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Edge fault-tolerance analysis of maximally edge-connected graphs and super edge-connected graphs [☆]

Shuang Zhao [a], Zongqing Chen [b], Weihua Yang [c], Jixiang Meng [a,*]

[a] College of Mathematics and System Sciences, Xinjiang University, Urumqi 830046, China
[b] School of Science, Tianjin Chengjian University, Tianjin 300384, China
[c] Department of Mathematics, Taiyuan University of Techology, Taiyuan 030024, China

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ABSTRACT

Edge fault-tolerance of interconnection network is of significant important to the design and maintenance of multiprocessor systems. A connected graph G is maximally edge-connected (maximally-λ for short) if its edge-connectivity attains its minimum degree. G is super edge-connected (super-λ for short) if every minimum edge-cut isolates one vertex. The edge fault-tolerance of the maximally-λ (resp. super-λ) graph G with respect to the maximally-λ (resp. super-λ) property, denoted by $m_\lambda(G)$ (resp. $S_\lambda(G)$), is the maximum integer m for which $G-S$ is still maximally-λ (resp. super-λ) for any edge subset S with $|S| \leq m$. In this paper, we give upper and lower bounds on $m_\lambda(G)$. Furthermore, we completely determine the exact values of $m_\lambda(G)$ and $S_\lambda(G)$ for vertex transitive graphs.

1. Introduction

It is well-known that a network can be modeled by a graph $G=(V(G), E(G))$, where the vertex set $V(G)$ represents the node-set of the network and the edge set $E(G)$ represents the link-set of the network. Then requirements in the design of the network correspond to graph measures. In [33], several measurements for building networks, such as diameter, hamiltonicity, connectivity and symmetry are discussed. Since link faults may happen when a network is put in use, apart from such network requirements, it is practically meaningful and important to consider faulty networks [17]. The problems of diameter and hamiltonicity have been solved on a variety of faulty networks, see [16–18,25,30,35]. Comparatively, there are few results about connectivity, concerning with the faulty networks.

As a traditional measurement for reliability and fault-tolerance of the network, the *edge-connectivity* of a connected graph G, denoted by $\lambda(G)$, is the minimum cardinality of a set of edges (named as *edge-cut*) of G, whose removal from G makes the remaining graph no longer connected. It is well-known that $\lambda(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G. Hence, a graph G with $\lambda(G) = \delta(G)$ is said to be *maximally edge-connected*, *maximally-λ* for short. To design more reliable networks, besides the requirement of maximal edge-connectivity, it is also desirable that the number of minimum edge-cuts is as small as possible. For this purpose, Bauer et al. [3] defined the super edge-connected graphs. A connected graph G is called *super edge-connected*, or simply *super-λ*, if every minimum edge-cut isolates one vertex.

A more refined measurement is the restricted edge-connectivity, proposed by Esfahanian and Hakimi [9]. The *restricted edge-connectivity* $\lambda'(G)$, is the minimum cardinality of a set of edges (named as *restricted edge-cut*) of G, if any, whose dele-

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* Corresponding author.
E-mail address: mjxxju@sina.com (J. Meng).

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tion makes G not connected and the remaining graph contains no isolated vertices. A graph G is said to be λ'-*connected*, if $\lambda'(G)$ exists. A λ'-connected graph G is called *super restricted edge-connected*, or simply *super-λ'*, if every minimum restricted edge-cut isolates an edge. Known results about $\lambda(G)$ and $\lambda'(G)$ are presented in [6,12,13,20–24,27,28,32,34,35].

