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A quadratic identity for the number of perfect matchings of plane graphs

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ABSTRACT

We present a quadratic identity on the number of perfect matchings of plane graphs by the method of graphical condensation, which generalizes the results found by Propp [J. Propp, Generalized domino-shuffling, Theoret. Comput. Sci. 303 (2003) 267–301], Kuo [E.H. Kuo, Applications of graphical condensation for enumerating matchings and tilings, Theoret. Comput. Sci. 319 (2004) 29–57], and Yan, Yeh, and Zhang [W.G. Yan, Y.-N. Yeh, F.J. Zhang, Graphical condensation of plane graphs: A combinatorial approach, Theoret. Comput. Sci. 349 (2005) 452–461].

1. Introduction

Throughout this paper, we suppose that $G = (V(G), E(G))$ is a simple graph with the vertex set $V(G) = \{1, 2, \dots, n\}$ and the edge set $E(G) = \{e_1, e_2, \dots, e_m\}$, if not specified. A perfect matching of G is a set of independent edges of G covering all vertices of G. Denote the number of perfect matchings of G by $M(G)$. If G is a weighted graph, the weight of a perfect matching P of G is defined to be the product of weights of edges in P. We also denote the sum of weights of perfect matchings of G by $M(G)$. Let $A = \{a_1, a_2, \dots, a_s\}$ (resp. $E_1 = \{e_{i_1}, e_{i_2}, \dots, e_{i_t}\}$) be a subset of the vertex set $V(G)$ (resp. a subset of the edge set $E(G)$). By $G - A$ or $G - a_1 - a_2 - \cdots - a_s$ (resp. $G - E_1$ or $G - e_{i_1} - e_{i_2} - \cdots - e_{i_t}$) we denote the induced subgraph of G by deleting all vertices in A and the incident edges from G (resp. by deleting all edges in E_1). It is well known that computing $M(G)$ of a graph G is an NP-complete problem (see [4]). In this paper, by using a Pfaffian identity and a previous result [18], we obtain a quadratic identity on the number of perfect matchings of plane graphs by using the method of graphical condensation, which generalizes the results by Propp [14], Kuo [8], and Yan, Yeh, and Zhang [17] as follows:

Theorem 1.1. *Let G be a plane weighted graph with $2n$ vertices. Let vertices $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ $(2 \le k \le n)$ appear in a cyclic order on a face of G, and let $A = \{a_1, a_2, \dots, a_k\}$, $B = \{b_1, b_2, \dots, b_k\}$. Then, for any $j = 1, 2, \dots, k$, we have*

$$M(G)M(G - A - B) = \sum_{i=1}^{k} M(G - a_j - b_i)M(G - \overline{\{a_j, b_i\}}) - \sum_{1 \le i \le k, i \ne j} M(G - a_i - a_j)M(G - \overline{\{a_i, a_j\}}),$$

where $\overline{\{a_j, b_i\}} = (A \cup B) \backslash \{a_j, b_i\}$ and $\overline{\{a_i, a_j\}} = (A \cup B) \backslash \{a_i, a_j\}$.

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The following result is immediate from the above theorem.

Corollary 1.1 (*Yan, Yeh, and Zhang [17]*). *Let $G = (U, V)$ be a plane weighted bipartite graph in which $U = \{u_i | 1 \le i \le n\}$ and $V = \{v_i | 1 \le i \le n\}$. Let vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order on a face of G. If $A = \{a_i | 1 \le i \le k\} \subseteq U$, and $B = \{b_i | 1 \le i \le k\} \subseteq V$, then*

$$M(G)M(G - A - B) = \sum_{i=1}^{k} M(G - a_j - b_i)M(G - (A \cup B)\backslash\{a_j, b_i\})$$

for any $j = 1, 2, \ldots, k$.

Corollary 1.2 (*Kenyon*). *Let G be a plane graph with four vertices a, b, c and d (in the cyclic order) adjacent to a single face. Then*

$$\begin{aligned} &M(G)M(G - a - b - c - d) + M(G - a - c)M(G - b - d) \\ &\quad = M(G - a - b)M(G - c - d) + M(G - a - d)M(G - b - c). \end{aligned}$$

Corollary 1.2, which was first reported by Kenyon in "Domino Forum" in an email (for details, see [17]), is the special case of Theorem 2.2 in [17].

