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# A palindromization map for the free group

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ABSTRACT

We define a self-map $\mathrm{Pal} : F_2 \to F_2$ of the free group on two generators $a, b$, using automorphisms of $F_2$ that form a group isomorphic to the braid group $B_3$. The map Pal restricts to de Luca's right iterated palindromic closure on the submonoid generated by $a, b$. We show that Pal is continuous for the profinite topology on $F_2$; it is the unique continuous extension of de Luca's right iterated palindromic closure to $F_2$. The values of Pal are palindromes and coincide with the elements $g \in F_2$ such that $abg$ and $bag$ are conjugate.



## 0. Introduction

To any word $w$ on an alphabet consisting of two letters $a$ and $b$, de Luca [11] associated a palindromic word $\mathrm{P}(w)$, called its *right iterated palindromic closure*. The element $\mathrm{P}(w) \in \{a, b\}^*$ is defined recursively by $\mathrm{P}(1) = 1$ and

$$\mathrm{P}(wx) = \big(\mathrm{P}(w)x\big)^+$$

for all $w \in \{a, b\}^*$ and $x \in \{a, b\}$; here $w^+$ is the unique shortest palindrome having $w$ as a prefix. De Luca showed that all words $\mathrm{P}(w)$ are *central* in the sense of [10], and that any central word is of this form. Moreover, de Luca's map P is injective, i.e., $\mathrm{P}(u) = \mathrm{P}(v)$ implies $u = v$.

In this paper we construct a self-map

$$\mathrm{Pal} : F_2 \to F_2$$

of the free group $F_2$ on $a, b$, whose restriction to the monoid $\{a, b\}^*$ is de Luca's map $\mathrm{P} : \{a, b\}^* \to \{a, b\}^*$, i.e., we have $\mathrm{Pal}(w) = \mathrm{P}(w)$ for all $w \in \{a, b\}^*$. The map Pal, which we call the *palindromization map*, is defined using certain automorphisms of $F_2$. These automorphisms form a group that is isomorphic to the *braid group* $B_3$ of braids on three strands. One of the most interesting properties of our map Pal is that it is continuous for the *profinite topology* on $F_2$; we actually prove that Pal is the unique continuous extension of P to $F_2$. As is the case with the map P, each image $\mathrm{Pal}(w)$ of the palindromization map is a palindrome, i.e., is fixed by the unique anti-automorphism of $F_2$ fixing $a$ and $b$. We also characterize the elements $g \in F_2$ belonging to the image of Pal as those for which $abg$ and $bag$ are conjugate elements. Contrary to P, our map Pal is not injective; we determine all pairs $(u, v)$ such that $\mathrm{Pal}(u) = \mathrm{Pal}(v)$; the result is expressed in terms of the braid group $B_3$.

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To shed light on the theory of Sturmian words and morphisms, it is convenient to put it into the context of the free group. In this paper we illustrate this idea on de Luca's right iterated palindromic closure. It was precisely by connecting the combinatorics of words and group theory that we were able to find our extension Pal. Indeed, Justin [6] proved that de Luca's map P satisfies a certain functional equation. We interpret this as expressing that P is a *cocycle* in the sense of Serre's non-abelian cohomology. In this language our main observation is that P is a *trivial* cocycle. This has two consequences, of which the second one is quite fortunate: firstly, to express the triviality of the cocycle P we are forced to work in the free group $F_2$; secondly, expressing P as a trivial cocycle yields *ipso facto* a formula for Pal.

Let us set out the contents of the paper. In Section 1 we associate an automorphism $R_w$ of $F_2$ to each $w \in F_2$; the automorphisms $R_w$ are exactly the automorphisms of $F_2$ fixing $aba^{-1}b^{-1}$. In Section 2 we show that the automorphisms $R_w$ form a subgroup that is isomorphic to the braid group $B_3$. In Section 3 we define the palindromization map $\mathrm{Pal} : F_2 \to F_2$ and we give it a cohomological interpretation. In Section 4 we show that each element $\mathrm{Pal}(w)$ is a palindrome in $F_2$ and that our map Pal extends de Luca's right iterated palindromic closure; we also compute the image of $\mathrm{Pal}(w)$ in the free abelian group $\mathbb{Z}^2$. As mentioned above, the map Pal is not injective; in Section 5 we determine all pairs $(u, v)$ such that $\mathrm{Pal}(u) = \mathrm{Pal}(v)$. In Section 6 we establish that Pal is continuous for the profinite topology on $F_2$. We characterize the elements of $F_2$ belonging to the image of Pal in Section 7. The paper concludes with a short appendix collecting the basic facts on non-abelian cohomology needed in Section 3.

## 1. The automorphisms $R_w$

Let $F_2$ be the free group generated by $a$ and $b$. Consider the automorphisms $R_a, R_b$ of $F_2$ defined by

$$R_a = \begin{cases} a \mapsto a, \\ b \mapsto ba, \end{cases} \quad \text{and} \quad R_b = \begin{cases} a \mapsto ab, \\ b \mapsto b. \end{cases} \tag{1.1}$$

Their inverses are given by

$$(R_a)^{-1} = \begin{cases} a \mapsto a, \\ b \mapsto ba^{-1}, \end{cases} \quad \text{and} \quad (R_b)^{-1} = \begin{cases} a \mapsto ab^{-1}, \\ b \mapsto b. \end{cases}$$

The automorphisms $R_a$ and $R_b$ are respectively denoted by $\widetilde{G}, \widetilde{D}$ in [10, Sect. 2.2.2] (see also [7, Eqn. (2.1)]). They are related by

$$E\, R_a = R_b\, E, \tag{1.2}$$

where $E$ is the involution of $F_2$ exchanging $a$ and $b$. (The automorphisms $R_a, R_b$ are instances of what Godelle calls transvections in [3].)

