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Deciding the point-to-fixed-point problem for skew tent maps on an interval

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ABSTRACT

We consider a family of skew tent maps f_a on the unit interval, determined by the parameter a, with $0 < a < 1$. We give a decision procedure, that on input a and a point x_0 in the unit interval, determines whether or not the sequence $x_0, f_a(x_0), f_a^2(x_0), \dots$ of iterates of f_a on x_0 reaches one of the two fixed points of f_a after a *finite* number of iterations.

1. Introduction

We consider a family of skew tent maps f_a on the unit interval, determined by the parameter a $(0 < a < 1)$ and defined as $f_a(x) = \frac{x}{a}$ for $0 \le x \le a$ and $f_a(x) = \frac{1-x}{1-a}$ for $a < x \le 1$ (illustrated, further on, in Fig. 1). The maps in this family have two fixed points (that is, points for which $f_a(x) = x$). One question about the dynamics of such maps concerns the decidability of the so-called *point-to-fixed-point* problem. This question asks for an algorithm to determine, on input a and a point x_0 in the unit interval, whether or not the sequence $x_0, f_a(x_0), f_a^2(x_0), \dots$ of iterates of f_a on x_0 reaches one of the two fixed points of f_a after a *finite* number of iterations. The main contribution of this paper is a decision algorithm for this problem for rational input values a and x_0.

This decision problem originates from dynamical system theory [2,3,5] but is also relevant to database theory [4]. In this context, iterates of functions $f : \mathbf{R}^n \to \mathbf{R}^n$ (by $\mathbf{R}$ we denote the real numbers) are studied and the decidability of properties such as "mortality", "nilpotency", "termination" and "point-to-fixed-point" is investigated.

A function $f : \mathbf{R}^n \to \mathbf{R}^n$ is called *mortal* if $f(\mathbf{0}) = \mathbf{0}$ and if for each $\mathbf{x} \in \mathbf{R}^n$ there exists a natural number $k \ge 1$ such that $f^k(\mathbf{x}) = \mathbf{0}$ (here $\mathbf{0}$ denotes the origin of $\mathbf{R}^n$) and a function $f : \mathbf{R}^n \to \mathbf{R}^n$ is called *nilpotent* if $f(\mathbf{0}) = \mathbf{0}$ and if there exists a natural number $k \ge 1$ such that for all $\mathbf{x} \in \mathbf{R}^n$, $f^k(\mathbf{x}) = \mathbf{0}$ [3]. Mortality and nilpotency are known to be undecidable for piecewise affine functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ and for functions from $\mathbf{R}$ to $\mathbf{R}$ the (un)decidability of these properties is open [3].

The transitive closure of the graph of a function $f : \mathbf{R}^n \to \mathbf{R}^n$, viewed as a binary relation over $\mathbf{R}^n$, can be computed by determining iteratively the $2n$-ary relations $TC_1(f), TC_2(f), TC_3(f), \dots,$ where $TC_1(f) = \text{graph}(f)$ and $TC_{i+1}(f) := TC_i(f) \cup \{(\mathbf{x}, \mathbf{y}) \in \mathbf{R}^{2n} \mid (\exists \mathbf{z})\, ((\mathbf{x}, \mathbf{z}) \in TC_i(f) \wedge f(\mathbf{z}) = \mathbf{y})\}$. We call a function f *terminating* if there exists a $k \ge 1$ such that $TC_{k+1}(f) = TC_k(f)$. Termination of functions from $\mathbf{R}^2$ to $\mathbf{R}^2$ is undecidable but termination of continuous semi-algebraic functions from $\mathbf{R}$ to $\mathbf{R}$ is decidable [4]. The decidability of this problem has implications in the area of database theory, where it is used to obtain extensions of first-order logics with recursion, based on a transitive-closure operator [4] for constraint databases [6].

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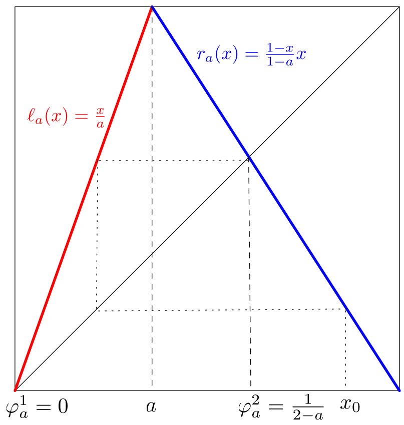

Fig. 1. The graph of the skew tent map f_a, with the graph of ℓ_a in red and the graph of r_a in blue. The two fixed points of f_a, φ_1^a or φ_2^a, are indicated and x_0 is an example of a number for which $f_a^2(x_0) = \varphi_2^a$. (For interpretation of the colours in the figure, the reader is referred to the web version of this article.)

The *point-to-fixed-point problem* is another decision problem in this context, which asks whether for a given algebraic point $\mathbf{x}$ and a given piecewise affine function $f : \mathbf{R}^n \to \mathbf{R}^n$, the sequence $\mathbf{x}, f(\mathbf{x}), f^2(\mathbf{x}), f^3(\mathbf{x}), \ldots$ reaches a fixed point, i.e., whether there exists a $k \geq 1$ such that $f^k(\mathbf{x}) = f^{k+1}(\mathbf{x})$ [2,5]. As in the case of mortality and nilpotency, the point-to-fixed-point problem is undecidable for piecewise affine functions from $\mathbf{R}^2$ to $\mathbf{R}^2$. The decidability of the point-to-fixed-point problem is open in dimension 1, even for piecewise linear functions with only two non-constant pieces [2,5]. The problem we address in this paper should be seen in this context and we propose a solution for a particular subclass of this problem in dimension 1. We study this problem in the more convenient setting of functions on an interval. A general solution for arbitrary linear functions with two pieces remains open. This is also the case for functions with three or more linear pieces. The decidability of the point-to-fixed-point problem has also implications in database theory. The termination of query evaluation in certain extensions of first-order logic with transitive closure operations depends on this problem [4].

