



Approximating partition functions of bounded-degree Boolean counting Constraint Satisfaction Problems [☆]



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ABSTRACT

We study the complexity of $\#\text{CSP}_\Delta(\Gamma)$, which is the problem of counting satisfying assignments to CSP instances with constraints from Γ and whose variables can appear at most Δ times. Our main result shows that: (i) if every function in Γ is affine, then $\#\text{CSP}_\Delta(\Gamma)$ is in FP for all Δ , (ii) otherwise, if every function in Γ is in a class called IM_2 , then for large Δ , $\#\text{CSP}_\Delta(\Gamma)$ is equivalent under approximation-preserving reductions to the problem of counting independent sets in bipartite graphs, (iii) otherwise, for large Δ , it is NP-hard to approximate $\#\text{CSP}_\Delta(\Gamma)$, even within an exponential factor.

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1. Introduction

Constraint Satisfaction Problems (CSPs), which originated in Artificial Intelligence [21] provide a general framework for modelling decision, counting and approximate counting problems. The paradigm is sufficiently general that applications from diverse areas such as database theory, scheduling and graph theory can all be captured (see, for example, [17,18,20]). Moreover, all graph homomorphism decision and counting problems [15] can be re-cast in the CSP framework and partition function problems from statistical physics [25] can be represented as counting CSPs. Given the usefulness of CSPs, the study of the complexity of CSPs is an extremely active area in computational complexity (for example, see [3] and the references therein).

In this paper, we will be concerned with Boolean counting CSPs. An instance $I = (V, \mathcal{C})$ of a Boolean counting CSP consists of a set V of variables and a set \mathcal{C} of constraints. An assignment $\sigma : V \rightarrow \{0, 1\}$ assigns a Boolean value called a “spin” to each variable. Each constraint associates a tuple (v_1, \dots, v_k) of variables with a Boolean relation which constrains the spins that can be assigned to v_1, \dots, v_k . In particular, the assignment σ is said to “satisfy” the constraint if the tuple $(\sigma(v_1), \dots, \sigma(v_k))$ is in the corresponding relation. An assignment is said to be “satisfying” if it satisfies all constraints. A Constraint Satisfaction Problem comes with two important parameters – the constraint language Γ is the set of all relations that may be used in constraints and the degree Δ is the maximum number of times that any variable $v \in V$ may be used in constraints in any instance. The number of satisfying assignments is denoted Z_I . The computational problem $\#\text{CSP}_\Delta(\Gamma)$ is the problem of computing Z_I , given a CSP instance I with constraints in Γ and degree at most Δ . We use $\#\text{CSP}(\Gamma)$ to denote the version of the problem in which the degree of instances is unconstrained.

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Although constraints are supported by Boolean relations, they can be used to code up weighted interactions such as those that arise in statistical physics. For example, let R be the “not-all-equal” relation of arity 3. Then consider the conjunction of $R(x, a, b)$ and $R(y, a, b)$. There are two satisfying assignments with $\sigma(x) = 0$ and $\sigma(y) = 1$ since $\sigma(a)$ and $\sigma(b)$ must differ. Similarly, there are two satisfying assignments with $\sigma(x) = 1$ and $\sigma(y) = 0$. On the other hand, there are three satisfying assignments with $\sigma(x) = \sigma(y) = 1$ and there are three satisfying assignments with $\sigma(x) = \sigma(y) = 0$. Thus, the induced interaction on the variables x and y is the same as the interaction of the ferromagnetic Ising model (at an appropriate temperature) – an assignment in which x and y have the same spin has weight 3, whereas an assignment where they have different spins has weight 2.

For every $\Delta \geq 3$, the work of Cai, Lu and Xia [6] completely classifies the complexity of exactly solving $\#\text{CSP}_\Delta(\Gamma)$, depending on the parameter Γ . If every relation in Γ is affine, then $\#\text{CSP}_\Delta(\Gamma)$ is solvable in polynomial time (so the problem is in the complexity class FP). Otherwise, it is $\#\text{P}$ -complete. The term “affine” will be defined in Section 2. Roughly, it means that the tuples in the relation are solutions to a linear system, so Gaussian elimination gives an appropriate polynomial-time algorithm. The characterisation of Cai, Lu and Xia is exactly the same classification that was obtained for the unbounded problem $\#\text{CSP}(\Gamma)$ by Creignou and Hermann [7]. Thus, as far as exact counting is concerned, the degree-bound Δ does not affect the complexity as long as $\Delta \geq 3$. As Cai, Lu and Xia point out, the dichotomy is false for $\Delta = 2$, where $\#\text{CSP}_2(\Gamma)$ is equivalent to the Holant problem $\text{Holant}(\Gamma)$ – see the references in [6] for more information about Holant problems.

Much less is known about the complexity of *approximately* solving $\#\text{CSP}_\Delta(\Gamma)$. In fact, even the *decision problem* is still open. While Schaefer [22] completely classified the complexity of the decision problem $\text{CSP}(\Gamma)$ – where the goal is to determine whether or not Z_I is 0 for an instance of $\#\text{CSP}(\Gamma)$ – the complexity of the corresponding decision problem $\text{CSP}_\Delta(\Gamma)$, where the instance has degree at most Δ , is still not completely resolved. For $\Delta \geq 3$, Dalmau and Ford [10] have solved the special case where Γ includes both of the relations $R_{\delta_0} = \{0\}$ and $R_{\delta_1} = \{1\}$. This special case is known as the “conservative case” in the CSP literature. For $\Delta \geq 6$, Dyer et al. [12] have classified the difficulty of the approximation problem:

- If every relation in Γ is affine, then $\#\text{CSP}_\Delta(\Gamma \cup \{R_{\delta_0}, R_{\delta_1}\})$ is in FP.
- Otherwise, if every relation in Γ is in a class called IM_2 (a class which will be defined in Section 2) then $\#\text{CSP}_\Delta(\Gamma \cup \{R_{\delta_0}, R_{\delta_1}\})$ is equivalent under approximation-preserving (AP) reductions to the counting problem $\#\text{BIS}$ (the problem of counting independent sets in bipartite graphs).
- Otherwise, there is no FPRAS for $\#\text{CSP}_\Delta(\Gamma \cup \{R_{\delta_0}, R_{\delta_1}\})$ unless $\text{NP} = \text{RP}$.

Dyer et al. made only partial progress on the cases where $\Delta \in \{3, 4, 5\}$. We refer the reader to [12,19] for a discussion of the partial classification. However, it is worth noting here that the complexity of $\#\text{CSP}_\Delta(\Gamma \cup \{R_{\delta_0}, R_{\delta_1}\})$ is closely related to the complexity of counting satisfying assignments of so-called read- d Monotone CNF Formulas. Crucial progress was made by Liu and Lu [19], who completely resolved the complexity of the latter problem. Given the work of Liu and Lu, a complete classification of $\#\text{CSP}_\Delta(\Gamma \cup \{R_{\delta_0}, R_{\delta_1}\})$ for $\Delta \in \{3, 4, 5\}$ may be in reach.

The restriction that R_{δ_0} and R_{δ_1} are contained in Γ is a severe one because it does not apply to many natural applications. On the other hand, we are a long way from a precise understanding of the complexity of $\#\text{CSP}_\Delta(\Gamma)$ without this restriction because there are specific, relevant parameters that we do not understand. For example, for a positive integer k , let Γ be the singleton set containing only the arity- k “not-all-spin-1” relation. Then satisfying assignments of an instance of $\#\text{CSP}_\Delta(\Gamma)$ correspond to independent sets of a k -uniform hypergraph with maximum degree Δ . The current state-of-the-art for this problem is that there is an FPRAS for $\Delta = O(2^{k/2})$ [16] and that the problem is NP-hard to approximate for $\Delta = \Omega(2^{k/2})$ [1]; the implicit constants in these bounds do not currently match and thus, for large k , there is a large range of Δ 's where we do not yet know the complexity of approximating $\#\text{CSP}_\Delta(\Gamma)$. If Γ instead contains (only) the arity- k “at-least-one-spin-0” relation then satisfying assignments of an instance of $\#\text{CSP}_\Delta(\Gamma)$ correspond to the so-called “strong” independent sets of a k -uniform hypergraph. Song, Yin and Zhao [23] have presented a barrier for hardness results, showing why current technology is unsuitable for resolving the cases where $\Delta \in \{4, 5\}$ (roughly, these cases are in “non-uniqueness”, but this is not realisable by finite gadgets).

The purpose of the present paper is to remove the severe restriction that R_{δ_0} and R_{δ_1} are contained in Γ in the approximate counting classification of $\#\text{CSP}_\Delta(\Gamma)$ from [12]. Since pinning down precise thresholds seems a long way out of reach, we instead focus on whether there is a “barrier” value Δ_0 such that, for all $\Delta \geq \Delta_0$, approximation is intractable. Since we wish to get the strongest possible inapproximability results (showing the hardness of approximating Z_I even within an exponential factor), we define the following computational problem, which has an extra parameter $c > 1$ that captures the desired accuracy.

Name $\#\text{CSP}_{\Delta,c}(\Gamma)$.

Instance An n -variable instance I of a CSP with constraint language Γ and degree at most Δ .

Output A number \tilde{Z} such that $c^{-n}Z_I \leq \tilde{Z} \leq c^n Z_I$.

Although we have not yet defined all of the terms, we can now at least state (a weak version of) our result.

Theorem 1. Let Γ be a Boolean constraint language. Then,

1. If every relation in Γ is affine then $\#\text{CSP}(\Gamma)$ is in FP.
2. Otherwise, if every relation in Γ is in the class IM_2 , then there exists an integer Δ_0 such that for all $\Delta \geq \Delta_0$, $\#\text{CSP}_\Delta(\Gamma)$ is $\#\text{BIS}$ -equivalent under AP-reductions.
3. Otherwise, there exists an integer Δ_0 such that for all $\Delta \geq \Delta_0$, there exists a real number $c > 1$ such that $\#\text{CSP}_{\Delta,c}(\Gamma)$ is NP-hard.

After defining all of the terms, we will state a stronger theorem, Theorem 6, which immediately implies Theorem 1. The stronger version applies to the $\#\text{CSP}$ problems that we have already introduced, but it also applies to other restrictions of these problems, which have even more applications.

We now explain the restriction. Note that in the CSP framework, as we have defined it, the variables that are constrained by a given constraint need not be distinct. Thus, if the arity-4 relation R is present in a constraint language Γ , then an instance of $\#\text{CSP}(\Gamma)$ with variables x and y may contain a constraint such as $R(x, x, y, x)$. This ability to repeat variables is equivalent to assuming that equality relations of all arities are present in Γ . This feature of the CSP definition is inconvenient for two reasons: (1) It does not fit well with some spin-system applications, and (2) In many settings, it obscures the nuanced complexity classification that arise.

As an example of (1), recall the application where Γ is the singleton set containing only the arity- k “not-all-spin-1” relation. As we noted earlier, satisfying assignments of a $\#\text{CSP}(\Gamma)$ instance correspond to independent sets of a k -uniform hypergraph. Here, hyperedges are size- k subsets of vertices and it does not make sense to allow repeated vertices!

The point (2) is well-known. In fact, the “equality is always present” assumption is the main feature that separates $\#\text{CSP}$ s from the more general Holant framework [4].

In our current setting, it turns out that adding equality functions to Γ does not change the complexity classification, but this is a result of our theorems rather than an a priori assumption – indeed, determining which constraint languages Γ can appropriately simulate equality functions is one of the difficulties – thus, throwing equalities in “for free” would substantially weaken our results! Our main result, Theorem 6, which will be presented in Section 2, applies both to the $\#\text{CSP}$ s that we have already defined, and to more refined versions, in which constraints may not repeat variables.

We wish now to discuss an important special case in which both the $\#\text{CSP}$ s and the refined versions have already been studied. This is the special case in which Γ consists of a single relation which is symmetric in its arguments. A symmetric relation that is not affine is not in IM_2 . Therefore, Item 2 in the statement of Theorem 1 never arises in this special case. Our earlier paper [14] shows that, in this case (where Γ consists of a single, symmetric, non-affine relation) there is an integer Δ_0 such that for all $\Delta \geq \Delta_0$, there exists a real number $c > 1$ such that $\#\text{CSP}_{\Delta,c}(\Gamma)$ is NP-hard.

While the work of [14] is important for this paper, note that the special case is far from general – in particular, it is easy to induce asymmetric constraints using symmetric ones. For example, suppose that R_1 is the (symmetric) arity-2 “not-all-spin-1” constraint, R_2 is the (symmetric) arity-2 “not the same spin” constraint and $R_3 = \{(0, 0), (0, 1), (1, 1)\}$ is the (asymmetric) arity-2 “Implies” constraint. Then the conjunction of $R_1(x, a)$ and $R_2(a, y)$ induces $R_3(x, y)$.

It is interesting that Theorem 1 is exactly the same classification that was obtained for the *unbounded* problem $\#\text{CSP}(\Gamma)$ by Dyer et al. [13]. In particular, they showed

1. If every relation in Γ is affine then $\#\text{CSP}(\Gamma)$ is in FP.
2. Otherwise, if every relation in Γ is in the class IM_2 , then $\#\text{CSP}(\Gamma)$ is $\#\text{BIS}$ -equivalent under AP-reductions.
3. Otherwise, $\#\text{CSP}(\Gamma)$ is $\#\text{SAT}$ -equivalent under AP-reductions, where $\#\text{SAT}$ is the problem of counting the satisfying assignments of a Boolean formula.

The inapproximability that we demonstrate in Item 3 of Theorem 1 is stronger than what was known in the unbounded case, both (obviously) because of the degree bound, but also because we show that it is hard to get within an exponential factor. (This strong kind of inapproximability was also missing from the results of [12].)

2. Definitions and statement of main result

Before giving formal definitions of the problems that we study, we introduce some notation. We use boldface letters to denote Boolean vectors. A *pseudo-Boolean* function is a function of the form $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$ for some positive integer k , which is called the *arity* of f .

Definition 2. Given a pseudo-Boolean function $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$, we use the notation R_f to denote the relation $R_f = \{\mathbf{x} \in \{0, 1\}^k \mid f(\mathbf{x}) > 0\}$, which is the relation underlying f .

If the range of f is $\{0, 1\}$ then f is said to be a *Boolean function* and of course in that case $R_f = \{\mathbf{x} \in \{0, 1\}^k \mid f(\mathbf{x}) = 1\}$.

In order to allow consistency with obvious generalisations, our formal definition of the Boolean Constraint Satisfaction Problem is in terms of Boolean functions (rather than, equivalently, using the underlying relations).

A *Constraint language* Γ is a set of pseudo-Boolean functions. It is a *Boolean constraint language* if all of the functions in it are Boolean functions. An instance $I = (V, \mathcal{C})$ of a CSP with constraint language Γ consists of a set V of variables and

a set \mathcal{C} of constraints. Each constraint $C_i \in \mathcal{C}$ is of the form $f_i(v_{i,1}, \dots, v_{i,k_i})$ where f_i is an arity- k_i function in Γ and $(v_{i,1}, \dots, v_{i,k_i})$ is a tuple of (not necessarily distinct) variables in V . The constraint C_i is said to be “Repeat-Free” if all of the variables are distinct. Each assignment $\sigma : V \rightarrow \{0, 1\}$ of Boolean values to the variables in V has a weight

$$w_I(\sigma) := \prod_{f_i(v_{i,1}, \dots, v_{i,k_i}) \in \mathcal{C}} f_i(\sigma(v_{i,1}), \dots, \sigma(v_{i,k_i})).$$

The partition function maps the instance I to the quantity

$$Z_I := \sum_{\sigma : V \rightarrow \{0,1\}} w_I(\sigma) = \sum_{\sigma : V \rightarrow \{0,1\}} \prod_{f_i(v_{i,1}, \dots, v_{i,k_i}) \in \mathcal{C}} f_i(\sigma(v_{i,1}), \dots, \sigma(v_{i,k_i})).$$

If Γ is a Boolean constraint language then it is easy to see that $w_I(\sigma) = 1$ if the assignment is satisfying and $w_I(\sigma) = 0$, otherwise. Thus, Z_I is the number of satisfying assignments of I .

When $Z_I > 0$, we will use $\mu_I(\cdot)$ to denote the Gibbs distribution corresponding to Z_I . This is the probability distribution on the set of assignments $\sigma : V \rightarrow \{0, 1\}$ such that

$$\mu_I(\sigma) = \frac{w_I(\sigma)}{Z_I} \text{ for all } \sigma : V \rightarrow \{0, 1\}.$$

The degree $d_v(C)$ of a variable v in a constraint C is the number of times that the variable v appears in the tuple corresponding to C and the degree d_v of the variable is $d_v = \sum_{C \in \mathcal{C}} d_v(C)$. Finally, the degree of the instance I is $\max_{v \in V} d_v$.

Definition 3. $\#CSP_{\Delta}(\Gamma)$ is the problem of computing Z_I , given a CSP instance I with constraints in Γ and degree at most Δ . $\#CSP(\Gamma)$ is the version of the problem in which the degree of instances is unconstrained. $\#CSP_{\Delta,c}(\Gamma)$ has an extra parameter $c > 1$ that captures the desired accuracy. The problem is to compute a number \widehat{Z} such that $c^{-n} Z_I \leq \widehat{Z} \leq c^n Z_I$, where n is the number of variables in the instance I . The problems $\#NoRepeatCSP_{\Delta}(\Gamma)$, $\#NoRepeatCSP(\Gamma)$ and $\#NoRepeatCSP_{\Delta,c}(\Gamma)$ are defined similarly, except that inputs are restricted so that all constraints are Repeat-Free.

Definition 4. A Boolean function $f : \{0, 1\}^k \rightarrow \{0, 1\}$ is *affine* if there is a $k \times k$ Boolean matrix \mathbf{A} and a length- k Boolean vector \mathbf{b} such that R_f is equal to the set of solutions \mathbf{x} of $\mathbf{Ax} = \mathbf{b}$ over $\text{GF}(2)$.

Definition 5 (The set of functions IM_2). A Boolean function $f : \{0, 1\}^k \rightarrow \{0, 1\}$ is in IM_2 if $f(x_1, \dots, x_k)$ is logically equivalent to a conjunction of (any number of) predicates of the form x_i , $\neg x_i$ or $x_i \Rightarrow x_j$.

We have now defined all of the terms in our main theorem apart from some well-known concepts from complexity theory, which we discuss next. FP is the class of computational problems (with numerical output) that can be solved in polynomial time. An FPRAS is a randomised algorithm that produces approximate solutions within specified relative error with high probability in polynomial time. For two counting problems $\#A$ and $\#B$, we say that $\#A$ is $\#B$ -easy if there is an approximation-preserving (AP)-reduction from $\#A$ to $\#B$. The formal definition of an AP-reduction can be found in [11]. It is a randomised Turing reduction that yields close approximations to $\#A$ when provided with close approximations to $\#B$. The definition of AP-reduction meshes with the definition of FPRAS in the sense that the existence of an FPRAS for $\#B$ implies the existence of an FPRAS for $\#A$. We say that $\#A$ is $\#B$ -hard if there is an AP-reduction from $\#B$ to $\#A$. Finally, we say that $\#A$ is $\#B$ -equivalent if $\#A$ is both $\#B$ -easy and $\#B$ -hard.