Some related work on graphical condensation for enumerating perfect matchings of plane graphs can be found in [14,8, 9,17–19].

2. A Pfaffian identity

Let $A = (a_{ij})_{n \times n}$ be a skew symmetric matrix of order n, where n is even. Suppose that $\pi = \{(s_1, t_1), (s_2, t_2), \ldots, (s_{\frac{n}{2}}, t_{\frac{n}{2}})\}$ is a partition of $[n]$, that is, $[n] = \{s_1, t_1\} \cup \{s_2, t_2\} \cup \cdots \cup \{s_{\frac{n}{2}}, t_{\frac{n}{2}}\}$, where $[n] = \{1, 2, \ldots, n\}$. Define:

$$b_\pi = \operatorname{sgn}(s_1 t_1 s_2 t_2 \ldots s_{\frac{n}{2}} t_{\frac{n}{2}}) \prod_{l=1}^{\frac{n}{2}} a_{s_l t_l},$$

where $\operatorname{sgn}(s_1 t_1 s_2 t_2 \ldots s_{\frac{n}{2}} t_{\frac{n}{2}})$ denotes the sign of the permutation $s_1 t_1 s_2 t_2 \ldots s_{\frac{n}{2}} t_{\frac{n}{2}}$. Note that b_π depends neither on the order in which the classes of the partition are listed nor on the order of the two elements of a class. So b_π indeed depends only on the choice of the partition π. The Pfaffian of A, denoted by $Pf(A)$, is defined as

$$Pf(A) = \sum_{\pi} b_\pi,$$

where the summation is over all partitions of $[n]$, which are of the form of π. For the sake of convenience, we define the Pfaffian of A to be zero if A is a skew symmetric matrix of odd order. The following result, which is called Cayley's Theorem, is well known:

Theorem 2.2 (*[1]*). *For any skew symmetric matrix $A = (a_{ij})_{n \times n}$ of order n, we have*

$$\det(A) = [Pf(A)]^2.$$

If I is a subset of $[n]$, we use A_I to denote the submatrix of a skew symmetric matrix A by deleting rows and columns indexed by I. If $I = \{i_1, i_2, \ldots, i_l\} \subseteq [n]$, we use $Pf_A(i_1 i_2 \ldots i_l) =: Pf_A(I)$ to denote the Pfaffian of $A_{[n]\backslash I}$.

Lemma 2.1 (*Wenzel [16], Dress and Wenzel [3], and Knuth [7]*). *For any two subsets $I_1, I_2 \subseteq [n]$ of odd cardinality and elements $i_1, i_2, \ldots, i_t \in [n]$ with $i_1 < i_2 \cdots < i_t$ and $\{i_1, i_2, \ldots, i_t\} = I_1 \triangle I_2 =: (I_1 \backslash I_2) \cup (I_2 \backslash I_1)$, if $A = (a_{ij})_{n \times n}$ is a skew symmetric matrix with n even, then*

$$\sum_{\tau=1}^{t} (-1)^\tau Pf_A(I_1 \triangle \{i_\tau\}) Pf(I_2 \triangle \{i_\tau\}) = 0.$$

A direct result of Lemma 2.1 is the following lemma.

Lemma 2.2 (*[18]*). *Suppose that $A = (a_{ij})_{n \times n}$ is a skew symmetric matrix with n even and α is a subset of $[n]$ of even cardinality. Let $\beta = \{i_1, i_2, \ldots, i_{2p}\} \subseteq [n]\backslash\alpha$, where $i_1 < i_2 < \cdots < i_{2p}$. Then, for any fixed $s \in [2p]$, we have*

$$Pf_A(\alpha)Pf_A(\alpha\beta) = \sum_{l=1}^{2p} (-1)^{l+s+1} Pf_A(\alpha i_s i_l) Pf_A(\alpha\beta \backslash i_s i_l),$$

where $Pf_A(\alpha i_s i_s) = 0$.