Let $w \mapsto R_w$ be the group homomorphism $F_2 \to \mathrm{Aut}(F_2)$ sending $a$ to $R_a$ and $b$ to $R_b$. In particular, if $w = 1$ is the neutral element of $F_2$, then $R_w = \mathrm{id}$ (the identity of $F_2$). Moreover, $R_{a^{-1}} = (R_a)^{-1}$ and $R_{b^{-1}} = (R_b)^{-1}$. It follows from (1.2) that for all $w \in F_2$,

$$E\, R_w = R_{E(w)}\, E. \tag{1.3}$$

**Lemma 1.1.** *Each automorphism $R_w$ fixes the commutator $aba^{-1}b^{-1}$.*

**Proof.** It is enough to verify that both $R_a$ and $R_b$ fix $aba^{-1}b^{-1}$. □

When $w = (ab^{-1}a)^i$ for $i = 1, 2, 4$, then an easy computation shows that $R_w$ is given by

$$R_{ab^{-1}a} = \begin{cases} a \mapsto b^{-1}, \\ b \mapsto bab^{-1}, \end{cases} \tag{1.4}$$

$$R_{(ab^{-1}a)^2} = \begin{cases} a \mapsto ba^{-1}b^{-1}, \\ b \mapsto bab^{-1}a^{-1}b^{-1}, \end{cases} \tag{1.5}$$

$$R_{(ab^{-1}a)^4} = \begin{cases} a \mapsto (bab^{-1})\, a\, (ba^{-1}b^{-1}), \\ b \mapsto (bab^{-1}a^{-1})\, b\, (aba^{-1}b^{-1}). \end{cases} \tag{1.6}$$

We can rephrase (1.4)–(1.6) as follows. Let $\tau$ be the automorphism of $F_2$ sending $a$ to $b^{-1}$ and $b$ to $a$; we have $\tau^2(a) = a^{-1}$ and $\tau^2(b) = b^{-1}$, so that $\tau^4 = \mathrm{id}$. Then for all $u \in F_2$,

$$R_{ab^{-1}a}(u) = b\, \tau(u)\, b^{-1}, \tag{1.7}$$

$$R_{(ab^{-1}a)^2}(u) = ba\, \tau^2(u)\, (ba)^{-1}, \tag{1.8}$$

$$R_{(ab^{-1}a)^4}(u) = (bab^{-1}a^{-1})\, u\, (bab^{-1}a^{-1})^{-1}. \tag{1.9}$$

In other words, $R_{(ab^{-1}a)^4}$ is the conjugation by the commutator $bab^{-1}a^{-1}$.

We next consider the abelianizations of the automorphisms $R_w$. Let $\pi : F_2 \to \mathbb{Z}^2$ be the canonical surjection sending the generators $a$ and $b$ of $F_2$ to the respective column-vectors

$$\begin{pmatrix}1\\0\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}0\\1\end{pmatrix} \in \mathbb{Z}^2.$$

In the sequel we identify $\mathrm{Aut}(\mathbb{Z}^2)$ with the matrix group $\mathrm{GL}_2(\mathbb{Z})$.

For any $w \in F_2$, let $M_w$ be the abelianization of the automorphism $R_w$, i.e., the unique automorphism $M_w$ of $\mathbb{Z}^2$ such that $\pi \circ R_w = M_w \circ \pi$. Since $w \mapsto R_w$ is a group homomorphism, so is the map $w \mapsto M_w$. The latter is determined by its values $M_a$ and $M_b$. It follows from (1.1) that $M_a$ and $M_b$ can be identified with the matrices

$$M_a = \begin{pmatrix}1 & 1\\0 & 1\end{pmatrix} \quad \text{and} \quad M_b = \begin{pmatrix}1 & 0\\1 & 1\end{pmatrix}.$$

Since $M_a$ and $M_b$ are of determinant one, we conclude that $M_w \in \mathrm{SL}_2(\mathbb{Z})$ for all $w \in F_2$. Formulas (1.4)–(1.6) imply that

$$M_{ab^{-1}a} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}, \quad M_{(ab^{-1}a)^2} = -I_2, \quad M_{(ab^{-1}a)^4} = I_2, \tag{1.10}$$

where $I_2$ denotes the unit $2 \times 2$ matrix.

## 2. The braid group $B_3$

Let $\mathcal{R}$ be the subgroup of $\mathrm{Aut}(F_2)$ consisting of all automorphisms $R_w$, where $w \in F_2$. By definition, the subgroup $\mathcal{R}$ is generated by $R_a$ and $R_b$. It follows from Lemma 1.1 that every element of $\mathcal{R}$ fixes the commutator $aba^{-1}b^{-1}$. The converse also holds.

**Proposition 2.1.** *The group $\mathcal{R}$ is the subgroup of $\mathrm{Aut}(F_2)$ of all automorphisms fixing $aba^{-1}b^{-1}$.*

**Proof.** Let $\varphi$ be an automorphism fixing $aba^{-1}b^{-1}$. Consider the anti-automorphism $\omega$ of $F_2$ such that $\omega(a) = a$ and $\omega(b) = b$. For any word $w \in F_2$, the word $\omega(w)$ is its mirror image. In particular, $\omega(aba^{-1}b^{-1}) = b^{-1}a^{-1}ba$. We have

$$(\omega \circ \varphi \circ \omega)(b^{-1}a^{-1}ba) = \omega\big(\varphi(aba^{-1}b^{-1})\big) = \omega(aba^{-1}b^{-1}) = b^{-1}a^{-1}ba.$$

Now it is well known (see [2, Sect. 3] or [5]) that the subgroup of $\mathrm{Aut}(F_2)$ fixing $b^{-1}a^{-1}ba$ is generated by the two automorphisms

$$L_a = \begin{cases} a \mapsto a, \\ b \mapsto ab, \end{cases} \quad \text{and} \quad L_b = \begin{cases} a \mapsto ba, \\ b \mapsto b. \end{cases} \tag{2.1}$$

(The automorphisms $L_a$ and $L_b$ coincide respectively with the automorphisms $G$ and $D$ of [10, Sect. 2.2.2]; see also [7, Eqn. (2.1)].) Therefore, $\varphi$ belongs to the subgroup of $\mathrm{Aut}(F_2)$ generated by $\omega \circ L_a \circ \omega$ and $\omega \circ L_b \circ \omega$. A simple computation based on (1.1) and (2.1) shows that

$$R_a = \omega \circ L_a \circ \omega \quad \text{and} \quad R_b = \omega \circ L_b \circ \omega. \tag{2.2}$$

This proves that $\varphi$ belongs to $\mathcal{R}$. $\square$

We next claim that the group $\mathcal{R}$ is isomorphic to the braid group $B_3$ of braids on three strands. Recall that the group $B_3$ has the following presentation (see [8, Sect. 1.1]):

$$B_3 = \langle\, \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \,\rangle. \tag{2.3}$$

**Proposition 2.2.** *There is a group homomorphism $i : B_3 \to \mathrm{Aut}(F_2)$ such that $i(\sigma_1) = R_a$ and $i(\sigma_2) = R_b^{-1}$. This homomorphism is injective and its image is $\mathcal{R}$.*

**Proof.** The existence of the homomorphism $i : B_3 \to \mathrm{Aut}(F_2)$ results from the relation

$$R_a\, R_b^{-1}\, R_a = R_b^{-1}\, R_a\, R_b^{-1},$$

which was observed in [7, Lemma 2.1]. The image of $i$ is clearly the subgroup $\mathcal{R}$ of $\mathrm{Aut}(F_2)$ generated by $R_a$ and $R_b$.