In [14], Hong, Meng and Zhang proposed a parameter, denoted by $S_\lambda(G)$, to consider super edge-connectivity in graphs with faulty edges. The index $S_\lambda(G)$ is the maximum integer m for which $G-S$ is still super-λ for any edge subset S with $|S| \le m$. In [14], the authors gave upper and lower bounds on $S_\lambda(G)$ and presented more refined bounds on $S_\lambda(G)$ for some special classes of graphs. Motivated by these, a quite natural problem can be introduced as follows. For a graph G with some connectivity property $\mathcal{P}$, how many faulty edges can be tolerated such that the remaining graph still has property $\mathcal{P}$? We can define a parameter with respect to property $\mathcal{P}$, denoted by $m_{\mathcal{P}}(G)$, as the maximum integer m for which $G-S$ still has property $\mathcal{P}$ for any edge subset S with $|S| \le m$. The parameter $m_{\mathcal{P}}(G)$ can be used to evaluate the reliability and fault-tolerance more reasonably. If one seeks to destroy a network in such a way that the damaged network no longer has property $\mathcal{P}$, then we have to destroy at least $m_{\mathcal{P}}(G)+1$ edges. Thus, the value of $m_{\mathcal{P}}(G)$ will provide a beneficial reference for engineers when designing or selecting networks to build parallel systems.

Wang and Lu [29] studied three families of super-λ networks with faulty edges. In [15], Hong and Xu investigated $m_{\mathcal{P}}(G)$ when $\mathcal{P}$ denotes the property of super-λ', and determined the exact value of $m_{\mathcal{P}}(G)$ for two families of networks. Wang et al. [31] considered super-λ' Cartesian product graphs with faulty edges. In [8], Cheng and Hsieh gave upper and lower bounds on $m_{\mathcal{P}}(G)$, where $\mathcal{P}$ denotes the property of super-$\lambda^{(k)}$ (see [8] for definition) for $k \ge 3$. This paper is concerned with this parameter, denoted by $m_\lambda(G)$, when $\mathcal{P}$ denotes the property of maximally-λ. In [26], Sun, Zhao and Meng studied $m_\lambda(G)$ for two families of networks.

The rest of this paper is organized as follows. In Section 2, we introduce some definitions and related results which are to be used in this paper. In Section 3, we present upper and lower bounds on $m_\lambda(G)$. In Section 4, we focus on giving values of $m_\lambda(G)$ under some conditions. In Section 5, we completely determine how many edges can be deleted such that the remaining graph is still super-λ for super-λ vertex transitive graphs.

2. Definitions and related results

In this section, we introduce some definitions and related results which will be used in the following sections. For graph-theoretical terminologies and notations not given here, we follow [5,11].

Let $G=(V(G),E(G))$ be a graph. The *order* of G is the size of $|V(G)|$. We use $N_G(v)$ to denote the neighbor set of the vertex $v \in V(G)$, and $N_G(A)=(\bigcup_{v\in A} N_G(v)) \setminus A$ for $A \subseteq V(G)$. We write $d_A(v)=|N_G(v)\cap A|$. For $X, X' \subseteq V(G)$, notations $[X,X']_G$ and $G[X]$ are used to denote the set of edges between X and X' in G, and the subgraph of G induced by X, respectively. When the graph under consideration is obvious, we use $\omega(X)$ instead of $[X,\overline{X}]$, where $\overline{X}=V(G)\setminus X$. A *clique* of G is a subset X of $V(G)$ such that $G[X]$ is complete. If, furthermore, $|X|=k$, then X is a *k-clique*. We use K_n, C_n and $K_{1,n-1}$ to denote the *complete graph*, the *cycle* and the *star* of order n.

