (hypotenuse) is definitely broken. It appears probable that the scribe began the drawing with the shorter segment, but his hand slipped and he decided to start again from the other vertex. Realizing that this line was not going to meet the previous line, he decided to leave them both unfinished. Once again, we may wonder whether something in scribal training militated against making corrections of innocent errors. Both the unfinished lines show discernible inward bowing, and the internal dotted lines show the true straight lines between the vertex and the tip of the partial line. The other sides of the triangle also show some waviness when viewed under magnification, but I have not indicated this feature in the edited version. When measuring the angles, I have used the postulated straight line segments from the two vertices. When measuring the sides, I measure along a hypothetical straight line between the two vertices, not along the broken line. The external dotted line indicates the edge of the damage to the papyrus surrounding the diagram.

This problem is the inverse of MMP, Problem 4.

Appendix II. Catalog of diagrams in demotic papyri

In this appendix, I catalog and edit diagrams from demotic fragments based on available published photographs of the papyri. This demotic material was produced more than a thousand years after the treatises of the Second Intermediate Period. Because most of these papyri are so fragmentary, it is not easy to make correlations with the diagrams found in the earlier mathematical treatises. We can, however, discern some similar features uniting these demotic diagrams with earlier hieratic diagrams. Most notably, we can say that circles continue to be distorted—in fact, the amount of distortion seems to increase over time. We notice that the circular segments found in several problems also have their arcs considerably distorted. Moreover, these segments are constructed with their straight sides horizontal. Rectilinear quadrilateral figures, by way of contrast, do not show much difference over time. In these demotic fragments, though, we can see that triangles tend to be constructed as equilateral, rather than isosceles, and they typically have a horizontal base. Triangles representing pyramids are not drawn with solid base, with double side lines, or with darkening at the apex. At the same time, we remark that the demotic tradition continues to use two-dimensional figures to represent three-dimensional objects. One change that occurs within the demotic papyri, however, is the more frequent use of nonrepresentational diagrams. (Nonrepresentational diagrams are discussed in Section 7.6 of the paper.) While some problems remain the same (such as calculation of circular areas and calculations related to pyramids), there are also new kinds of problems that appear in the demotic documents, such as calculation of the volume of a "mast" and changing the dimension of a piece of cloth while still preserving the total area [Imhausen, 2008, 48–49]. The inclusion of metrical labels continues to be a staple feature of these demotic diagrams.[39] In this appendix, I follow the problem numeration used by Parker [1972], although his combining of the disparate fragments into a single numbering system ignores the chronological gaps between various testimonia.

[39] For reading demotic numeral forms, I have followed the guide prepared by Parker [1972, 85–86].

As in Appendix I, we begin discussion of each diagram entry with a statement of the problem to which it is attached and follow with references to the location of reproductions of the original diagrams. We then discuss the edited diagram as shown in a numbered figure. We conclude with any other remarks on features of the diagram and its editing.

P. Cairo 89137, Problem 32

Calculate the diameter of a circular piece of land having an area of 100 square cubits [Parker, 1972, Plate 11].

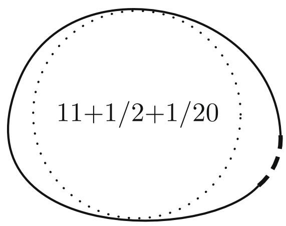

Fig. 24. Major axis = 6.8; minor axis = 5.2.

The break at the right side of the circle appears to result from ink flaking off the papyrus, rather than damage to the fabric of the papyrus itself. The postulated restoration is indicated by a dashed line. The diagram differs considerably from the circle it is intended to represent. For comparison purposes, a true circle is indicated in the diagram using a dotted line. The dimensions of this comparison circle are only a guess, since it is impossible to know where the scribe intended to place the center or what point on the oval should be taken to provide the radius of the comparison circle. In most cases such as this, I opt for making the diameter of the circle approximate the minor axis of the oval figure.

P. Cairo 89137, Problem 33

Calculate the diameter of a circular piece of land having an area of 10 square cubits [Parker, 1972, Plate 11].

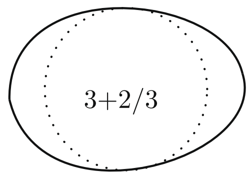

Fig. 25. Major axis = 5.8; minor axis = 4.0.

In this diagram, also, the scribe has produced an oval figure rather than a diagram more nearly approximating a circle. For comparison purposes, a true circle is indicated by a dotted line in the edited diagram. Since it is impossible to know where the scribe intended to place the center or what point on the oval should be taken to provide the radius of the comparison circle, I have made the diameter of the circle approximate the minor axis of the oval.

P. Cairo 89140, Problem 36

Calculate the component areas of a circular piece of land in which an equilateral triangle is laid out having each side 12 "divine-cubits" [Parker, 1972, Plate 12].[40]

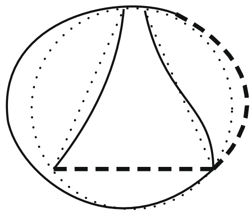

Fig. 26. Major axis = 5.9; minor axis = 4.9. I estimate the perimeter measures of the triangle, beginning from the base and proceeding clockwise, using the postulated straight lines connecting the apparent vertices: 4, 4.4, 4.3. The altitude (measuring a perpendicular from the point on the scribe's "circle" between the apparent ends of the sides to the base) = 3.8, giving an altitude-to-base ratio a little less than 1:1.

The first diagram (Fig. 26) given in this problem illustrates the general situation described in the problem statement. The diagram is in poor condition, although it is not clear from the photographs whether this damage is due to the fabric of the papyrus or simply flaking of the ink. The lower portion of the scribe's "circle" appears more like a series of dots than a connected line. I have constructed the curve to run through these remaining dots of ink. A larger gap on the right-hand side, for which there remain no visible traces, has been reconstructed assuming a continuous curve between the two ends of the gap. My postulated reconstruction is shown using a dashed line. A true circle, indicated with a dotted line and having its diameter approximately equal to the minor axis, has been inserted for comparison purposes.

The two sides of the triangle are considerably bowed, but whether that results from the condition of the papyrus or careless construction is impossible to decide from the photographic plates. True straight lines connecting the vertices, shown as dotted lines, are included for comparison purposes. The sides do not appear to form an apex at the circumference as they should, but whether they originally extended beyond the "circle" or simply terminated at its curve is impossible to tell. The base of the triangle has completely disappeared and has been reconstructed by joining the two lowest points of the sides of the triangle, as shown with the dashed line.

This general situation is then broken into two diagrams, an equilateral triangle and a segment of a circle, to show the calculations of the areas of the individual pieces of the circular plot of land. The diagram of this triangle (Fig. 27), too, has experienced considerable degradation, and the dashed lines indicate my reconstructions. There was probably some bowing of the sides, since the reconstructed portions do not align neatly with the existing line segments.