In order to prove that $i$ is injective, we use further results of [7, Sect. 2]. Let $B_4$ be the braid group on four strands; it has a presentation with generators $\sigma_1, \sigma_2, \sigma_3$ and relations

$$\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \quad \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \quad \sigma_1\sigma_3 = \sigma_3\sigma_1.$$

First observe that the natural homomorphism $j : B_3 \to B_4$ determined by $j(\sigma_1) = \sigma_1$ and $j(\sigma_2) = \sigma_2$ is injective. Indeed, there is an homomorphism $q : B_4 \to B_3$ such that $q \circ j = \mathrm{id}$; it is given by $q(\sigma_1) = q(\sigma_3) = \sigma_1$ and $q(\sigma_2) = \sigma_2$; see also [8, Cor. 1.14].

In [7, Lemma 2.6] we constructed a homomorphism $f : B_4 \to \mathrm{Aut}(F_2)$ whose kernel is the center of $B_4$ and such that

$$i(\sigma_1) = R_a = \tau^2 f(\sigma_1)\,\tau^{-2} \quad \text{and} \quad i(\sigma_2) = R_b^{-1} = \tau^2 f(\sigma_2)\,\tau^{-2},$$

where $\tau^2$ is the square of the automorphism $\tau$ of $F_2$ introduced in Section 1. Hence, $i(\beta) = \tau^2 f(\beta)\,\tau^{-2}$ for all $\beta \in B_3$. Pick $\beta \in B_3$ in the kernel of $i$. It follows from the previous relation that $f(\beta) = 1$, which implies that $\beta$ belongs to the center of $B_4$. The group $B_3$ being embedded in $B_4$, we conclude that $\beta$ belongs to the center of $B_3$. Now the center of $B_3$ is generated by $(\sigma_1\sigma_2\sigma_1)^2$. Therefore, $\beta = (\sigma_1\sigma_2\sigma_1)^{2k}$ for some $k \in \mathbb{Z}$ and hence $i(\beta) = R_{(ab^{-1}a)^{2k}}$. To complete the proof, it suffices to check that $R_{(ab^{-1}a)^{2k}}$ is the identity only if $k = 0$. Consider the abelianization $M_{(ab^{-1}a)^{2k}}$ of $R_{(ab^{-1}a)^{2k}}$. Since $M_{(ab^{-1}a)^2} = -I_2$ by (1.10), we have $M_{(ab^{-1}a)^{2k}} = (-1)^k I_2$. The latter being the identity, $k$ must be even. Set $k = 2\ell$ for some integer $\ell$. It follows from (1.9) that $R_{(ab^{-1}a)^{2k}} = R_{((ab^{-1}a)^4)^\ell}$ is the conjugation by $(bab^{-1}a^{-1})^\ell$ in $F_2$. Such a conjugation is the identity only if $\ell = 0$; therefore, $k = 0$, which implies that $\beta = 1$. $\square$

As a consequence of Proposition 2.2 we obtain the following result, which had been observed in [7, Remark 2.14 (c)].

**Corollary 2.3.** *There is a group isomorphism* $\mathcal{R} \cong B_3$.

## 3. The palindromization map and a cohomological interpretation

We use the automorphisms $R_w$ of Section 1 to define for each $w \in F_2$ an element $\mathrm{Pal}(w) \in F_2$ by the formula

$$\mathrm{Pal}(w) = b^{-1}a^{-1}\,R_w(ab). \tag{3.1}$$

Formula (3.1) defines a map $\mathrm{Pal} : F_2 \to F_2\,;\ w \mapsto \mathrm{Pal}(w)$, which we call the *palindromization map*.

As a consequence of the definition and of (1.1)–(1.4), we easily obtain the following values of Pal:

$$\mathrm{Pal}(w) = w \quad \text{if} \quad w = a^r \text{ or } b^r \text{ for some } r \in \mathbb{Z}, \tag{3.2}$$

$$\begin{aligned}
&\mathrm{Pal}(ab) = aba, \quad \mathrm{Pal}(ba) = bab,\\
&\mathrm{Pal}(ba^{-1}) = a^{-1}, \quad \mathrm{Pal}(ab^{-1}) = b^{-1},\\
&\mathrm{Pal}(a^{-1}b) = a^{-1}ba^{-1}, \quad \mathrm{Pal}(b^{-1}a) = b^{-1}ab^{-1},\\
&\mathrm{Pal}(aba^{-1}) = \mathrm{Pal}(ab^{-1}a^{-1}) = \mathrm{Pal}(bab^{-1}) = \mathrm{Pal}(ba^{-1}b^{-1}) = 1,\\
&\mathrm{Pal}(ab^{-1}a) = \mathrm{Pal}(b^{-1}ab^{-1}) = b^{-2},\\
&\mathrm{Pal}(ba^{-1}b) = \mathrm{Pal}(a^{-1}ba^{-1}) = a^{-2},
\end{aligned} \tag{3.3}$$

Thus the map Pal is not injective. In Section 5 we will characterize the pairs $(u, v)$ of elements of $F_2$ such that $\mathrm{Pal}(u) = \mathrm{Pal}(v)$.

We now prove a few properties of the palindromization map. First, we give another formula defining it and show that Pal is invariant under the exchange automorphism $E$.

**Lemma 3.1.** *For all* $w \in F_2$,

$$\mathrm{Pal}(w) = a^{-1}b^{-1}\,R_w(ba) \tag{3.4}$$

*and*

$$E\big(\mathrm{Pal}(w)\big) = \mathrm{Pal}\big(E(w)\big).$$

**Proof.** Since by Lemma 1.1, $R_w$ fixes $aba^{-1}b^{-1}$, we have

$$\begin{aligned}
aba^{-1}b^{-1}R_w(ba) &= R_w(aba^{-1}b^{-1})\,R_w(ba) = R_w(aba^{-1}b^{-1}ba)\\
&= R_w(ab) = ab\,\mathrm{Pal}(w).
\end{aligned}$$

Canceling on the left by $ab$, we obtain (3.4).