This paper is organised as follows. In Section 2, we give the necessary definitions and state the main result. Preliminary considerations and properties are given in Section 3. A decision procedure for the point-to-fixed-point problem for skew tent maps is described in Section 4. In Section 5, we end this paper with a discussion of possible extensions of the proposed methods.

2. Definitions, notations, and main result

Let a be a number, with $0 < a < 1$, and let the function $f_a : [0, 1] \to [0, 1]$ be defined as

$$f_a(x) := \begin{cases} \frac{x}{a} & \text{if } 0 \leq x \leq a, \\ \frac{1-x}{1-a} & \text{if } a < x \leq 1. \end{cases}$$

The function f_a is a *skew tent map* (where the adjective “skew” can be dropped only when $a = \frac{1}{2}$) with top at $(a, 1)$. We use the abbreviations $\ell_a(x) := \frac{x}{a}$ and $r_a(x) := \frac{1-x}{1-a}$ for the left and right part of the function f_a. If $f_a(x) = x$ for some $x \in [0, 1]$, we call x a *fixed point* of the function f_a. The function f_a has two fixed points, namely $\varphi_1^a = 0$ and $\varphi_2^a = \frac{1}{2-a}$. Fig. 1 gives an illustration of the graph of the function f_a along with its two fixed points.

We denote the set of the natural numbers by $\mathbf{N}$ and the set of the real numbers by $\mathbf{R}$. By $\mathbf{N}_0$ we denote the set $\mathbf{N} \setminus \{0\}$. We use the notation $f_a^0(x) := x$ and $f_a^{i+1}(x) := f_a(f_a^i(x))$, for $i \in \mathbf{N}$, to denote the iterates of f_a on $x \in [0, 1]$. We also use the notions of forward and backward orbit, as follows: for $x, y \in [0, 1]$, the *forward orbit* of x (under f_a), denoted $\mathrm{Orb}^+(f_a, x)$, is the set $\{f_a^n(x) \mid n \in \mathbf{N}\}$ and the *backward orbit* of y (under f_a), denoted $\mathrm{Orb}^-(f_a, y)$, is the set $\{x \in [0, 1] \mid \text{there is an } n \in \mathbf{N} \text{ such that } f_a^n(x) = y\}$. If $x \in \mathrm{Orb}^-(f_a, y)$, we say “$x$ *reaches* y (under f_a)” or “f *reaches* y *from* x”. For an overview of such concepts, we refer to [1,7,8].

For example, the point x_0, shown in Fig. 1, reaches the fixed point φ_2^a after two iterations of f_a, that is, $f_a^2(x_0) = \varphi_2^a$.

In this paper, we are interested in algorithmically deciding whether a point $x_0 \in [0, 1]$ reaches a fixed point of f_a after a *finite* number of iterations of f_a on x_0. This decision problem can be viewed as deciding the language PtoFP (abbreviating *“point-to-fixed-point”*), with

$$\mathsf{PtoFP} = \{\langle a, x_0\rangle \mid 0 \leq x_0 \leq 1 \text{ and } 0 < a < 1 \text{ and } x_0 \in \mathrm{Orb}^-(f_a, \varphi_1^a) \cup \mathrm{Orb}^-(f_a, \varphi_2^a)\},$$

where $\langle a, x_0\rangle$ represents a finite encoding of the numbers a and x_0. For reasons of finite representability, we assume a and x_0 to be rational numbers. We agree that a rational number A is encoded as pair (p, q), with $p, q \in \mathbf{N}$ (given in binary), $q \neq 0$, p and q relatively prime and $A = \frac{p}{q}$. Obviously, other encodings may be considered.

The main result of this paper is summarised in the following theorem.

Theorem 1. *There is a decision procedure that, on input two rational numbers a and x_0 (encoded as described before), decides whether* $\langle a, x_0\rangle \in$ PtoFP. □

3. Preliminary considerations and properties

Our decision procedure is called PtoFP(a, x_0) and it is described in Section 4. Obviously, the order conditions $0 \le x_0 \le 1$ and $0 < a < 1$ are easily checked by comparing the natural numbers that encode these two rational numbers. So, we focus on the non-trivial part, namely, deciding the existence of a $n \in \mathbf{N}_0$ such that $f_a^n(x_0)$ is a fixed point of f_a.