The three circle segments are identical, so one diagram (Fig. 27) is sufficient to illustrate their area computation. The baseline in the diagram is distinctly bowed outward and has been approximated using a curve. The true straight line between the two endpoints of the arc, indicated by a dotted line, is included for comparison purposes. The arc is notice-

[40] The relation of "divine cubits" to "short" and "long" cubits [Clagett, 1999, 9–11] is not clear to me.

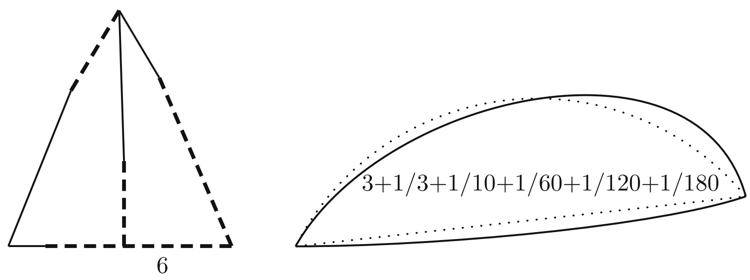

Fig. 27. The components of Problem 36: the equilateral triangle (left) and one of the segments of the circle (right). Perimeter of the equilateral triangle, beginning from the base and moving clockwise: 6, 7.1, 7.1. (Measurements are taken from vertex to vertex, assuming straight lines joining them.) Angle measures, based on my hypothesized reconstructions, beginning from lower left and moving clockwise: 69°, 54°, 66°. Altitude = 6.5. The line leans slightly to the left at 92°. Altitude-to-base ratio is a little greater than 1:1. Base of the circular segment = 8.6 as measured along the straight line connecting the two ends of the curve.

ably noncircular and has been approximated using a curve. A true circular arc, shown as a dotted line, is included for comparison purposes. Its dimensions are somewhat speculative, but I have assumed the arc would pass through the two endpoints of the horizontal line.

P. Cairo 89141, Problem 37

Calculate the area of a square inscribed in a circular field whose total area is 675 square cubits and whose diameter is 30 cubits. Then calculate the area of the four segments [Parker, 1972, Plate 14].

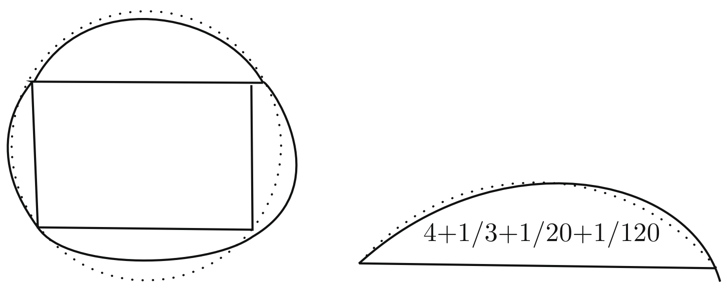

Fig. 28. The square inside a circular field (left) and one of the segments of the circle (right). Major axis = 4.9; minor axis = 4.1. Angle measures of the inner square figure, beginning from the lower left and proceeding clockwise: 92°, 89°, 91°, 92°. Perimeter measures of the square figure, beginning from the base and proceeding clockwise: 5.0, 3.4, 5.2, 3.4. Base of the circular segment = 6.1.

The first diagram of the problem illustrates the square within the circle. The circle is considerably distorted, so its circumference has been approximated by four Bézier curves, constructed to follow as closely as possible the outer edge of the scribe's curve, insofar as that can be distinguished. The joins between these curves are somewhat noticeable. A true dotted circle, whose radius was chosen to make the circle pass through the two lower vertices of the circumscribed quadrilateral, has been inserted for comparison purposes.

Although the statement of the problem calls for a square, the scribe has produced a rectangle whose length-to-width ratio is approximately 3:2. Throughout Egyptian history, one rarely finds a true square (that is, a quadrilateral having all four angles and sides equal).

Usually, the diagrams will show a rectangle, as we see here and in RMP, Problem 44 (Fig. 7).

There are four identical segments of a circle formed when the square is inscribed. They are represented in the papyrus by a single diagram. In this diagram, the arc of the segment is considerably distorted and is approximated by a Bézier curve. For comparison purposes, a true circular arc, constructed to pass through the endpoints of the straight line segment, is indicated as a dotted line.

P. Cairo 89141, Problem 38

Calculate the areas of a circular plot in which there is laid out an equilateral triangle 10 cubits on a side [Parker, 1972, Plate 14].

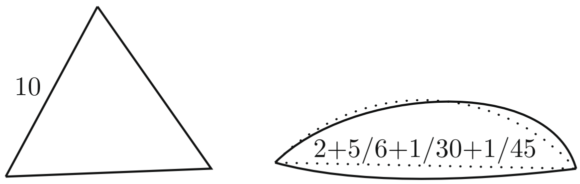

Fig. 29. The components of Problem 38: an equilateral triangle (left) and one of the segments of the circle (right). Angle measures of the triangle, beginning from the lower left and proceeding clockwise: 60°, 64°, 56°. Perimeter measures of the triangle, beginning from the base and proceeding clockwise: 4.9, 4.5, 4.6. Altitude (measured on the perpendicular from the apex to the base of the triangle) = 4.0, yielding an altitude-to-base ratio of nearly 5:4.

This problem is parallel to the demotic P. Cairo 89140, Problem 36. In this problem, however, the composite triangle within a circle diagram has been omitted. The scribe has inserted only the diagram of the equilateral triangle, without an explicit altitude line. There is a space below this triangular figure that Parker [1972, 49] hypothesizes once contained sketches of the three segments. I do not find anything recognizable in this space, although the fact that the text lines are indented to the left of the space implies that there should be something there. Such sketches showing all three circular segments are not found with similar problems in the same papyrus, however. The condition of the fragment does not permit a definite resolution to this question.

A clear diagram of one of these circular segments is found later in the problem. This segment is somewhat distorted in the diagram, and both the arc and the baseline, which bows outward, have been rendered with Bézier curves. The postulated straight line connecting the vertices is shown as a dotted line. For comparison purposes, a dotted circular arc, constructed to pass through the two endpoints of the baseline, shows the true circular curve of the segment.

P. Cairo 89143, Problem 39

Calculate the distance from the center of any side of a pyramid whose height is 300 cubits and whose side is 500 cubits in length to the apex [Parker, 1972, Plate 14].

This calculation is closely related to the calculation of the *seked* of a pyramid. Here, however, the scribe is calculating the actual length of the side, whereas the *seked* calculation refers to the slope of the side, or its rise and run.

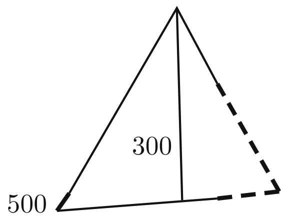

Fig. 30. Angle measures, beginning from the lower left and proceeding clockwise: 55°, 60°, 65°. Perimeter measures, beginning from the base and proceeding clockwise: 5.4, 5.8, 5.1. Altitude length (measured on the perpendicular from the apex to the base of the triangular figure) = 4.7, giving an altitude-to-base ratio a little greater than 1:1. The triangle's altitude line leans to the right at 88°.