3. Pfaffian orientations of graphs

Let G be a weighted graph with the vertex set $V(G) = \{1, 2, \dots, n\}$, and let the weight of each edge e in G be ω_e. Suppose $\vec{G}$ is an orientation of G. The skew adjacency matrix of $\vec{G}$ (see [11]), denoted by $A(\vec{G}) = (a_{ij})_{n\times n}$, is defined as follows:

$$a_{ij} = \begin{cases} \omega_{e_{ij}} & \text{if } (i,j) \text{ is an arc of } \vec{G}, \\ -\omega_{e_{ij}} & \text{if } (j,i) \text{ is an arc of } \vec{G}, \\ 0 & \text{otherwise,} \end{cases}$$

where e_{ij} denotes the edge of G joining vertices i and j. Obviously, $A(\vec{G})$ is a skew symmetric matrix. It is not difficult to see that the Pfaffian $Pf(A(\vec{G}))$ of $A(\vec{G})$ can be defined as

$$Pf(A(\vec{G})) = \sum_{\pi\in\mathcal{M}(G)} b_\pi,$$

where the summation is over all perfect matchings $\pi = \{(s_1,t_1),(s_2,t_2),\dots,(s_{\frac{n}{2}},t_{\frac{n}{2}})\}$ in $\mathcal{M}(G)$ (the set of perfect matchings of G), and $b_\pi = \operatorname{sgn}(s_1t_1s_2t_2s_{\frac{n}{2}}t_{\frac{n}{2}})\prod_{i=1}^{\frac{n}{2}}\omega_{(s_i,t_i)}$.

If $\vec{G}$ is an orientation of a graph G and C is a cycle of even length, we say that C is oddly oriented in $\vec{G}$ if C contains odd number of edges that are directed in $\vec{G}$ in the direction of each orientation of C. We say that $\vec{G}$ is a Pfaffian orientation of G if every nice cycle of even length of G is oddly oriented in $\vec{G}$ (a cycle C in G is nice if $G - C$ has perfect matchings). It is well known that if a graph G contains no subdivision of $K_{3,3}$ then G has a Pfaffian orientation (see [10]). McCuaig [12], McCuaig et al. [13], and Robertson et al. [15] found a polynomial–time algorithm to show whether a bipartite graph has a Pfaffian orientation.

Proposition 3.1 ([6,11]). *Let $\vec{G}$ be a Pfaffian orientation of a graph G. Then*

$$[M(G)]^2 = \det(A(\vec{G})),$$

where $A(\vec{G})$ is the skew adjacency matrix of $\vec{G}$.

Remark 3.1. Let $\vec{G}$ be a Pfaffian orientation of a graph G and $A(\vec{G})$ the skew adjacency matrix of $\vec{G}$. By Cayley's Theorem and Proposition 3.1, we have

$$M(G) = \pm Pf(A(\vec{G})),$$

which implies that, for two arbitrary perfect matchings π_1 and π_2 of G, both b_{π_1} and b_{π_2} have the same sign.

Proposition 3.2 (*Kasteleyn's Theorem, [5,6,11]*). *Every plane graph G has an orientation $\vec{G}$ such that every boundary face—except possibly the unbounded face—has an odd number of edges oriented clockwise. Furthermore, such an orientation is a Pfaffian orientation.*

4. Proof of the main result

In [18] some new identities on Pfaffians related to the Plücker relation were obtained. As an application of one of these new identities on Pfaffians, Yeh and the first author of this paper proved the following result which plays a key role in the proof of our main result:

Theorem 4.3 (*Yan and Yeh [18]*). *Suppose G is a plane weighted graph with an even number of vertices and the weight of every edge e in G is denoted by ω_e. Let $e_1 = a_1b_1, e_2 = a_2b_2, \dots, e_k = a_kb_k$ ($k \geq 2$) be k independent edges in the boundary of a face f of G, and let vertices $a_1, b_1, a_2, b_2, \dots, a_k, b_k$ appear in a cyclic order on f and let $X = \{e_i \mid i = 1, 2, \dots, k\}$. Then, for any $j = 1, 2, \dots, k$,*

$$\begin{aligned} M(G)M(G-X) = {} & M(G-e_j)M(G-X\backslash\{e_j\}) + \omega_{e_j}\sum_{1\le i\le k, i\ne j}\omega_{e_i}(M(G-b_j-a_i)M(G-X-a_j-b_i) \\ & - M(G-b_j-b_i)M(G-X-a_j-a_i)). \end{aligned}$$