The problem of counting satisfying assignments of a Boolean formula is denoted by $\#SAT$. Every counting problem in $\#P$ is AP-reducible to $\#SAT$, so $\#SAT$ is said to be complete for $\#P$ with respect to AP-reductions. It is known that there is no FPRAS for $\#SAT$ unless $\text{RP} = \text{NP}$. The problem of counting independent sets in a bipartite graph is denoted by $\#BIS$. The problem $\#BIS$ appears to be of intermediate complexity: there is no known FPRAS for $\#BIS$ (and it is generally believed that none exists) but there is no known AP-reduction from $\#SAT$ to $\#BIS$. Indeed, $\#BIS$ is complete with respect to AP-reductions for a complexity class $\#RH\Gamma_1$.

Given all of these definitions, we now formally state the stronger version of Theorem 1 promised in the introduction. The proof can be found in Section 9.

Theorem 6. Let Γ be a Boolean constraint language. Then,

1. If every function in Γ is affine then $\#CSP(\Gamma)$ and $\#NoRepeatCSP(\Gamma)$ are both in FP.
2. Otherwise, if $\Gamma \subseteq IM_2$, then there exists an integer Δ_0 such that for all $\Delta \geq \Delta_0$, $\#CSP_{\Delta}(\Gamma)$ and $\#NoRepeatCSP_{\Delta}(\Gamma)$ are both $\#BIS$ -equivalent under AP-reductions, and
3. Otherwise, there exists an integer Δ_0 such that for all $\Delta \geq \Delta_0$, there exists a real number $c > 1$ such that $\#CSP_{\Delta,c}(\Gamma)$ and $\#NoRepeatCSP_{\Delta,c}(\Gamma)$ are both NP-hard.

3. Overview of the proof of Theorem 6

In this section, we give a non-technical overview of the proof of Theorem 6. Our objective is to illustrate the main ideas and obstacles without delving into the more detailed definitions. A more technical overview can be found in Section 5. Our focus in this section will be on the case where Γ consists of a single Boolean function $f : \{0, 1\}^k \rightarrow \{0, 1\}$. As will be clear in Section 9, this case is the main ingredient in the proof of the theorem.

A typical approach for showing that a counting CSP is intractable is to use an instance of the CSP to build a “gadget” which simulates an intractable binary 2-spin constraint. This was the approach used in [14], which proved the intractability of $\#\text{NoRepeatCSP}_\Delta(\{f\})$ for any *symmetric* non-affine Boolean function f by constructing an instance I of $\#\text{NoRepeatCSP}_\Delta(\{f\})$, along with variables x and y , such that for all spins $s_x \in \{0, 1\}$ and $s_y \in \{0, 1\}$ the marginal distribution $\mu_I(x, y)$ satisfies

$$\mu_I(\sigma(x) = s_x, \sigma(y) = s_y) = \frac{g(s_x, s_y)}{g(0, 0) + g(0, 1) + g(1, 0) + g(1, 1)}, \tag{1}$$

where g is a binary function that codes up the interaction of an intractable anti-ferromagnetic 2-spin system. We will not need to give detailed definitions of 2-spin systems in this paper. Instead, we give a sufficient condition for intractability.

Definition 7. A binary function $g : \{0, 1\}^2 \rightarrow \mathbb{R}_{\geq 0}$ is said to be “hard” if all of the following hold:

$$\begin{aligned} g(0, 0) + g(1, 1) &> 0, \\ \min\{g(0, 0), g(1, 1)\} &< \sqrt{g(0, 1)g(1, 0)}, \\ \max\{g(0, 0), g(1, 1)\} &\leq \sqrt{g(0, 1)g(1, 0)}. \end{aligned}$$

It was established in [14] that the ability to “simulate” a hard function g in the sense of (1) ensures that $\#\text{NoRepeatCSP}_\Delta(\{f\})$ is NP-hard to approximate, even within an exponential factor.

A key feature of *symmetric* Boolean functions f which facilitated such simulation in [14] was the fact that the class of relevant hard functions g is well-behaved, and it turned out that it suffices to encode such a hard binary function with only ϵ -accuracy, for some sufficiently small $\epsilon > 0$, and this was enough to ensure the NP-hardness of $\#\text{CSP}_{\Delta,c}(\{f\})$.

The main obstacle in adapting the approach of [14] to the case where f need not be symmetric in its arguments arises when f is in IM_2 . It is unlikely that such a function f can simulate a hard function g in the sense of (1) – indeed such a simulation would prove the (very surprising) result that $\#\text{BIS}$ does not have an FPRAS (unless $\text{NP} = \text{RP}$). Thus, for $f \in IM_2$, we need instead to encode a binary function which will allow us to connect the problem $\#\text{NoRepeatCSP}_\Delta(\{f\})$ to $\#\text{BIS}$.

Now consider the binary Boolean function *Implies* whose underlying relation $R_{\text{Implies}} = \{(0, 0), (0, 1), (1, 1)\}$ contains all (x, y) satisfying $x \Rightarrow y$. Obviously, *Implies* is not symmetric, and it is not hard according to Definition 7. On bipartite instances, however, the symmetry can be restored by interpreting differently the spins 0 and 1 on the two parts of the graph, and this leads to a connection with $\#\text{BIS}$. In particular, it is well-known [13] that $\#\text{CSP}(\{\text{Implies}\})$ is equivalent to $\#\text{BIS}$ under AP-reductions. This connection was extended to the bounded-degree setting by [5].

Unfortunately, the symmetrisation which connects $\#\text{CSP}(\{\text{Implies}\})$ to $\#\text{BIS}$ is not very robust. For example, suppose that a (non-symmetric) Boolean function f can be used to simulate, in the sense of (1), a binary function g which is very close to *Implies*. In particular, suppose that for some $\epsilon > 0$ and $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ satisfying $|\epsilon_i| \leq \epsilon$ for $i = 1, 2, 3, 4$, we have

$$\begin{aligned} g(0, 0) &= 1 + \epsilon_1, & g(0, 1) &= 1 + \epsilon_2, \\ g(1, 0) &= \epsilon_3, & g(1, 1) &= 1 + \epsilon_4. \end{aligned}$$

Such a close approximation is about the best that can be expected using the kind of approximate encodings that are available. However, the complexity of asymmetric 2-spin systems is not sufficiently well understood to exploit such a simulation. Surprisingly, for *any* arbitrarily small constant $\epsilon > 0$, it is not known even whether the unbounded degree version $\#\text{CSP}(\{g\})$ is $\#\text{BIS}$ -hard, and certainly nothing is known in our bounded-degree setting! The trouble is that the symmetrisation that works for *Implies* (i.e., when $\epsilon_i = 0$ for $i = 1, 2, 3, 4$) is no longer guaranteed to symmetrise the imperfect version with the ϵ_i ’s, so the swapping of spin-0 and spin-1 values on one side of the bipartite graph leads to an *asymmetric* 2-spin system on bipartite graphs and this does not fall into the scope of known results [5] concerning bounded-degree bipartite 2-spin systems.

Our approach to handle this problem for $f \in IM_2$ is to carefully ensure that there is no accuracy error ϵ in encoding the function *Implies*. In other words, we show that, using $f \in IM_2$, we can encode *Implies* *perfectly*, a task which is surprisingly intricate in the repeat-free setting. Our main technical theorem, Theorem 17, achieves this goal. Namely, it shows that, for every non-affine Boolean function f , either f simulates a hard function (with arbitrarily small accuracy-error ϵ , which leads to the desired intractability of $\#\text{NoRepeatCSP}_\Delta(\{f\})$) or else f “supports perfect equality” – a concept which will be defined later, but essentially means that f can be used to perfectly simulate the binary function EQ with underlying relation $R_{\text{EQ}} = \{(0, 0), (1, 1)\}$. Using EQ, it is possible to simulate repeated variables in constraints, so the $\#\text{BIS}$ -hardness of $\#\text{CSP}_\Delta(\{f\})$ follows from [13]. When $f \notin IM_2$ but f supports perfect equality, instead of reducing to the work in [13], we work somewhat harder to make sure that we also get the strong (exponential factor) inapproximability given in Theorem 6.

4. Pinning, equality and simulating functions

We will often be interested in the case where Γ contains a single function $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$. In this case, we can we simplify the notation because the constraints in an instance I are in one-to-one correspondence with k -tuples of variables (there is no need to repeat the name of the function f in each constraint). So, for convenience, we make the following definitions.

A k -tuple hypergraph $H = (V, \mathcal{F})$ consists of a set V of vertices, together with a set \mathcal{F} of hyperarcs, where every hyperarc in \mathcal{F} is a k -tuple of distinct vertices in V . The degree of H is the maximum, over all vertices $v \in V$, of the number of hyperarcs that contain v . Given a function $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$, we let $I_f(H)$ denote the instance of $\#\text{NoRepeatCSP}(\{f\})$ whose constraints correspond to the hyperarcs of H . Given an assignment $\sigma : V \rightarrow \{0, 1\}$ we define $w_{f;H}(\sigma) := \prod_{(v_1, \dots, v_k) \in \mathcal{F}} f(\sigma(v_1), \dots, \sigma(v_k))$ and $Z_{f;H} := \sum_{\sigma : V \rightarrow \{0, 1\}} w_{f;H}(\sigma)$, so $Z_{I_f(H)} = Z_{f;I_f(H)}$. By analogy to the Gibbs distribution on satisfying assignments, when $Z_{f;H} > 0$, we use $\mu_{f;H}(\cdot)$ to denote the probability distribution in which, for all assignments $\sigma : V \rightarrow \{0, 1\}$, $\mu_{f;H}(\sigma) = w_{f;H}(\sigma)/Z_{f;H}$. Given a function $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$, a positive integer Δ , and a real number $c > 1$, the following computational problems are equivalent to $\#\text{NoRepeatCSP}_{\Delta,c}(\{f\})$ and $\#\text{NoRepeatCSP}_{\Delta,c}(\{f\})$, respectively.

Name $\#\text{Multi2Spin}_{\Delta}(f)$.

Instance A k -tuple hypergraph H with degree at most Δ .

Output The partition function $Z_{f;H}$.

Name $\#\text{Multi2Spin}_{\Delta,c}(f)$.

Instance An n -vertex k -tuple hypergraph H with degree at most Δ .

Output A number \widehat{Z} such that $c^{-n} Z_{f;H} \leq \widehat{Z} \leq c^n Z_{f;H}$.

The name $\#\text{Multi2Spin}_{\Delta}(f)$ indicates that the problem is to compute the partition function of a 2-spin system with multi-body interactions specified by f and degree-bound Δ .

4.1. Supporting pinning and equality

Let k be a positive integer and let $H = (V, \mathcal{F})$ be a k -tuple hypergraph. Given a configuration $\sigma : V \rightarrow \{0, 1\}$ and a subset $T \subseteq V$, we will use σ_T to denote the restriction of σ to vertices in T . For a vertex $v \in V$, we will also use σ_v to denote the spin $\sigma(v)$ of vertex v in σ . The following definitions are generalisations of definitions from [14].

Definition 8. Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$. Suppose that $\epsilon \geq 0$ and $s \in \{0, 1\}$. The k -tuple hypergraph H is an ϵ -realisation of pinning-to- s if there exists a vertex v of H such that $\mu_{f;H}(\sigma_v = s) \geq 1 - \epsilon$.

Definition 9. Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$ and $s \in \{0, 1\}$. We say that f supports pinning-to- s if, for every $\epsilon > 0$, there is a k -tuple hypergraph which is an ϵ -realisation of pinning-to- s . We say that f supports perfect pinning-to- s if there is a k -tuple hypergraph which is a 0-realisation of pinning-to- s .

We now define what it means for a function f to support (perfect) equality which was already discussed in Section 3.

Definition 10. Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$ and $\epsilon \geq 0$. The k -tuple hypergraph H is an ϵ -realisation of equality if there exist distinct vertices v_1 and v_2 of H such that, for each $s \in \{0, 1\}$,

$$\mu_{f;H}(\sigma_{v_1} = \sigma_{v_2} = s) \geq (1 - \epsilon)/2.$$

Definition 11. Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$. The function f supports equality if, for every $\epsilon > 0$, there is a k -tuple hypergraph which is an ϵ -realisation of equality. The function f supports perfect equality if there is a k -tuple hypergraph which is a 0-realisation of equality.

4.2. Realising conditional distributions induced by pinning and equality

Given a set S of vertices, it will be convenient to follow [14] as follows. We write $\sigma_S = \mathbf{0}$ to denote the event that all vertices in S are assigned the spin 0 under the assignment σ . We similarly write $\sigma_S = \mathbf{1}$ to denote the event that all vertices in S are assigned the spin 1 under the assignment σ . Finally, we use σ_S^{eq} to denote the event that all vertices in S have the same spin under σ (the spin could be 0 or 1). The following definition is a generalisation of Definition 16 of [14] except that we have changed the notation slightly for convenience.

Definition 12 ([14, Definition 16]). Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$. Let $H = (V, \mathcal{F})$ be a k -tuple hypergraph. Let $\mathcal{V} = (V_{\text{pin}0}, V_{\text{pin}1}, \mathcal{V}_{\text{eq}})$ where $V_{\text{pin}0}$ and $V_{\text{pin}1}$ are disjoint subsets of V and \mathcal{V}_{eq} is a (possibly empty) set of disjoint subsets of $V \setminus (V_{\text{pin}0} \cup V_{\text{pin}1})$. Suppose that: (i) $V_{\text{pin}0} = \emptyset$ if f does not support pinning-to-0, (ii) $V_{\text{pin}1} = \emptyset$ if f does not support pinning-to-1, (iii) $\mathcal{V}_{\text{eq}} = \emptyset$ if f does not support equality, (iv) it holds that $\mu_{f;H}(\sigma_{V_{\text{pin}0}} = \mathbf{0}, \sigma_{V_{\text{pin}1}} = \mathbf{1}, \bigcap_{W \in \mathcal{V}_{\text{eq}}} \sigma_W^{\text{eq}}) > 0$. We will then say that “ \mathcal{V} is admissible for H with respect to f ” and we will denote by $\mu_{f;H}^{\text{cond}(\mathcal{V})}$ the probability distribution $\mu_{f;H}(\cdot \mid \sigma_{V_{\text{pin}0}} = \mathbf{0}, \sigma_{V_{\text{pin}1}} = \mathbf{1}, \bigcap_{W \in \mathcal{V}_{\text{eq}}} \sigma_W^{\text{eq}})$.

Remark 13. Frequently, instead of formally specifying \mathcal{V} , we will specify \mathcal{V} implicitly by just saying “consider the conditional distribution $\mu_{f;H}^{\text{cond}(\mathcal{V})}$ where the vertices in $V_{\text{pin}0}$ are pinned to 0, the vertices in $V_{\text{pin}1}$ are pinned to 1 and for all $W \in \mathcal{V}_{\text{eq}}$, all the vertices in W are forced to be equal”.

4.3. Simulating hard functions and inapproximability results

We can now give a formal definition of “simulation”, along the lines that was informally discussed in Section 3 (Equation (1)).

Definition 14. Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$ and $g : \{0, 1\}^t \rightarrow \mathbb{R}_{\geq 0}$. The function f simulates the function g if there is a k -tuple hypergraph H , an admissible set \mathcal{V} for H with respect to f , and t vertices v_1, v_2, \dots, v_t of H such that, for all $(s_1, s_2, \dots, s_t) \in \{0, 1\}^t$,

$$\mu_{f;H}^{\text{cond}(\mathcal{V})}(\sigma(v_1) = s_1, \sigma(v_2) = s_2, \dots, \sigma(v_t) = s_t) = \frac{g(s_1, s_2, \dots, s_t)}{\sum_{(s'_1, s'_2, \dots, s'_t) \in \{0, 1\}^t} g(s'_1, s'_2, \dots, s'_t)}.$$

If $\mathcal{V} = (\emptyset, \emptyset, \emptyset)$, then we say that f perfectly simulates g . More generally, we say that f simulates a set of functions \mathcal{G} if f simulates every $g \in \mathcal{G}$.

The connection between “hard” as defined in Definition 7 and intractability is given in the following lemma.

Lemma 15 ([14, Lemma 18]). Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$. If f simulates a hard function, then for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard. \square

Remark 16. [14, Lemma 18] is stated for symmetric functions, but the proof in [14] also works for asymmetric functions.

5. Proof sketch

In this section, for a Boolean function $f : \{0, 1\}^k \rightarrow \{0, 1\}$, we consider the complexity of the problems $\#\text{Multi2Spin}_{\Delta}(f)$ and $\#\text{Multi2Spin}_{\Delta,c}(f)$. Classifying the complexity of these problems is the most important step in the proof of Theorem 6. Namely, to obtain Theorem 6, it suffices to show that for every non-affine function f , we have that:

- If f is in IM_2 , then for all sufficiently large Δ , $\#\text{Multi2Spin}_{\Delta}(f)$ is #BIS-equivalent.
- If f is not in IM_2 , then for all sufficiently large Δ , there exists a real number $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard.

Our main technical theorem to prove this is the following classification of Boolean functions, which asserts that every non-affine function either supports perfect equality or simulates a hard function.

Theorem 17. Let $k \geq 2$ and let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function. Then at least one of three following propositions is true:

1. f is affine;
2. f supports perfect equality;
3. f simulates a hard function.

Theorem 17 is proved in Section 7. When f simulates a hard function, using Lemma 15, we can immediately conclude that for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard. As we already discussed in Section 3, it is important that, in the case where f does not simulate a hard function, Theorem 17 guarantees that f supports perfect equality (rather than simple imperfect equality); this allows us to recover the connection to #BIS for those $f \in IM_2$. In fact, when f supports perfect equality, we can effectively carry out (a strengthening of) the program in [13] to obtain the following classification which perfectly aligns with Theorem 6.

Theorem 18. Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function that is not affine. Suppose that f supports perfect equality.

1. If f is in IM_2 , then for all sufficiently large Δ , $\#Multi2Spin_\Delta(f)$ is #BIS-equivalent.
2. If f is not in IM_2 , then for all sufficiently large Δ , there exists a real number $c > 1$ such that $\#Multi2Spin_{\Delta,c}(f)$ is NP-hard.

Theorem 18 is proved in Section 8. Thus, Theorems 17 and 18 together achieve the desired classification of $\#Multi2Spin_\Delta(f)$ when $f \in IM_2$ as well as the strong inapproximability results when $f \notin IM_2$. Before delving into the proofs of Theorems 17 and 18 however, it will be instructive to give the main ideas behind the proofs, especially of the more critical Theorem 17.

To prove Theorem 17, our proof departs from the previous approaches in the related works [13] and [14]. In these works, f was used to directly encode a binary function which was feasible because of the presence of equality in [13] and the symmetry of f in [14]. Instead, we take a much more painstaking combinatorial approach by using induction on the arity of the function f .

The base case of the induction (proving Theorem 17 for arity-2 functions) is fairly simple to handle, so let us focus on the induction step. The rough idea, to put the induction hypothesis to work, is to study whether f supports pinning-to-0 or pinning-to-1; then, provided that at least one these pinnings is available, we need to pin appropriately some arguments of f to obtain a function h of smaller arity. Our goal is then to ensure that h is non-affine; then, we can invoke the induction hypothesis and obtain that h either supports perfect equality or simulates a hard function. From there, since h was obtained by pinning some arguments of f , we will obtain by a transitivity argument (cf. Lemma 33) that f either supports perfect equality or f simulates the same hard function as h . (A detail here is that, in the case where h supports perfect equality, to conclude that f supports perfect equality from Lemma 33, we need to ensure that the pinnings of f used to obtain h were perfect.)