Using (1.3), (3.1), and (3.4), we obtain

$$\begin{aligned}
E\big(\mathrm{Pal}(w)\big) &= E\big(b^{-1}a^{-1}R_w(ab)\big) = a^{-1}b^{-1}\,E\big(R_w(ab)\big)\\
&= a^{-1}b^{-1}\,R_{E(w)}\big(E(ab)\big) = a^{-1}b^{-1}\,R_{E(w)}(ba)\\
&= \mathrm{Pal}\big(E(w)\big). \quad \square
\end{aligned}$$

**Corollary 3.2.** (a) *For* $w \in F_2$,

$$\mathrm{Pal}(w) = 1 \iff R_w(ab) = ab \iff R_w(ba) = ba.$$

(b) *The set* $\mathrm{Pal}^{-1}(1) = \{w \in F_2 \mid \mathrm{Pal}(w) = 1\}$ *is a subgroup of* $F_2$.

**Proof.** Part (a) is an immediate consequence of (3.1) and (3.4). Part (b) follows from Part (a). $\square$

The palindromization satisfies the following important functional equation.

**Proposition 3.3.** *For all $u, v \in F_2$, we have*

$$\mathrm{Pal}(uv) = \mathrm{Pal}(u)\, R_u\big(\mathrm{Pal}(v)\big). \tag{3.5}$$

**Proof.** We have

$$\begin{aligned} ab\,\mathrm{Pal}(uv) &= R_{uv}(ab) = R_u\big(R_v(ab)\big) \\ &= R_u(ab\,\mathrm{Pal}(v)) = R_u(ab)\,R_u(\mathrm{Pal}(v)) \\ &= ab\,\mathrm{Pal}(u)\,R_u(\mathrm{Pal}(v)). \end{aligned}$$

We conclude by canceling on the left by $ab$. □

Eq. (3.5) has an interesting interpretation in the language of Serre's *non-abelian cohomology*, whose definition we recall in the Appendix. The free group $F_2$ acts on itself via the group homomorphism

$$F_2 \to \mathrm{Aut}(F_2)\,;\ \ w \mapsto R_w.$$

It follows from (3.1) that the function $\mathrm{Pal} : F_2 \to F_2$ is a *trivial cocycle* in the sense of the Appendix; see (A.3) with $X = ab$. Therefore Pal satisfies a cocycle condition of the form (A.1), which in our case is nothing else that Eq. (3.5).

Let $F_2 \rtimes F_2$ be the semi-direct product associated to the action $w \mapsto R_w$ of $F_2$ on itself; it is the set $F_2 \times F_2$ equipped with the product

$$(w_1, u)\,(w_2, v) = (w_1\,R_u(w_2),\, uv)$$

for all $w_1, w_2 \in F_2$ and $u, v \in F_2$. Now consider the map

$$\widehat{\mathrm{Pal}} = (\mathrm{Pal}, \mathrm{id}) : F_2 \to F_2 \rtimes F_2\,;\ \ u \mapsto \big(\mathrm{Pal}(u), u\big).$$

Since Pal satisfies the cocycle equation (3.5), it follows from a direct computation or from (A.4) that $\widehat{\mathrm{Pal}}$ is a group homomorphism, i.e.,

$$\widehat{\mathrm{Pal}}(uv) = \widehat{\mathrm{Pal}}(u)\,\widehat{\mathrm{Pal}}(v)$$

for all $u, v \in F_2$. We have even better: Pal being a trivial cocycle, it follows from (3.1) and (A.5) that for all $u \in F_2$,

$$\widehat{\mathrm{Pal}}(u) = (ab, 1)^{-1}\,(1, u)\,(ab, 1)$$

in the semi-direct product $F_2 \rtimes F_2$.

## 4. Properties

In this section we show that each element $\mathrm{Pal}(w)$ is a palindrome in $F_2$ and that our map Pal extends de Luca's right iterated palindromic closure; we also compute the image of $\mathrm{Pal}(w)$ in the free abelian group $\mathbb{Z}^2$.

By definition, a *palindrome* in $F_2$ is an element fixed by the anti-automorphism $\omega$ of $F_2$ introduced in the proof of Proposition 2.1. The first property of Pal that we prove in this section is the following.

**Proposition 4.1.** *We have $\omega\big(\mathrm{Pal}(w)\big) = \mathrm{Pal}(w)$ for all $w \in F_2$.*

Let $L_a$ and $L_b$ be the automorphisms defined by (2.1). One easily checks that for all $w \in F_2$,

$$L_a(w) = a\,R_a(w)\,a^{-1} \quad \text{and} \quad L_b(w) = b\,R_b(w)\,b^{-1}. \tag{4.1}$$

Let $w \mapsto L_w$ be the homomorphism $F_2 \to \mathrm{Aut}(F_2)$ sending $a$ to $L_a$ and $b$ to $L_b$. We have the following lemma reminiscent of (3.4).

**Lemma 4.2.** *We have $L_w(ba) = \mathrm{Pal}(w)\,ba$ for all $w \in F_2$.*

**Proof.** We prove the lemma by induction on the length of a reduced word representing $w \in F_2$. If $w = 1$, then the lemma holds trivially. Suppose that $L_w(ba) = \mathrm{Pal}(w)\,ba$ for some $w \in F_2$ and let us prove the lemma for $a^{\pm 1}w$ and $b^{\pm 1}w$. We have

$$\begin{aligned} L_{a^{\pm 1}w}(ba) &= L_{a^{\pm 1}}\big(L_w(ba)\big) = L_{a^{\pm 1}}\big(\mathrm{Pal}(w)\,ba\big) \\ &= L_{a^{\pm 1}}(\mathrm{Pal}(w))\,L_{a^{\pm 1}}(ba) = L_{a^{\pm 1}}(\mathrm{Pal}(w))\,a^{\pm 1}ba \\ &= a^{\pm 1}\,R_{a^{\pm 1}}(\mathrm{Pal}(w))\,a^{\mp 1}a^{\pm 1}ba = \mathrm{Pal}(a^{\pm 1})\,R_{a^{\pm 1}}(\mathrm{Pal}(w))\,ba \\ &= \mathrm{Pal}(a^{\pm 1}w)\,ba. \end{aligned}$$