Let $G = (V(G), E(G))$ be a weighted graph and $e = ab$ an edge of G. Define a new weighted graph $G' = (V(G'), E(G'))$ from G as follows. Delete the edge $e = ab$ from G and add three edges $aa', a'b', b'b$ with the weights $\sqrt{\omega_e}$, 1, and $\sqrt{\omega_e}$, where ω_e denotes the weight of edge e. The resulting weighted graph is G'. Hence $V(G') = \{a', b'\} \cup V(G)$ and $E(G') = \{aa', a'b', b'b\} \cup E(G)\backslash\{e\}$. Fig. 1(a) and (b) illustrate this procedure.

Lemma 4.3 (*Ciucu [2]*). *Let G be a weighted graph and $e = ab$ an edge of G, and let G' be the weighted graph defined above. Then*

$$M(G) = M(G').$$

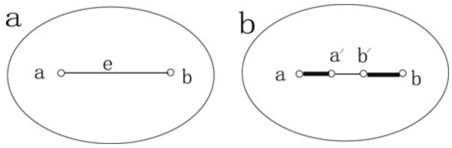

Fig. 1. (a) The weighted graph G in Lemma 4.3. (b) The weighted graph G' obtained from G in Lemma 4.3.

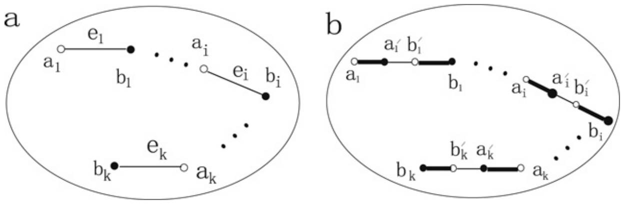

Fig. 2. (a) The weighted graph G in the proof of Lemma 4.4. (b) The weighted graph G' obtained from G in the proof of Lemma 4.4.

We first use the above theorem to prove the following result which is a special case of our main result:

Lemma 4.4. *Suppose G is a plane weighted graph with an even number of vertices and the weight of every edge e in G is denoted by ω_e. Let $e_1 = a_1b_1, e_2 = a_2b_2, \ldots, e_k = a_kb_k$ ($k \geq 2$) be k independent edges in the boundary of a face f of G ($\omega_{e_i} \neq 0$ for $1 \leq i \leq k$), where vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order on f. Then, for any $j = 1, 2, \ldots, k$, we have*

$$M(G)M(G-A-B) = \sum_{i=1}^{k} M(G-a_j-b_i)M(G-\overline{\{a_j,b_i\}}) - \sum_{i=1}^{k} M(G-a_i-a_j)M(G-\overline{\{a_i,a_j\}}),$$

where $A = \{a_i | 1 \leq i \leq k\}$, $B = \{b_i | 1 \leq i \leq k\}$, $\overline{\{a_j, b_i\}} = (A \cup B)\backslash\{a_j, b_i\}$, and $\overline{\{a_i, a_j\}} = (A \cup B)\backslash\{a_i, a_j\}$.

Proof. Let G' be the graph obtained from G by deleting k edges $e_1, e_2, \ldots, e_k$ and adding $3k$ edges a_ia_i', $a_i'b_i'$, $b_i'b_i$ with the weights $\sqrt{\omega_{e_i}}$, 1, $\sqrt{\omega_{e_i}}$ for $i = 1, 2, \ldots, k$, and leaving all other weights unchanged. Hence, the vertex set of G', denoted by $V(G')$, is $\{a_i', b_i' | 1 \leq i \leq k\} \cup V(G)$, and the edge set of G', denoted by $E(G')$, is $\{a_ia_i', a_i'b_i', b_i'b_i | i = 1, 2, \ldots, k\} \cup E(G)\backslash\{e_i | 1 \leq i \leq k\}$, where $V(G)$ and $E(G)$ are the vertex set and the edge set of G, respectively. For the sake of convenience, denote the edge $a_i'b_i'$ by $e_i' = a_i'b_i'$ for $i = 1, 2, \ldots, k$. Fig. 2(a) and (b) show this procedure.