Determining which arguments of f need to be pinned is the most challenging aspect of this scheme. Our method for reducing the number of functions under consideration is to symmetrise f in a natural way and obtain a new function f^* which is now symmetric (see Definition 20). Then, it turns out that there are seven possibilities for the function f^* which we need to consider in detail (the functions are given in Definition 22). That is, when the symmetrisation of f is one of these seven functions, we have to figure out whether f supports perfect equality and, if not, work out the combinatorial structure of f and pinpoint which arguments are suitable to be pinned. The details of the argument can be found in Section 7.2.

The proof of Theorem 18, where f supports perfect equality, basically follows the approach of [13]. However, to get the stronger inapproximability results, we have to take a detour studying self-dual functions (functions whose value does not change when we complement their arguments). We show that if f is self-dual then it simulates a hard function (Theorem 46). The problem with self-dual functions is that they do not support pinning-to-0 or pinning-to-1, so we are not able to use the relevant results from [13]. After proving Theorem 46 and doing some preparatory work in Section 8.1 to ensure that “implementations in CSPs” work in the repeat-free setting when f supports perfect equality (see Lemma 42), the techniques of [13] can be adapted to get Theorem 18.

6. Notation and results from the literature

6.1. Notation

For a vector \mathbf{x} , we use x_i to denote the i 'th entry of \mathbf{x} . Further, for vectors \mathbf{x} and \mathbf{y} of the same length, $\mathbf{x} \oplus \mathbf{y}$ will denote the coordinate-wise addition of \mathbf{x} and \mathbf{y} over $GF(2)$. More generally, for any binary Boolean operator \otimes , we will denote by $\mathbf{x} \otimes \mathbf{y}$ the vector whose i -th entry is given by $x_i \otimes y_i$. We will use $\mathbf{0}, \mathbf{1}$ to denote the vectors whose entries are all zeros and all ones, respectively (the length of these vectors will be clear from context). Finally, for a Boolean vector \mathbf{x} , $\bar{\mathbf{x}}$ will denote the coordinate-wise “negation” of \mathbf{x} , i.e., $\bar{\mathbf{x}} = \mathbf{x} \oplus \mathbf{1}$. For a positive integer k , $[k]$ denotes $\{1, \dots, k\}$.

Definition 19 (Ω_f, χ_S). Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$. For $S \subseteq [k]$, χ_S denotes the characteristic vector of S , which is the length- k Boolean vector such that, for all $i \in [k]$, the i -th bit of χ_S is 1 if and only if $i \in S$. Finally, $\Omega_f = \{S \subseteq [k] \mid \chi_S \in R_f\}$, where R_f is the relation underlying f , defined at the beginning of Section 2.

Definition 20 (The symmetrisation f^*). Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$. We denote by f^* the symmetrisation of f obtained as follows. Let S_k denote the set of all permutations $\pi : [k] \rightarrow [k]$. Then $f^* : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$ is the function defined by

$$f^*(x_1, \dots, x_k) = \prod_{\pi \in S_k} f(x_{\pi(1)}, \dots, x_{\pi(k)}).$$

6.2. Affine functions

The following well-known characterisation of affine functions (cf. Definition 4) is instructive and will be useful later. For a proof, see, for example, Lemma 4.10 of [9] or Lemma 11 of [13].

Lemma 21. Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function. Then:

1. f is affine iff for every $\mathbf{a}, \mathbf{b}, \mathbf{c} \in R_f$, it holds that $\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \in R_f$.
2. If f is not affine, then for every $\mathbf{a} \in R_f$, there are $\mathbf{b}, \mathbf{c} \in R_f$ such that $\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \notin R_f$. \square

The set of affine symmetric Boolean functions $f : \{0, 1\}^k \rightarrow \{0, 1\}$ is given by the following set $\text{EASY}(k)$.

Definition 22. For $k \geq 2$, let $\text{EASY}(k)$ be the set containing the following seven functions.

$$\begin{aligned} f_{\text{zero}}^{(k)}(x_1, \dots, x_k) &= 0, & f_{\text{one}}^{(k)}(x_1, \dots, x_k) &= 1, & f_{\text{allzero}}^{(k)}(x_1, \dots, x_k) &= \mathbf{1}\{x_1 = \dots = x_k = 0\}, \\ f_{\text{allone}}^{(k)}(x_1, \dots, x_k) &= \mathbf{1}\{x_1 = \dots = x_k = 1\}, & f_{\text{EQ}}^{(k)}(x_1, \dots, x_k) &= \mathbf{1}\{x_1 = \dots = x_k\}, \\ f_{\text{even}}^{(k)}(x_1, \dots, x_k) &= \mathbf{1}\{x_1 \oplus \dots \oplus x_k = 0\}, & f_{\text{odd}}^{(k)}(x_1, \dots, x_k) &= \mathbf{1}\{x_1 \oplus \dots \oplus x_k = 1\}. \end{aligned}$$

6.3. A characterisation of IM_2

In the language of universal algebra, Creignou, Kolaitis, and Zanuttini [9] have shown that IM_2 (see Definition 5) is precisely the “co-clone” corresponding to the “clone” M_2 in Post’s lattice (see [2]). Defining clones and co-clones would be a bit of a distraction from this paper, but the only fact that we need is the following (which follows directly from the definitions of clones and co-clones and from the fact that IM_2 is the co-clone corresponding to M_2).

Lemma 23. Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function. Then, the function f is in IM_2 iff for every $\mathbf{x}, \mathbf{y} \in R_f$ it holds that $\mathbf{x} \vee \mathbf{y} \in R_f$ and $\mathbf{x} \wedge \mathbf{y} \in R_f$. \square

6.4. The case where f is symmetric: extensions to the asymmetric case

In this section, we state a few results from [14] which were stated for the case where f is a symmetric Boolean function, but whose proof works just as well even when f is asymmetric.

The following lemma, which is Lemma 12 of [14], gives sufficient conditions for pinning-to-0, pinning-to-1 and equality. The statement of the lemma in [14] is restricted to symmetric functions f , but the proof applies to all functions (with the trivial modification that the vertices in the hyperarcs in the constructed k -tuple hypergraph H must be ordered appropriately).

Lemma 24 ([14, Lemma 12]). Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$ and let H be a k -tuple hypergraph.

1. If there is a vertex v in H such that $\mu_{f;H}(\sigma_v = 0) > \mu_{f;H}(\sigma_v = 1)$, then f supports pinning-to-0.
2. If there is a vertex v in H such that $\mu_{f;H}(\sigma_v = 1) > \mu_{f;H}(\sigma_v = 0)$, then f supports pinning-to-1.
3. If there are vertices x, y in H such that $\mu_{f;H}(\sigma_x = \sigma_y = 0) = \mu_{f;H}(\sigma_x = \sigma_y = 1)$ and $\mu_{f;H}(\sigma_x = \sigma_y) > \mu_{f;H}(\sigma_x \neq \sigma_y)$, then f supports equality. \square

Lemma 25 ([14, Lemma 17]). Let $f : \{0, 1\}^k \rightarrow \mathbb{R}_{\geq 0}$. Let $H = (V, \mathcal{F})$ be a k -tuple hypergraph and let S be a subset of V . Let \mathcal{V} be admissible for H with respect to f . Then, for every $\epsilon > 0$, there is a k -tuple hypergraph $H' = (V', \mathcal{F}')$ with $V \subseteq V'$ and $\mathcal{F} \subseteq \mathcal{F}'$ such that, for every $\tau : S \rightarrow \{0, 1\}$, it holds that

$$|\mu_{f;H'}(\sigma_S = \tau) - \mu_{f;H}^{\text{cond}(\mathcal{V})}(\sigma_S = \tau)| \leq \epsilon,$$

where $\mu_{f;H}^{\text{cond}(\mathcal{V})}(\cdot)$ is as in Definition 12. \square

We will also use the following result from [14] which applies to symmetric Boolean functions.

Lemma 26 ([14, Proof of Theorem 3]). Let $k \geq 2$ and let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a symmetric Boolean function such that $f \notin \text{EASY}(k)$. Then either f simulates a hard function or f supports perfect equality (or both).

Proof. We briefly overview the proof in [14], the relevant parts are in [14, Section 4].

1. [14, Lemma 13] shows that every function $f \notin \text{EASY}(k)$ supports one of pinning-to-0, pinning-to-1 or equality.
2. In [14, Section 4.1], the case where f supports both pinning-to-0 and pinning-to-1 is considered. Then, [14] shows that f simulates a hard function.

3. In [14, Section 4.2], the case where f supports equality but neither pinning-to-0 nor pinning-to-1 is considered. The proof splits into cases depending on whether $f(\mathbf{0}) = 0$ or $f(\mathbf{0}) = 1$. When $f(\mathbf{0}) = 0$ ([14, Section 4.2.2]), [14] shows that f supports perfect equality ([14, Lemma 28]). When $f(\mathbf{0}) = 1$ ([14, Section 4.2.3]), [14] shows that f simulates a hard function.
4. In [14, Section 4.3], the case where f supports pinning-to-0 is considered. Then, [14] shows that f simulates a hard function. (The case where f supports pinning-to-1 is identical by switching the spins 0 and 1.)

Thus, for every symmetric function $f : \{0, 1\}^k \rightarrow \{0, 1\}$ such that $f \notin \text{EASY}(k)$, the results of [14] show that either f simulates a hard function or f supports perfect equality. \square

7. Non-affine Boolean functions either support perfect equality or simulate a hard function

In this section, we prove Theorem 17, i.e., that every non-affine Boolean function either supports perfect equality or simulates a hard function. Before proving the theorem, we will need a few technical lemmas.

7.1. A few preparatory lemmas

Lemma 27. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f^* = f_{\text{allone}}$. Then f supports perfect pinning-to-1.*

Proof. Let $H = (V, \mathcal{F})$ be the k -tuple hypergraph with $V = \{v_1, v_2, \dots, v_k\}$ and $\mathcal{F} = \{e_\pi \mid \pi \in S_k\}$ where $e_\pi = (v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)})$. Since $f^* = f_{\text{allone}}$, we have that for all $\sigma : V \rightarrow \{0, 1\}$ it holds that $w_{f,H}(\sigma) > 0$ if and only if $\sigma(v_1) = \sigma(v_2) = \dots = \sigma(v_k) = 1$. Thus, f supports perfect pinning-to-1. \square

Completely analogously, we have the following pinning lemma when $f^* = f_{\text{allzero}}$.

Lemma 28. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f^* = f_{\text{allzero}}$. Then f supports perfect pinning-to-0. \square*

For any function f such that $f^* = f_{\text{zero}}$, we have the following pinning lemma.

Lemma 29. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f \neq f_{\text{zero}}$ and $f^* = f_{\text{zero}}$. Then at least one of the following two propositions is true:*

1. f supports perfect pinning-to-0 and perfect pinning-to-1;
2. f supports perfect equality.

Proof. Note that the conditions in the lemma imply that $k \geq 2$. Let S_k denote the set of all permutations $\pi : [k] \rightarrow [k]$ and let $\text{id} \in S_k$ denote the identity permutation. For any subset $A \subseteq S_k$, let f_A be the function defined by $f_A(w_1, \dots, w_k) := \prod_{\pi \in A} f(w_{\pi(1)}, \dots, w_{\pi(k)})$. Note that $f_{S_k} = f^* = f_{\text{zero}}$. Also, for any $\pi \in S_k$ we have $f_{\{\pi\}} \neq f_{\text{zero}}$ (since $f \neq f_{\text{zero}}$). By iteratively removing permutations from S_k we will thus obtain a subset $T \subseteq S_k$ with $|T| > 1$ such that $f_T = f_{\text{zero}}$ and, for every $\pi \in T$, it holds that $f_{T \setminus \{\pi\}} \neq f_{\text{zero}}$. By renaming the variables if necessary, we may assume that $\text{id} \in T$.

Let $H_0 = (V_0, \mathcal{F}_0)$ be the k -tuple hypergraph with vertex set $V_0 = \{x_1, \dots, x_k\}$ and hyperarc set $\mathcal{F}_0 = \cup_{\pi \in T \setminus \{\text{id}\}} \{(x_{\pi(1)}, \dots, x_{\pi(k)})\}$. By the choice of the set T , we have that $Z_{f;H_0} > 0$. For $i = 1, \dots, k$, let $H_i = (V_i, \mathcal{F}_i)$ be the k -tuple hypergraph with vertex set $V_i = V_0 \cup \{y_{i+1}, \dots, y_k\}$ and hyperarc set $\mathcal{F}_i = \mathcal{F}_0 \cup \{(x_1, \dots, x_i, y_{i+1}, \dots, y_k)\}$. Again by the choice of the set T we have that $Z_{f;H_k} = 0$. Thus, there exists $0 \leq j < k$ such that $Z_{f;H_j} > 0$ and $Z_{f;H_{j+1}} = 0$. Note that for every assignment $\sigma : V_j \rightarrow \{0, 1\}$ with $w_{f;H_j}(\sigma) > 0$ it holds that $\sigma(x_{j+1}) \neq \sigma(y_{j+1})$; otherwise, for the assignment $\sigma' = \sigma_{V_{j+1}}$ (i.e., the restriction of the assignment σ to the set V_{j+1}), it would hold that $w_{f;H_{j+1}}(\sigma') > 0$, contradicting that $Z_{f;H_{j+1}} = 0$.

For $s_1, s_2 \in \{0, 1\}$, let

$$Z_{s_1, s_2} := \sum_{\substack{\sigma : V_j \rightarrow \{0, 1\}; \\ \sigma(x_{j+1}) = s_1, \sigma(y_{j+1}) = s_2}} w_{f;H_j}(\sigma)$$

By the argument above, we have that $Z_{00} = Z_{11} = 0$. Since $Z_{f;H_j} > 0$, we have that at least one of Z_{01} and Z_{10} is non-zero. In fact, we may assume that both are non-zero, since otherwise f supports both perfect pinning-to-0 and perfect pinning-to-1 so proposition 1 in the statement of the lemma is satisfied (for example, if $Z_{10} = 0$, then $\mu_{f;H_j}(\sigma(x_{j+1}) = 0) = 1$ and $\mu_{f;H_j}(\sigma(y_{j+1}) = 1) = 1$).

Let J_1, J_2 be two disjoint copies of H_j . Denote by u_1, u_2 the vertices corresponding to x_{j+1} in J_1, J_2 , respectively. Also, denote by v_1, v_2 the vertices corresponding to y_{j+1} in J_1, J_2 . Let $J = (V, \mathcal{F})$ be the k -tuple hypergraph obtained by taking the union of J_1 and J_2 and identifying the vertices u_2 and v_1 into a new vertex w (i.e., we merge the vertex corresponding to x_{j+1} in J_2 and the vertex corresponding to y_{j+1} in J_1).

For $s_1, s_2 \in \{0, 1\}$, let

$$Z'_{s_1, s_2} := \sum_{\substack{\sigma: V \rightarrow \{0,1\}; \\ \sigma(u_1)=s_1, \sigma(v_2)=s_2}} w_{f;J}(\sigma).$$

By considering the spin of the vertex w , we obtain that

$$Z'_{s_1, s_2} = Z_{s_1,0} Z_{0, s_2} + Z_{s_1,1} Z_{1, s_2},$$

which gives that

$$Z'_{00} = Z_{01} Z_{10}, \quad Z'_{01} = 0, \quad Z'_{10} = 0, \quad Z'_{11} = Z_{10} Z_{01}.$$

Since $Z_{01}, Z_{10} \neq 0$, we obtain that

$$\mu_{f;J}(\sigma(u_1) = \sigma(v_2) = 0) = \mu_{f;J}(\sigma(u_1) = \sigma(v_2) = 1) = \frac{1}{2},$$

and hence f supports perfect equality. \square

For any function f , we can show that if f^* is f_{EQ} , f_{odd} or f_{even} then f supports perfect equality.

Lemma 30. Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f^* = f_{EQ}$. Then f supports perfect equality.

Proof. Let $H = (V, \mathcal{F})$ be the k -tuple hypergraph with $V = \{v_1, v_2, \dots, v_k\}$ and hyperarc set $\mathcal{F} = \{e_\pi \mid \pi \in S_k\}$ where $e_\pi = (v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)})$. Since $f^* = f_{EQ}$, we have that for all $\sigma : V \rightarrow \{0, 1\}$ it holds that $w_{f,H}(\sigma) > 0$ iff $\sigma(v_1) = \sigma(v_2) = \dots = \sigma(v_k) = 1$ or $\sigma(v_1) = \sigma(v_2) = \dots = \sigma(v_k) = 0$. Thus, f supports perfect equality. \square

Lemma 31. Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f^* \in \{f_{odd}, f_{even}\}$. Then f supports perfect equality.

Proof. Let $H = (V, \mathcal{F})$ be the k -tuple hypergraph with $V = \{v_1, v_2, \dots, v_{k+1}\}$ and $\mathcal{F} = \{e_\pi \mid \pi \in S_k\} \cup \{e'_\pi \mid \pi \in S_k\}$ where $e_\pi = (v_{\pi(1)}, v_{\pi(2)}, \dots, v_{\pi(k)})$ and $e'_\pi = (v_{\pi(1)+1}, v_{\pi(2)+1}, \dots, v_{\pi(k)+1})$ (note that H has $k + 1$ vertices and $2k!$ hyperarcs). Since f^* is either f_{odd} or f_{even} , for all $\sigma : V \rightarrow \{0, 1\}$ with $w_{f,H}(\sigma) > 0$, we have that the parity of number of ones among $\sigma(v_1), \sigma(v_2), \dots, \sigma(v_k)$ and the parity of number of ones among $\sigma(v_2), \sigma(v_3), \dots, \sigma(v_{k+1})$ must be the same and thus $\sigma(v_1) = \sigma(v_{k+1})$. Furthermore, for $s \in \{0, 1\}$, there are exactly 2^{k-1} assignments $\sigma : V \rightarrow \{0, 1\}$ such that $w_{f,H}(\sigma) > 0$, $\sigma(v_1) = \sigma(v_{k+1})$ and $\sigma(v_2) \oplus \sigma(v_3) \oplus \dots \oplus \sigma(v_k) = s$. It follows that

$$\mu_{f;H}(\sigma(v_1) = \sigma(v_{k+1}) = 0) = \mu_{f;H}(\sigma(v_1) = \sigma(v_{k+1}) = 1) = \frac{1}{2},$$

which means that f supports perfect equality. \square

By the above lemmas, we can show that some functions can be either dealt with directly, or reduced to other functions with smaller arity.

Definition 32. For $s \in \{0, 1\}$, let $\delta_s : \{0, 1\} \rightarrow \{0, 1\}$ be the Boolean function defined by $\delta_s(s) = 1$ and $\delta_s(1 \oplus s) = 0$. Define $f_{i \rightarrow s}$ to be the function obtained from f by pinning the i -th argument of f to s , i.e.

$$f_{i \rightarrow s}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_k) = \sum_{x_i \in \{0,1\}} f(x_1, \dots, x_k) \cdot \delta_s(x_i).$$

Similarly, for disjoint $S, T \subseteq [k]$, let $f_{S \rightarrow 0, T \rightarrow 1}$ be the $(k - |S \cup T|)$ -ary function obtained from f by pinning the arguments in S to 0 and the arguments in T to 1 so if \mathbf{x}' denotes the $|S \cup T|$ -ary vector containing all x_i with $i \in S \cup T$ and \mathbf{x}'' denotes the $k - |S \cup T|$ -ary vector containing all x_i with $i \in [k] \setminus S \cup T$,

$$f_{S \rightarrow 0, T \rightarrow 1}(\mathbf{x}'') = \sum_{\mathbf{x}' \in \{0,1\}^{|S \cup T|}} f(x_1, \dots, x_k) \cdot \prod_{i \in S} \delta_0(x_i) \cdot \prod_{j \in T} \delta_1(x_j).$$

If $S = \emptyset$ or $T = \emptyset$, we will omit $S \rightarrow 0$ or $T \rightarrow 1$ from the notation.