Because a repair incorrectly overlay the diagram, Parker [1972, 1] found it necessary to cut the photograph and realign the pieces. Since his cut passes directly through the figure, it is difficult to edit the diagram. The perimeter of the triangle has also been damaged in several places and has been reconstructed using dashed lines. The numerical value for 300 is difficult to make out distinctly in the photograph, but the numeral 500 is quite clearly legible. That the interior numeral should be read as 300 is confirmed by the text and mathematics of the problem.

P. Cairo 89143, Problem 40

Calculate the volume of a pyramid whose height is 10 cubits and whose base is 10 cubits on a side [Parker, 1972, Plate 14].

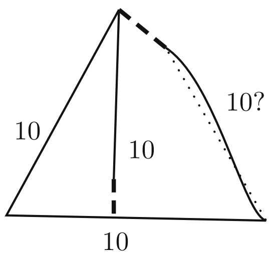

Fig. 31. Angle measures at the base: 63° (left) and 59° (right). Because of the distortion in the right side, it is not possible to obtain an accurate measure of the apex angle. Base length: 6.2; Left side length: 5.6. The distortion in the figure makes it impossible to form an accurate estimate of the length of the right side. The altitude line leans approximately 1° to the left when measured from the base of the triangle. Length of the altitude line (measured from the apex to the base of the figure) = 5.0, giving an altitude-to-base ratio a little greater than 5:6.

There is also considerable distortion in this diagram due to the problematic repair of the papyrus. The right side is bowed outward somewhat in the photograph. A comparison dotted straight line between the vertex and the end of the visible side line is indicated in the edited diagram. The sign for 10 on the right side of the diagram is not easy to make out, and is indicated in the edited diagram by the use of the question mark following the number. The dashed portions represent reconstructions of lines that are no longer visible in the papyrus.

BM Papyrus 10520, Problem 64

Calculate the area of a rectangular piece of land whose sides are 12 cubits and 10 cubits [Parker, 1972, Plate 24; Imhausen, 2008, 50].

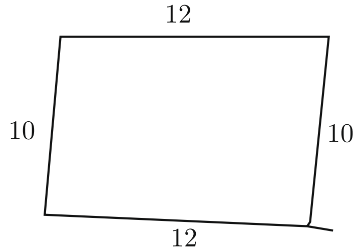

Fig. 32. Angle measures, beginning at the lower left and continuing clockwise: 88°, 96°, 85°, 92°. Perimeter measures, beginning from the base and proceeding clockwise: 6.4, 4.2, 6.5, 4.5. Length to width ratio is slightly greater than 4:3.

This diagram illustrates many of the difficulties one finds in attempting to edit ancient Egyptian diagrams. Although the image is clear in the photograph, the diagram requires many editorial decisions. Judging from the varying thickness of the diagram line, it appears that the scribe began the drawing at the lower right, drawing the vertical side, continuing without raising his pen (producing a quite rounded corner) across the top side, down the left side, and across the base to the beginning point in one long sweeping stroke. One sees that his line begins very thick and dark, probably indicating that he has just charged his rush or reed with ink, and the line rapidly becomes thinner and fainter as he nears the ending point. His line does not neatly stop at the beginning of the vertical line, but in the photograph the extension is less noticeable in comparison to the thickness of the line with which he began his drawing. In the edited diagram, however, the extension appears far more obvious since all the lines are the same breadth. Our inability to show variations in the thickness of lines in the editing process forces our angle and side measures to be somewhat arbitrary in this diagram, since different editorial decisions will yield diagrams that differ slightly from one another.

P. Carlsberg 30r, Problem 69

The nature of this problem, with three diagrams (Figs. 33–35), is unknown since only two partial lines of text are preserved [Parker, 1972, Plate 25].

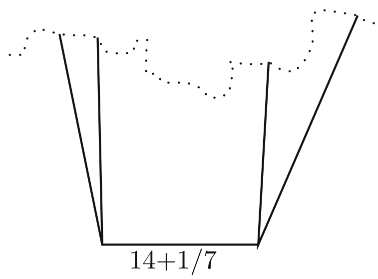

Fig. 33. Angle measurements, beginning from the left and proceeding toward the right: 103°, 92°, 93°, 114°. Base measure: 3.8. The intended lengths of the remaining lines in the diagram cannot be guessed.

The only definite information connecting the three diagrams is that each seems to involve in some way the value 14 + 1/7 [Parker, 1972, 74]. The three diagrams would be consistent with a proposition about areas, but it is difficult to speculate. The diagram in Fig. 33, at the moment, remains inscrutable. The dotted line in the edited diagram represents the break in the papyrus.

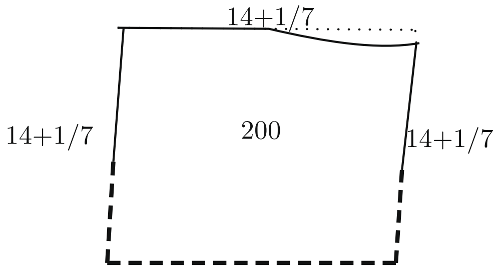

Fig. 34. Angle measures: 95° (left). It is not possible to get an accurate measure of the right-hand angle given the ambiguities in the diagram. Although I have reconstructed a closed figure, its dimensions are purely conjectural so no perimeter or angle measures can be given.

The values placed on the sides of the diagrams in Figs. 34 and 35 seem to imply square figures [Parker, 1972, 74]. There is a distinct inward bowing in the top line of the diagram in Fig. 34. Since it is not known for certain which level of the line was ultimately intended by the scribe, I have included dotted lines to indicate the potential extension of the higher portion of the line. Presuming the diagram to be intended as a square, I have postulated a completion of the figure using dashed lines. Of course, we cannot know for certain what the actual metric of the incomplete and missing sides might have been. Given the apparent antipathy to drawing true square figures in ancient Egypt, I have reconstructed a somewhat rectangular figure in Fig. 34.

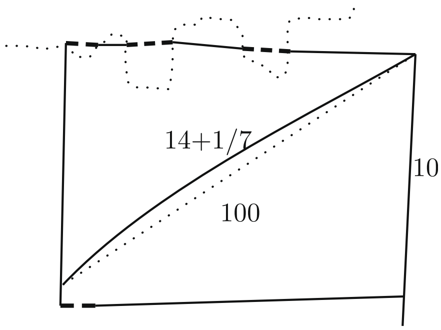

Fig. 35. Angle measures: 89° (lower left) and 94° (lower right). The lower left angle is estimated based on the hypothesis that the left side and baseline would meet at a vertex. Because of the apparent nonlinearity of the top line, the upper angles and top line length are impossible to estimate with any degree of certainty. Line measures: base = 8.2, left side = 6.2, right side = 5.8. The baseline and left side are measured on the hypothesis that the two lines would meet at a vertex. Using these values, we obtain a length-to-width ratio between 6:5 and 7:5, depending on which of the vertical sides one considers.