Hence, by Lemma 4.3, we have

$$M(G) = M(G'). \tag{1}$$

Obviously, by the definition of G', G' is a plane weighted graph with an even number of vertices. Furthermore, vertices $a_1', b_1', a_2', b_2', \ldots, a_k', b_k'$ appear in a cyclic order on a face of G', and $e_1' = a_1'b_1', e_2' = a_2'b_2', \ldots, e_k' = a_k'b_k'$ ($k \geq 2$) are k independent edges in the boundary of a face f of G'. Let $X' = \{e_i' | i = 1, 2, \ldots, k\}$. Note that the weight of each edge e_j' in G', for $j = 1, 2, \ldots, k$, equals 1. Then, by Theorem 4.3, for any $j = 1, 2, \ldots, k$,

$$\begin{aligned} M(G')M(G'-X') &= M(G'-e_j')M(G'-X'\backslash\{e_j'\}) + \sum_{1\leq i\leq k, i\neq j} [M(G'-b_j'-a_i')M(G'-X'-a_j'-b_i') \\ &\quad - M(G'-b_j'-b_i')M(G'-X'-a_j'-a_i')]. \end{aligned} \tag{2}$$

It is not difficult to see that the following identities hold:

$$M(G'-X') = \left(\prod_{i=1}^{k} \omega_{e_i}\right) M(G-A-B); \tag{3}$$

$$M(G'-e_j') = \omega_{e_j} M(G-a_j-b_j); \tag{4}$$

$$M(G'-X'\backslash\{e_j'\}) = \left(\prod_{1\leq i\leq k, i\neq j} \omega_{e_i}\right) M(G-(A\cup B)\backslash\{a_j,b_j\}); \tag{5}$$

$$M(G'-b_j'-a_i') = \sqrt{\omega_{e_i}\omega_{e_j}} M(G-a_j-b_i); \tag{6}$$

$$M(G'-X'-a_j'-b_i') = \sqrt{\omega_{e_i}\omega_{e_j}} \left(\prod_{1\leq s\leq k, s\neq i,j} \omega_{e_s}\right) M(G-(A\cup B)\backslash\{a_j,b_i\}); \tag{7}$$

$$M(G'-b_j'-b_i') = \sqrt{\omega_{e_i}\omega_{e_j}} M(G-a_i-a_j); \tag{8}$$

$$M(G'-X'-a_j'-a_i') = \sqrt{\omega_{e_i}\omega_{e_j}} \left(\prod_{1\leq s\leq k, s\neq i,j} \omega_{e_s}\right) M(G-(A\cup B)\backslash\{a_i,a_j\}). \tag{9}$$

Hence the lemma follows from (1)–(9). ■

Now we can start to prove our main result.

Proof of Theorem 1.1. Since G is a plane graph, for an arbitrary face F of G there exists a planar embedding of G such that the face F is the unbounded one. Hence we may assume that vertices $a_1, b_1, a_2, b_2, \ldots, a_k, b_k$ appear in a cyclic order in the boundary of the unbounded face of G. We construct a plane graph G_1 from G by adding edge (a_i, b_i) with weight 1 if (a_i, b_i) is not an edge of G for $1 \le i \le k$ such that all edges (a_i, b_i)'s are in the boundary of the unbounded face of G_1. If all (a_i, b_i)'s are edges of G, then $G = G_1$. Hence, by Lemma 4.4, the theorem holds. So we may assume $G \neq G_1$. Note that edges $e_1 = (a_i, b_i), e_2 = (a_2, b_2), \ldots, (a_k, b_k)$ are k independent edges in the boundary of the unbounded face of G_1. By Lemma 4.4, we have

$$M(G_1)M(G_1 - A - B) = \sum_{i=1}^{k} M(G_1 - a_j - b_i)M(G_1 - \overline{\{a_j, b_i\}}) - \sum_{i=1}^{k} M(G_1 - a_i - a_j)M(G_1 - \overline{\{a_i, a_j\}}). \tag{10}$$