Using Definition 32, we have the following lemma:

Lemma 33. Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function. Suppose that S_0 and S_1 are disjoint subsets of $[k]$ such that, for $a \in \{0, 1\}$, S_a is empty if f does not support perfect pinning-to- a . Let $h = f_{S_0 \rightarrow 0, S_1 \rightarrow 1}$.

1. If h supports equality, then f also supports equality. Further, if h supports perfect equality, then f also supports perfect equality.
2. If h supports pinning-to- s for some $s \in \{0, 1\}$, then f also supports pinning-to- s .
3. If h simulates a function $g : \{0, 1\}^2 \rightarrow \mathbb{R}_{\geq 0}$ that is not $f_{\text{zero}}^{(2)}$ then f simulates g as well. Also, if h perfectly simulates g then f perfectly simulates g as well.

Proof. Without loss of generality, we assume that the arity of h is n , and that $S_0 \cup S_1 = \{n + 1, n + 2, \dots, k\}$. For each $a \in \{0, 1\}$, if S_a is non-empty, then by assumption f supports perfect pinning-to- a , so there exists a k -tuple hypergraph $H_a = (V_a, \mathcal{F}_a)$ with a vertex $w_a \in V_a$ such that $\mu_{f; H_a}(\sigma_{w_a} = a) = 1$.

We now give a general construction which takes any n -tuple hypergraph $H = (V, \mathcal{F})$ and produces a new k -tuple hypergraph $H' = (V', \mathcal{F}')$. To do this, we take $k - n$ new vertices v'_{n+1}, \dots, v'_k that are not in V and let $V' = V \cup \{v'_{n+1}, \dots, v'_k\}$. The hyperarcs of H' are in one-to-one correspondence with those in H : For each hyperarc (u_1, u_2, \dots, u_n) in H , we add the hyperarc $(u_1, u_2, \dots, u_n, v'_{n+1}, v'_{n+2}, \dots, v'_k)$ to H' . Moreover, for $i \in S_0$, add a distinct copy of H_0 to H' by identifying v'_i with the vertex w_0 in H_0 . Also, for $i \in S_1$, add a distinct copy of H_1 to H' by identifying v'_i with the vertex w_1 in H_1 .

Say that an assignment $\sigma : V' \rightarrow \{0, 1\}$ is *relevant* if, for each $a \in \{0, 1\}$ and each $i \in S_a$, $\sigma(v_i) = a$. The copies of H_0 and H_1 ensure that, for every assignment $\sigma : V' \rightarrow \{0, 1\}$ with $w_{f; H'}(\sigma) > 0$, σ is relevant. The definition of h ensures that, for any relevant assignment σ ,

$$w_{f; H'}(\sigma) = w_{h; H}(\sigma_V). \tag{2}$$

We now use (2) to establish the three items in the statement of the lemma.

1. Suppose that h supports equality. For any $\epsilon \in (0, 1)$, there is an n -tuple hypergraph $H = (V, \mathcal{F})$ and two vertices x and y of H such that, for every $s \in \{0, 1\}$, $\mu_{h; H}(\sigma_x = \sigma_y = s) \geq (1 - \epsilon)/2$. Construct H' from H using the general construction above. From (2), we conclude that, for any $s \in \{0, 1\}$, $\mu_{f; H'}(\sigma_x = \sigma_y = s) = \mu_{h; H}(\sigma_x = \sigma_y = s) \geq (1 - \epsilon)/2$, so f supports equality. If h supports perfect equality, then we can take $\epsilon = 0$ in this argument, obtaining the conclusion that f also supports perfect equality.
2. Suppose that h supports pinning-to- s . For any $\epsilon \in (0, 1)$ there is an n -tuple hypergraph $H = (V, \mathcal{F})$ and a vertex x of H such that $\mu_{h; H}(\sigma_x = s) \geq 1 - \epsilon$. Construct H' from H using the general construction above. From (2), we conclude that, $\mu_{f; H'}(\sigma_x = s) = \mu_{h; H}(\sigma_x = s) \geq 1 - \epsilon$, so f supports pinning-to- s .
3. Let $g : \{0, 1\}^2 \rightarrow \mathbb{R}_{\geq 0}$ be a function that is not $f_{\text{zero}}^{(2)}$. Suppose first that h simulates g . By the definition of “simulates”, there exists an n -tuple hypergraph H with admissible \mathcal{V} (with respect to h) and two vertices u and v in H such that, for every $s, t \in \{0, 1\}$, it holds that

$$\mu_{h; H}^{\text{cond}(\mathcal{V})}(\sigma(u) = s, \sigma(v) = t) = \frac{g(s, t)}{\sum_{i, j \in \{0, 1\}} g(i, j)}. \tag{3}$$

Since $g \neq f_{\text{zero}}^{(2)}$, the expression in (3) is well-defined.

Construct H' from H using the general construction above. From Items 1 and 2 of the lemma, if h supports equality or pinning-to-0 or pinning-to-1 then so does f . Thus, \mathcal{V} is admissible for H' with respect to f . It follows from (2) that

$$\begin{aligned} \mu_{h; H}^{\text{cond}(\mathcal{V})}(\sigma(u) = s, \sigma(v) = t) &= \mu_{f; H'}^{\text{cond}(\mathcal{V})}(\sigma(u) = s, \sigma(v) = t \mid \wedge_{i \in S_0} \sigma(v_i) = 0, \wedge_{i \in S_1} \sigma(v_i) = 1) \\ &= \mu_{f; H'}^{\text{cond}(\mathcal{V})}(\sigma(u) = s, \sigma(v) = t), \end{aligned}$$

so, using (3), we obtain that f simulates g , as wanted. If h perfectly simulates g then we can take $\mathcal{V} = (\emptyset, \emptyset, \emptyset)$, so the argument shows that f perfectly simulates g . \square

7.2. Proof that every non-affine Boolean functions either supports perfect equality or simulates a hard function

Definition 34. A function $f : \{0, 1\}^k \rightarrow \{0, 1\}$ is *semi-trivial* if and only if there exists a set $S \subseteq [k]$ such that $\Omega_f = \{T \mid S \subseteq T \subseteq [k]\}$ or $\Omega_f = \{T \mid T \subseteq S\}$.

Remark 35. Every semi-trivial function f is affine since R_f equals the solution set of the system of equations of the form $\{x_i = 1\}_{i \in S}$ or $\{x_i = 0\}_{i \in S}$ where $S \subseteq [k]$ is as in Definition 34.

Lemma 36. Let $k \geq 2$ and let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f \neq f_{\text{allone}}$ and $f^* = f_{\text{allone}}$. Let S be a set in Ω_f such that $S \neq [k]$. Then at least one of the following propositions is true:

1. $\forall T \supseteq S$, we have $T \in \Omega_f$;
2. f supports perfect equality;
3. f simulates a hard function.

Proof. Without loss of generality (by re-numbering the variables), let $S = \{n + 1, n + 2, \dots, k\}$ for some integer $n \geq 1$. By Lemma 27, f supports perfect pinning-to-1. Let $h(x_1, x_2, \dots, x_n) = f(x_1, \dots, x_k)_{S \rightarrow 1}$. Note that $h^*(\mathbf{0}) = 1$ and $h^*(\mathbf{1}) = 1$. We may assume that $n \geq 2$ (otherwise $\forall T \supseteq S$, we have $T \in \Omega_f$).

Case 1. $h^* \notin \text{EASY}(n)$. In this case, Lemma 26 ensures that either h simulates a hard function or h supports perfect equality (or both). If h simulates a hard function, then by Item 3 of Lemma 33, f also simulates a hard function. If h supports perfect equality, then, by Item 1 of Lemma 33, f also supports perfect equality.

Case 2. $h^* \in \text{EASY}(n)$. Then $h^* \in \{f_{\text{one}}, f_{\text{even}}, f_{\text{EQ}}\}$ since $h^*(\mathbf{0}) = 1$ and $h^*(\mathbf{1}) = 1$. If $h^* = f_{\text{one}}$, we have that $h(\mathbf{x}) = 1$ for all $\mathbf{x} \in \{0, 1\}^n$. Since $h(\mathbf{x}) = f_{S \rightarrow 1}$, we obtain that $T \in \Omega_f$ for all $T \supseteq S$.

If $h^* \in \{f_{\text{even}}, f_{\text{EQ}}\}$, then h supports perfect equality by Lemmas 30 and 31. Since f supports perfect pinning-to-1, by Item 1 of Lemma 33 we obtain that f supports perfect equality as well.

This concludes the proof. \square

Lemma 37. Let $k \geq 2$ and let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f \neq f_{\text{allone}}$ and $f^* = f_{\text{allone}}$. Then at least one of the four following propositions is true:

1. f is semi-trivial;
2. there exists $t \in [k]$ such that $f_{t \rightarrow 1}$ is not affine;
3. f supports perfect equality;
4. f simulates a hard function.

Proof. If $k = 2$, since $f \neq f_{\text{allone}}$ and $f^* = f_{\text{allone}}$, we have $f(1, 1) = 1$, $f(0, 0) = 0$ and exactly one of $f(0, 1)$ and $f(1, 0)$ is one, so f is semi-trivial. Thus, for the rest of the proof we may assume that $k \geq 3$.

Since $f^* = f_{\text{allone}}$, we have that $f(\mathbf{1}) = 1$ and $f(\mathbf{0}) = 0$. Further, since $f \neq f_{\text{allone}}$, there exists $S \in \Omega_f$ with $|S| < k$.

Case 1. Every $S \in \Omega_f$ satisfies $|S| \geq k - 1$.

If there is only one set S in Ω_f with $|S| = k - 1$, then we have that f is semi-trivial (since $f(\mathbf{1}) = 1$). Otherwise, there are distinct sets $S, S' \in \Omega_f$ with $|S| = |S'| = k - 1$, so $|S \cap S'| = k - 2$ and thus $S \cap S' \neq \emptyset$ and $S \cap S' \notin \Omega_f$. Let $t \in S \cap S'$. We claim that $h = f_{t \rightarrow 1}$ is not affine; to see this, note that $f(\chi_S) = f(\chi_{S'}) = f(\chi_{[k]}) = 1$ and $f(\chi_S \oplus \chi_{S'} \oplus \chi_{[k]}) = f(\chi_{S \cap S'}) = 0$. Since $h = f_{t \rightarrow 1}$ and $t \in S \cap S'$, we obtain that

$$S \setminus \{t\}, S' \setminus \{t\}, [k] \setminus \{t\} \in \Omega_h \text{ but } (S \cap S') \setminus \{t\} \notin \Omega_h.$$

By Item 1 of Lemma 21, it thus follows that h is not affine, as wanted.

Case 2. There exists $S \in \Omega_f$ with $|S| \leq k - 2$.

Let S be a set in Ω_f with the smallest cardinality among the sets in Ω_f . By Lemma 36, either f satisfies proposition 3 or 4, in the statement of the lemma (so we are finished), or every $Q \supseteq S$ satisfies $Q \in \Omega_f$. Thus, for the rest of the proof we may assume that for every $Q \supseteq S$ it holds that $Q \in \Omega_f$.

Let $\Psi = \{W \in \Omega_f \mid S \setminus W \neq \emptyset\}$. If Ψ is empty then f is semi-trivial, so it satisfies proposition 1 in the statement of the lemma (and we are finished). So assume that Ψ is non-empty and choose $T \in \Psi$ with cardinality as small as possible. By the choice of S, T cannot be a strict subset of S , so $T \setminus S$ is not empty. Applying Lemma 36 to the set T , we may assume that $\forall Q \supseteq T$ it holds that $Q \in \Omega_f$ (otherwise, f will satisfy proposition 3 or 4, in the statement of the lemma, so we are finished). Since $f(\mathbf{0}) = 0$ and S has minimum cardinality among the sets in Ω_f , we have $1 \leq |S| \leq |T|$.

Case 2a. $|T| = 1$, which implies $|S| = 1$. Suppose $S = \{s\}$ and $T = \{t\}$. Consider a set $Q \subseteq [k]$ with $|Q| = k - 2$. By the above assumptions, we have that if $s \in Q$ or $t \in Q$ then $Q \in \Omega_f$. This accounts for all but one sets $Q \subseteq [k]$ with $|Q| = k - 2$; for the remaining set $Q = [k] \setminus \{s, t\}$, it must be the case that $Q \notin \Omega_f$, otherwise all sets Q with $|Q| = k - 2$ are in Ω_f , which contradicts the fact that $f^* = f_{\text{allone}}$. Now let's consider Ω_f . The number of sets $W \in \Omega_f$ which contain both s and t is 2^{k-2} . Similarly, the number of sets $W \in \Omega_f$ which contain s but not t is 2^{k-2} and the number of sets $W \in \Omega_f$ which contain t but not s is 2^{k-2} . But the number of sets $W \in \Omega_f$ which contain neither s nor t is less than 2^{k-2} . So the k -tuple hypergraph with the single hyperarc (v_1, \dots, v_k) induces a hard function on the two vertices v_s and v_t and therefore f simulates a hard function.

Case 2b. $|T| \geq 2$ and $S \cap T \neq \emptyset$. Since $S \setminus T \neq \emptyset$, we have $|S| > |S \cap T|$ and thus $S \cap T \notin \Omega_f$ by the minimality of S . Let $r \in S \cap T$. Now we know that $S \in \Omega_f, T \in \Omega_f$ and $S \cup T \in \Omega_f$ by the assumptions above. But $S \cap T \notin \Omega_f$ and $\chi_S \oplus \chi_T \oplus \chi_{S \cup T} = \chi_{S \cap T}$, so by Item 1 of Lemma 21, $f_{r \rightarrow 1}$ is not affine.

Case 2c. $|T| \geq 2$ and $S \cap T = \emptyset$. Since $T \setminus S \neq \emptyset$, let $r \in T \setminus S$. By the above assumptions, we have that $S \cup \{r\}$ and $S \cup T$ are in Ω_f . Note that $\{r\} \notin \Omega_f$; otherwise, we would obtain a contradiction to the choice of the set T , since

$T' = \{r\}$ satisfies $T' \in \Omega_f$, $S \setminus T' = S \neq \emptyset$ and $|T'| < |T|$. Now we know that $S \cup \{r\}$, T , $S \cup T \in \Omega_f$ and $\{r\} \notin \Omega_f$. Note that since $S \cap T = \emptyset$, it holds that $\chi_{S \cup \{r\}} \oplus \chi_T \oplus \chi_{S \cup T} = \chi_{\{r\}}$, so by Item 1 of Lemma 21 we have that $f_{r \rightarrow 1}$ is not affine.

This concludes the proof of Lemma 37. \square

Similarly, by switching the spins 0 and 1, we obtain the following lemma when $f^* = f_{\text{allzero}}$.

Lemma 38. *Let $k \geq 2$ and let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f \neq f_{\text{allzero}}$ and $f^* = f_{\text{allzero}}$. Then at least one of the four following propositions is true:*

1. f is semi-trivial;
2. there exists $t \in [k]$ such that $f_{t \rightarrow 0}$ is not affine;
3. f supports perfect equality;
4. f simulates a hard function.

Proof. Suppose f is a Boolean function such that $f^* = f_{\text{allzero}}$ and $f \neq f_{\text{allzero}}$. Let g be the function defined by $g(\mathbf{x}) = f(\bar{\mathbf{x}})$ for all $\mathbf{x} \in \{0, 1\}^k$. Now it holds that $g^* = f_{\text{allone}}$ and $g^* \neq f_{\text{allone}}$. So g satisfies one of the four propositions in Lemma 37. We then have

1. If g is semi-trivial, f is semi-trivial.
2. If $g_{t \rightarrow 1}$ is not affine for some $t \in [k]$, $f_{t \rightarrow 0}$ is not affine either.
3. If g supports perfect equality, f supports perfect equality too.
4. If g simulates a hard function, f simulates the bitwise complement of the hard function, which is also hard. \square

For every function f such that $f^* = f_{\text{zero}}$ and $f \neq f_{\text{zero}}$, we still have a similar reduction lemma, but the proof is more complicated.

Lemma 39. *Let $k \geq 3$ and let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f^* = f_{\text{zero}}$ and $f \neq f_{\text{zero}}$. Let $S \in \Omega_f$. Then, at least one of the four following propositions is true:*

1. $h = f_{\bar{S} \rightarrow 0}$ is semi-trivial;
2. there exists $T \subseteq [k]$ such that $f_{\bar{S} \rightarrow 0, T \rightarrow 1}$ is not affine;
3. f supports perfect equality;
4. f simulates a hard function.

Proof. By Lemma 29, we have that either f supports perfect equality or f supports both perfect pinning-to-0 and perfect pinning-to-1. We assume that the latter holds (otherwise we are done).

Let $h = f_{\bar{S} \rightarrow 0}$. Since $f(\mathbf{0}) = f^*(\mathbf{0}) = 0$ and $S \in \Omega_f$, we have that $h(\mathbf{0}) = 0$ and $h(\mathbf{1}) = 1$. Note that h has arity $q := |S|$. We may assume that $q > 1$; otherwise, h is semi-trivial (proposition 1 in the statement of the lemma). There are two cases to consider: $h^* \notin \text{EASY}(q)$ or $h^* \in \text{EASY}(q)$.

- Case 1. $h^* \notin \text{EASY}(q)$.

Case 1a. $q = 2$. In this case, $h^*(0, 0) = 0$ and $h^*(0, 1) = h^*(1, 0) = h^*(1, 1) = 1$, so $h^* = \text{OR}$ which is a hard function. We have already assumed (in the first line of the proof) that f supports perfect pinning-to-0. Also, by definition, h perfectly simulates itself. By Item 3 of Lemma 33, f perfectly simulates h as well, so f simulates a hard function (proposition 4 in the statement of the lemma).

Case 1b. $q > 2$. By Lemma 26, either h simulates a hard function or h supports perfect equality (or both). If h simulates a hard function then by Item 3 of Lemma 33, f simulates the same hard function (proposition 4 in the statement of the lemma). On the other hand, if h supports perfect equality then by Item 1 of Lemma 33 so does f (proposition 3 in the statement of the lemma).

- Case 2. $h^* \in \text{EASY}(q)$. Since $h(\mathbf{0}) = 0$ and $h(\mathbf{1}) = 1$, we have that h^* is f_{odd} or f_{allone} .

Case 2a. $h^* = f_{\text{odd}}$. By Lemma 31, h supports perfect equality and thus f supports perfect equality by Item 1 of Lemma 33 (proposition 3 in the statement of the lemma).

Case 2b. $h^* = f_{\text{allone}}$. If $h = f_{\text{allone}}$, h is semi-trivial (proposition 1 in the statement of the lemma). Otherwise, note that $q > 1$, so by Lemma 37, h is semi-trivial (proposition 1 in the statement of the lemma), or there exists $t \in [k]$ such that $h_{t \rightarrow 1}$ is not affine or h supports perfect equality or h simulates a hard function. If there exists $t \in [k]$ such that $h_{t \rightarrow 1}$ is not affine then taking $T = \{t\}$, f satisfies proposition 2 in the statement of the lemma. Finally, if h supports perfect equality then so does f (like Case 2a) and if h simulates a hard function, then so does f (like Case 1b). \square

Lemma 40. Let $k \geq 2$ and let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a k -ary Boolean function. Suppose that $f \neq f_{\text{zero}}$ and $f^* = f_{\text{zero}}$. Then at least one of the four following propositions is true:

1. f is affine;
2. there exist $S, T \subseteq [k]$ such that $f_{S \rightarrow 0, T \rightarrow 1}$ is not affine;
3. f supports perfect equality;
4. f simulates a hard function.