Our approach is, given an input $\langle a, x_0\rangle$, to establish an *upper bound* M^{a,x_0} for the values of n for which $f_a^n(x_0)$ can be a fixed point of f_a. Once this upper bound M^{a,x_0} is determined, it remains to be checked whether one of the numbers $f_a^1(x_0), f_a^2(x_0), \ldots, f_a^{M^{a,x_0}}(x_0)$ actually is a fixed point of f_a. In fact, we determine an upper bound M_1^{a,x_0} for fixed point φ_1^a and an upper bound M_2^{a,x_0} for fixed point φ_2^a and M^{a,x_0} is then defined to be the largest of these two values. The determination of the upper bound M^{a,x_0} depends on the uniqueness of a particular form in which a rational number can be written. This form is derived by observing that, when f_a is repeatedly applied to x_0, this repetition involves alternating applications of powers of ℓ_a and r_a and our results rely on the general form that such an alternation of applications of ℓ_a and r_a can produce. An iteration $f_a^n(x)$ of f_a on some $x \in [0,1]$ is therefore of the form

$$r_a^{j_k}\ell_a^{i_k}r_a^{j_{k-1}}\ell_a^{i_{k-1}}\cdots r_a^{j_1}\ell_a^{i_1}(x),$$

for some $k \in \mathbf{N}_0$, where $i_1 > 0$ if $0 \le x \le a$ and $i_1 = 0$ if $a < x \le 1$, $j_k \ge 0$, $j_1, i_2, j_2, \ldots, i_{k-1}, j_{k-1} > 0$ and $i_1 + j_1 + \cdots + i_k + j_k = n$.

In this section, we give an explicit formula for such an alternating application of two affine functions on some real point. Throughout this paper, we use the notation $[A]^i$, to abbreviate the sum $1 + A + A^2 + \cdots + A^{i-1}$, for $A \in \mathbf{R}$ and $i \in \mathbf{N}$. This means that

$$[A]^i = \begin{cases} i & \text{if } A = 1 \text{ and} \\ \frac{1-A^i}{1-A} & \text{if } A \neq 1, \end{cases}$$

for $i \in \mathbf{N}$.

Now, we introduce some abbreviations for sums of exponents, that are used throughout this paper.

Notation 1. Let $i_0, i_1, i_2, \ldots$ and $j_0, j_1, j_2, \ldots$ be two sequences of natural numbers, and let $n, m \in \mathbf{N}$. For $n \le m$, we define $I_n^m := i_n + i_{n+1} + \cdots + i_m$ and $J_n^m := j_n + j_{n+1} + \cdots + j_m$. For $m < n$, we define $I_n^m := 0$ and $J_n^m := 0$. □

The following property gives the general form of an alternated application of powers of two linear functions F and G on some real point x. Its straightforward induction proof is given, for completeness, in the Appendix.

Property 1. *Let $A, B, C, D \in \mathbf{R}$. Let $F : \mathbf{R} \to \mathbf{R} : x \mapsto Ax + B$ and $G : \mathbf{R} \to \mathbf{R} : x \mapsto Cx + D$ be affine functions. If $k \in \mathbf{N}_0$ and $i_1, \ldots, i_k, j_1, \ldots, j_k \in \mathbf{N}$, then*

$$G^{j_k}F^{i_k}G^{j_{k-1}}F^{i_{k-1}}\cdots G^{j_1}F^{i_1}(x) = A^{I_1^k}C^{J_1^k}x + \sum_{\nu=1}^{k}\{A^{I_{\nu+1}^k}C^{J_{\nu+1}^k}(B[A]^{i_\nu}C^{j_\nu} + D[C]^{j_\nu})\}. \quad \square \tag{$\dagger_1$}$$

We remark that, for $j_k = 0$, the equality ($\dagger_1$) becomes

$$F^{i_k}G^{j_{k-1}}F^{i_{k-1}}\cdots G^{j_1}F^{i_1}(x) = A^{I_1^k}C^{J_1^{k-1}}x + \sum_{\nu=1}^{k-1}\{A^{I_{\nu+1}^k}C^{J_{\nu+1}^{k-1}}(B[A]^{i_\nu}C^{j_\nu} + D[C]^{j_\nu})\} + B[A]^{i_k}. \tag{$\dagger_2$}$$

4. A decision procedure for the point-to-fixed-point problem

In this section, we describe the decision procedure PtoFP(a, x_0), which accepts the input (a, x_0), when f_a reaches φ_1^a from x_0 or f_a reaches φ_2^a from x_0. The first test is described in Section 4.1 and the second is described in Section 4.2.

4.1. The points that reach the fixed point $\varphi_1^a = 0$

The backward orbit of $\varphi_1^a = 0$ contains infinitely many points besides 0 and 1 since $f_a(a) = 1$ and $f_a(a^2) = a$, $f_a(1 - a + a^2) = a$, etc. In fact, $\mathrm{Orb}^-(f_a, \varphi_1^a)$ certainly contains $a, a^2, a^3, \ldots$, which reach a under ℓ_a, besides infinitely many points (like $1 - a + a^2$) from the domain of r_a.

The following theorem implies a decision procedure to establish whether a point $x_0 \in [0, 1]$ is in $\mathrm{Orb}^-(f_a, \varphi_1^a)$, as we explain following its proof.