The diagram in Fig. 35 is relatively complete, although some crucial sections, such as the lower left corner, are unfortunately damaged and so necessitate some editorial decisions regarding the reconstruction of the figure. Should we assume that the diagonal line will meet the left side, as it appears destined to do in the surviving part of the diagram? Or should we assume that it will meet the base at the vertex, which would entail a very distinct bowing of either the baseline or the diagonal in the short space that is now missing? I have opted for the assumption that the diagonal would have been intended to be straight, although this seems to require that it does not meet the opposite vertex but falls distinctly above it. All lines of the diagram have noticeable bowing in places, but I have indicated the departure from rectilinearity only in the case of the diagonal by inserting a straight dotted line between its the two endpoints. The external dotted lines indicate breakage in the papyrus as it now exists.

P. Carlsberg 30r, Problem 70

The problem cannot be deciphered from the fragmentary remains of the text [Parker, 1972, Plate 25]. It is accompanied by a small "in-line drawing" of a triangle with the number 3 on its base.

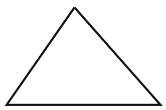

Fig. 36. Angle measures, beginning from lower left and proceeding clockwise: 56°, 77°, 48°. Perimeter measures, beginning from the base and proceeding clockwise: 3.7, 2.8, 3.0. Altitude (measured on the perpendicular from the apex to the base) = 2.5, yielding an altitude-to-base ratio of approximately 5:7.

Appendix III. Comparison with diagrams from other cultural traditions

This appendix takes a somewhat more speculative direction to look at premodern diagrams across cultural and linguistic boundaries. Since geometrical diagrams also exist in other early mathematical traditions, I wondered to what extent the diagrams in these various mathematical traditions differed from one another, either in their construction or in their usage within mathematics. Thus I will use this appendix to point out some similarities and contrasts between the Egyptian diagrams catalogued in the previous two appendixes and diagrams used in other early traditions of geometric discourse. I have limited this preliminary comparison/contrast to diagrams from ancient Mesopotamian mathematics and from early Sanskrit mathematics. This initial inquiry, although not exhaustive, does point to interesting convergences and divergences that suggest the potential fruitfulness of further studies of diagram traditions.

When assessing any diagram tradition, one must always bear in mind the potential complicating factor of scribal idiosyncracies in shaping one's perceptions of diagram conventions. What diagram features represent a valid tradition and what features derive from the personal aptitudes and aesthetics of the copyist is almost impossibly difficult to sort out unless one has available a reasonably large body of relatively contemporaneous exemplars. Hence the first step in carrying further comparative studies such as this will be the preparation of much more complete catalogs of diagrams from the various mathematical traditions.

Equally important, we must bear in mind the specific limitations imposed by the writing technology used to record the mathematics and how these practicalities impact the constructing of mathematical diagrams. Diagrams in ancient Egypt were recorded on papyrus using a reed or rush pen to apply the ink. Contemporaneously in Mesopotamia, diagrams were inscribed on the surface of a clay tablet using a stylus. And, although most surviving manuscripts date from considerably later, at least some traditional Sanskrit mathematical discussions were inscribed on palm-leaf codices using a *kanta* or thick metal needle with a sharp point to engrave letters onto the surfaces of the specially prepared palm leaves. After the letters were inscribed on the leaves, coal powder was usually applied to make the inscribed letters more visible. While it may be difficult for us to assess the full implications of these various writing methods for the diagrams produced from them, certain similarities and differences in the diagrams can be imaginatively related to these disparate writing technologies.

For this introduction to cross-cultural diagram studies, I have used several examples from Mesopotamian tablets (most derived from images published on the Internet) and some examples of traditional Sanskrit diagrams published recently in this journal [Keller, 2005].[41] I have chosen to focus on Mesopotamian diagrams, since the Mesopotamian and Egyptian mathematical traditions flourished at approximately the same time and we know that there were regular if not routine economic and political contacts between the two states. A brief comparison between Egyptian and Mesopotamian diagrams permits a preliminary assessment of possible lines of influence and interaction. In my discussion, I shall combine these various examples under the general rubric "Mesopotamian", intending by that term the geographical region lying between and on either side of the Tigris and Euphrates rivers, rather than attempting to differentiate the various historical and cultural periods. In large part, use of this broad geographical definition, to the neglect of more specific historical and cultural divisions, is a reflection on the logistical difficulty in obtaining sample illustrations of Mesopotamian diagrams.[42]

Photographs of diagrams from traditional Sanskrit mathematical manuscripts are almost impossible to locate in standard library resources or on the Internet. Therefore,

[41] One could, of course, extend the study almost indefinitely. The significance of diagrams in mathematical treatises from China has already received some study and discussion [Chemla, 2005; Gillon, 1977; Qu, 1977]. Most studies of early mathematical traditions, if they even discuss diagrams, have not focused on techniques of diagram construction, but instead discuss their use in interpretation of the texts. For example, a recently published sourcebook of early non-Western mathematical writings [Katz, 2008] provides an invaluable guide for introducing the English reading public to the riches of several premodern traditions. Yet it disappointingly has almost nothing to say about traditional diagrams, their constructional features, or their possible role in creating mathematics. Most of the diagrams illustrating these translations have been redrawn and their relation to the original diagrams is not made explicit.

[42] Robson [2008b] provides a concise introduction to the history of Mesopotamian mathematics. There are a far greater number of mathematical texts surviving from Mesopotamian civilization than from ancient Egypt, with correspondingly many more examples of mathematical diagrams available for study. I have consciously chosen diagrams that incorporate specific classes of figures which parallel the figures found in Egyptian mathematical diagrams. Anyone wishing to undertake a more comprehensive survey of diagrams in the Mesopotamian mathematical tradition must begin with the classic studies by Neugebauer [1935–1937] or Neugebauer and Sachs [1945]. I regret that these standard academic resources were unavailable to me for this study.

my study is limited entirely to selected photographs published by Keller [2005], which come from only one manuscript, KUOML (Kerala University Oriental Manuscripts Library), Co 1712, a copy of Bhāskara's seventh century commentary on the *Āryabhaṭīya*.[43] The roughly sketched diagrams occurring in this manuscript, probably copied in a palm-leaf codex between the 17th and 19th centuries C.E., are from a Sanskrit tradition centuries later than the Mesopotamian.[44] Its diagrams generally appear to be quite small, almost "in-line" drawings, and are crudely done when compared to the relative precision of many Mesopotamian diagrams.

Keller [2005, 278] notes that four manuscripts were consulted during her study of Bhāskara I's commentary and that the diagrams varied from one manuscript to another. The precise nature of that variation is not discussed, but clearly there is a need for further study of the development of diagrams in Indian mathematical traditions also. The sample diagrams published by Keller, however, show some interesting similarities to and differences from those produced in both ancient Babylon and ancient Egypt. Like diagrams of ancient Egypt and Mesopotamia, these Sanskrit diagrams carry only numerical labels [Keller, 2005, 282–283] and are often intended to make visible certain aspects of problems that are essentially algorithmic in nature [Keller, 2005, 280]. On the other hand, Sanskrit diagrams also accompany problems, such as the "Hawk and Rat" questions [Keller, 2005, 280], which discuss spatial relationships different from those found in the more ancient Egyptian and Mesopotamian traditions. These Indian diagrams are more akin, for example, to the diagrams used in medieval and Renaissance mathematical natural philosophy to represent relationships in the phenomenal world [Roche, 1993].