Proof. If $k = 2$, we have $f(0, 0) = f(1, 1) = 0$, so $|\Omega_f| \leq 2$ and thus f is affine (cf. Item 1 of Lemma 21) so it satisfies proposition 1 in the statement of the lemma.

Now suppose $k \geq 3$. By Lemma 39, we can assume that for all $W \in \Omega_f$, $f_{\overline{W} \rightarrow 0}$ is a semi-trivial function (otherwise f satisfies at least one of propositions 2, 3 or 4).

Choose $S \in \Omega_f$ such that $|S|$ is as large as possible. Let $h = f_{\overline{S} \rightarrow 0}$. Since h is semi-trivial (by taking $W = S$ above), we claim that there is a T satisfying $\emptyset \subset T \subseteq S$ such that $\Omega_h = \{U \mid T \subseteq U \subseteq S\}$. (To see this, note that the definition of semi-trivial implies that there is a subset T of S such that either $\Omega_h = \{U \mid U \subseteq T\}$ or $\Omega_h = \{U \mid T \subseteq U \subseteq S\}$. The former is impossible since $\emptyset \notin \Omega_h$ since $h(\mathbf{0}) = f(\mathbf{0})$ and $f(\mathbf{0}) = 0$ since $f^* = f_{\text{zero}}$. Also, in the latter case, T is not empty because, once again, $\emptyset \notin \Omega_h$.)

Case 1. Suppose that $\forall X \in \Omega_f, T \subseteq X$: Recall that T is non-empty. Also, for every $i \in T$, $\{i\} \cup \Omega_{f_{i \rightarrow 1}} = \Omega_f$ so either f is affine (proposition 1 in the statement of the lemma) or $f_{i \rightarrow 1}$ is not affine (proposition 2 in the statement of the lemma).

Now, if Case 1 does not hold then there is an $X \in \Omega_f$ such that $T \setminus X$ is non-empty. Since $\Omega_h = \{U \mid T \subseteq U \subseteq S\}$ we conclude that $X \notin \Omega_h$. Since $h = f_{\overline{S} \rightarrow 0}$ we conclude that $X \setminus S$ is non-empty. Thus, the only other case to consider is as follows.

Case 2. Suppose that there is an $X \in \Omega_f$ such that $T \setminus X$ and $X \setminus S$ are both non-empty:

Let $\Psi = \{X \in \Omega_f \mid T \setminus X \neq \emptyset \text{ and } X \setminus S \neq \emptyset\}$. Let $a = \min\{|T \setminus X| : X \in \Psi\}$, and $b = \min\{|X \setminus S| : X \in \Psi \text{ and } |T \setminus X| = a\}$. Choose $R \in \Psi$ with $|T \setminus R| = a$ and $|R \setminus S| = b$.

Now before proceeding, we use the sets S, T and R to partition k .

$$A = \{i \in [k] \mid i \in S, i \in T, i \notin R\},$$

$$B = \{i \in [k] \mid i \in S, i \in T, i \in R\},$$

$$C = \{i \in [k] \mid i \in S, i \notin T, i \notin R\},$$

$$D = \{i \in [k] \mid i \in S, i \notin T, i \in R\},$$

$$E = \{i \in [k] \mid i \notin S, i \notin T, i \notin R\},$$

$$F = \{i \in [k] \mid i \notin S, i \notin T, i \in R\}.$$

It is clear from the definitions that the sets A, B, C, D, E and F are disjoint. Also, since $T \subseteq S$, they partition $[k]$. From the definitions, $A = T \setminus R$ and $F = R \setminus S$ so, by the choice of R , A and F are non-empty. Let $g = f_{C \cup E \rightarrow 0, B \cup D \rightarrow 1}$.

By definition, every element of Ω_g is a subset of $A \cup F$. Also, for $Y \subseteq A \cup F$, “ $Y \in \Omega_g$ ” means the same thing as “ $Y \cup B \cup D \in \Omega_f$ ”. We establish some facts before dividing the analysis into sub-cases.

Fact 1: $A \in \Omega_g$. We have $\Omega_h = \{U \mid T \subseteq U \subseteq S\}$ and $T = A \cup B$ so $A \cup B \cup D \in \Omega_h$. Since $A \cup B \cup D \subseteq S$, this means $A \cup B \cup D \in \Omega_f$. Equivalently, $A \in \Omega_g$.

Fact 2: $F \in \Omega_g$. From the definition of R , $R \in \Omega_f$. Also, $R = B \cup D \cup F$ so $F \cup B \cup D \in \Omega_f$. Equivalently, $F \in \Omega_g$.

Fact 3: If $Y \in \Omega_g$ then either $Y \cap A \in \{\emptyset, A\}$ or $Y \cap F = \emptyset$ (or both). Suppose for contradiction that $\emptyset \subset Y \cap A \subset A$ and $Y \cap F$ is non-empty. Note that $R = B \cup D \cup F$. Let $R' = B \cup D \cup Y$. Note that $T \setminus R = A$ and $T \setminus R' = A \setminus Y \subset A$ so $|T \setminus R'| < |T \setminus R|$. We will show a contradiction to the choice of R by showing that $R' \in \Psi$. First, since $Y \in \Omega_g$, $R' \in \Omega_f$. Also, $T \setminus R' = A \setminus Y$ is non-empty and $R' \setminus S = Y \cap F$ is non-empty.

Fact 4: If $Y \in \Omega_g$ and $Y \cap A = \emptyset$ then $Y \in \{\emptyset, F\}$. Suppose for contradiction that $\emptyset \subset Y \subset F$. As in the proof of Fact 3, let $R' = B \cup D \cup Y$. Note that $T \setminus R = T \setminus R' = A$. Also, $R \setminus S = F$ and $R' \setminus S = Y$ so $|R \setminus S| > |R' \setminus S|$. Once again, we will show a contradiction to the choice of R by showing that $R' \in \Psi$. As in the proof of Fact 3, since $Y \in \Omega_g$, $R' \in \Omega_f$. Also, $T \setminus R'$ is non-empty since $T \setminus R$ is. Finally, $R' \setminus S = Y$, which is non-empty.

Fact 5: If $Y \in \Omega_g$ and $Y \cap F = \emptyset$ then $Y = A$. Since $Y \in \Omega_g$, we have $Y \cup B \cup D \in \Omega_f$. But since $Y \subseteq A$, we have $Y \cup B \cup D \subseteq S$, so $Y \cup B \cup D \in \Omega_h$. Since $\Omega_h = \{U \mid T \subseteq U \subseteq S\}$ we have $T \subseteq Y \cup B \cup D$ so $A \subseteq Y$.

Given Facts 1–5, we have only the following sub-cases.

Case 2a: $\Omega_g = \{A, F\}$. In this case, we will show that f supports perfect equality so it satisfies proposition 3 in the statement of the lemma. Using Lemma 29, we conclude that either f supports perfect equality (in which case we are finished) or f supports perfect pinning-to-0 and also perfect pinning-to-1, which we now assume. Let H_0 be a k -tuple hypergraph, with a vertex u_0 such that $\mu_{f; H_0}(\sigma_{u_0} = 0) = 1$. Let H_1 be a k -tuple hypergraph, with a vertex u_1 such that

$\mu_{f;H_1}(\sigma_{u_1} = 0) = 1$. We have already noted that A is non-empty. Suppose, without loss of generality, that $1 \in A$ (otherwise, we simply re-order the arguments of $[k]$). Now let H' be the k -tuple hypergraph with vertices v_0, v_1, \dots, v_k and hyperarcs (v_0, v_2, \dots, v_k) and (v_1, v_2, \dots, v_k) . Construct H from H' by doing the following:

- For every $i \in C \cup E$, take a new copy of H_0 and identify vertex u_0 with v_i .
- For every $i \in B \cup D$, take a new copy of H_1 and identify vertex u_1 with v_i .

Now since $\Omega_g = \{A, F\}$, $\mu_{f;H}(\sigma(v_0) = \sigma(v_1) = 0) = \mu_{f;H}(\sigma(v_0) = \sigma(v_1) = 1) = 1/2$. Thus, f supports perfect equality, so we have finished Case 2a.

Case 2b: $\exists Y \in \Omega_g$ such that $Y \cap A = A$ and $Y \cap F$ is non-empty. We will show that f satisfies proposition 2 in the statement of the lemma. Specifically, consider some $t \in A$. We will show that $f_{t \rightarrow 1}$ is not affine.

Let $Y' = Y \cap F$ so that $Y = A \cup Y'$. Let

$$S_1 := B \cup D \cup Y = B \cup D \cup A \cup Y', \quad S_2 := A \cup B = T, \quad S_3 := A \cup B \cup C.$$

We claim that $S_1 \setminus \{t\}, S_2 \setminus \{t\}, S_3 \setminus \{t\} \in \Omega_{f_{t \rightarrow 1}}$. Since $t \in S_1, S_2, S_3$ (from $t \in A$), the claim will follow by showing that $S_1, S_2, S_3 \in \Omega_f$. Indeed, since $Y \in \Omega_g$, we have that $S_1 \in \Omega_f$. Also, since $S_2 = T$, we have that $S_2 \in \Omega_h$ so $S_2 \in \Omega_f$. Finally, since $T = A \cup B \subseteq S_3 \subseteq A \cup B \cup C \cup D = S$, we have that $S_3 \in \Omega_h$ so $S_3 \in \Omega_f$.

Let $S' := A \cup B \cup C \cup D \cup Y' = S \cup Y'$ and note that $\chi_{S'} = \chi_{S_1} \oplus \chi_{S_2} \oplus \chi_{S_3}$ (see Section 6.1 for the relevant notation) by the disjointness of A, B, C, D, E, F . Since Y' is non-empty by assumption, we obtain that S' is not in Ω_f by maximality of S . Note that $t \in S'$, so we have that $S' \setminus \{t\} \notin \Omega_{f_{t \rightarrow 1}}$.

To sum up, we have shown that

$$S_1 \setminus \{t\}, S_2 \setminus \{t\}, S_3 \setminus \{t\} \in \Omega_{f_{t \rightarrow 1}} \text{ but } S' \setminus \{t\} \notin \Omega_{f_{t \rightarrow 1}}$$

Since $\chi_{S_1 \setminus \{t\}} \oplus \chi_{S_2 \setminus \{t\}} \oplus \chi_{S_3 \setminus \{t\}} = \chi_{S' \setminus \{t\}}$, by Item 1 of Lemma 21, $f_{t \rightarrow 1}$ is not affine.

This concludes the proof of Lemma 40. \square

Now we can prove Theorem 17, which we restate here for convenience.

Theorem 17. Let $k \geq 2$ and let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function. Then at least one of three following propositions is true:

1. f is affine;
2. f supports perfect equality;
3. f simulates a hard function.

Proof. We prove this Theorem by induction on the arity of f .

- $k = 2$. So $R_f \subseteq \{00, 01, 10, 11\}$. If f is not affine, then $|R_f| = 3$. If $00 \notin R_f$ or $11 \notin R_f$, let G be a graph with two vertices u and v and an edge (u, v) . Then either $\mu_{f;G}(\sigma_u = 1, \sigma_v = 1) = \mu_{f;G}(\sigma_u = 0, \sigma_v = 1) = \mu_{f;G}(\sigma_u = 1, \sigma_v = 0) = \frac{1}{3}$ or $\mu_{f;G}(\sigma_u = 0, \sigma_v = 0) = \mu_{f;G}(\sigma_u = 0, \sigma_v = 1) = \mu_{f;G}(\sigma_u = 1, \sigma_v = 0) = \frac{1}{3}$. So f simulates a hard function. If $01 \notin R_f$ or $10 \notin R_f$, f^* will be f_{EQ} and thus f supports perfect equality by Lemma 30.
- $k \geq 3$. Suppose that for all $2 \leq k' < k$, all k' -ary functions f' satisfy at least one of the three propositions in the statement. We now prove that an arbitrary $f : \{0, 1\}^k \rightarrow \{0, 1\}$ satisfies at least one of the propositions as well. If f is affine, then it satisfies proposition 1 in the statement of the lemma, so we assume that f is not affine, so $f \notin \text{EASY}(k)$. We have the following case analysis.
 - Case 1. $f^* \notin \text{EASY}(k)$. By Lemma 26, f^* either simulates a hard function in which case f simulates the same hard function as well or f^* supports perfect equality in which case f supports perfect equality as well.
 - Case 2. $f^* \in \text{EASY}(k)$. There are 6 sub-cases to consider:
 - Case 2a. $f^* = f_{EQ}$. By Lemma 30, f supports perfect equality.
 - Case 2b. $f^* = f_{\text{odd}}$ or $f^* = f_{\text{even}}$. By Lemma 31, f supports perfect equality.
 - Case 2c. $f^* = f_{\text{allone}}$. By Lemma 27, f supports perfect pinning-to-1. By Lemma 37, f is semi-trivial (and thus f is affine), or f supports perfect equality or simulates a hard function, or there exists $t \in [k]$ such that $f_{t \rightarrow 1}$ is not affine. If $f_{t \rightarrow 1}$ is not affine for some $t \in [k]$, $f_{t \rightarrow 1}$ must support perfect equality or simulate a hard function by the induction hypothesis. So f supports or simulates the same function by Lemma 33.
 - Case 2d. $f^* = f_{\text{allzero}}$. The proof for this case is completely analogous to the case 2c by switching the spins 0 and 1 (cf. Lemma 38).
 - Case 2e. $f^* = f_{\text{one}}$. $f^* = f_{\text{one}}$ means $f(\mathbf{x}) = 1$ for all $\mathbf{x} \in \{0, 1\}^k$, so f is affine.
 - Case 2f. $f^* = f_{\text{zero}}$. By Lemma 29, we have that either f supports perfect equality or f supports both perfect pinning-to-0 and perfect pinning-to-1. We assume that the latter is the case (otherwise we are done). By Lemma 40, f is affine, or f supports perfect equality or simulates a hard function, or there exists some

$S, T \subseteq [k]$ such that $f_{S \rightarrow 0, T \rightarrow 1}$ is not affine. The only case where we aren't immediately finished is the final one. In this case, the arity of $f_{S \rightarrow 0, T \rightarrow 1}$ must be at least 2 since every unary function is affine. Thus, since $f_{S \rightarrow 0, T \rightarrow 1}$ is not affine, it must support perfect equality or simulate a hard function g by the induction hypothesis. Then, f either supports perfect equality or simulates the hard function g by Lemma 33.

This concludes the case analysis and the proof of Theorem 17. \square

8. The case where f supports perfect equality

In this section, we assume that f is not affine but that it supports perfect equality. In this case, due to the presence of perfect equality, we will be able to employ results and techniques from [13] to show Theorem 18.

8.1. Constraint satisfaction problems and implementations

In the introduction to this paper, we illustrated how Boolean relations can implement more complicated interactions by considering the “not-all-equal” relation of arity 3 and using it to “implement” ferromagnetic Ising interactions. At this point, it is useful to make the notion of “implement” more precise. There are various notions in the literature of *implementations*. We use (a generalisation of) the notion from [13], which is essentially the “faithful, perfect” variant of “implementation” from [8].

Definition 41. Let Γ be a Boolean constraint language. The language Γ *implements* a t -ary function $g : \{0, 1\}^t \rightarrow \mathbb{R}_{\geq 0}$, if for some $t' \geq t$ there is a CSP instance I with variables $x_1, \dots, x_{t'}$ and constraint language Γ such that for every tuple $(s_1, \dots, s_t) \in \{0, 1\}^t$, there are precisely $g(s_1, s_2, \dots, s_t)$ satisfying assignments σ of I with $\sigma(x_1) = s_1, \dots, \sigma(x_t) = s_t$.¹

When f supports perfect equality, we will use the following “transitivity” lemma, which will allow us to use some known implementations.

Lemma 42. Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function which supports perfect equality. Let Γ be a Boolean constraint language and let $g : \{0, 1\}^t \rightarrow \mathbb{R}_{\geq 0}$ be a t -ary function such that g is not $f_{\text{zero}}^{(t)}$ and Γ implements g . Then, if f perfectly simulates Γ , f also perfectly simulates the function g .

Proof. Since Γ implements g , there exists some $t' \geq t$ and a CSP instance I with variables $X := \{x_1, \dots, x_{t'}\}$ and constraints in Γ such that for every tuple $(s_1, \dots, s_t) \in \{0, 1\}^t$, there are precisely $g(s_1, s_2, \dots, s_t)$ satisfying assignments σ of I with $\sigma(x_1) = s_1, \dots, \sigma(x_t) = s_t$. Since $g \neq f_{\text{zero}}^{(t)}$, we conclude that, for all $s_1, \dots, s_t \in \{0, 1\}$,

$$\mu_I(\sigma(x_1) = s_1, \dots, \sigma(x_t) = s_t) = \frac{g(s_1, s_2, \dots, s_t)}{\sum_{(s'_1, s'_2, \dots, s'_t) \in \{0, 1\}^t} g(s'_1, s'_2, \dots, s'_t)}. \tag{4}$$

Since f supports perfect equality, there exists a k -tuple hypergraph $H_{\text{eq}} = (V_{\text{eq}}, \mathcal{F}_{\text{eq}})$ and vertices $y, z \in V_{\text{eq}}$ such that

$$\mu_{f; H_{\text{eq}}}(\sigma(y) = \sigma(z) = 0) = \mu_{f; H_{\text{eq}}}(\sigma(y) = \sigma(z) = 1) = 1/2. \tag{5}$$

We will use the CSP instance I and the hypergraph H_{eq} to construct a k -tuple hypergraph $H = (V, \mathcal{F})$ with vertices $v_1, \dots, v_{t'}$ in V satisfying

$$\mu_{f; H}(\sigma(v_1) = s_1, \dots, \sigma(v_{t'}) = s_{t'}) = \mu_I(\sigma(x_1) = s_1, \dots, \sigma(x_{t'}) = s_{t'}) \tag{6}$$

for all $s_1, \dots, s_{t'} \in \{0, 1\}$. From this, the lemma follows since we can sum over the values of $s_{t+1}, \dots, s_{t'} \in \{0, 1\}$ to obtain that

$$\mu_{f; H}(\sigma(v_1) = s_1, \dots, \sigma(v_t) = s_t) = \mu_I(\sigma(x_1) = s_1, \dots, \sigma(x_t) = s_t)$$

for all $s_1, \dots, s_t \in \{0, 1\}$, which in conjunction with (4) yields that f perfectly simulates g .

To formally construct the k -tuple hypergraph H , we will need some notation. Suppose that I has m constraints and for $j \in [m]$ write the j 'th constraint as $f_j(x_{j,1}, \dots, x_{j,w(j)})$, where $w(j)$ is the arity of $f_j \in \Gamma$ and, for all $i \in [w(j)]$, $x_{j,i} \in \{x_1, \dots, x_{t'}\}$. Since f perfectly simulates Γ and every f_j is in Γ , for every constraint $C_j = f_j(x_{j,1}, \dots, x_{j,w(j)})$, there is a k -tuple hypergraph $H_j = (V_j, \mathcal{F}_j)$ and vertices $v_{j,1}, \dots, v_{j,w(j)}$ of H_j such that for all $s_1, \dots, s_{w(j)} \in \{0, 1\}$, it holds that

¹ See also the relevant equation (4).

$$\mu_{f;H_j}(\sigma(v_{j,1}) = s_1, \dots, \sigma(v_{j,w(j)}) = s_{w(j)}) = \frac{f_j(s_1, \dots, s_{w(j)})}{|R_{f_j}|}. \tag{7}$$

Note that the expression $|R_{f_j}|$ in the denominator in (7) is not zero because the constraint C_j has a satisfying assignment, since I does (which follows from the fact that $g \neq f_{\text{zero}}^{(t)}$ and from the definition of I).