The second equality holds by induction, the fourth by (2.1), the fifth by (4.1), the sixth by (3.2), and the seventh by (3.5). One proves the lemma for $b^{\pm 1}w$ in a similar fashion. □

**Proof of Proposition 4.1.** By (3.1), (2.2), and Lemma 4.2,

$$\begin{aligned} ab\,\omega\big(\mathrm{Pal}(w)\big) &= \omega\big(\mathrm{Pal}(w)\,ba\big) \\ &= \omega\big(L_w(ba)\big) = R_w\big(\omega(ba)\big) \\ &= R_w(ab) = ab\,\mathrm{Pal}(w) \end{aligned}$$

for all $w \in F_2$. The conclusion follows immediately. □

We next show that the map Pal satisfies an equation established by Justin [6] for de Luca's map $\mathrm{P} : \{a, b\}^* \to \{a, b\}^*$.

**Proposition 4.3.** *For all $u, v \in F_2$, we have*

$$\mathrm{Pal}(uv) = L_u\big(\mathrm{Pal}(v)\big)\,\mathrm{Pal}(u). \tag{4.2}$$

**Proof.** By Proposition 4.1, $\mathrm{Pal}(u)$, $\mathrm{Pal}(v)$, and $\mathrm{Pal}(uv)$ are palindromes. We thus obtain

$$\begin{aligned} L_u\big(\mathrm{Pal}(v)\big)\,\mathrm{Pal}(u) &= \omega\Big(R_u\big(\omega(\mathrm{Pal}(v))\big)\Big)\,\omega\big(\mathrm{Pal}(u)\big) \\ &= \omega\big(\mathrm{Pal}(u)\,R_u(\mathrm{Pal}(v))\big) \\ &= \omega\big(\mathrm{Pal}(uv)\big) \\ &= \mathrm{Pal}(uv). \end{aligned}$$

The first equality follows from (2.2) and the third one from (3.5). □

Since $L_a$ and $L_b$ obviously preserve the (free) submonoid $\{a, b\}^*$ of $F_2$, it follows from (4.2) that the restriction of Pal to $\{a, b\}^*$ takes its values in $\{a, b\}^*$.

**Corollary 4.4.** *The restriction of* Pal *to the monoid $\{a, b\}^*$ coincides with de Luca's right iterated palindromic closure* $\mathrm{P} : \{a, b\}^* \to \{a, b\}^*$.

**Proof.** De Luca's map P is by [6] the unique map $\{a, b\}^* \to \{a, b\}^*$ fixing $a, b$ and satisfying (4.2). □

**Remark 4.5.** Our palindromization map Pal cannot be obtained as a right iterated palindromic closure as is the case when $w \in \{a, b\}^*$. Indeed, the right iterated palindromic closure in the free monoid $\{a, b, a^{-1}, b^{-1}\}^*$ of the word $ab^{-1}a$ is

$$\big((a^+b^{-1})^+a\big)^+ = (ab^{-1}aa)^+ = ab^{-1}aab^{-1}a,$$

whereas $\mathrm{Pal}(ab^{-1}a) = b^{-2}$ by (3.3). Moreover, since $b^+ = b$ and

$$\big((ba)^+a^{-1}\big)^+ = (baba^{-1})^+ = baba^{-1}bab \neq b,$$

we see that an iterated palindromic closure cannot be defined on $F_2$ in a naive way.

In [1, Sect. 3], the image $\pi\big(\mathrm{P}(w)\big)$ in $\mathbb{Z}^2$ was computed for $w \in \{a, b\}^*$. This computation extends easily to the whole group $F_2$ as an immediate consequence of the definition of Pal.

**Proposition 4.6.** *For all $w \in F_2$, we have*

$$\pi\big(\mathrm{Pal}(w)\big) = \big(M_w - I_2\big)\begin{pmatrix}1\\1\end{pmatrix}.$$

**Proof.** Since $\pi \circ R_w = M_w \circ \pi$, we have

$$\begin{aligned} \pi\big(\mathrm{Pal}(w)\big) &= \pi\big((ab)^{-1}R_w(ab)\big) = \pi\big((ab)^{-1}\big) + \pi\big(R_w(ab)\big) \\ &= \begin{pmatrix}-1\\-1\end{pmatrix} + M_w\begin{pmatrix}1\\1\end{pmatrix} = \big(M_w - I_2\big)\begin{pmatrix}1\\1\end{pmatrix}. \quad \square \end{aligned}$$

**Remark 4.7.** By (3.1) and (3.4), $X = ab$ and $X = ba$ are solutions of the equations

$$\mathrm{Pal}(w) = X^{-1}\,R_w(X) \qquad (w \in F_2). \tag{4.3}$$

Using Proposition 4.6, it is easy to check that any solution $X$ of (4.3) necessarily satisfies $\pi(X) = (1, 1) \in \mathbb{Z}^2$. Therefore, $X = ab$ and $X = ba$ are the only solutions of (4.3) in the monoid $\{a, b\}^*$. Indeed, $\pi(X) = (1, 1)$ for $X \in \{a, b\}^*$ means that the word $X$ has exactly one occurrence of $a$ and one occurrence of $b$. In the free group $F_2$, Eq. (4.3) has infinitely many solutions such as $X_r = (aba^{-1}b^{-1})^r ab$, where $r \in \mathbb{Z}$. Note that $X_0 = ab$ and $X_{-1} = ba$.

## 5. Elements with equal palindromization

The aim of this section is to characterize all pairs $(u, v)$ of elements of $F_2$ such that $\mathrm{Pal}(u) = \mathrm{Pal}(v)$. We start with the following observation.