In the following overview, I have deliberately sought diagrams that incorporate specific classes of figures which parallel those found in Egyptian mathematical diagrams. I shall follow the same general arrangement that was used to discuss categories of ancient Egyptian diagrams in Section 7 of the paper. I shall typically begin with descriptions of Mesopotamian examples, and then proceed to consider relevant Sanskrit diagrams.

Squares and rectangles

Squares and rectangles are used in problems of the Egyptian tradition to illustrate the calculation of areas (Figs. 8, 10, 28, and 32). As we have seen, diagrams from ancient Egypt, even if intending to illustrate square areas, are constructed as rectangles with their longer sides horizontal. Similar geometrical entities occur also in the statement of Mesopotamian mathematical problems. These Mesopotamian problems are also often accompanied by diagrams, but these diagrams are usually constructed more precisely than were those of ancient Egypt.

[43] There are no diagrams in manuscripts of the *Āryabhaṭīya* itself. Diagrams appear only in manuscripts of the commentary by Bhāskara I (seventh century C.E.) [Keller, 2005, 277].

[44] Since palm-leaf codices rapidly deteriorate in India's moist climate, the surviving manuscripts were copied hundreds of years after the original commentary was written. It is difficult to correlate these surviving diagrams with what might have existed in the autograph of Bhāskara [Keller, 2005, 277–278]. My own experience with diagrams in the medieval Islamic tradition [De Young, 2005], however, suggests that the basic features of the diagrams probably changed relatively little over time.

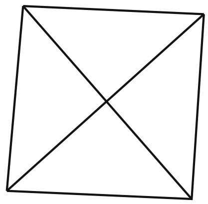

Fig. 37. British Museum Tablet 15285, detail of a square divided by two diagonals. Angle measures, beginning from the lower left and proceeding clockwise: 90°, 92°, 89°, 91°.

Several Mesopotamian diagrams involving squares are found on Tablet BM 15285.[45] One of these diagrams (Fig. 37) is a square divided by its two diagonals. The sides of the square are very nearly equal, but the angles are somewhat less exactly constructed. Because the angles are not precisely formed right angles, the square as a whole is tilted slightly to the right, making its baseline not quite parallel to the text of the tablet.

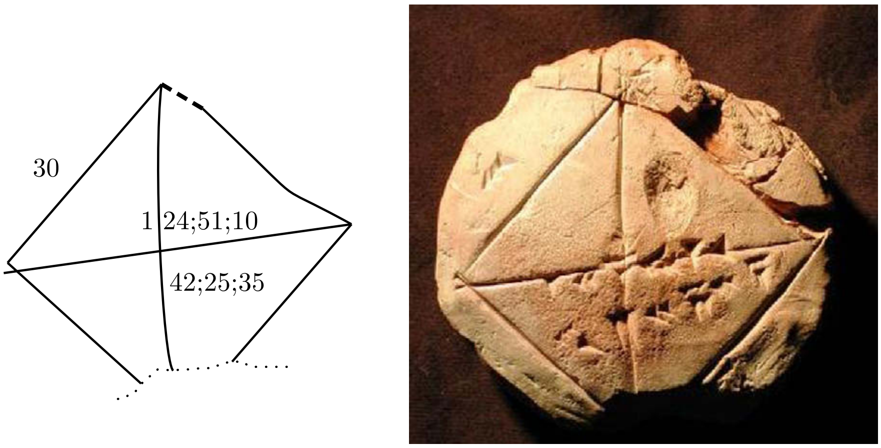

Fig. 38. YBC 7289 obverse—edited diagram (left) and photograph of the tablet, courtesy of W. Casselman (right). Used with permission of Yale University Babylonian Collection. The dotted line indicates where a portion of the tablet (lower vertex) is completely missing. There has also been damage to the surface, as is evident in the photograph, so that not all the lines are completely clear. The dashed line indicates my suggested reconstruction of the upper right side of the figure.

[45] The tablet gives only the statement of the problems, not the calculation process or the solutions and thus seems to lie outside the pedagogical tradition. Thus it seems to have many features in common with what have been called "riddle" texts. (BM 13901 is another example of the genre [Høyrup, 2001, 84–88].) Now in several pieces, the tablet contains an extensive collection of diagrams involving squares. Some of its diagrams have been described and discussed by Saggs [1960, 141–145] and Robson [2008b, 93–100]. These diagrams involve a variety of plane figures, some simple, some rather more complex, each bounded by a square. A direct inspection of the tablet, it is claimed, will reveal "the partially erased construction lines and compass marks . . . which the scribe used in making the drawings" [Robson, 1999, 208]. At least the scribe's compass marks can be clearly discerned in some of the unauthorized photos of the tablet currently posted on the Internet.

Another diagram, again a square bisected by two diagonals, appears on the obverse of tablet YBC 7289 (Fig. 38).[46] The tablet, an example of "rough work," appears to have been reused and contains only the diagram and its accompanying numerical values, leaving us to speculate about the original problem. It has often been taken to represent calculation of the diagonal of a square [Robson, 2008b, 143]. Since this calculation is equivalent to use of the Pythagorean Theorem, the drawing has also sparked some discussion about the origins of Euclid's best-known theorem.[47] The lines do not meet neatly at all the vertices, and there is some slight bowing in several of the lines, although this may be an artifact of the slightly curved surface of the tablet. The line of damage to the tablet obscuring the diagram at the lower vertex is indicated by dotted lines.

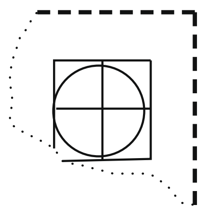

Fig. 39. British Museum Tablet 15285, detail of a circle inscribed in a square.

Another diagram from Tablet BM 15285 (Fig. 39) consists of a small square within a larger square and, within the smaller square, an inscribed circle divided by horizontal and vertical diameters. The circle is slightly misshapen and is approximated in the edited diagram by an ellipse. As the edited diagram shows, the circle is not quite tangent to the top or right side of the bounding square. The line of the larger surrounding square is shown as dashed, and the broken edge of the tablet is shown by a dotted line. Although like the diagram accompanying RMP, Problem 48 (Fig. 8) in that it is a circle inscribed within a square, in this Babylonian diagram, the inner square has straighter and more equal lines and the circle is distinctly closer to a true circle than in the Egyptian diagram.