Consider now the k -tuple hypergraph $H' = (V', \mathcal{F}')$ which is simply the disjoint union of H_1, \dots, H_m (i.e., $V' = \cup_{j=1}^m V_j$ and $\mathcal{F}' = \cup_{j=1}^m \mathcal{F}_j$). Note that, for every subset $S \subseteq V'$ and every assignment $\tau : S \rightarrow \{0, 1\}$, it holds that

$$\mu_{f;H'}(\sigma_S = \tau) = \prod_{j=1}^m \mu_{f;H_j}(\sigma_{S \cap V_j} = \tau_{S \cap V_j}). \tag{8}$$

To complete the construction of the desired H , we need some further notation. For a variable $x_i \in \{x_1, \dots, x_{t'}\}$ of the CSP instance I , let $U_i \subseteq V'$ denote the subset of vertices of H' which correspond to occurrences of the variable x_i in the CSP instance I . More precisely, assume that the variable x_i has d occurrences in I for some integer $d \geq 1$, and let C_{j_1}, \dots, C_{j_d} be the constraints in which x_i appears (note that the indices j_1, \dots, j_d are not necessarily distinct). Further, let t_1, \dots, t_d denote the indices of the positions where x_i appears in C_{j_1}, \dots, C_{j_d} respectively. Then $U_i := \{v_{j_1, t_1}, \dots, v_{j_d, t_d}\}$ is precisely the subset of vertices of H' which correspond to occurrences of the variable x_i in the CSP instance I . Let $U := \cup_{i \in [t']} U_i$ (note that in general $U \neq V'$ since H' may contain vertices that do not correspond to occurrences of variables of I).

We are now ready to complete the construction of the desired k -tuple hypergraph $H = (V, \mathcal{F})$. Start by setting H equal to H' . Then, for each $i \in [t']$ and each pair of vertices $u, u' \in U_i$, add to H a distinct copy of the k -tuple hypergraph H_{eq} , identifying the vertices y and z of H_{eq} with the vertices u and u' . Having defined H , we next choose the specified vertices $v_1, \dots, v_{t'}$. In fact, it suffices, for each $i \in [t']$, to let v_i be an arbitrary vertex in U_i .

It remains to prove that (6) holds. We call an assignment $\tau : U \rightarrow \{0, 1\}$ *relevant* if for every $i \in [t']$ there exists $s_i \in \{0, 1\}$ such that for every vertex $v \in U_i$, it holds that $\tau(v) = s_i$. For relevant assignments τ , we will refer to the tuple $(s_1, \dots, s_{t'})$ as the CSP assignment corresponding to τ . For non-relevant τ , the copies of H_{eq} on top of the sets $U_1, \dots, U_{t'}$ ensure that $\mu_{f;H}(\sigma_U = \tau) = 0$. For all relevant $\tau : U \rightarrow \{0, 1\}$, we have from (5) that

$$\mu_{f;H}(\sigma_U = \tau) = \mu_{f;H'}(\sigma_U = \tau)$$

and, hence, using (8), we have that

$$\mu_{f;H}(\sigma_U = \tau) = \prod_{j=1}^m \mu_{f;H_j}(\sigma_{U \cap V_j} = \tau_{U \cap V_j}). \tag{9}$$

Note that for every $j \in [m]$ we have $U \cap V_j = \{v_{j,1}, \dots, v_{j,w(j)}\}$ and, hence, (7) gives

$$\mu_{f;H_j}(\sigma_{U \cap V_j} = \tau_{U \cap V_j}) = \frac{f_j(\tau(v_{j,1}), \dots, \tau(v_{j,w(j)}))}{|R_{f_j}|}. \tag{10}$$

It follows from (9) and (10) that

$$\mu_{f;H}(\sigma_U = \tau) \propto \prod_{j=1}^m f_j(\tau(v_{j,1}), \dots, \tau(v_{j,w(j)})) \text{ for all relevant } \tau. \tag{11}$$

For a relevant $\tau : U \rightarrow \{0, 1\}$, let $(s_1, \dots, s_{t'})$ be the CSP assignment corresponding to τ . Then, the product in the r.h.s. of (11) is 1 iff $(s_1, \dots, s_{t'})$ encodes a satisfying assignment of the CSP instance I . Since the relevant $\tau : U \rightarrow \{0, 1\}$ and assignments to the CSP instance I are in 1-1 correspondence, we obtain (6), as wanted. This concludes the proof of Lemma 42. \square

Lemma 43. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ and $g : \{0, 1\}^t \rightarrow \{0, 1\}$ be Boolean functions such that f simulates g , and $g \neq f_{\text{zero}}^{(t)}$. Suppose that g supports pinning-to- s for some $s \in \{0, 1\}$. Then f supports pinning-to- s as well.*

Proof. Without loss of generality, we assume that $s = 0$. Suppose that the function g supports pinning-to-0. Our goal is to show that f supports pinning-to-0 as well.

First, let $Z_g := \sum_{(s_1, s_2, \dots, s_t) \in \{0, 1\}^t} g(s_1, s_2, \dots, s_t)$. Since $g \neq f_{\text{zero}}^{(t)}$, $Z_g > 0$. Since g supports pinning-to-0, by Definition 9, there exists a t -tuple hypergraph $H_0 = (V_0, \mathcal{F}_0)$ and a vertex $v_0 \in V_0$ such that

$$\mu_{g;H_0}(\sigma_{v_0} = 0) \geq 9/10. \tag{12}$$

(The choice of the constant 9/10 is arbitrary, any constant greater than 1/2 would work. Also, $Z_{g;H_0} > 0$.) For all $\eta : V_0 \rightarrow \{0, 1\}$ define $A_\eta := \frac{w_{g;H_0}(\eta)}{(Z_g)^{|\mathcal{F}_0|}}$ and define $M := \sum_{\eta: V_0 \rightarrow \{0, 1\}} A_\eta = \frac{Z_{g;H_0}}{(Z_g)^{|\mathcal{F}_0|}}$. Since $Z_{g;H_0}$ and Z_g are positive, $M > 0$. Also,

$$\mu_{g;H_0}(\eta) = \frac{A_\eta}{M}. \tag{13}$$

Now let $\epsilon := \min\{M/8, 1/8\}$, $\epsilon_1 := \epsilon/(|\mathcal{F}_0| 2^{2|V_0|})$ and $\epsilon_2 := \epsilon/(2^{|V_0|})$. Since f simulates the function g , by Definition 14 and Lemma 25, there exists a k -tuple hypergraph $H_g = (V_g, \mathcal{F}_g)$ and t vertices v_1, v_2, \dots, v_t of H_g such that, for all $(s_1, s_2, \dots, s_t) \in \{0, 1\}^t$,

$$\left| \mu_{f;H_g}(\sigma(v_1) = s_1, \sigma(v_2) = s_2, \dots, \sigma(v_t) = s_t) - \frac{g(s_1, s_2, \dots, s_t)}{Z_g} \right| \leq \epsilon_1. \tag{14}$$

Construct the k -tuple hypergraph $H = (V, \mathcal{F})$ as follows. For every hyperarc e of H_0 , say $e = (u_1, \dots, u_t) \in \mathcal{F}_0$, take a distinct copy of H_g , which we will denote by $H_g^{(e)}$, and identify the vertices $u_1, \dots, u_t \in V_0$ with the vertices v_1, \dots, v_t of H_g . Note that H is a union of copies of H_g which intersect only at vertices in V_0 . Now for all $\eta : V_0 \rightarrow \{0, 1\}$ define $A'_\eta := \prod_{e \in \mathcal{F}_0} \mu_{f;H_g^{(e)}}(\sigma_e = \eta_e)$ and $M' := \sum_{\eta:V_0 \rightarrow \{0,1\}} A'_\eta$. Then

$$\mu_{f;H}(\sigma_{V_0} = \eta) = \frac{A'_\eta}{M'}. \tag{15}$$

By (14), for every $e = (u_1, \dots, u_t) \in \mathcal{F}_0$, it holds that

$$\left| \mu_{f;H_g^{(e)}}(\sigma_e = \eta_e) - \frac{g(\eta(u_1), \dots, \eta(u_t))}{Z_g} \right| \leq \epsilon_1. \tag{16}$$

Recall that for real numbers $a_1, \dots, a_n \in [0, 1]$ and $b_1, \dots, b_n \in [0, 1]$, it holds that $|\prod_{i=1}^n a_i - \prod_{i=1}^n b_i| \leq \sum_{i=1}^n |a_i - b_i|$. Thus, using (16), we obtain that, for every $\eta : V_0 \rightarrow \{0, 1\}$, it holds that

$$|A'_\eta - A_\eta| = \left| \prod_{e \in \mathcal{F}_0} \mu_{f;H_g^{(e)}}(\sigma_e = \eta_e) - \frac{w_{g;H_0}(\eta)}{(Z_g)^{|\mathcal{F}_0|}} \right| \leq \epsilon_1 |\mathcal{F}_0|. \tag{17}$$

Summing this over all $\eta : V_0 \rightarrow \{0, 1\}$, we obtain that

$$|M' - M| = \left| \sum_{\eta:V_0 \rightarrow \{0,1\}} A'_\eta - \sum_{\eta:V_0 \rightarrow \{0,1\}} A_\eta \right| \leq \epsilon_1 |\mathcal{F}_0| 2^{|V_0|} = \epsilon_2. \tag{18}$$

Note that the expression $\epsilon_1 |\mathcal{F}_0|$ in (17) is at most ϵ_2 . Also, for all $\eta : V_0 \rightarrow \{0, 1\}$, $A_\eta \leq M$ and $A'_\eta \leq M'$. The bounds in (17) and (18) yield that $A_\eta - \epsilon_2 \leq A'_\eta \leq A_\eta + \epsilon_2$ and $M - \epsilon_2 \leq M' \leq M + \epsilon_2$. By the choice of ϵ , we have $M - \epsilon > M/2$ and hence $M - \epsilon_2 > M/2$ as well. Further, we have the bound

$$\left| \frac{A_\eta}{M} - \frac{A'_\eta}{M'} \right| \leq \max \left\{ \frac{A_\eta}{M} - \frac{A_\eta - \epsilon_2}{M + \epsilon_2}, \frac{A_\eta + \epsilon_2}{M - \epsilon_2} - \frac{A_\eta}{M} \right\} \leq \frac{\epsilon_2(A_\eta + M)}{M(M - \epsilon_2)} \leq \frac{2\epsilon_2}{M - \epsilon_2} \leq \frac{4\epsilon_2}{M} \leq \frac{1}{2^{|V_0|+1}}. \tag{19}$$

From (13) and (15) and (19), we thus obtain that for every $\eta : V_0 \rightarrow \{0, 1\}$, it holds that

$$|\mu_{f;H}(\sigma_{V_0} = \eta) - \mu_{g;H_0}(\eta)| \leq 1/2^{|V_0|+1}.$$

Summing this over the $2^{|V_0|-1}$ possible values of $\eta_{V_0 \setminus \{v_0\}}$, we obtain that for $s \in \{0, 1\}$ it holds that

$$|\mu_{f;H}(\sigma_{v_0} = s) - \mu_{g;H_0}(\sigma_{v_0} = s)| \leq 1/4.$$

Combining this with (12), we obtain that

$$\mu_{f;H}(\sigma_{v_0} = 0) > 1/2 > \mu_{f;H}(\sigma_{v_0} = 1).$$

Thus, by Lemma 24, we obtain that f supports pinning-to-0. This concludes the proof of Lemma 43. \square

The following lemma is similar to Lemma 42 except that, instead of assuming that f perfectly simulates Γ , we only assume that f simulates Γ so instead of concluding that f perfectly simulates g , we only conclude that f simulates g .

Lemma 44. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function which supports equality. Let Γ be a Boolean constraint language and let $g : \{0, 1\}^t \rightarrow \mathbb{R}_{\geq 0}$ be a t -ary function such that g is not $f_{\text{zero}}^{(t)}$ and Γ implements g . Then, if f simulates Γ , f also simulates the function g .*

Proof. The proof is similar to the proof of Lemma 42, but the imperfect nature of the simulation adds technical details. Since Γ implements g we can follow the proof of Lemma 42 to define the CSP instance I with variables $\{x_1, \dots, x_{t'}\}$ and constraints in Γ satisfying (4).

We will use the CSP instance I and the fact that f supports equality to construct a k -tuple hypergraph $H = (V, \mathcal{F})$ with an admissible set \mathcal{V}^* for H with respect to f and vertices $v_1, \dots, v_{t'}$ in V satisfying

$$\mu_{f;H}^{\text{cond}(\mathcal{V}^*)}(\sigma(v_1) = s_1, \dots, \sigma(v_{t'}) = s_{t'}) = \mu_I(\sigma(x_1) = s_1, \dots, \sigma(x_{t'}) = s_{t'}) \tag{20}$$

for all $s_1, \dots, s_{t'} \in \{0, 1\}$. From this, the lemma follows since we can sum over the values of $s_{t+1}, \dots, s_{t'} \in \{0, 1\}$ to obtain that

$$\mu_{f;H}^{\text{cond}(\mathcal{V}^*)}(\sigma(v_1) = s_1, \dots, \sigma(v_t) = s_t) = \mu_I(\sigma(x_1) = s_1, \dots, \sigma(x_t) = s_t)$$

for all $s_1, \dots, s_t \in \{0, 1\}$, which in conjunction with (4) yields that f simulates g .

To formally construct the k -tuple hypergraph H , we will need some notation. As in the proof of Lemma 42, suppose that I has m constraints and for $j \in [m]$ write the j 'th constraint as $f_j(x_{j,1}, \dots, x_{j,w(j)})$, where $w(j)$ is the arity of $f_j \in \Gamma$ and, for all $i \in [w(j)]$, $x_{j,i} \in \{x_1, \dots, x_{t'}\}$. Since f simulates Γ and every f_j is in Γ , for every constraint $C_j = f_j(x_{j,1}, \dots, x_{j,w(j)})$, there is a k -tuple hypergraph $H_j = (V_j, \mathcal{F}_j)$, an admissible collection $\mathcal{V}^j = (V_{\text{pin}0}^j, V_{\text{pin}1}^j, \mathcal{V}_{\text{eq}}^j)$ for H_j with respect to f_j and vertices $v_{j,1}, \dots, v_{j,w(j)}$ of H_j such that for all $s_1, \dots, s_{w(j)} \in \{0, 1\}$, it holds that

$$\mu_{f;H_j}^{\text{cond}(\mathcal{V}^j)}(\sigma(v_{j,1}) = s_1, \dots, \sigma(v_{j,w(j)}) = s_{w(j)}) = \frac{f_j(s_1, \dots, s_{w(j)})}{|R_{f_j}|}. \tag{21}$$

Consider now the k -tuple hypergraph $H = (V, \mathcal{F})$ which is simply the disjoint union of H_1, \dots, H_m (i.e., $V = \cup_{j=1}^m V_j$ and $\mathcal{F} = \cup_{j=1}^m \mathcal{F}_j$). Further, let $\mathcal{V} = (V_{\text{pin}0}, V_{\text{pin}1}, \mathcal{V}_{\text{eq}})$, where $V_{\text{pin}0} = \cup_{j=1}^m V_{\text{pin}0}^j$, $V_{\text{pin}1} = \cup_{j=1}^m V_{\text{pin}1}^j$ and $V_{\text{eq}} = \cup_{j=1}^m \mathcal{V}_{\text{eq}}^j$. We wish to argue that \mathcal{V} is admissible for H with respect to f . The various disjointness constraints in Definition 12 are satisfied since H_1, \dots, H_m are disjoint (using the fact that each \mathcal{V}^j is admissible for H_j with respect to f_j). We have assumed, in the statement of the lemma, that f supports equality. To show that \mathcal{V} is admissible for H with respect to f , we need to show that if some f_j supports pinning-to- s for some $s \in \{0, 1\}$ then so does f . This follows from Lemma 43 since, by assumption, f simulates f_j . Thus, \mathcal{V} is admissible for H with respect to f .

Note that, for every subset $S \subseteq V$ and every assignment $\tau : S \rightarrow \{0, 1\}$, it holds that

$$\mu_{f;H}^{\text{cond}(\mathcal{V})}(\sigma_S = \tau) = \prod_{j=1}^m \mu_{f;H_j}^{\text{cond}(\mathcal{V}^j)}(\sigma_{S \cap V_j} = \tau_{S \cap V_j}). \tag{22}$$

Having completed the construction of the desired H , to recover (6), it remains to specify \mathcal{V}^* and the vertices $v_1, \dots, v_{t'}$. For each $i \in [t']$, define U_i as in the proof of Lemma 42. Also, let $U := \cup_{i \in [t']} U_i$. The main idea is that \mathcal{V}^* is the same as \mathcal{V} except that the sets $U_1, \dots, U_{t'}$ are added to V_{eq} because we want to condition on the fact that the variables in each of these sets are equal. In order to formally specify $\mathcal{V}^* = (V_{\text{pin}0}^*, V_{\text{pin}1}^*, V_{\text{eq}}^*)$ there is a slight technical difficulty because $V_{\text{pin}0}^*$ and $V_{\text{pin}1}^*$ have to be disjoint from each other and from all sets in V_{eq}^* . In order to deal with this (rather unimportant, but technical) detail, we give an algorithm for defining \mathcal{V}^* . Let $\mathcal{V}^0 = \mathcal{V}$. Then, for $i = 1, \dots, t'$ define $\mathcal{V}^i = (V_{\text{pin}0}^i, V_{\text{pin}1}^i, \mathcal{V}_{\text{eq}}^i)$ as follows.

- Let $\mathcal{V}^i = \mathcal{V}^{i-1}$.
- Note that no vertex in U_i is in $V_{\text{pin}0} \cap V_{\text{pin}1}$. This follows since I is satisfiable (since g is not the always-zero function f_{zero}).
- If any vertex in U_i is in $V_{\text{pin}0}$ then replace $V_{\text{pin}0}^i$ with $V_{\text{pin}0}^{i-1} \cup U_i$.
- Otherwise, if any vertex in U_i is in $V_{\text{pin}1}$ then replace $V_{\text{pin}1}^i$ with $V_{\text{pin}1}^{i-1} \cup U_i$.
- Otherwise, if U_i does not intersect any sets in $\mathcal{V}_{\text{eq}}^{i-1}$ then replace $\mathcal{V}_{\text{eq}}^i$ with $\mathcal{V}_{\text{eq}}^{i-1} \cup \{U_i\}$.
- Otherwise, let W_1, \dots, W_z be the sets in $\mathcal{V}_{\text{eq}}^{i-1}$ that intersect U_i and replace $\mathcal{V}_{\text{eq}}^i$ with $(\mathcal{V}_{\text{eq}}^{i-1} \setminus \{W_1, \dots, W_z\}) \cup \{U_i \cup W_1 \cup \dots \cup W_z\}$.

Finally, let $\mathcal{V}^* = \mathcal{V}^{t'}$. Then choose the vertices $v_1, \dots, v_{t'}$ to be arbitrary vertices in $U_1, \dots, U_{t'}$, respectively.