**Proposition 5.1.** *We have* $\mathrm{Pal}(u) = \mathrm{Pal}(v)$ *if and only if* $u^{-1}v \in \mathrm{Pal}^{-1}(1)$.

**Proof.** By (3.1), $\mathrm{Pal}(u) = \mathrm{Pal}(v)$ if and only if $R_u(ab) = R_v(ab)$, which is equivalent to $R_{u^{-1}v}(ab) = ab$, hence to $\mathrm{Pal}(u^{-1}v) = 1$. □

We are thus reduced to describing the subgroup $\mathrm{Pal}^{-1}(1)$ of $F_2$. To this end we consider the injective homomorphism $i : B_3 \to \mathrm{Aut}(F_2)$ of Proposition 2.2 and the homomorphism $\beta : F_2 \to B_3$ defined by

$$\beta(a) = \sigma_1 \quad \text{and} \quad \beta(b) = \sigma_2^{-1}.$$

It follows from the definitions that $R_w = i\big(\beta(w)\big)$ for all $w \in F_2$. Therefore, if $\beta(u) = \beta(v)$, then $R_u = R_v$, hence $\mathrm{Pal}(u) = \mathrm{Pal}(v)$ by (3.1). In other words, $\mathrm{Pal}(w)$ depends only on the image of $w$ in $B_3$.

Let $N$ be the subgroup of $B_3$ generated by $\sigma_1\sigma_2^{-1}\sigma_1^{-1}$. Since $B_3$ is torsion-free, $N$ is infinite cyclic. We now give the promised description.

**Proposition 5.2.** *We have* $\mathrm{Pal}^{-1}(1) = \beta^{-1}(N)$.

**Proof.** By Corollary 3.2 it is enough to check that

$$N_0 = \{\, \beta(w) \in B_3 \mid R_w(ab) = ab \,\} = \beta\big(\mathrm{Pal}^{-1}(1)\big)$$

coincides with $N$.

First, observe that $R_{aba^{-1}} = R_a R_b R_{a^{-1}}$ sends $a$ to $aba$ and $b$ to $a^{-1}$. Therefore, $R_{aba^{-1}}(ab) = (aba)a^{-1} = ab$. Since $\beta(aba^{-1}) = \sigma_1\sigma_2^{-1}\sigma_1^{-1}$ generates $N$, the latter is contained in $N_0$.

Conversely, let $w \in F_2$ be such that $\mathrm{Pal}(w) = 1$. Then by Corollary 3.2, $R_w$ fixes $ab$; hence, $M_w$ fixes $(1, 1) \in \mathbb{Z}^2$. An easy computation shows that any matrix in $\mathrm{SL}_2(\mathbb{Z})$ fixing $(1, 1)$ is of the form

$$C_r = \begin{pmatrix} 1+r & -r \\ r & 1-r \end{pmatrix},$$

where $r \in \mathbb{Z}$. Since $r \mapsto C_r$ is a group homomorphism, we have $C_r = (C_1)^r$. Now,

$$C_1 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = M_{aba^{-1}}.$$

Therefore, $M_w = M_{(aba^{-1})^r}$. It is well known (see [8, App. A]) that $\sigma_1 \mapsto M_a$ and $\sigma_2 \mapsto M_b^{-1}$ defines a surjective group homomorphism $B_3 \to \mathrm{SL}_2(\mathbb{Z})$ whose kernel is the infinite cyclic group generated by $(\sigma_1\sigma_2\sigma_1)^4$. Since $(\sigma_1\sigma_2\sigma_1)^4$ is central in $B_3$, we conclude that

$$\beta(w) = (\sigma_1\sigma_2\sigma_1)^{4p}\,(\sigma_1\sigma_2^{-1}\sigma_1^{-1})^r$$

for some $p \in \mathbb{Z}$. Consequently,

$$R_w = \mathrm{i}\big(\beta(w)\big) = (R_{(ab^{-1}a)^4})^p\,(R_{aba^{-1}})^r.$$

Since $R_{aba^{-1}}$ fixes $ab$ and $R_w$ is assumed to fix $ab$, we obtain

$$ab = R_w(ab) = (R_{(ab^{-1}a)^4})^p(ab).$$

By (1.9), $R_{(ab^{-1}a)^4}$ is the conjugation by $bab^{-1}a^{-1}$; hence, $(R_{(ab^{-1}a)^4})^p$ is the conjugation by $(bab^{-1}a^{-1})^p$. We thus obtain

$$ab = (bab^{-1}a^{-1})^p\,ab\,(bab^{-1}a^{-1})^{-p}.$$

If $p < 0$, then the right-hand side is a reduced word different from $ab$. If $p > 0$, then the right-hand side represents the same element of $F_2$ as the reduced word $(bab^{-1}a^{-1})^{p-1}\,ba\,(bab^{-1}a^{-1})^{-p}$, which is also different from $ab$. It follows that necessarily $p = 0$ and hence $\beta(w) = (\sigma_1\sigma_2^{-1}\sigma_1^{-1})^r$. We have thus proved that $N_0 \subset N$. □

Observe that $\mathrm{Pal}^{-1}(1) = \beta^{-1}(N)$ is the product (in $F_2$) of the cyclic group generated by $aba^{-1}$ by the kernel $\mathrm{Ker}(\beta)$ of $\beta : F_2 \to B_3$. In view of the presentation (2.3), $\mathrm{Ker}(\beta)$ is the normal subgroup of $F_2$ generated by $ab^{-1}aba^{-1}b$ (or equivalently by $(aba^{-1})(bab^{-1})$).

By [11], if $u, v \in \{a, b\}^*$, then $\mathrm{Pal}(u) = \mathrm{Pal}(v)$ if and only if $u = v$. This sharply contrasts with the previous results.

## 6. The profinite topology

The *profinite topology* on $F_2$ is the coarsest topology such that every group homomorphism from $F_2$ into a finite group is continuous. This topology was introduced by Marshall Hall in [4] and is sometimes called the Hall topology; see [13,15] for applications to automata theory and semigroup theory.

**Theorem 6.1.** *The map* Pal : $F_2 \to F_2$ *is continuous for the profinite topology.*

Thus the map Pal yields an example of a non-trivial combinatorially-defined continuous function on $F_2$. For other examples of such functions, see the subword functions in [12].

**Proof.** Since $\mathrm{Pal}(w) = b^{-1}a^{-1}R_w(a)\,R_w(b)$, it suffices to prove that the map $F_2 \to F_2 \times F_2\,;\ \ w \mapsto (R_w(a), R_w(b))$ is continuous. The latter is equivalent to the continuity of

$$w \mapsto \big((\varphi \circ R_w)(a), (\varphi \circ R_w)(b)\big)$$

for all group homomorphisms $\varphi : F_2 \to G$ into a finite group $G$ (equipped with the discrete topology). It thus suffices to check that for each $(g, h) \in G \times G$, the set $X(g, h)$ of $w \in F_2$ such that

$$((\varphi \circ R_w)(a), (\varphi \circ R_w)(b)) = (g, h)$$

is a union of cosets of subgroups of finite index of $F_2$.