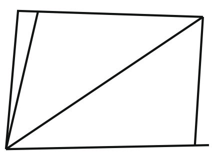

Fig. 40. Diagram from KUOML, Co 1712, folio 60r, showing a rectangular figure divided by a diagonal. Angle measures, beginning from the lower left and proceeding clockwise: 85°, 94°, 89°, 94°. Perimeter measures, beginning from the base and proceeding clockwise: 4.6, 3.4, 4.6, 3.1, giving a length-to-width ratio of approximately 3:2. (I assume that the more vertical of the two lines on the left side represent the actual wishes of the scribe. I ignore the extension of the baseline beyond the lower right vertex.)

[46] The diagram on this tablet is best known for its approximation of $\sqrt{2}$, given in sexagesimal values along the horizontal diagonal of the quadrilateral.

[47] The typical diagram of a Pythagorean triangle (*Elements* I, 47) in the Byzantine Greek and medieval Arabic Euclidean transmission is also an isosceles right triangle [Saito, 2006, 143]. Although the existence of an isosceles right triangle in this diagram is intriguing, there appears to be no other evidence to postulate a causative connection between the two diagram traditions.

A diagram from Sanskrit manuscript KUOML, Co 1712 [Keller, 2005, 204], shows a rectangle (although not drawn to modern standards of precision) cut by a diagonal (Fig. 40).

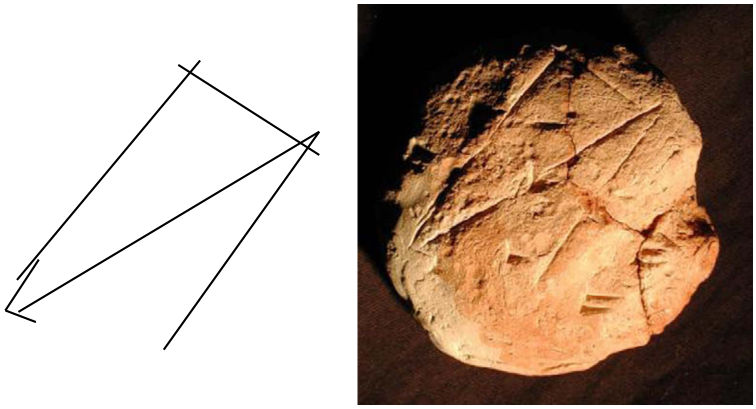

Fig. 41. YBC 7289 reverse—edited diagram (left) and photograph of the tablet, courtesy of W. Casselman (right). Used with permission of Yale University Babylonian Collection. I have omitted the numerals from my edited diagram, since many of them are not clearly legible. The irregularities in the diagram make it difficult to obtain useful perimeter and angle measures, but we can estimate the length-to-width ratio of the rectangle as roughly 2:1.

The basic diagrammatic structure of this rectangle seems in many ways little different from the diagrams of squares and rectangles in the demotic Egyptian (Fig. 35) or Mesopotamian traditions.[48] What is different is the relative crudity of this sketch compared to the majority of Egyptian or Mesopotamian diagrams. The lines in this Sanskrit diagram are little more than rough free-hand sketches, although the copyist has apparently made some effort to correct the left-hand side of the rectangle. The relatively crude diagram on the reverse of "rough work" tablet YBC 7289 (Fig. 41)[49] has the basic lines essentially straight (perhaps drawn with the aid of a straightedge?), with the exception of the broken line at the top, although the lines do not necessarily meet precisely at the vertices. Although there are examples of rectangles (Figs. 10 and 32), there are no diagrams precisely like either the Sanskrit or Mesopotamian rectangles found in the ancient Egyptian tradition. The closest example is the incomplete diagram in demotic P. Carlsberg 30, which appears to have a rectangle divided by a diagonal (Fig. 35).

[48] For additional Mesopotamian examples of rectangles with diagonals, see the redrawn diagrams that accompany Robson's translation of tablets BM96957-VAT 6598 [Robson, 2008b, 137–141]. These redrawn diagrams present a length-to-width ratio of approximately 2:1.

[49] Robson [2008b, 143] interprets the diagram to represent calculation of the diagonal of a rectangle constructed from two 3–4–5 triangles. For an example of typical Mesopotamian calculations related to the diagonal of a quadrilateral, see the translation of tablet Db_2-146 [Robson, 2008b, 100]. The rectangle on Robson's redrawn diagram of this tablet presents a length-to-width ratio of approximately 7:5.

Triangles

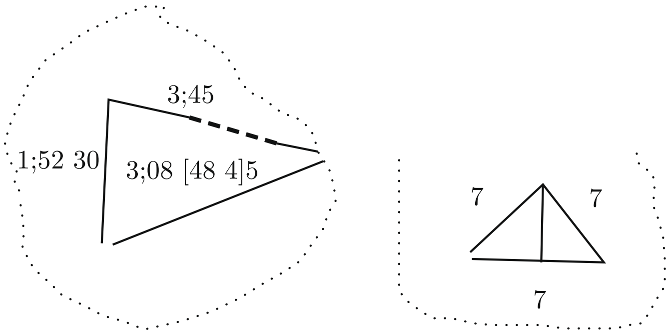

Fig. 42. Babylonian tablet Ash 1931.91 (left), following the redrawing of Robson, showing an isosceles triangle with horizontal axis of symmetry. The incompleteness of the diagram makes it difficult to obtain an accurate altitude-to-base ratio, but it appears to be approximately 3:2. KUOML, Co 1712, folio 36r (right), showing an equilateral triangle with vertical axis. Angle measures, beginning from the left vertex and proceeding clockwise: 45?°, 86°, 50°. (Because of the incompleteness of the left angle, I have estimated on the assumption that the side and base were intended to meet.) Perimeter measures, beginning from the base and proceeding clockwise: 7.2, 5.6, 5. These measures are again based on estimating that the left side and base would meet if rectilinearly extended. The altitude line is 3.9, giving an altitude-to-base ratio of approximately 2:5.

Triangles are also well represented in Mesopotamian mathematics, although I have been able to find few published photographs to use in editing.[50] One example, Ash 1931.91, appears to be a problem requiring calculation of the area of the triangle given two of its sides [Robson, 2008b, 143]. As redrawn by Robson, the diagram (Fig. 42) shows an isosceles triangle with its axis of symmetry placed horizontally and its vertical "base" standing at the left of the diagram.[51] This tablet is another example of "rough work" and contains only the diagram and numerical values for the two sides and the (incorrectly) calculated area. The centerpart of the face has suffered some damage, so part of the upper line must be reconstructed (shown as a dashed line). Several of the numerical values have also been obscured, as indicated by the bracketed numbers. The dotted line indicates the present outline of the tablet, which has obviously been damaged, so that the apex of the triangle is now missing.

The diagrams in two problems from ancient Egypt (Figs. 12 and 20) appear similar in construction and orientation, but are more elongated than this Mesopotamian example. In each case, we find what appears to be an isosceles triangle with its axis of symmetry positioned horizontally and its apex located to the right.