Theorem 2. *Let f_a be a skew tent map, as before. Let $a = \frac{p}{q}$, with $p, q \in \mathbf{N}_0$, $\gcd(p, q) = 1$ and $0 < p < q$. Then f_a reaches its fixed point $\varphi_1^a = 0$ from $x_0 \in [0, 1]$ if and only if $x_0 = 0$ or if there exists an $n \in \mathbf{N}_0$ such that $\frac{2q-p}{q} x_0$ is of the form $\frac{N}{q^n}$, with $N \in \mathbf{N}_0$ and $\gcd(q, N) = 1$ and $f_a^n(x_0) = 0$.*

Proof. Let $a = \frac{p}{q}$, with $p, q \in \mathbf{N}_0$, $\gcd(p, q) = 1$ and $0 < p < q$. Let f_a be a skew tent map with left part ℓ_a and right part r_a, as explained in Section 2. Therefore, we have

$$f_a(x) = \begin{cases} \frac{qx}{p} & \text{if } 0 \le x \le a, \\ \frac{q(1-x)}{q-p} & \text{if } a < x \le 1. \end{cases}$$

We first prove the only-if direction. Since $f_a(0) = \ell_a(0) = 0$, we do not have to consider $x_0 = 0$ any further. We remark that also for $x_0 = 1$, $\frac{2q-p}{q} x_0$ is of the requested form $\frac{N}{q^k}$ with $N = 2q - p$ and $k = 1$, since $\gcd(2q - p, q) = 1$ follows from the assumption $\gcd(p, q) = 1$. The same holds for $x_0 = a$, in which case $\frac{2q-p}{q} x_0 = \frac{(2q-p)p}{q^2}$ and we have $N = (2q - p)p$ and $k = 2$. Also here, $\gcd(p, q) = 1$ implies $\gcd((2q - p)p, q) = 1$.

Assume that $x_0 \in \mathrm{Orb}^-(f_a, 0)$ and $x_0 \notin \{0, 1, a\}$. Then there exists an $n \in \mathbf{N}_0$, with $n > 2$, such that $f_a^n(x_0) = 0$. This implies that there exists a $k \in \mathbf{N}_0$ and $i_1, j_1, \ldots, i_k, j_k \in \mathbf{N}$ such that

$$r_a^{j_k} \ell_a^{i_k} r_a^{j_{k-1}} \ell_a^{i_{k-1}} \cdots r_a^{j_1} \ell_a^{i_1} (x_0) = 0,$$

where $i_1 > 0$ if $0 \le x \le a$ and $i_1 = 0$ if $a < x \le 1$ and $i_1 + j_1 + \cdots + i_k + j_k = n$ and $j_1, i_2, .., i_k, j_k > 0$. Since 0 can only be reached via a and 1 (that is, $r_a(\ell_a(a)) = r_a(1) = 0$), we also know that $j_k = 1$ and $i_k \ge 1$.

If then we apply (†1) from Property 1, with $A = \frac{1}{a}$, $B = 0$, $C = \frac{1}{a-1}$ and $D = \frac{-1}{a-1}$, we obtain

$$(\frac{1}{a})^{I_1^k} (\frac{1}{a-1})^{J_1^k} x_0 + \sum_{\nu=1}^{k} \{(\frac{1}{a})^{I_{\nu+1}^k} (\frac{1}{a-1})^{J_{\nu+1}^k} (\frac{-1}{a-1} [\frac{1}{a-1}]^{j_\nu})\} = 0.$$

When we substitute $\frac{p}{q}$ for a, the above equation becomes

$$(\frac{q}{p})^{I_1^k} (\frac{q}{p-q})^{J_1^k} x_0 + \sum_{\nu=1}^{k} \{(\frac{q}{p})^{I_{\nu+1}^k} (\frac{q}{p-q})^{J_{\nu+1}^k} (\frac{-q}{p-q} [\frac{q}{p-q}]^{j_\nu})\} = 0.$$

We remark that $\frac{q}{p-q} \ne 1$. Indeed, if we assume $\frac{q}{p-q} = 1$, we get $2q = p$, which is impossible, since $p < q$ (or $a < 1$). Therefore, $\frac{-q}{p-q} [\frac{q}{p-q}]^{j_\nu} = \frac{-q}{2q-p} \frac{q^{j_\nu} - (p-q)^{j_\nu}}{(p-q)^{j_\nu}}$. If we use this fact, then the above equality, after dividing both sides by $(\frac{q}{p})^{I_1^k} (\frac{q}{p-q})^{J_1^k}$, becomes

$$\frac{2q-p}{q} x_0 = \frac{1}{q^{I_1^k + J_1^k}} \sum_{\nu=1}^{k} \{q^{I_{\nu+1}^k + J_{\nu+1}^k} p^{I_1^\nu} (p-q)^{J_1^{\nu-1}} (q^{j_\nu} - (p-q)^{j_\nu})\}$$

or $\frac{2q-p}{q} x_0 = \frac{N}{q^{I_1^k + J_1^k}}$ with

$$N = \sum_{\nu=1}^{k} \{q^{I_{\nu+1}^k + J_{\nu+1}^k} p^{I_1^\nu} (p-q)^{J_1^{\nu-1}} (q^{j_\nu} - (p-q)^{j_\nu})\}.$$

Since, $j_k = 1$ and $i_k \ge 1$, we have, for $\nu < k$, that $I_{\nu+1}^k + J_{\nu+1}^k \ge i_k + j_k \ge 2$ and thus $q \mid q^{I_{\nu+1}^k + J_{\nu+1}^k}$ (and even $q^2 \mid q^{I_{\nu+1}^k + J_{\nu+1}^k}$). From this observation follows that $\gcd(q, N) = 1$. Indeed, let $d \in \mathbf{N}$ be a common divisor of q and N (that is, $d \mid q$ and $d \mid N$), then $d \mid p^{I_1^k} (p-q)^{J_1^{k-1}} (q^{j_k} - (p-q)^{j_k})$ and thus $d \mid p^{I_1^k + J_1^k}$. From $\gcd(p, q) = 1$, $d \mid q$ and $d \mid p^{I_1^k + J_1^k}$, $d \mid 1$ follows. So, we can conclude that $\gcd(q, N) = 1$.