It remains to prove that (20) holds. As in the proof of Lemma 42, we call an assignment $\tau : U \rightarrow \{0, 1\}$ *relevant* if for every $i \in [t']$ there exists $s_i \in \{0, 1\}$ such that for every vertex $v \in U_i$, it holds that $\tau(v) = s_i$. For relevant assignments τ , we will refer to the tuple $(s_1, \dots, s_{t'})$ as the CSP assignment corresponding to τ . Clearly, for non-relevant τ , we have that $\mu_{f;H}^{\text{cond}(\mathcal{V}^*)}(\sigma_U = \tau) = 0$ since \mathcal{V}^* forces equality on each of the sets $U_1, \dots, U_{t'}$. For all relevant $\tau : U \rightarrow \{0, 1\}$, we have that

$$\mu_{f;H}^{\text{cond}(\mathcal{V}^*)}(\sigma_U = \tau) = \frac{\mu_{f;H}^{\text{cond}(\mathcal{V})}(\sigma_U = \tau)}{\mu_{f;H}^{\text{cond}(\mathcal{V})}(\sigma_{U_1}^{\text{eq}}, \dots, \sigma_{U_{t'}}^{\text{eq}})}$$

and hence

$$\mu_{f;H}^{\text{cond}(\mathcal{V}^*)}(\sigma_U = \tau) \propto \mu_{f;H}^{\text{cond}(\mathcal{V})}(\sigma_U = \tau) \text{ for all relevant } \tau. \tag{23}$$

Using (22), we have that

$$\mu_{f;H}^{\text{cond}(\mathcal{V})}(\sigma_U = \tau) = \prod_{j=1}^m \mu_{f;H_j}^{\text{cond}(\mathcal{V}^j)}(\sigma_{U \cap V_j} = \tau_{U \cap V_j}). \tag{24}$$

Note that for every $j \in [m]$ we have $U \cap V_j = \{v_{j,1}, \dots, v_{j,w(j)}\}$ and, hence, (21) gives

$$\mu_{f;H_j}^{\text{cond}(\mathcal{V}^j)}(\sigma_{U \cap V_j} = \tau_{U \cap V_j}) = \frac{f_j(\tau(v_{j,1}), \dots, \tau(v_{j,w(j)}))}{|R_{f_j}|}. \tag{25}$$

It follows from (23), (24) and (25) that

$$\mu_{f;H}^{\text{cond}(\mathcal{V}^*)}(\sigma_U = \tau) \propto \prod_{j=1}^m f_j(\tau(v_{j,1}), \dots, \tau(v_{j,w(j)})) \text{ for all relevant } \tau. \tag{26}$$

For a relevant $\tau : U \rightarrow \{0, 1\}$, let $(s_1, \dots, s_{\ell'})$ be the CSP assignment corresponding to τ . Then, the product in the r.h.s. of (26) is 1 iff $(s_1, \dots, s_{\ell'})$ encodes a satisfying assignment of the CSP instance I . Since the relevant $\tau : U \rightarrow \{0, 1\}$ and assignments to the CSP instance I are in 1-1 correspondence, we obtain (20), as wanted.

This concludes the proof of Lemma 44. \square

The following Boolean functions, which were considered in [13], will be important in what follows: δ_0 and δ_1 (defined in Definition 32), and XOR, Implies, NAND, OR. For convenience, we state the corresponding relations here.

- $R_{\delta_0} = \{(0)\}$ and $R_{\delta_1} = \{(1)\}$ (these correspond to satisfying assignments of $\neg x$ and x , respectively).
- $R_{\text{XOR}} = \{(0, 1), (1, 0)\}$ (corresponds to satisfying assignments of $x \neq y$).
- $R_{\text{Implies}} = \{(0, 0), (0, 1), (1, 1)\}$ (corresponds to satisfying assignments of $x \Rightarrow y$).
- $R_{\text{NAND}} = \{(0, 0), (0, 1), (1, 0)\}$ (corresponds to satisfying assignments of $\neg x \vee \neg y$).
- $R_{\text{OR}} = \{(0, 1), (1, 0), (1, 1)\}$ (corresponds to satisfying assignments of $x \vee y$).

8.2. The case of self-dual functions

A Boolean function f is said to be *self-dual* if, for all \mathbf{x} , $f(\mathbf{x}) = f(\bar{\mathbf{x}})$. In this section, we show (Theorem 46, below) that if f is a self-dual Boolean function which is not affine, and f supports perfect equality, then f simulates a hard function. First, we establish a useful lemma.

Lemma 45. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a self-dual function Boolean with $f \neq f_{\text{zero}}^{(k)}$ and $f(\mathbf{0}) = 0$. Further, suppose that f supports perfect equality. Then, f perfectly simulates XOR.*

Proof. From $f(\mathbf{0}) = 0$ and self-duality, we have that $f(\mathbf{1}) = 0$. Since $f \neq f_{\text{zero}}^{(k)}$, there must be some $\mathbf{x} \notin \{\mathbf{0}, \mathbf{1}\}$ such that $f(\mathbf{x}) = 1$. By self-duality, we have that $f(\bar{\mathbf{x}}) = 1$ as well. Let $U_0 = \{i \in [k] \mid x_i = 0\}$ and $U_1 = \{i \in [k] \mid x_i = 1\}$.

Since f supports perfect equality, there exists a k -tuple hypergraph $H_{\text{eq}} = (V_{\text{eq}}, \mathcal{F}_{\text{eq}})$ and vertices $y, z \in V_{\text{eq}}$ such that

$$\mu_{f;H_{\text{eq}}}(\sigma(y) = \sigma(z) = 0) = \mu_{f;H_{\text{eq}}}(\sigma(y) = \sigma(z) = 1) = 1/2.$$

Construct the k -tuple hypergraph H as follows. First, take a single hyperarc (v_1, \dots, v_k) . Then, for every $s \in \{0, 1\}$ and every i, j such that i and j are both in U_s , add a new copy of H_{eq} , identifying y with v_i and z with v_j . Finally, choose $v_1 \in U_0$ and $v_2 \in U_1$. Then

$$\mu_{f;H}(\sigma(v_1) = 0, \sigma(v_2) = 1) = \mu_{f;H}(\sigma(v_1) = 1, \sigma(v_2) = 0) = 1/2,$$

so f perfectly simulates XOR. \square

Theorem 46. *Suppose that f is a self-dual Boolean function which is not affine and supports perfect equality. Then f simulates a hard function.*

Proof. Let k be the arity of f . The proof has two cases depending on whether $f(\mathbf{0}) = 0$ or $f(\mathbf{0}) = 1$. We begin with the case where $f(\mathbf{0}) = 1$ (we will reduce the proof for the other case to this one).

So, assume first that $f(\mathbf{0}) = 1$. Since f is not affine, by applying Item 2 of Lemma 21 to $\mathbf{a} = \mathbf{0}$, we obtain that there exist $\mathbf{b}, \mathbf{c} \in \{0, 1\}^k$ such that $f(\mathbf{b}) = f(\mathbf{c}) = 1$ but $f(\mathbf{b} \oplus \mathbf{c}) = 0$. By self-duality, we also have that $f(\overline{\mathbf{b}}) = f(\overline{\mathbf{c}}) = 1$. Note that

$$\mathbf{b} \neq \mathbf{c}, \quad \mathbf{b} \neq \mathbf{0}, \mathbf{1}, \quad \mathbf{c} \neq \mathbf{0}, \mathbf{1}.$$

Indeed, it cannot be the case that $\mathbf{b} = \mathbf{c}$ since then $f(\mathbf{b} \oplus \mathbf{c}) = f(\mathbf{0}) = 1$. Analogously, $\mathbf{b} = \mathbf{0}$ would give that $f(\mathbf{b} \oplus \mathbf{c}) = f(\mathbf{c}) = 1$. Similarly, $\mathbf{b} = \mathbf{1}$ would give that $f(\mathbf{b} \oplus \mathbf{c}) = f(\overline{\mathbf{c}}) = 1$. By symmetry between \mathbf{b} and \mathbf{c} , we have that $\mathbf{c} \neq \mathbf{0}, \mathbf{1}$.

Let w, x, y, z be Boolean variables. For $i \in [k]$, let

$$r_i = \begin{cases} w, & \text{if } b_i = 0, c_i = 0, \\ x, & \text{if } b_i = 0, c_i = 1, \\ y, & \text{if } b_i = 1, c_i = 0, \\ z, & \text{if } b_i = 1, c_i = 1. \end{cases}$$

Let $V := \{r_1, \dots, r_k\}$ (note that V has at most 4 elements). Also, consider the Boolean function $h : \{0, 1\}^{|V|} \rightarrow \{0, 1\}$ defined by $h = f(r_1, \dots, r_k)$.

We next study in more detail the function h . Observe that

- V must contain at least one of x, y since $\mathbf{b} \neq \mathbf{c}$.
- V must contain at least one of w, x since $\mathbf{b} \neq \mathbf{1}$.
- V must contain at least one of w, y since $\mathbf{c} \neq \mathbf{1}$.
- V must contain at least one of y, z since $\mathbf{b} \neq \mathbf{0}$.
- V must contain at least one of x, z since $\mathbf{c} \neq \mathbf{0}$.

Thus, the cases to consider are $V = \{w, x, y, z\}$, $|V| = 3$, or $V = \{x, y\}$. However, $V = \{x, y\}$ is not possible since then $\mathbf{b} \oplus \mathbf{c} = \mathbf{1}$ and $f(\mathbf{1}) = 1$ (contradicting that $f(\mathbf{b} \oplus \mathbf{c}) = 0$). We now consider the function h (and the corresponding relation R_h) in each of the possible cases.

- Case 1. $V = \{x, y, z\}$.

Note that $(x, y, z) = (0, 0, 0) \in R_h$ since $f(\mathbf{0}) = 1$. Also, $(0, 1, 1) \in R_h$ since $f(\mathbf{b}) = 1$. Also, $(1, 0, 1) \in R_h$ since $f(\mathbf{c}) = 1$. By self-duality $(1, 1, 1), (1, 0, 0), (0, 1, 0)$ are also in R_h . Then $(x, y, z) = (1, 1, 0)$ is not in R_h since $f(\mathbf{b} \oplus \mathbf{c}) = 0$ and by self-duality neither is $(0, 0, 1)$. So $h(x, y, z)$ is completely determined. Then, for the function $g(x, y) := \sum_z h(x, y, z)$, we have that

$$g(0, 0) = g(1, 1) = 1 \quad \text{and} \quad g(0, 1) = g(1, 0) = 2,$$

which is a hard function.

- Case 2. $V = \{w, x, y\}$. This case is similar to Case 1 by switching the spins 0 and 1.
- Case 3. $V = \{w, x, z\}$. $(w, x, z) = (0, 0, 0)$ is in R_h since $f(\mathbf{0}) = 1$. $(0, 0, 1)$ is in R_h since $f(\mathbf{b}) = 1$. $(0, 1, 1)$ is in R_h since $f(\mathbf{c}) = 1$. By self-duality, $(1, 1, 1), (1, 1, 0)$ and $(1, 0, 0)$ are also in R_h . $(0, 1, 0)$ is not in R_h since $f(\mathbf{b} \oplus \mathbf{c}) = 0$. By self-duality, $(1, 0, 1)$ is not in R_h . Then, for the function $g(w, z) = \sum_x h(w, x, z)$, we have that

$$g(0, 0) = g(1, 1) = 1 \quad \text{and} \quad g(0, 1) = g(1, 0) = 2,$$

which is a hard function.

- Case 4. $V = \{w, y, z\}$. This case follows from Case 3 by switching \mathbf{b} and \mathbf{c} .
- Case 5. $V = \{w, x, y, z\}$

Similarly to the other cases, we have the following tuples in R_h : $(w, x, y, z) = (0, 0, 0, 0), (0, 0, 1, 1), (0, 1, 0, 1)$, and their complements and we know that $(w, x, y, z) = (0, 1, 1, 0)$ and its complement are not in R_h . Let $h_0 = h(0, x, y, z)$. Let

$$C = \{(0, 0, 1), (0, 1, 0), (1, 0, 0), (1, 1, 1)\}.$$

For every possible subset S of C , we have to consider the possibility that $R_{h_0} = S \cup \{(0, 0, 0), (0, 1, 1), (1, 0, 1)\}$. This is a lot of cases, but fortunately, some of them can be combined.

- Case 5a. $(x, y, z) = (0, 1, 0)$ is in S but $(1, 1, 1)$ is not. Then, for the function $g(x, y) := \sum_w h(w, x, y, x)$, we have

$$g(0, 0) = g(1, 1) = 1 \quad \text{and} \quad g(0, 1) = g(1, 0) = 2,$$

which is a hard function.

- Case 5b. $(x, y, z) = (1, 1, 1)$ is in S but $(0, 1, 0)$ is not. Then, for the function $g(w, x) := \sum_y h(w, x, y, x)$, we have

$$g(0, 0) = g(1, 1) = 1 \quad \text{and} \quad g(0, 1) = g(1, 0) = 2,$$

which is a hard function.

- Case 5c. $(x, y, z) = (1, 0, 0)$ is in S but $(1, 1, 1)$ is not. This case is symmetric to Case 5a.
- Case 5d. $(x, y, z) = (1, 1, 1)$ is in S but $(1, 0, 0)$ is not. This case is symmetric to Case 5b.
- Case 5e. $(x, y, z) = (0, 1, 0)$ and $(1, 0, 0)$ are both in S .
Then, for the function $g(x, y) := \sum_w h(w, x, y, w)$, we have

$$g(0, 0) = g(1, 1) = 1 \quad \text{and} \quad g(0, 1) = g(1, 0) = 2,$$

which is a hard function.

- Case 5f. $S = \emptyset$. Then, for the function $g(x, y) := \sum_{w,z} h(w, x, y, z)$, we have

$$g(0, 0) = g(1, 1) = 1 \quad \text{and} \quad g(0, 1) = g(1, 0) = 2,$$

which is a hard function.

- Case 5g. $S = \{(0, 0, 1)\}$. Then, for the function $g(w, z) := \sum_{x,y} R_h(w, x, y, z)$, we have

$$g(0, 0) = g(1, 1) = 1 \quad \text{and} \quad g(0, 1) = g(1, 0) = 3,$$

which is a hard function.

It remains to argue that each of the functions g used in Cases 1–5 can be simulated using the function f . This is direct. $\{f\}$ implements the function h and $\{h\}$ implements the function g , so $\{f\}$ implements g . Since f supports perfect equality, we can apply Lemma 42 taking $\Gamma = \{f\}$. Since (trivially) f perfectly simulates Γ , we find that f perfectly simulates g . This completes the proof for the case where $f(\mathbf{0}) = 1$.

We next argue for the case where $f(\mathbf{0}) = 0$. Since f is not affine, we have that $f \neq f_{\text{zero}}^{(k)}$, so there exists $\mathbf{t} \neq \mathbf{0}$ such that $f(\mathbf{t}) = 1$. Let $S := \{i \in [k] \mid t_i = 1\}$ and note that $S \neq \emptyset$.

Consider the function f' defined by $f'(\mathbf{x}) := f(\mathbf{x} \oplus \mathbf{t})$ for all $\mathbf{x} \in \{0, 1\}^k$. Note that

- $f'(\mathbf{0}) = 1$, since $f(\mathbf{t}) = 1$.
- f' is self-dual. Indeed, for $\mathbf{x} \in \{0, 1\}^k$ we have

$$f'(\bar{\mathbf{x}}) = f(\mathbf{x} \oplus \mathbf{1} \oplus \mathbf{t}) = f(\mathbf{x} \oplus \mathbf{t}) = f'(\mathbf{x}),$$

where the middle equality follows from the self-duality of f .

- f' is not affine. Since f is not affine, we know from Lemma 21(1) that there are $\mathbf{a}, \mathbf{b}, \mathbf{c}$ such that $f(\mathbf{a}) = f(\mathbf{b}) = f(\mathbf{c}) = 1$ and $f(\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c}) = 0$. Let $\mathbf{a}' = \mathbf{a} \oplus \mathbf{t}$, $\mathbf{b}' = \mathbf{b} \oplus \mathbf{t}$ and $\mathbf{c}' = \mathbf{c} \oplus \mathbf{t}$. Then by the definition of f' , $f'(\mathbf{a}') = f'(\mathbf{b}') = f'(\mathbf{c}') = 1$. But $f'(\mathbf{a}' \oplus \mathbf{b}' \oplus \mathbf{c}') = f'(\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c} \oplus \mathbf{t}) = f(\mathbf{a} \oplus \mathbf{b} \oplus \mathbf{c}) = 0$, so f' is not affine.

By the previous argument, we thus have $\{f'\}$ implements a hard function g . We will show that f simulates g . Indeed, observe that the constraint language $\{f, \text{XOR}\}$ implements f' (just apply XOR to the bits of f which correspond to non-zero entries of the vector \mathbf{t}). Since $f(\mathbf{0}) = 0$ and f supports perfect equality, by Lemma 45 we have that f perfectly simulates $\{f, \text{XOR}\}$. Applying Lemma 42 with $\Gamma = \{f, \text{XOR}\}$ and the g of Lemma 42 as f' , we find that f perfectly simulates f' . Then applying Lemma 42 again with $\Gamma = \{f'\}$, and the g of Lemma 42 as g , we obtain that f simulates the hard function g , as wanted. This concludes the proof of Theorem 46. \square

8.3. #BIS-easiness

The goal of Section 8 is to prove Theorem 18. The required #BIS-easiness results follow directly from [13].

Lemma 47 ([13, Lemma 9]). *Let Γ be a constraint language such that every relation in Γ belongs to IM_2 . Then, $\#\text{CSP}(\Gamma)$ is #BIS-easy. \square*

8.4. #BIS-hardness

We next prove the required #BIS-hardness results (cf. Lemma 51 below). We will use the following results from the literature.

Lemma 48 ([5, Corollary 3]). *Let $\Delta \geq 6$. It is #BIS-hard to count the number of independent sets in bipartite graphs of maximum degree Δ . \square*

The following lemma is from Lemma 13 of [13]. We take the lemma from there since we use the notation of [13]. However, the proof is originally from Lemmas 5.24 and 5.25 of [8].

Lemma 49 ([8]). *If f is a Boolean function that is not self-dual, then $\{f\}$ implements either δ_0 or δ_1 .*

Proof. We just need to explain the terminology in [13, Lemma 13]. It will be then apparent that Items (i)–(iv) in [13, Lemma 13] show that $\{f\}$ implements δ_0 or δ_1 . “0-valid” in [13] means that $\mathbf{0} \in R_f$, “1-valid” means that $\mathbf{1} \in R_f$ and “complement-closed” means self-dual. \square

Lemma 50 ([13, Lemma 15], see also [8]). *If f is a Boolean function that is not affine, then $\{f, \delta_0\}$ implements one of OR, Implies, NAND. The same is true for $\{f, \delta_1\}$.* \square

We are now ready to show that, for every f which supports perfect equality and is not affine, it holds that, for all sufficiently large Δ , $\#\text{Multi2Spin}_\Delta(f)$ is $\#\text{BIS-hard}$.

Lemma 51. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function which supports perfect equality. Suppose that f is not affine. Then, for all sufficiently large Δ , $\#\text{Multi2Spin}_\Delta(f)$ is $\#\text{BIS-hard}$.*

Proof. Assume first that f is self-dual. Then, by Theorem 46 (note that f is not affine and supports perfect equality by assumption), f perfectly simulates a hard function. By Lemma 15, we obtain that for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard. Now, recall that every problem in $\#\text{P}$ admits an FPRAS using an NP-oracle [24]. Since $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard, we can use it as an oracle to obtain an FPRAS for $\#\text{BIS}$.