[50] Tablet Strasbourg 364 contains a series of problems involving a "wedge" divided by "rivers" in which one is asked to calculate various areas. The redrawn diagrams that accompany the translation by Robson [2008b, 111–113] are all isosceles triangles, each having its horizontal axis pointing toward the right. As redrawn, the diagrams present an altitude-to-base ratio of approximately 2:1. How these redrawings correspond in shape or orientation to the diagrams on the tablet is not made clear. As drawn, they have many features in common with the isosceles triangle diagram of RMP, Problem 53 (Fig. 14).

[51] I have been unable to find a photograph of this tablet. For my edited diagram, I have reluctantly used Robson's redrawing of the tablet.

Independent triangles, whether scalene, isosceles, or equilateral, in the diagrams of Bhāskara's commentary consistently appear oriented with the axis of symmetry positioned vertically, that is, having the base horizontal at the bottom of the diagram and the apex at the top. My edited example, from KUOML, Co 1712 (Fig. 42), is clearly labeled to show that it was intended to be equilateral [Keller, 2005, 295], although not all independent triangles are shown as equilateral [Keller, 2005, 283]. The dotted line indicates the line demarcating the diagram from the surrounding text.

This Sanskrit tradition of diagram construction has many features in common with the style observed in the demotic papyri (Figs. 28, 30, and 36) and the majority of the extant Byzantine Greek manuscripts. Keller [2005, 289] reports that pyramids are discussed in Bhāskara I's commentary. Although no examples are reproduced, it is implied that they are represented in diagrams as two-dimensional triangles, as was also the case in the diagrams of pyramids from ancient Egypt (Figs. 15–18 and 30–31).

Trapezia

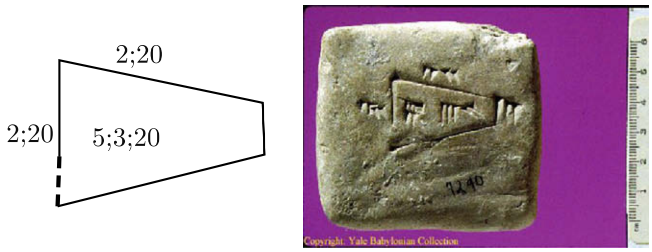

Fig. 43. Edited diagram of YBC 7290 (left) with a photograph of the tablet (right). Used with permission of Yale University Babylonian Collection.

Trapezia (or truncated triangles) are found on several Mesopotamian tablets. YBC 7290 (Fig. 43) is an example of "rough work" containing only the outline of a trapezium on the obverse, along with several numerical values apparently related to calculation of area [Robson, 2008b, 144]. Its reverse holds the diagram of another trapezium, even more crudely drawn, but without numerical values [Robson, 2008b, 144]. Other examples, redrawn by Robson [2008b, 144–145], occur on tablets YBC 11126 and Ash 1922.168. In these latter examples, the only truncated triangle diagram in the RMP, Problem 52 (Fig. 13), is also oriented horizontally with its "base" on the left, leaning from the vertical. Perhaps because the "base" is not vertical, the two sides are set at distinctly different angles so that the figure as a whole does not appear isosceles. The Mesopotamian truncated triangle diagrams in Robson's examples are typically drawn as isosceles triangles with horizontal axis of symmetry. In each, the "base" is placed vertically at the left, so that both sides are equally inclined to the base. [52] Another trapezium, representing a frustum or truncated pyramid in MMP, Problem 14 (Fig. 22), is oriented vertically, presumably because it represents a solid object. Its left side stands vertically on the base while its right side slopes upward toward the left.

Diagrams of trapezia are also included in Bhāskara's commentary. In KUOML, Co 1712, a trapezium (Fig. 44) is shown oriented vertically, with sides typically sloping approximately equally (rather like truncated isosceles triangles) as they rise toward the upper face

[52] Yuste [2006] implies, through several redrawn figures, that diagrams of trapezia on other Mesopotamian tablets are not necessarily positioned with horizontal axes.

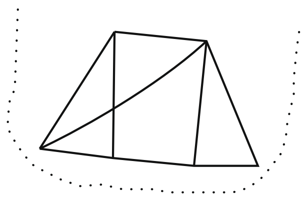

Fig. 44. KUOML, Co 1712, folio 42v, with an isosceles trapezium oriented along a vertical axis. The line separating the diagram from the surrounding text is indicated by the dotted line. The numerical values have been omitted from the edited diagram.

[Keller, 2005, 282]. Bhāskara describes another trapezium (whether it was also diagrammed is not clear) as having its axis of symmetry horizontally oriented and the "base" and upper surface vertical. Keller [2005, 285] notes that the modern editor of Bhāskara's commentary has redrawn this diagram with the base located to the left.

Circular figures

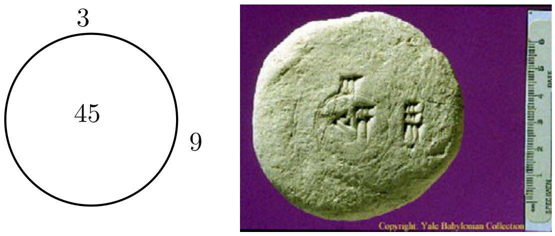

Fig. 45. YBC Tablet 7302, the edited diagram (left) with a photograph of the tablet (right). Used with permission of Yale University Babylonian Collection. The outer circle has been omitted from the edited diagram. Because the two circles are so closely placed, it proved technically too difficult to include it.

Although circles used in Mesopotamian diagrams are rarely perfect, they are typically much closer to circular than are those found in the diagrams surviving from ancient Egypt (Figs. 5, 24, and 26). Tablet YBC 7302 (Fig. 45), for example, apparently illustrates the calculation of a circular area. There is no text accompanying the diagram—only the numbers inscribed on the clay tablet [Robson, 2008b, 142]. The tablet is most probably reused and may represent a kind of pedagogical exercise.[53] It contains two closely spaced and nearly concentric circular figures. The outer circle, incomplete because of minor damage to the face of the tablet, bulges somewhat toward the right. Perhaps the double circle results from a redrawing with a slightly smaller radius because the outer circle was perceived as imperfect. Neither is, in fact, perfectly circular, but the divergence of the inner figure from a true circle is too small to be detected visually. Most circular figures in Mesopotamian tablets appear to be quite accurately constructed, which may imply use of some sort of tool to construct them.

[53] It resembles in some respects the working sketch of the dimensions of an arch or barrel vault, inscribed on a limestone ostracon from Dynasty III [Clagett, 1999, 462; Clarke and Engelbach, 1930, 52–53, 56], which also contains no text but only numerical values.

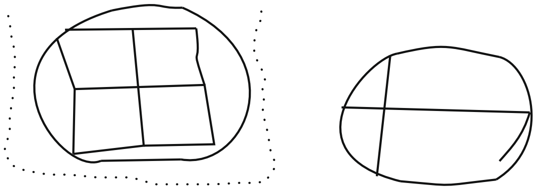

Fig. 46. Circle diagrams in KUOML, Co 1712. From folio 44v (left), a flattened circle (major axis = 7.6, minor axis = 5.5) circumscribing a square. The dotted line indicates the line separating the diagram from the text. From folio 59v (right), a flattened circle (major axis = 5.6; minor axis = 4.0) from a "hawk and rat" problem.