We see that from the assumption $f_a^n(x_0) = 0$, it follows that $\frac{2q-p}{q} x_0 = \frac{N}{q^{I_1^k + J_1^k}} = \frac{N}{q^n}$, with $N \in \mathbf{N}$ and $\gcd(q, N) = 1$.

The if-direction is clear. □

To see that this theorem implies a decision procedure to test whether 0 can be reached from a given point $x_0 \in (0,1]$, we need the following property on the unique expression of some rational numbers.

Property 2. *Let $q \in \mathbf{N}$ with $q > 1$. If α is a rational number, then there exists at most one $k \in \mathbf{N}$ such that $\alpha = \frac{N}{q^k}$ with $N \in \mathbf{N}_0$ and* $\gcd(q, N) = 1$.

Proof. Let α be a rational number. If α cannot be expressed in the form $\frac{N}{q^k}$ (for instance, when α is 0 or negative), then the statement is true. Suppose, for the sake of contradiction that $\alpha = \frac{N}{q^k}$ and $\alpha = \frac{N'}{q^{k'}}$, with $\gcd(q, N) = 1$, $\gcd(q, N') = 1$. In the case $k = k'$, we have $N = N'$ which gives uniqueness. For $k < k'$, we obtain $q^{k'-k}N = N'$ from these two equalities. Since $k' - k \geq 1$, this implies that $q \mid N'$. But then $\gcd(q, N') = q > 1$, contradicting the assumption $\gcd(q, N') = 1$. So, in all cases α can be expressed in at most one way as a fraction of the form $\frac{N}{q^k}$. □

Theorem 2 implies that if $\frac{2q-p}{q}x_0$ cannot be written in the form $\frac{N}{q^n}$ for some $n \in \mathbf{N}$ and $N \in \mathbf{N}_0$, with $\gcd(q, N) = 1$, then x_0 does not reach the fixed point 0 of f_a.

On the other hand, if, for $x_0 \neq 0$, $\frac{2q-p}{q}x_0$ can be written in the form $\frac{N}{q^n}$ for some $n, N \in \mathbf{N}$ with $\gcd(q, N) = 1$, then by Property 2, this n is unique and it suffices to check whether $f_a^n(x_0)$ equals 0. Following this observation, we give in the following corollary the upper bound M_1^{a,x_0}, as introduced in Section 3, on the number of iterations of f_a on x_0 to reach the fixed point $\varphi_1^a = 0$.

Corollary 1. *Let $a = \frac{p}{q}$, with $p, q \in \mathbf{N}_0$, $\gcd(p, q) = 1$ and $0 < p < q$ and let $x_0 \in (0, 1]$. If $\frac{2q-p}{q}x_0$ can be written in the form $\frac{N}{q^n}$, then we have $M_1^{a,x_0} = n$.*

This concludes the description of the part of <u>PtoFP</u>(a, x_0) that tests whether x_0 reaches φ_1^a under f_a.

We end this section by remarking that the condition that $\frac{2q-p}{q}x_0$ can be written in the form $\frac{N}{q^n}$ for some $n, N \in \mathbf{N}$ with $\gcd(q, N) = 1$ is a necessary but not sufficient condition. We consider the example of $a = \frac{1}{2}$ (with $p = 1$ and $q = 2$). For $x_0 = \frac{1}{12}$, we have $\frac{2q-p}{q}x_0 = \frac{1}{2^3}$, which is of the required form, but x_0 does not reach the fixed point 0 (rather, it reaches the second fixed point $\varphi_2^{\frac{1}{2}} = \frac{2}{3}$, since $\frac{1}{12} \xrightarrow{\ell_a} \frac{1}{6} \xrightarrow{\ell_a} \frac{1}{3} \xrightarrow{\ell_a} \frac{2}{3}$). On the other hand, for $x_0 = \frac{1}{4}$, we have $\frac{2q-p}{q}x_0 = \frac{3}{2^3}$, which again is of the required form, and in this case x_0 reaches the fixed point $\varphi_1^{\frac{1}{2}} = 0$, since $\frac{1}{4} \xrightarrow{\ell_a} \frac{1}{2} \xrightarrow{\ell_a} 1 \xrightarrow{r_a} 0$.

4.2. The points that reach the fixed point $\varphi_2^a = \frac{1}{2-a}$

The following theorem implies a decision procedure to establish whether a point $x_0 \in (0, a)$ reaches φ_2^a under f_a, as we explain following its proof. There, we also explain, how this procedure can be extended to $x_0 \in [0, 1]$.

Theorem 3. *Let f_a be a skew tent map, as before. Let $a = \frac{p}{q}$, with $p, q \in \mathbf{N}_0$, $\gcd(p, q) = 1$ and $0 < p < q$. Then f_a reaches its fixed point $\varphi_2^a = \frac{1}{2-a}$ from $x_0 \in (0, a)$ if and only if there exists an $n \in \mathbf{N}_0$ such that $\frac{2q-p}{q}x_0$ is of the form $\frac{N}{q^n}$, with $N \in \mathbf{N}_0$ and $\gcd(q, N) = 1$ and $f_a^n(x_0) = \varphi_2^a$.*

Proof. Let $a = \frac{p}{q}$, with $p, q \in \mathbf{N}_0$, $\gcd(p, q) = 1$ and $0 < p < q$. Let f_a be a skew tent map with left part ℓ_a and right part r_a, as explained in Section 2. Let $x_0 \in (0, a)$.