Since $F_2$ acts on the left on itself by $w \mapsto R_w$, it acts on the right on the set $\mathrm{Hom}(F_2, G)$ of group homomorphisms of $F_2$ into $G$ by $\varphi \cdot w = \varphi \circ R_w$ for all $\varphi \in \mathrm{Hom}(F_2, G)$ and $w \in F_2$. Now, $\mathrm{Hom}(F_2, G)$ is in bijection with $G \times G$ via the map $\varphi \mapsto (\varphi(a), \varphi(b))$. Hence, $F_2$ acts on the right on $G \times G$ in such a way that

$$\big(\varphi(a), \varphi(b)\big) \cdot w = \big((\varphi \circ R_w)(a), (\varphi \circ R_w)(b)\big)$$

for all $\varphi \in \mathrm{Hom}(F_2, G)$ and $w \in F_2$. Therefore, the above-defined set $X(g, h)$ coincides with the set of $w \in F_2$ such that $(\varphi(a), \varphi(b)) \cdot w = (g, h)$. This set is a coset in $F_2$ of the stabilizer $G(g, h) = \{w \in F_2 \mid (g, h) \cdot w = (g, h)\}$. We conclude by observing that each stabilizer of an action of a group $F$ on a *finite* set is necessarily a finite index subgroup of $F$. □

**Corollary 6.2.** *The map* Pal : $F_2 \to F_2$ *is the unique continuous extension to* $F_2$ *of de Luca's right iterated palindromic closure* P : $\{a, b\}^* \to \{a, b\}^*$.

**Proof.** It is well known that the submonoid $\{a, b\}^*$ is dense in $F_2$ for the profinite topology. Indeed, since $(w^{n!})_n$ converges to 1, the sequence $(w^{n!-1})_n$ converges to $w^{-1}$. Applying this remark to $w = a, b$, we see that $a^{-1}$ and $b^{-1}$ are limits of sequences of elements of $\{a, b\}^*$. Therefore the map P : $\{a, b\}^* \to \{a, b\}^*$ has at most one continuous extension to $F_2$. We conclude with Corollary 4.4 and Theorem 6.1. □

**Remark 6.3.** Though Pal is continuous for the profinite topology, it is *not* continuous for the pro-$p$-finite topology on $F_2$, where $p$ is a prime number. Recall that the *pro-$p$-finite topology* is the coarsest topology such that every group homomorphism from $F_2$ into a finite $p$-group is continuous. By definition of the pro-$p$-finite topology, the sequence $(w^{p^n})_n$ converges to 1 for any $w \in F_2$. We claim that if $w = a^{-1}b$, then the sequence $\mathrm{Pal}(w^{p^n})$ does not converge to $\mathrm{Pal}(1) = 1$. It is enough to check that that the vector-valued sequence $\pi(\mathrm{Pal}(w^{p^n})) \in \mathbb{Z}^2$ does not converge to $\pi(1)$, which is the zero vector.

The matrix $M_w \in \mathrm{SL}_2(\mathbb{Z})$ corresponding to $w = a^{-1}b$ is equal to

$$M_w = M_a^{-1}M_b = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

which is of order six. So $M_w^k$ takes only six values when $k$ runs over the positive integers. One checks that if $k$ is not divisible by 6, then

$$\big(M_w^k - I_2\big)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since no power $p^n$ of a prime number is divisible by 6, by the previous observation and by Proposition 4.6,

$$\pi\big(\mathrm{Pal}(w^{p^n})\big) = \big(M_w^{p^n} - I_2\big)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This shows that the sequence $\pi(\mathrm{Pal}(w^{p^n}))$ does not converge to $\pi(1)$, and hence Pal is not continuous for the pro-$p$-finite topology.

**Remark 6.4.** As a consequence of Theorem 6.1, de Luca's right iterated palindromic closure P : $\{a, b\}^* \to \{a, b\}^*$ is continuous for the restriction of the profinite topology to $\{a, b\}^*$. Nevertheless, $w \mapsto w^+$ is not continuous. Indeed, we have seen in the proof of Corollary 6.2 that for any $w \in F_2$, the sequence $(w^{n!})_n$ converges to 1 in the profinite topology. In particular, the sequence $(ab^{n!})_n$ converges to $a$. Now, $(ab^{n!})^+ = ab^{n!}a$. Therefore, the sequence $(ab^{n!})^+$ converges to $aa$, which is different from $a^+$.

## 7. A characterization of the image of Pal

The aim of this section is to characterize the elements of $F_2$ belonging to the image $\mathrm{Pal}(F_2)$ of Pal. The notation $g \sim h$ used in the sequel means that $g$ and $h$ are conjugate elements of $F_2$.

By (3.1) and (3.4), for any $w \in F_2$,

$$\begin{aligned} ab\ \mathrm{Pal}(w) &= R_w(ab) = R_w(a)\, R_w(ba)\, R_w(a)^{-1} \\ &= R_w(a)\big(ba\ \mathrm{Pal}(w)\big)R_w(a)^{-1}. \end{aligned}$$

Thus, $abg \sim bag$ for all $g \in \mathrm{Pal}(F_2)$. In [14], Pirillo proved that, if $A$ is an alphabet and $g$ is a word in $A$ such that $abg \sim bag$ for two distinct letters $a, b \in A$, then $g$ is a central word in the alphabet $\{a, b\}$; hence by de Luca [11], $g = \mathrm{Pal}(w)$ for some $w \in \{a, b\}^*$. We have the following extension of Pirillo's result to our palindromization map.

**Theorem 7.1.** *An element $g \in F_2$ belongs to* $\mathrm{Pal}(F_2)$ *if and only if $abg$ and $bag$ are conjugate in $F_2$.*

**Proof.** We assume that $g$ is a reduced word in $F_2$ such that $abg \sim bag$. We shall prove by induction on the length of $g$ that $g = \mathrm{Pal}(w)$ for some $w \in F_2$.