Circular figures in Bhāskara's commentary seem usually to occur in conjunction with other figures, rather than as independent entities. The circles drawn in KUOML, Co 1712, are typically crude sketches, more oval than circular. Like Egyptian circular diagrams (Figs. 5 and 24), these are often flattened at the poles and elongated on the horizontal axis. The circumscribed square [Keller, 2005, 284] on folio 44v (Fig. 46) shows the circle considerably flattened at the poles and elongated horizontally. This diagram accompanies a problem requiring an area calculation similar to that found in demotic Problem 37 [Parker, 1972, 47–49]. The diagram in Bhāskara's commentary, however, is even more crudely produced than its Egyptian counterpart (Fig. 28). Similarly, the diagram on folio 59v, accompanying a typical "hawk and rat" problem [Keller, 2005, 298], shows a distinctly flattened circle, which has been approximated using four Bézier curves (Fig. 46). It appears almost certain that the scribe began the circle at the lower right quadrant and completed it in one stroke of the pen, although the end of the stroke did not meet the beginning. The flattened ovoid in the Egyptian hieratic Lahun fragment UCL 32160 (Fig. 5) manifests what appears to be a similar construction technique.

Robson's translation [2008b, 131–137] of the 35 problems found on Tablet BM 85194 is accompanied by four redrawn diagrams of circles or portions of circles. The only reproduction of this tablet that I have seen shows only the obverse, at the lower left corner of which is a diagram of two concentric circular figures related to a problem concerning excavation of a circular moat [Robson, 2008b, 132]. Presumably, the remaining three diagrams are to be found on the reverse of the tablet. The fourth of these diagrams accompanies a problem requiring calculation of the area of a "crescent moon" (or a segment of a circle) when the arc and the chord length are given [Robson, 2008b, 136]. The problem is similar to portions of demotic Problems 36–38 [Parker, 1972, 44–51]. In her translation of the Babylonian problem, however, Robson has positioned her redrawn diagram so that the segment has its chord vertically oriented with the arc to the left, whereas the Egyptian diagrams (Figs. 27–29) are all drawn with the chord placed horizontally and the arc above. The Egyptian diagrams are rather crudely worked; we are unable to make any evaluation of the Babylonian example, however, on the basis of this redrawn diagram.

Summary

From this very brief comparative survey, I would draw only two tentative observations. First, circles seem to have been difficult for at least some scribes in both the Egyptian and Sanskrit mathematical traditions. Perhaps this difficulty arises from the type of traditional writing technology used by the scribes. On the other hand, most circles found in diagrams from ancient Mesopotamia are well formed. Perhaps their writing technology allowed a better utilization of drawing aids. Second, hieratic Egyptian and Mesopotamian diagrams seem to have a predilection for positioning the major axis of figures horizontally. This does not seem to be the case in either demotic Egyptian or Sanskrit manuscript Co 1712, where triangles and trapezia are more likely to be positioned with their baseline horizontal and their axis vertical. Whether this variation is somehow related to influences arising from the spread of the ancient Greek tradition of diagram construction must remain, for the moment, a matter for speculation.

Postscript: Egyptian and Greek figures compared

The differences, in terms of diagram construction, between ancient Egypt and the classical Greek tradition (as epitomized by Euclid's *Elements*) give one the distinct impression that there has been a mathematical or intellectual disjunction. Egyptian diagrams are labeled only with numbers taken from the specific data values in the problem to which the diagram is attached. The calculation of a specific quantity is the primary focus of the Egyptian mathematical exercise. Greek geometrical diagrams are typically labeled with letters attached to the key geometric elements in the drawn diagram.[54] The use of such labels has been argued to play a crucial role in the abstract and logical demonstration of geometrical truths that is the primary focus of the Greek mathematical effort in geometry [Netz, 1999].

Looking beyond this difference in labeling techniques, there are also differences in the orientation of diagrams. For example, Greek diagrams using figures such as isosceles triangles typically have the axis of symmetry oriented vertically, rather than horizontally as in the hieratic mathematical papyri of ancient Egypt. Or we might say that the "base" of hieratic (but not demotic) Egyptian triangles is placed vertically (unless the diagram represents a physical structure, such as a pyramid) while the "base" of Greek diagrams of triangles is placed horizontally. In addition to this change in orientation, the altitude-to-base ratio in Greek figures is noticeably smaller than that found in the hieratic papyri.

We have already noted that the use of overspecified, either isosceles (hieratic) or equilateral (demotic), triangles is typical of Egyptian geometric diagrams. A similar overspecification appears also to be endemic in Greek geometry.[55] On the other hand, in yet another important contrast to Greek diagrams (if we can take Byzantine and medieval Arabic dia-

[54] Although this is a valid description of the broad corpus of manuscript evidence that has come down to us, two of the earliest known examples of diagrams from Euclid appear to have been constructed without labels. The first, preserved on a fragment of papyrus found in a rubbish heap at Oxyrhynchus and dating from about the end of the first century of the Common Era, contains most of Proposition II, 5 along with its diagram [Fowler, 1990, 212 and Plates 1–3; Grenfell and Fell, 1898, I, 58]. The second, P. Berol. 17469, containing diagrams from Euclid I, 8–10 and dating from the second century of the Common Era, was recently studied by Brashear [1994, 29–30].

[55] That the creation of isosceles triangles in Byzantine Greek manuscript diagrams was not accidental but intentional is clear from the numerals inserted into some diagrams of Euclid's Proposition I,47, such as in Vat. Gr. 190 (folio 39r), where the sides of the right triangle are labeled $5, 5, \sqrt{50}$, the last value given in its sexagesimal equivalent. In another manuscript, Oxford, Bodleian Library ms. D'Orville 301 (folio 31r), the diagram for the same proposition, although visually constructed with an isosceles triangle, has its sides labeled with the Greek alphanumeric values 3, 4, 5. The tradition of overspecification seems to have continued well into the Islamic period as well.

grams as representative of the ancient Greek mathematical tradition), we observe that Byzantine copyists and illustrators consistently attempt the use of linear perspective to represent three-dimensional figures, as in Books XI–XIII of the *Elements*, and some were able to achieve remarkable results. From our surviving evidence, the Egyptian tradition of diagram construction seems never to have moved beyond purely two-dimensional figures, even when portraying solid physical objects such as a pyramid or a granary.

Recently there has been some recognition among historians of mathematics that our understanding of diagrams and their historical role requires deepening [Grattan-Guinness, 1996, note 10; Mahoney, 1985, note 12], but the response of the scholarly community to these pleas is only beginning to develop. There does not yet exist a history of mathematical diagrams comparable to Cajori's [1928–29] history of mathematical notations. I hope that this study will make a contribution to the goal of understanding the development and role of diagrams in ancient Egyptian mathematics and that other historians may be inspired to contribute to our understanding of diagrams in other mathematical traditions and time periods.

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