In the design of network topology, highly symmetric graphs, for instance vertex transitive graphs or edge transitive graphs, are popular due to their desirable properties [2,19,33]. If for any two vertices $u, v \in V(G)$, there is an automorphism $\phi \in Aut(G)$ such that $\phi(u)=v$, then G is *vertex transitive*, where $Aut(G)$ denotes the automorphism group of G. A vertex transitive graph is always regular. It is known that vertex transitive graphs are maximally-λ [22]. Circulant graphs are examples of vertex transitive graphs. Let $\mathbb{Z}_n$ be the ring of integers modulo n and $1 \le a_1 < a_2 < \ldots < a_k \le \frac{n}{2}$ be k integers. The *circulant graph* $G(n; a_1, a_2, \ldots, a_k)$ is a graph with vertex set $\mathbb{Z}_n$, and for any $i \in \mathbb{Z}_n$, i is adjacent to $i \pm a_1$, $i \pm a_2$, $\ldots$, $i \pm a_k$ $(mod\ n)$. And $G(n; a_1, a_2, \ldots, a_k)$ is connected if and only if $gcd(n, a_1, a_2, \ldots, a_k)=1$. G is *edge transitive* if for any two edges $e, f \in E(G)$, there is an automorphism $\phi \in Aut(G)$ such that $\phi(e)=f$. Since edge transitive graph G satisfies $\kappa(G)=\delta(G)$ [32], where $\kappa(G)$ is the vertex-connectivity of G, combining with $\kappa(G) \le \lambda(G) \le \delta(G)$, we see that edge transitive graph is also maximally-λ.

Now, we introduce the definition of the edge fault-tolerance of a maximally-λ graph G with respect to the maximally-λ property.

Definition 2.1. A maximally-λ graph G is *m-maximally edge-connected* (*m-maximally-λ* for short) if $G-S$ is still maximally-λ for any edge subset $S \subseteq E(G)$ with $|S| \le m$. The edge fault-tolerance of a maximally-λ graph G with respect to the maximally-λ property, denoted by $m_\lambda(G)$, is the maximum integer m such that G is m-maximally-λ.

Example 2.2. $m_\lambda(K_3)=1$. Clearly, K_3 is 0-maximally-λ. For any edge e in K_3, $K_3-\{e\}$ is still maximally-λ. Thus K_3 is 1-maximally-λ. However, K_3 is no longer 2-maximally-λ, since $K_3-\{e,f\}$ is not connected for any edge e, f in K_3.

In fact, 0-maximally-λ is exactly maximally-λ. Hence, m-maximally-λ is a generalization of maximally-λ. It is known that any two vertices are connected by a unique path in a tree. So, the following proposition follows.

Proposition 2.3. *For any tree T, $m_\lambda(T)=0$. And, $m_\lambda(C_n)=1$.*

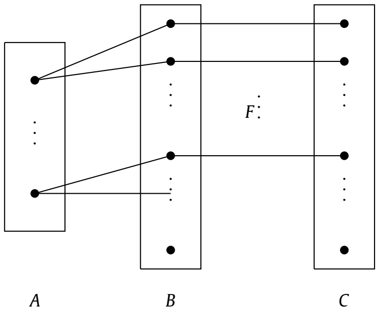

Fig. 1. The graph G in Remark 3.4.

3. Upper and lower bounds for $m_\lambda(G)$

In this section, we will establish some upper and lower bounds on $m_\lambda(G)$ for a graph G. For an edge subset $S \subseteq E(G)$, to measure whether $G - S$ is maximally-λ, we need the following necessary and sufficient condition for a graph to be maximally-λ.

Lemma 3.1. *A connected graph G is maximally-λ if and only if $|\omega(A)| \geq \delta(G)$ holds for any nonempty vertex subset $A \subset V(G)$.*

Proof. In fact, $\lambda(G) = \min\{|\omega(A)| : \emptyset \neq A \subset V(G)\}$. The lemma follows from the definition of maximally-λ graphs. $\square$

Lemma 3.2. *For $S \subseteq E(G)$, suppose A is a nonempty proper vertex subset of $V(G)$ such that $|\omega_{G-S}(A)| < \delta(G-S)$ and $|A|$ is as small as possible, then*
(1) *$G[A]$ is connected;*
(2) *$|A| \geq 2$.*

Proof. (1) Suppose to the contrary that $G[A]$ is not connected. Let C be a component of $G[A]$, then $\omega_{G-S}(V(C)) \subseteq \omega_{G-S}(A)$, which implies that $|\omega_{G-S}(V(C))| \leq |\omega_{G-S}(A)| < \delta(G-S)$. Thus $V(C)$ is a smaller nonempty proper vertex subset of $V(G)$ satisfying the assumption, a contradiction.