If $g$ is of length zero, then $g = 1 = \mathrm{Pal}(1)$. Now, suppose that $g$ is of length $> 0$. If there is no cyclic cancellation in $abg$, then there is none in $bag$ since $abg \sim bag$; in this case, $abg$ and $bag$ are conjugate in the free monoid $\{a, b, a^{-1}, b^{-1}\}^*$. Applying the above-mentioned theorem by Pirillo, we conclude that $g = \mathrm{Pal}(w)$ for some $w \in \{a, b\}^*$.

If the reduced word $g$ starts by $b^{-1}$, write $g = b^{-1}g'$, where $g'$ is a reduced word. We claim that $g'$ ends with $b^{-1}$. Indeed,

$$ag' = abg \sim bag = bab^{-1}g'.$$

If $g'$ did not end with $b^{-1}$, then $bab^{-1}g'$ would be cyclically reduced and conjugate to $ag'$, which is shorter than $bab^{-1}g'$; this is impossible and thus establishes the claim. It follows that $g = b^{-1}hb^{-1}$ for some shorter reduced word $h$. Let us show that

$$ab^{-1}h \sim b^{-1}ah. \tag{7.1}$$

Indeed, $ab^{-1}h = agb \sim bag \sim abg = ahb^{-1} \sim b^{-1}ah$. To (7.1) we apply the inverse $\tau^{-1}$ of the automorphism $\tau$ of $F_2$ introduced in Section 1, thus obtaining

$$ba\,\tau^{-1}(h) = \tau^{-1}(ab^{-1}h) \sim \tau^{-1}(b^{-1}ah) = ab\,\tau^{-1}(h).$$

Since $\tau^{-1}(h)$ is a reduced word of length less than the length of $g$, we may apply the induction hypothesis to $\tau^{-1}(h)$ and deduce that $\tau^{-1}(h) = \mathrm{Pal}(u)$ for some $u \in F_2$. Let us compute $\mathrm{Pal}(ab^{-1}au)$. Using (3.5), (3.3), and (1.7) successively, we obtain

$$\begin{aligned} \mathrm{Pal}(ab^{-1}au) &= \mathrm{Pal}(ab^{-1}a)\, R_{ab^{-1}a}\big(\mathrm{Pal}(u)\big) \\ &= b^{-2}\, R_{ab^{-1}a}\big(\tau^{-1}(h)\big) \\ &= b^{-2}\,(bhb^{-1}) = g. \end{aligned}$$

The case where $g$ starts by $a^{-1}$ is treated in a similar manner. □

**Corollary 7.2.** *The subset* $\mathrm{Pal}(F_2)$ *is closed in $F_2$ for the profinite topology.*

**Proof.** By Theorem 7.1, it is enough to check that the subset of elements $g \in F_2$ such that $abg \sim bag$ is closed in $F_2$. It suffices to consider two sequences $(u_n)$ and $(v_n)$ converging in $F_2$ respectively to $u$ and $v$ and such that $u_n \sim v_n$ for all $n$ and to show that $u$ and $v$ are conjugate in $F_2$. In this situation there are elements $x_n \in F_2$ such that $u_n x_n = x_n v_n$. Since the profinite completion of $F_2$ is compact, there is a subsequence of $(x_n)$ that converges to an element $x$ of the completion such that $ux = xv$. Therefore, $u$ and $v$ are conjugate in the completion, hence in all finite quotients of $F_2$. By [9, Prop. I.4.8], $u$ and $v$ are conjugate in $F_2$. □

**Question 7.3.** De Luca [11] gives a simple algorithm to recover $w \in \{a, b\}^*$ from $g = \mathrm{P}(w)$, namely $w$ is the sequence of the letters of $g$ that immediately follow all palindromic prefixes (including the empty word) of $g$. It is possible to extract from the proof of Theorem 7.1 an algorithm producing $w \in F_2$ out of $g = \mathrm{Pal}(w)$; the element $w$ obtained in this way is not necessarily of shortest length. We raise the question of finding a simple constructive procedure that produces a (unique?) element of shortest length out of its image under Pal.

## Appendix. Serre's non-abelian cohomology

Let $G$ be a group acting on another group $E$ via a group homomorphism $G \to \mathrm{Aut}(E)\,;\ u \mapsto R_u$ (the group $E$ is not assumed to be abelian). Following [16, Chap. 1, Sect. 5.1], we call *cocycle of $G$ with values in $E$* any map $\varphi : G \to E$ verifying the *cocycle condition*

$$\varphi(uv) = \varphi(u)\, R_u(\varphi(v)) \tag{A.1}$$

for all $u, v \in G$.

Two cocycles $\varphi, \varphi' : G \to E$ are said to be *cohomologous* if there exists $X \in E$ such that

$$\varphi'(u) = X^{-1}\,\varphi(u)\, R_u(X) \tag{A.2}$$

for all $u \in G$. A cocycle $\varphi$ is said to be *trivial* if it is cohomologous to the cocycle sending each $u \in G$ to the neutral element 1 of $E$; it follows from (A.2) that $\varphi$ is trivial if and only if there is $X \in E$ such that for all $u \in G$,

$$\varphi(u) = X^{-1} R_u(X). \tag{A.3}$$

Let $E \rtimes G$ be the semi-direct product associated to the action of $G$ on $E$; it is the set $E \times G$ equipped with the product

$$(X, u)\,(Y, v) = (X\,R_u(Y), uv)$$

for all $X, Y \in E$ and $u, v \in G$. To a map $\varphi : G \to E$ we associate the map $\widehat{\varphi} = (\varphi, \mathrm{id}) : G \to E \rtimes G$. Then $\varphi$ satisfies the cocycle equation (A.1) if and only if $\widehat{\varphi}$ is a group homomorphism, i.e.,

$$\widehat{\varphi}(uv) = \widehat{\varphi}(u)\,\widehat{\varphi}(v) \quad (u, v \in G). \tag{A.4}$$

It is also easy to check that, if $\varphi, \varphi' : G \to E$ and $X \in E$ satisfy (A.2), then $\widehat{\varphi}$ and $\widehat{\varphi'}$ are conjugate in the group $E \rtimes G$; more precisely,

$$\widehat{\varphi'}(u) = (X, 1)^{-1}\,\widehat{\varphi}(u)\,(X, 1) \in E \rtimes G$$

for all $u \in G$. In particular, if $\varphi$ is a trivial cocycle with $X \in E$ as in (A.3), then for all $u \in G$,

$$\widehat{\varphi}(u) = (X, 1)^{-1}\,(1, u)\,(X, 1). \tag{A.5}$$

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