We first prove the only-if direction. We assume that $x_0 \in \mathrm{Orb}^-(f_a, \varphi_2^a)$. Then there exists an $n \in \mathbf{N}_0$, such that $f_a^n(x_0) = \varphi_2^a$. This implies that there exists a $k \in \mathbf{N}_0$ and $i_1, j_1, \ldots, i_k \in \mathbf{N}_0$ such that

$$\ell_a^{i_k} r_a^{j_{k-1}} \ell_a^{i_{k-1}} \cdots r_a^{j_1} \ell_a^{i_1}(x_0) = \varphi_2^a,$$

where $i_1 + j_1 + \cdots + i_k = n$. We remark that $i_1 > 0$ because $x_0 \in (0, a)$ and that $i_k > 0$ because φ_2^a can only be reached via ℓ_a.

If then we apply (†2) from Property 1, with $A = \frac{1}{a}$, $B = 0$, $C = \frac{1}{a-1}$ and $D = \frac{-1}{a-1}$, we obtain

$$(\frac{1}{a})^{I_1^k}(\frac{1}{a-1})^{J_1^{k-1}}x_0 + \sum_{\nu=1}^{k-1}\{(\frac{1}{a})^{I_{\nu+1}^k}(\frac{1}{a-1})^{J_{\nu+1}^{k-1}}(\frac{-1}{a-1}[\frac{1}{a-1}]^{j_\nu})\} = \varphi_2^a.$$

Since $a \neq 2$, $\frac{-1}{a-1}[\frac{1}{a-1}]^{j_\nu} = -\varphi_2^a((\frac{1}{a-1})^{j_\nu} - 1)$ and thus the above equality, using $a = \frac{p}{q}$, becomes

$$(\frac{q}{p})^{I_1^k}(\frac{q}{p-q})^{J_1^{k-1}}x_0 = \varphi_2^a(1 + \sum_{\nu=1}^{k-1}\{(\frac{q}{p})^{I_{\nu+1}^k}(\frac{q}{p-q})^{J_{\nu+1}^{k-1}}(\frac{q^{j_\nu} - (p-q)^{j_\nu}}{(p-q)^{j_\nu}})\}).$$

After dividing both sides by $(\frac{q}{p})^{I_1^k}(\frac{q}{p-q})^{J_1^{k-1}}$ and by $\varphi_2^a = \frac{q}{2q-p}$, we obtain

$$\frac{2q-p}{q}x_0 = \frac{N}{q^{I_1^k + J_1^{k-1}}}$$

with

$$N = p^{I_1^k}(p-q)^{J_1^{k-1}} + \sum_{\nu=1}^{k-1}\{q^{I_{\nu+1}^k + J_{\nu+1}^{k-1}} p^{I_1^\nu}(p-q)^{J_1^{\nu-1}}(q^{j_\nu} - (p-q)^{j_\nu})\}.$$

Since, $I_{\nu+1}^k + J_{\nu+1}^{k-1} \geq i_k > 0$ for all $\nu \leq k-1$, we see that $q \mid q^{I_{\nu+1}^k + J_{\nu+1}^{k-1}}$. From this observation follows that $\gcd(q, N) = 1$. Indeed, let $d \in \mathbf{N}$ be a common divisor of q and N, then $d \mid p^{I_1^k}(p-q)^{J_1^{k-1}}$ and thus $d \mid p^{I_1^k + J_1^{k-1}}$. From $\gcd(p, q) = 1$, $d \mid q$ and $d \mid p^{I_1^k + J_1^{k-1}}$, $d \mid 1$ follows. So, we can conclude that $\gcd(q, N) = 1$.

We see that from the assumption $f_a^n(x_0) = \varphi_2^a$, it follows that $\frac{2q-p}{q}x_0 = \frac{N}{q^{I_1^k + J_1^k}} = \frac{N}{q^n}$, with $N \in \mathbf{N}_0$ and $\gcd(q, N) = 1$.

The if-direction is clear. □

Theorem 3 implies a decision procedure to test whether φ_2^a can be reached from a given point $x_0 \in (0, a)$. Indeed, equivalently, this theorem says that if $\frac{2q-p}{q}x_0$ cannot be written in the form $\frac{N}{q^n}$ for some $n, N \in \mathbf{N}$, with $\gcd(q, N) = 1$, then x_0 does not reach φ_2^a.

On the other hand, if, for $x_0 \in (0, a)$, $\frac{2q-p}{q}x_0$ can be written in the form $\frac{N}{q^n}$ for some $n, N \in \mathbf{N}$ with $\gcd(q, N) = 1$, then by Property 2, this n is unique and it suffices to check whether $f_a^n(x_0)$ is φ_2^a. Indeed, we know that if $f_a^{n'}(x_0) = \varphi_1$ for an $n' > n$, then $\frac{2q-p}{q}x_0$ can also be written as $\frac{N'}{q^{n'}}$ for some $n' \in \mathbf{N}$ and $N' \in \mathbf{N}_0$, with $\gcd(q, N') = 1$, contradicting Property 2.

We already know that 0, a and 1 reach 0 (and thus do not reach φ_2^a) and that φ_2^a reaches itself. Therefore, what remains is to give a procedure to determine whether $x_0 \in (a, 1) \setminus \{\varphi_2^a\}$ reaches φ_2^a.

Hereto, we define the sequence $\alpha_i := r_a^{-i}(a)$, for $i \in \mathbf{N}$. The following property gives an expression for α_i.