On the other hand, by Proposition 3.2 plane graph G_1 has a Pfaffian orientation $\vec{G}_1$ such that every boundary face–except possibly the unbounded face—has an odd number of edges oriented clockwise. Let $A(\vec{G}_1)$ be the skew adjacency matrix of $\vec{G}_1$. Without loss of generality, we suppose that $a_i = n - 2k + 2i - 1$ and $b_i = n - 2k + 2i$ for $i = 1, 2, \ldots, k$. Let $\beta = \{n - 2k + 1, n - 2k + 2, \ldots, n\} = \{a_i | 1 \le i \le k\} \cup \{b_i | 1 \le i \le k\} = A \cup B$, $\alpha = [n] \backslash \beta$, and $s = 2j - 1$. Then, by Lemma 2.2, we have

$$Pf_{A(\vec{G}_1)}(\alpha)Pf_{A(\vec{G}_1)}([n]) = \sum_{i=1}^{k} Pf_{A(\vec{G}_1)}(\alpha a_j b_i)Pf_{A(\vec{G}_1)}([n]\backslash\{a_j, b_i\}) - \sum_{i=1}^{k} Pf_{A(\vec{G}_1)}(\alpha a_j a_i)Pf_{A(\vec{G}_1)}([n]\backslash\{a_j, a_i\}), \tag{11}$$

which is equivalent to

$$\begin{aligned} Pf(A(\vec{G}_1 - A - B))Pf(A(\vec{G}_1)) &= \sum_{i=1}^{k} Pf(A(\vec{G}_1 - (A \cup B)\backslash\{a_j, b_i\}))Pf(A(\vec{G}_1 - a_j - b_i)) \\ &\quad - \sum_{i=1}^{k} Pf(A(\vec{G}_1 - (A \cup B)\backslash\{a_i, a_j\}))Pf(A(\vec{G}_1 - a_i - a_j)), \end{aligned} \tag{11$'$}$$

where $A(\vec{G}_1 - A - B)$ denotes the skew adjacency matrix of $\vec{G}_1 - A - B$. Note that all orientations $\vec{G}_1 - A - B$, $\vec{G}_1 - (A \cup B)\backslash\{a_j, b_i\}$, $\vec{G}_1 - a_j - b_i$, $\vec{G}_1 - (A \cup B)\backslash\{a_i, a_j\}$, and $\vec{G}_1 - a_i - a_j$ satisfy the condition in Proposition 3.2 (since all edges (a_i, b_i)'s are in the boundary of the unbounded face of G_1), and hence are Pfaffian orientations. By Remark 3.1, we have

$$M(G_1) = \pm Pf(A(\vec{G}_1)), \qquad M(G_1 - A - B) = \pm Pf(A(\vec{G}_1 - A - B)), \tag{12}$$

$$M(G_1 - a_j - b_i) = \pm Pf(A(\vec{G}_1 - a_j - b_i)), \tag{13}$$

$$M(G_1 - \overline{\{a_j, b_i\}}) = \pm Pf(A(\vec{G}_1 - (A \cup B)\backslash\{a_j, b_i\})), \tag{14}$$

$$M(G_1 - a_i - a_j) = \pm Pf(A(\vec{G}_1 - a_i - a_j)), \tag{15}$$

$$M(G_1 - \overline{\{a_i, a_j\}}) = \pm Pf(A(\vec{G}_1 - (A \cup B)\backslash\{a_i, a_j\})). \tag{16}$$

By (10) and (11′), and (12)–(16), we have the following:

Claim 1. *All terms $Pf(A(\vec{G}_1 - A - B))Pf(A(G_1^e))$, $Pf(A(\vec{G}_1 - (A \cup B)\backslash\{a_j, b_i\}))Pf(A(\vec{G}_1 - a_j - b_i))$, and $Pf(A(\vec{G}_1 - (A \cup B)\backslash\{a_i, a_j\}))Pf(A(\vec{G}_1 - a_i - a_j))$ have the same sign.*