Assume next that f is not self-dual. By Lemma 49 we have that $\{f\}$ implements either δ_0 or δ_1 . We only need to consider the case where $\{f\}$ implements δ_0 , the case of δ_1 follows by just switching the spins 0 and 1. First, by Lemma 42 with $\Gamma = \{f\}$ and $g = \delta_0$, f perfectly simulates δ_0 , so f perfectly simulates $\{f, \delta_0\}$. Recall that f is not affine. By Lemma 50, it thus follows that $\{f, \delta_0\}$ implements one of OR, NAND, Implies. Using Lemma 42 again, it follows that f perfectly simulates one of OR, NAND, Implies.

Note that OR and NAND correspond to hard functions, so when f perfectly simulates either OR or NAND, we obtain from Lemma 15 that for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard. Thus, as in the case of self-dual functions, we may conclude that for all sufficiently large Δ , $\#\text{Multi2Spin}_{\Delta,c}(f)$ is $\#\text{BIS-hard}$.

Thus, it remains to consider the case where f perfectly simulates Implies. By Definition 14, this means that there exists a k -tuple hypergraph $H' = (V', \mathcal{F}')$ and vertices x, y in H' such that

$$\frac{Z_{00}}{Z_{f;H'}} = \frac{1}{3}, \quad \frac{Z_{01}}{Z_{f;H'}} = \frac{1}{3}, \quad \frac{Z_{11}}{Z_{f;H'}} = \frac{1}{3}, \quad \frac{Z_{10}}{Z_{f;H'}} = 0, \tag{27}$$

where, for $s_1, s_2 \in \{0, 1\}$, we denote

$$Z_{s_1 s_2} := \sum_{\substack{\sigma: V' \rightarrow \{0,1\}; \\ \sigma_x = s_1, \sigma_y = s_2}} w_{f;H'}(\sigma).$$

Let Δ' be the degree of H' . We will show that for all $\Delta \geq 6\Delta'$, $\#\text{Multi2Spin}_{\Delta,c}(f)$ is $\#\text{BIS-hard}$.

We will use Lemma 48. In particular, let $G = (V_1 \cup V_2, E)$ be a bipartite graph of maximum degree 6 where V_1, V_2 denote the parts of G in its partition. Let $H = (V, \mathcal{F})$ be the k -tuple hypergraph obtained from G as follows. Start by putting all of the vertices in $V_1 \cup V_2$ into V . Then add additional vertices and hyperarcs as follows. For every edge $(v_1, v_2) \in E$ such that $v_1 \in V_1$ and $v_2 \in V_2$, take a distinct copy of H' and identify vertex x in H' with v_1 and vertex y in H' with v_2 . Note that $V_1 \cup V_2 \subseteq V$ and that the degree of H is $6\Delta'$.

Let \mathcal{I}_G denote the set of independent sets of G . Then, we claim that

$$Z_{f;H} = |\mathcal{I}_G| \cdot (Z_{f;H'}/3)^{|E|}. \tag{28}$$

Before proving (28), note that an oracle call to $\#\text{Multi2Spin}_\Delta(f)$ for $\Delta \geq 6\Delta'$ with input H and relative error $\epsilon > 0$ yields via (28) an estimate for the number of independent sets in bipartite graphs of maximum degree 6 which is within relative error ϵ from the true value. Thus, using Lemma 48, we obtain an AP-reduction from $\#\text{BIS}$ to $\#\text{Multi2Spin}_\Delta(f)$ for all $\Delta \geq 6\Delta'$, as wanted.

To show (28), let $\sigma : V \rightarrow \{0, 1\}$ be an assignment such that $w_{f;H}(\sigma) > 0$. The copies of H' ensure that for every edge $(v_1, v_2) \in E$ such that $v_1 \in V_1$ and $v_2 \in V_2$ it holds that either $\sigma(v_1) \neq 1$ or $\sigma(v_2) \neq 0$. Thus, the set $(\sigma^{-1}(1) \cap V_1) \cup (\sigma^{-1}(0) \cap V_2)$ is an independent set of G . Conversely, for every independent set I of G , consider

$$\Omega_I = \{\sigma : V \rightarrow \{0, 1\} \mid \sigma_{I \cap V_1} = \mathbf{1}, \sigma_{I \cap V_2} = \mathbf{0}, \sigma_{V_1 \setminus I} = \mathbf{0}, \sigma_{V_2 \setminus I} = \mathbf{1}\}.$$

Then, using (27), we have that the number of assignments $\sigma \in \Omega_I$ such that $w_{f;H}(\sigma) > 0$ is equal to

$$\prod_{(v_1, v_2) \in E} Z_{\sigma(v_1)\sigma(v_2)} = (Z_{f;H'}/3)^{|E|}.$$

Summing this over all $I \in \mathcal{I}_G$, we obtain (28), thus completing the proof of Lemma 51. \square

8.5. NP-hardness

In this section, we show (Lemma 54 below) that if f supports perfect equality, and it is not affine, and is not in IM_2 then, for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard. To do this, we need some preparation. The ideas behind the following lemma are essentially from [13].

Lemma 52. *If f perfectly simulates Implies, then f simulates $\{\delta_0, \delta_1\}$ (not necessarily perfectly).*

Proof. We first show that f supports both pinning-to-0 and pinning-to-1. Since f perfectly simulates Implies, there exists a k -tuple hypergraph H and vertices v_1, v_2 in H such that

$$\mu_{00} = 1/3, \quad \mu_{01} = 1/3, \quad \mu_{11} = 1/3, \quad \mu_{10} = 0, \tag{29}$$

where, for $s_1, s_2 \in \{0, 1\}$, we denote $\mu_{s_1 s_2} := \mu_{f;H}(\sigma_{v_1} = s_1, \sigma_{v_2} = s_2)$.

Note that

$$\begin{aligned} \mu_{f;H}(\sigma_{v_1} = 0) &= \mu_{00} + \mu_{01}, & \mu_{f;H}(\sigma_{v_1} = 1) &= \mu_{10} + \mu_{11}, \\ \mu_{f;H}(\sigma_{v_2} = 0) &= \mu_{00} + \mu_{10}, & \mu_{f;H}(\sigma_{v_2} = 1) &= \mu_{01} + \mu_{11}. \end{aligned}$$

It follows that

$$\begin{aligned} \mu_{f;H}(\sigma_{v_1} = 0) &= 2/3, & \mu_{f;H}(\sigma_{v_1} = 1) &= 1/3, \\ \mu_{f;H}(\sigma_{v_2} = 0) &= 1/3, & \mu_{f;H}(\sigma_{v_2} = 1) &= 2/3. \end{aligned} \tag{30}$$

Then, using (30), we obtain that H and its vertices v_1, v_2 satisfy the assumptions of Lemma 24 (v_1 satisfies Item 1 and v_2 Item 2) and hence we obtain that f supports pinning-to-0 and pinning-to-1.

To conclude that f simulates $\{\delta_0, \delta_1\}$, consider the k -tuple hypergraph H as above and consider the conditional distribution $\mu_{f;H}^{\nu_0}$ where we pin the vertex v_2 to 0 (this is allowed since f supports pinning-to-0). Then,

$$\mu_{f;H}^{\nu_0}(\sigma(v_1) = 0) = 1, \quad \mu_{f;H}^{\nu_0}(\sigma(v_1) = 1) = 0$$

so f simulates δ_0 . Analogously, by pinning the vertex v_1 to 1, we also obtain that f simulates δ_1 , concluding the proof. \square

Lemma 53 ([13, Proof of Lemma 19]). *If f is a Boolean function that is not in IM_2 , then $\{f, \text{Implies}, \delta_0, \delta_1\}$ implements either OR or NAND. \square*

Lemma 54. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a function which supports perfect equality. Suppose that f is not affine and is not in IM_2 . Then, for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard.*

Proof. The proof is similar in structure to the proof of Lemma 51.

Assume first that f is self-dual. Then, by Theorem 46 (note that f is not affine and supports perfect equality by assumption), f perfectly simulates a hard function. By Lemma 15, we obtain that for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard.

Assume next that f is not self-dual. By Lemma 49 we have that $\{f\}$ implements either δ_0 or δ_1 . We only need to consider the case where $\{f\}$ implements δ_0 , the case of δ_1 follows by switching the spins 0 and 1. First, by Lemma 42 with $\Gamma = \{f\}$ and $g = \delta_0$, f perfectly simulates δ_0 , so f perfectly simulates $\{f, \delta_0\}$. Recall that f is not affine. By Lemma 50, it thus follows that $\{f, \delta_0\}$ implements one of OR, NAND, Implies. Using Lemma 42 again, it follows that f perfectly simulates one of OR, NAND, Implies. OR and NAND correspond to hard functions, so when f perfectly simulates either OR or NAND, we obtain from Lemma 15 that for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard. Thus, it remains to consider the case where f perfectly simulates Implies.

Since f perfectly simulates Implies, by Lemma 52, we obtain that f simulates $\{\delta_0, \delta_1\}$. Thus, f simulates $\{f, \text{Implies}, \delta_0, \delta_1\}$. By Lemma 53, using that f is not in IM_2 , we have that $\{f, \text{Implies}, \delta_0, \delta_1\}$ implements either OR or NAND. By Lemma 44, we thus obtain that f simulates either OR or NAND. Hence, as above, we can use Lemma 15 to conclude that for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard.

This concludes the proof. \square

8.6. Proof of Theorem 18

We are ready to prove Theorem 18, which we restate here for convenience.

Theorem 18. *Let $f : \{0, 1\}^k \rightarrow \{0, 1\}$ be a Boolean function that is not affine. Suppose that f supports perfect equality.*

1. If f is in IM_2 , then for all sufficiently large Δ , $\#\text{Multi2Spin}_\Delta(f)$ is $\#\text{BIS}$ -equivalent.
2. If f is not in IM_2 , then for all sufficiently large Δ , there exists a real number $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard.

Proof. Item 1 is a consequence of Lemma 47 and Lemma 51. Item 2 is a consequence of Lemma 54. \square

9. Proof of Theorem 6

In this section, we combine the pieces to prove Theorem 6. We will need the following lemma.

Lemma 55. Let $f_1 : \{0, 1\}^{k_1} \rightarrow \{0, 1\}$ and $f_2 : \{0, 1\}^{k_2} \rightarrow \{0, 1\}$ be Boolean functions such that f_1 is not affine and f_2 is not in IM_2 . Then, the function f defined by $f(\mathbf{x}, \mathbf{y}) = f_1(\mathbf{x})f_2(\mathbf{y})$ is neither affine nor does it belong to IM_2 .

Proof. We first prove that f is not affine. Since f_1 is not affine, by Item 1 of Lemma 21, there exist $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \mathbf{x}^{(3)} \in R_{f_1}$ such that $\mathbf{x}^{(1)} \oplus \mathbf{x}^{(2)} \oplus \mathbf{x}^{(3)} \notin R_{f_1}$. Let \mathbf{y} be such that $f_2(\mathbf{y}) = 1$ (such a \mathbf{y} exists, otherwise f_2 would belong to IM_2).

For each $i = 1, 2, 3$, consider the vector $\mathbf{z}^{(i)}$ of length $k_1 + k_2$ obtained by concatenating the vectors $\mathbf{x}^{(i)}$ and \mathbf{y} . Since $\mathbf{x}^{(i)} \in R_{f_1}$ and $\mathbf{y} \in R_{f_2}$, we have that $f(\mathbf{z}^{(i)}) = f_1(\mathbf{x}^{(i)})f_2(\mathbf{y}) = 1$, so $\mathbf{z}^{(i)} \in R_f$ for $i = 1, 2, 3$. Observe that $f(\mathbf{z}^{(1)} \oplus \mathbf{z}^{(2)} \oplus \mathbf{z}^{(3)}) = f_1(\mathbf{x}^{(1)} \oplus \mathbf{x}^{(2)} \oplus \mathbf{x}^{(3)})f_2(\mathbf{y}) = 0$, so $\mathbf{z}^{(1)} \oplus \mathbf{z}^{(2)} \oplus \mathbf{z}^{(3)} \notin R_f$. Thus,

$$\mathbf{z}^{(1)}, \mathbf{z}^{(2)}, \mathbf{z}^{(3)} \in R_f \text{ but } \mathbf{z}^{(1)} \oplus \mathbf{z}^{(2)} \oplus \mathbf{z}^{(3)} \notin R_f,$$

so by Item 1 of Lemma 21, we have that f is not affine.

We next show that f does not belong to IM_2 . Since $f_2 \notin IM_2$, by Lemma 23, there exist $\mathbf{y}^{(1)}, \mathbf{y}^{(2)} \in R_{f_2}$ such that either $\mathbf{y}^{(1)} \vee \mathbf{y}^{(2)} \notin R_{f_2}$ or $\mathbf{y}^{(1)} \wedge \mathbf{y}^{(2)} \notin R_{f_2}$. Assume that $\mathbf{y}^{(1)} \vee \mathbf{y}^{(2)} \notin R_{f_2}$, the other case is completely analogous and actually follows by duality (switching the spins 0 and 1). Let \mathbf{x} be such that $f_1(\mathbf{x}) = 1$ (such an \mathbf{x} exists, otherwise f_1 would be affine).

For each $i = 1, 2$, consider the vector $\mathbf{w}^{(i)}$ of length $k_1 + k_2$ obtained by concatenating the vectors \mathbf{x} and $\mathbf{y}^{(i)}$. Since $\mathbf{x} \in R_{f_1}$ and $\mathbf{y}^{(i)} \in R_{f_2}$, we have that $f(\mathbf{w}^{(i)}) = f_1(\mathbf{x})f_2(\mathbf{y}^{(i)}) = 1$, so $\mathbf{w}^{(i)} \in R_f$ for $i = 1, 2$. Observe that $f(\mathbf{w}^{(1)} \vee \mathbf{w}^{(2)}) = f_1(\mathbf{x})f_2(\mathbf{y}^{(1)} \vee \mathbf{y}^{(2)}) = 0$, so $\mathbf{w}^{(1)} \vee \mathbf{w}^{(2)} \notin R_f$. Thus,

$$\mathbf{w}^{(1)}, \mathbf{w}^{(2)} \in R_f \text{ but } \mathbf{w}^{(1)} \vee \mathbf{w}^{(2)} \notin R_f,$$

so by Lemma 23, we have that f does not belong to IM_2 .

This concludes the proof. \square

Theorem 6. Let Γ be a Boolean constraint language. Then,

1. If every function in Γ is affine then $\#\text{CSP}(\Gamma)$ and $\#\text{NoRepeatCSP}(\Gamma)$ are both in FP.
2. Otherwise, if $\Gamma \subseteq IM_2$, then there exists an integer Δ_0 such that for all $\Delta \geq \Delta_0$, $\#\text{CSP}_\Delta(\Gamma)$ and $\#\text{NoRepeatCSP}_\Delta(\Gamma)$ are both $\#\text{BIS}$ -equivalent under AP-reductions, and
3. Otherwise, there exists an integer Δ_0 such that for all $\Delta \geq \Delta_0$, there exists a real number $c > 1$ such that $\#\text{CSP}_{\Delta,c}(\Gamma)$ and $\#\text{NoRepeatCSP}_{\Delta,c}(\Gamma)$ are both NP-hard.

Proof. We consider each of the three cases.

1. If every function in Γ is affine (cf. Definition 4), then Z_Γ can be computed exactly in polynomial time using Gaussian elimination. This was already noted in the exact-counting dichotomy of Creignou and Hermann [7].
2. Suppose that $\Gamma \subseteq IM_2$ and that Γ includes a function f which is not affine. By the unbounded-degree $\#\text{BIS}$ -easiness result of [13], which is stated here as Lemma 47, it follows that for all positive integers Δ , $\#\text{CSP}_\Delta(\Gamma)$ is $\#\text{BIS}$ -easy. Clearly, every instance of $\#\text{NoRepeatCSP}_\Delta(\Gamma)$ is an instance of $\#\text{CSP}_\Delta(\Gamma)$, from which we obtain that $\#\text{NoRepeatCSP}_\Delta(\Gamma)$ is $\#\text{BIS}$ -easy as well.

If f supports perfect equality, then by Theorem 18, for all sufficiently large Δ , the problem $\#\text{Multi2Spin}_\Delta(f)$ is $\#\text{BIS}$ -hard. As we noted in Section 2, the problem $\#\text{Multi2Spin}_\Delta(f)$ is equivalent to $\#\text{NoRepeatCSP}_\Delta(\{f\})$ from which we obtain that $\#\text{NoRepeatCSP}_\Delta(\Gamma)$ is $\#\text{BIS}$ -hard as well. Note that $\#\text{NoRepeatCSP}_\Delta(\Gamma)$ is a restricted version of $\#\text{CSP}_\Delta(\Gamma)$ (the restriction being that constraints may not repeat variables), so it follows immediately that $\#\text{CSP}_\Delta(\Gamma)$ is $\#\text{BIS}$ -hard. There is a final case that does not arise if $\#\text{BIS}$ is not NP-hard to approximate, but we include it to make the proof complete. In particular, if f does not support perfect equality, then by Theorem 17, it simulates a hard function. So, by Lemma 15, for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard. As observed in the proof of Lemma 51, this implies that $\#\text{Multi2Spin}_\Delta(f)$ is $\#\text{BIS}$ -hard. Then, as in the previous case, we obtain that $\#\text{NoRepeatCSP}_\Delta(\Gamma)$ and $\#\text{CSP}_\Delta(\Gamma)$ are $\#\text{BIS}$ -hard.

3. Suppose that there are functions $f_1, f_2 \in \Gamma$ such that f_1 is not affine and f_2 is not in IM_2 (it might be the case that $f_1 = f_2$). Then, consider the function $f(\mathbf{x}, \mathbf{y})$ defined by $f(\mathbf{x}, \mathbf{y}) = f_1(\mathbf{x})f_2(\mathbf{y})$. By Lemma 55, we have that f is neither affine nor does it belong to IM_2 .

Thus, if f supports perfect equality, then by Theorem 18, for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard, which is equivalent to saying that $\#\text{NoRepeatCSP}_{\Delta,c}(\{f\})$ is NP-hard. Now note that there is an easy reduction from $\#\text{NoRepeatCSP}_{\Delta,c}(\{f\})$ to $\#\text{NoRepeatCSP}_{\Delta,c}(\Gamma)$ – given an instance I of $\#\text{CSP}_{\Delta,c}(\{f\})$, every constraint involving f is re-written as two constraints involving f_1 and f_2 . Thus, $\#\text{NoRepeatCSP}_{\Delta,c}(\Gamma)$ is also NP-hard. Since $\#\text{NoRepeatCSP}_{\Delta}(\Gamma)$ is a restricted version of $\#\text{CSP}_{\Delta}(\Gamma)$, we have that $\#\text{CSP}_{\Delta,c}(\Gamma)$ is NP-hard as well. Otherwise, by Theorem 17, f simulates a hard function. So, by Lemma 15, for all sufficiently large Δ , there exists $c > 1$ such that $\#\text{Multi2Spin}_{\Delta,c}(f)$ is NP-hard. As in the previous paragraph, this implies that $\#\text{NoRepeatCSP}_{\Delta,c}(\Gamma)$ and $\#\text{CSP}_{\Delta,c}(\Gamma)$ are NP-hard. \square

CRedit authorship contribution statement

Andreas Galanis: Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing - original draft, Writing - review & editing. **Leslie Ann Goldberg:** Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing - original draft, Writing - review & editing. **Kuan Yang:** Conceptualization, Formal analysis, Investigation, Methodology, Validation, Writing - original draft, Writing - review & editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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