Property 3. *For $i \in \mathbf{N}$, we have $\alpha_i = (a-1)^i(a - \varphi_2^a) + \varphi_2^a$.*

Proof. Clearly, we have $r_a^{-1}(x) = (a-1)x + 1$. By Lemma 2 in the Appendix, we obtain that $r_a^{-i}(x) = (a-1)^i x + [a-1]^i$. Since $\varphi_2^a = \frac{1}{2-a}$ we have $[a-1]^i = -\varphi_2^a((a-1)^i - 1)$. So, we get $r_a^{-i}(x) = (a-1)^i x - \varphi_2^a((a-1)^i - 1) = (a-1)^i(x - \varphi_2^a) + \varphi_2^a$. This implies that $\alpha_i = r^{-i}(a) = (a-1)^i(a - \varphi_2^a) + \varphi_2^a$. □

We have $\alpha_0 = a$ and we can extend the sequence by setting $\alpha_{-1} := r_a(a) = 1$ (taking $r_a(a) = \ell_a(a)$). Since $a < \varphi_2^a$, we derive $\alpha_{2i} < \varphi_2^a$ and $\varphi_2^a < \alpha_{2i+1}$, for $i \in \mathbf{N}_0$, from this property. Also from this property and the observation that $0 < (a-1)^2 < 1$ for $0 < a < 1$, a straightforward calculation gives the following ordering of the α_i:

$$a = \alpha_0 < \alpha_2 < \alpha_4 < \alpha_6 < \cdots < \varphi_2^a < \cdots < \alpha_5 < \alpha_3 < \alpha_1 < \alpha_{-1} = 1.$$

We observe that all α_i eventually reach a and thus 0 under f_a and will never reach φ_2^a. For the other $x_0 \in (a, 1)$, we first determine between which values in the above ordering x_0 is situated to test whether x_0 reaches φ_2^a. If $x_0 \in (\alpha_1, 1)$, then $f_a(x_0) = r_a(x_0) \in (0, a)$. Therefore, $x_0 \in (\alpha_1, 1)$ reaches φ_2^a if and only if $f_a(x_0)$ reaches φ_2^a. If $x_0 \in (a, \alpha_2)$, then $f_a(x_0) = r_a(x_0) \in (\alpha_1, 1)$ and this case reduces to the previous one. For $k \in \mathbf{N}_0$, if $x_0 \in (\alpha_{2k}, \alpha_{2k+2})$, then $f_a^{2k}(x_0) = r^{2k}(x_0) \in (a, \alpha_2)$ and if $x_0 \in (\alpha_{2k+1}, \alpha_{2k-1})$, then $f_a^{2k}(x_0) = r^{2k}(x_0) \in (\alpha_1, 1)$ and these cases also reduce to the previous ones.

Following these observations, we give in the following corollary the upper bound M_2^{a,x_0}, as introduced in Section 3, on the number of iterations of f_a on x_0 to reach the fixed point φ_2^a.

Corollary 2. *Let $a = \frac{p}{q}$, with $p, q \in \mathbf{N}_0$, $\gcd(p, q) = 1$ and $0 < p < q$ and let $x_0 \in (0, 1]$. If $x_0 \in (0, a)$ and $\frac{2q-p}{q}x_0$ can be written in the form $\frac{N}{q^n}$, then we have $M_2^{a,x_0} = n$. For $x_0 \in (a, 1)$, we distinguish between the following cases:*

- *if $x_0 = \varphi_2^a$, then we have $M_2^{a,x_0} = 1$;*
- *if $x_0 \in (\alpha_{2k}, \alpha_{2k+2})$, for some $k \in \mathbf{N}$, and if $\frac{2q-p}{q} f_a^{2k+2}(x_0)$ can be written in the form $\frac{N}{q^n}$, then we have $M_2^{a,x_0} = 2k + 2 + M_2^{a, f_a^{2k+2}(x_0)} = 2k + 2 + n$; and*
- *if $x_0 \in (\alpha_{2k+1}, \alpha_{2k-1})$, for some $k \in \mathbf{N}$, and if $\frac{2q-p}{q} f_a^{2k+1}(x_0)$ can be written in the form $\frac{N}{q^n}$, then we have $M_2^{a,x_0} = 2k + 1 + M_2^{a, f_a^{2k+1}(x_0)} = 2k + 1 + n$.*

These observations complete the description of the part of PtoFP(a, x_0) that tests whether x_0 reaches φ_2^a under f_a.

5. Discussion

We discuss some possible extensions of the proposed methods. Another class of tent maps that can be considered is the family of skew tent maps $f_{a,b}$ defined by $f_{a,b}(x) = \frac{bx}{a}$ for $0 \le x \le a$ and $f_a(x) = \frac{b(1-x)}{1-a}$ for $a < x \le 1$, with an extra parameter b. In this paper, we discuss the case $b = 1$. On the unit interval, the interesting case occurs when $a < b \le 1$. The techniques of this paper can also be used in this setting, when $a = \frac{p}{q}$, $b = \frac{u}{v}$, with $\gcd(p, q) = 1$, $\gcd(u, v) = 1$ and the denominators q and v have not too many factors in common. When $\gcd(q, v) < q$ our techniques still work (using an extension of Property 2), but when (a power of) q divides v, this is no longer clear. A decision procedure for point-to-fixed-point problem for the family of skew tent maps $f_{a,b}$ would bring us close to a solution for arbitrary linear functions with two pieces, since many cases can be reduced to this case via topological conjugacy [1,7,8]. The general case of tent maps with $b \le 1$ is also connected to the case of unimodal maps [1], which (under certain conditions) are semi-conjugate to such tent maps. The one-dimensional case for functions with three or more linear pieces remains open.