Let $\vec{G}$ be the orientation of G obtained from $\vec{G}_1$ by deleting the arcs whose end vertices are a_i and b_i such that (a_i, b_i) is not an edge of G for $1 \le i \le k$. Let $A(\vec{G})$ be the skew adjacency matrix of $\vec{G}$. Similar to (11) and (11′), we have

$$Pf_{A(\vec{G})}(\alpha)Pf_{A(\vec{G})}([n]) = \sum_{i=1}^{k} Pf_{A(\vec{G})}(\alpha a_j b_i)Pf_{A(\vec{G})}([n]\backslash\{a_j, b_i\}) - \sum_{i=1}^{k} Pf_{A(\vec{G})}(\alpha a_j a_i)Pf_{A(\vec{G})}([n]\backslash\{a_j, a_i\}), \tag{17}$$

which is equivalent to

$$\begin{aligned} Pf(A(\vec{G} - A - B))Pf(A(\vec{G})) &= \sum_{i=1}^{k} Pf(A(\vec{G} - (A \cup B)\backslash\{a_j, b_i\}))Pf(A(\vec{G} - a_j - b_i)) \\ &\quad - \sum_{i=1}^{k} Pf(A(\vec{G} - (A \cup B)\backslash\{a_i, a_j\}))Pf(A(\vec{G} - a_i - a_j)), \end{aligned} \tag{17$'$}$$

where $A(\vec{G} - A - B)$ denotes the skew adjacency matrix of $\vec{G} - A - B$.

Note that all edges (a_i, b_i)'s are in the boundary of the unbounded face of G_1. Thus $\vec{G}$ satisfies the condition in Proposition 3.2. Hence it is a Pfaffian orientation. Similarly, all orientations $\vec{G} - A - B$, $\vec{G} - (A \cup B)\backslash\{a_j, b_i\}$, $\vec{G} - a_j - b_i$, $\vec{G} - (A \cup B)\backslash\{a_i, a_j\}$, and $\vec{G} - a_i - a_j$ satisfy the condition in Proposition 3.2, and hence are also Pfaffian orientations. Thus

$$M(G) = \pm Pf(A(\vec{G})), \qquad M(G - A - B) = \pm Pf(A(\vec{G} - A - B)), \tag{18}$$

$$M(G - a_j - b_i) = \pm Pf(A(\vec{G} - a_j - b_i)), \tag{19}$$

$$M(G - \overline{\{a_j, b_i\}}) = \pm Pf(A(\vec{G} - (A \cup B)\backslash\{a_j, b_i\})), \tag{20}$$

$$M(G - a_i - a_j) = \pm Pf(A(\vec{G} - a_i - a_j)), \tag{21}$$

$$M(G - \overline{\{a_i, a_j\}}) = \pm Pf(A(\vec{G} - (A \cup B)\backslash\{a_i, a_j\})), \tag{22}$$

In view of (17′) and (18)–(22), to finish the proof of the theorem, it suffices to prove the following:

Claim 2. *All terms $Pf(A(\vec{G} - A - B))Pf(A(\vec{G}))$, $Pf(A(\vec{G} - (A \cup B)\backslash\{a_j, b_i\}))Pf(A(\vec{G} - a_j - b_i))$, and $Pf(A(\vec{G} - (A \cup B)\backslash\{a_i, a_j\}))Pf(A(\vec{G} - a_i - a_j))$ have the same sign.*

Note that each perfect matching $\pi = \{(s_1, t_1), (s_2, t_2), \ldots, (s_{\frac{n}{2}}, t_{\frac{n}{2}})\}$ of G is also a perfect matching of G_1. By the definitions of $\vec{G}$ and $\vec{G_1}$, the term b_π in $Pf(A(\vec{G}))$ has the same sign as b_π in $Pf(A(\vec{G_1}))$. Hence both $Pf(A(\vec{G}))$ and $Pf(A(\vec{G_1}))$ have the same sign. Similarly, so have both $Pf(A(\vec{G} - A - B))$ and $Pf(A(\vec{G_1} - A - B))$, both $Pf(A(\vec{G} - (A \cup B)\backslash\{a_j, b_i\}))$ and $Pf(A(\vec{G_1} - (A \cup B)\backslash\{a_j, b_i\}))$, both $Pf(A(\vec{G} - a_j - b_i))$ and $Pf(A(\vec{G_1} - a_j - b_i))$, both $Pf(A(\vec{G} - (A \cup B)\backslash\{a_i, a_j\}))$ and $Pf(A(\vec{G_1} - (A \cup B)\backslash\{a_i, a_j\}))$, and both $Pf(A(\vec{G} - a_i - a_j))$ and $Pf(A(\vec{G_1} - a_i - a_j))$. Claim 2 follows from Claim 1.