In this paper, we describe a decision procedure for the case where a and x_0 are rational. The restriction to rational takes care of the finite representability of the input. Obviously a wider class of real numbers in the unit interval can be encoded in a finite way. We can think of the Turing-computable real numbers or the more restricted class of the algebraic real numbers. It is not obvious how the techniques of this paper can be extended to these settings.

We conclude by remarking that the proposed techniques can also be used to decide the "point-to-point" problem (instead of "point-to-fixed-point"), where both the initial and final point are two arbitrary points of the unit interval given as inputs, along with the parameter a.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix A

In this appendix, we give the proof of Property 1. We start with two straightforward lemmas. The first lemma follows directly from the definition.

Lemma 1. *We have $[A]^0 = 0$ and $[A]^{i+1} = A[A]^i + 1$, for $A \in \mathbf{R}$ and $i \in \mathbf{N}$.* □

The following lemma gives an expression for the result of iterating a linear function F i times on a point x.

Lemma 2. *Let $F : \mathbf{R} \to \mathbf{R} : x \mapsto Ax + B$, with $A, B \in \mathbf{R}$, be an affine function. For $i \in \mathbf{N}$ and $x \in \mathbf{R}$, we have $F^i(x) = A^i x + B[A]^i$.* □

Proof. We prove this by induction on i. For $i = 0$, we have $F^0(x) = x$ and $A^0 x + B[A]^0 = x$. For the induction step, we have $F^{i+1}(x) = F(F^i(x))$, which equals $A(A^i x + B[A]^i) + B$ by the induction hypothesis. This equals $A^{i+1}x + B(A[A]^i + 1)$ and this is $A^{i+1}x + B[A]^{i+1}$ by Lemma 1. This concludes the induction proof. □

We are now ready for the proof of Property 1.

Proof of Property 1. We prove this lemma by induction on k. For $k = 1$ and $j_1 = 0$, clearly, $F^{i_1}(x) = A^{i_1}x + B[A]^{i_1}$ by Lemma 2 and this is the desired expression, since the empty sum in ($\dagger_2$) equals zero. For $k = 1$ and $j_1 > 0$, we get $G^{j_1}F^{i_1}(x) = G^{j_1}(F^{i_1}(x)) = C^{j_1}(A^{i_1}x + B[A]^{i_1}) + D[C]^{j_1} = A^{i_1}C^{j_1}x + B[A]^{i_1}C^{j_1} + D[C]^{j_1}$, again using Lemma 2. Since $A^{I_2^1}C^{J_2^1} = A^0C^0 = 1$, this is the desired expression.

For the induction step, we assume that the property holds for $k \ge 1$ and we have, for $j_{k+1} = 0$, $F^{i_{k+1}}G^{j_k}F^{i_k} \cdots G^{j_1}F^{i_1}(x) = F^{i_{k+1}}(G^{j_k}F^{i_k} \cdots G^{j_1}F^{i_1}(x))$ which, using Lemma 2 and the induction hypothesis ($\dagger_1$), is $A^{i_{k+1}}(A^{I_1^k}C^{J_1^k}x + \sum_{\nu=1}^{k}\{A^{I_{\nu+1}^k}C^{J_{\nu+1}^k}(B[A]^{i_\nu}C^{j_\nu} + D[C]^{j_\nu})\}) + B[A]^{i_{k+1}}$. This expression equals $A^{I_1^{k+1}}C^{J_1^k}x + \sum_{\nu=1}^{k}\{A^{I_{\nu+1}^{k+1}}C^{J_{\nu+1}^k}(B[A]^{i_\nu}C^{j_\nu} + D[C]^{j_\nu})\} + B[A]^{i_{k+1}}$, which is ($\dagger_2$) for the value $k + 1$.

For $j_{k+1} > 0$, we have, using the previous expression and Lemma 2, that $G^{j_{k+1}}F^{i_{k+1}}G^{j_k}F^{i_k} \cdots G^{j_1}F^{i_1}(x)$, which equals $G^{j_{k+1}}(F^{i_{k+1}}G^{j_k}F^{i_k} \cdots G^{j_1}F^{i_1}(x))$ is equal to

$$C^{j_{k+1}}(A^{I_1^{k+1}}C^{J_1^k}x + \sum_{\nu=1}^{k}\{A^{I_{\nu+1}^{k+1}}C^{J_{\nu+1}^k}(B[A]^{i_\nu}C^{j_\nu} + D[C]^{j_\nu})\} + B[A]^{i_{k+1}}) + D[C]^{j_{k+1}}.$$

This expression equals

$$A^{I_1^{k+1}}C^{J_1^{k+1}}x + \sum_{\nu=1}^{k}\{A^{I_{\nu+1}^{k+1}}C^{J_{\nu+1}^{k+1}}(B[A]^{i_\nu}C^{j_\nu} + D[C]^{j_\nu})\} + B[A]^{i_{k+1}}C^{j_{k+1}} + D[C]^{j_{k+1}}.$$

Since $A^{I_{k+2}^{k+1}} C^{J_{k+2}^{k+1}} = 1$, we get ($\dagger_1$) for the value $k+1$. This is the desired result and the induction proof is finished. □

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