Hence we have finished the proof of the theorem. ∎

5. Some remarks

Propp [14] and Kuo [8] first found the method of graphical condensation for enumerating perfect matchings of plane bipartite graphs. For the plane graph (not necessarily bipartite), in an email sent to "Domino Forum" Propp wrote that Kenyon recently told him about the identity of Pfaff's (Corollary 1.2) in combination with Kasteleyn's Pfaffian method. Professor Krattenthaler told one of the current authors by an email that one could use directly the Plücker relation on Pfaffians to obtain some Pfaffian identities. In this paper, by using a Pfaffian identity and a result in [18] we generalize Kenyon's identity. It is natural to ask whether one can use directly the Plücker relation on Pfaffians to prove Theorem 1.1.

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References

[1] A. Cayley, Sur les Déterminants gauches, J. Reine Angew. Math. 38 (1848) 93–96; or Collected Mathematical Papers I, Cambridge U.P, Cambridge, 1889–97, pp. 410–413.
[2] M. Ciucu, Enumeration of perfect matchings of cellular graphs, J. Algebraic Combin. 5 (1996) 87–103.
[3] A.W.M. Dress, W. Wenzel, A simple proof of an identity concerning Pfaffians of skew symmetric matrices, Adv. Math. 112 (1995) 120–134.
[4] M. Jerrum, Two-dimensional monomer–dimer systems are computationally intractable, J. Stat. Phys. 48 (1987) 121–134.
[5] P.W. Kasteleyn, Dimer statistics and phase transition, J. Math. Phys. 4 (1963) 287–293.
[6] P.W. Kasteleyn, in: F. Harary (Ed.), Graph Theory and Crystal Physics, Graph Theory and Theoretical Physics, Academic Press, 1967, pp. 43–110.
[7] D.E. Knuth, Overlapping Pfaffians, Electron. J. Combin. 3 (1996) R5.
[8] E.H. Kuo, Applications of graphical condensation for enumerating matchings and tilings, Theoret. Comput. Sci. 319 (2004) 29–57.
[9] E.H. Kuo, Graphical condensation generalizations involving Pfaffians and determinants, preprint.
[10] C.H.C. Little, A characterization of convertible (0, 1)-matrices, J. Combin. Theory Ser. B 18 (1975) 187–208.
[11] L. Lovász, M. Plummer, Matching Theory, in: Ann. Discrete Math., vol. 29, North-Holland, New York, 1986.
[12] W. McCuaig, Pólya's permanent problem, Electron. J. Combin. 11 (1) (2004) R79.
[13] W. McCuaig, N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations, and even directed circuits (Extended abstract), in: Proc. Symp. on the Theory of Computing, STOC97, 1997.
[14] J. Propp, Generalized domino-shuffling, Theoret. Comput. Sci. 303 (2003) 267–301.
[15] N. Robertson, P.D. Seymour, R. Thomas, Permanents, Pfaffian orientations, and even directed circuits, Ann. Math. 150 (1999) 929–975.
[16] W. Wenzel, Geometric Algebra of Δ-Matroids and Related Combinatorial Geometries, Habilitationsschrift, Bielefeld, 1991.
[17] W.G. Yan, Y.-N. Yeh, F.J. Zhang, Graphical condensation of plane graphs: A combinatorial approach, Theoret. Comput. Sci. 349 (2005) 452–461.
[18] W.G. Yan, Y.-N. Yeh, Replacing Pfaffians and applications, Adv. Appl. Math. 39 (2007) 121–140.
[19] W.G. Yan, F.J. Zhang, Graphical condensation for enumerating perfect matchings, J. Combin. Theory Ser. A 110 (2005) 113–125.