(2) Suppose A has only one vertex, say x, then $|\omega_{G-S}(A)| = d_{G-S}(x) \geq \delta(G-S)$, a contradiction. $\square$

In [9], the authors proved that a connected graph G of order $n \geq 4$ is λ'-connected if and only if $G \ncong K_{1,n-1}$. Obviously, if G is maximally-λ but not λ'-connected with order $n \geq 4$, then $G \cong K_{1,n-1}$. By Proposition 2.3, $m_\lambda(G) = 0$. Hence, we only need to consider $m_\lambda(G)$ for graph G being maximally-λ and $G \ncong K_{1,n-1}$ with $n \geq 4$. For simplicity, we sometimes write $\lambda' = \lambda'(G)$, $\lambda = \lambda(G)$, $\delta = \delta(G)$ and $m_\lambda = m_\lambda(G)$. A bipartite graph is said to be *biregular* if all vertices in each part have the same degree. The following result presents upper and lower bounds for m_λ. We point out here that we obtained the same bounds as that of [7], but use a different method from [7]. For the sake of completeness, we also give the complete proof in our paper. Note that this proof basically follows the steps of Theorem 3.2 in [14].

Theorem 3.3. *Let G be a maximally-λ graph with $G \ncong K_{1,n-1}$ and $n \geq 4$. Then $\min\{\lambda' - \delta, \delta - 1\} \leq m_\lambda \leq \delta - 1$. If in addition G is a regular or a biregular graph, then $\min\{\lambda' - \delta + 1, \delta - 1\} \leq m_\lambda \leq \delta - 1$.*

Proof. Let S be a minimum edge-cut of G. Then $|S| = \delta$. Clearly, $G - S$ is not connected, and by the definition of m_λ, we have that $m_\lambda \leq |S| - 1 = \delta - 1$. The upper bound is proved.

To prove the lower bound, let $m = \min\{\lambda' - \delta, \delta - 1\}$. It suffices to show that $G - S$ is still maximally-λ for any $S \subseteq E(G)$ with $|S| \leq m$. Since $|S| \leq m \leq \delta - 1 = \lambda - 1$, we see that $G - S$ is connected. Suppose, on the contrary, that $G - S$ is no longer maximally-λ for some $S \subseteq E(G)$ with $|S| \leq m$. Then by Lemmas 3.1 and 3.2, there exists a vertex subset $A \subset V(G-S)$ with $|\omega_{G-S}(A)| < \delta(G-S)$, $|A|$ is as small as possible, $G[A]$ is connected and $|A| \geq 2$. We claim that $\omega_G(A)$ is a restricted edge-cut of G. If $G[\overline{A}]$ is connected, since $|\overline{A}| \geq |A| \geq 2$, then the claim is obvious. Now, suppose that $G[\overline{A}]$ is not connected. For any component C of $G[\overline{A}]$, $|\omega_{G-S}(V(C))| \leq |\omega_{G-S}(\overline{A})| = |\omega_{G-S}(A)| < \delta(G-S)$. And by the minimality of $|A|$, it follows that $|V(C)| \geq |A| \geq 2$, and the claim follows. Thus $|\omega_G(A)| \geq \lambda'(G)$, which implies that

$$\delta(G-S) > |\omega_{G-S}(A)| \geq |\omega_G(A)| - |S| \geq \lambda'(G) - m \geq \delta = \delta(G),$$

contradicting that $\delta(G-S) \leq \delta(G)$ for any $S \subseteq E(G)$.

Next suppose that G is a regular graph or a biregular graph. Then, every edge of G is incident with some vertex of degree δ, and thus $\delta(G-S) \leq \delta(G) - 1$ for any S with $1 \leq |S| \leq \delta - 1$. Using this observation and a similar argument as above, we can show that $m_\lambda \geq \min\{\lambda' - \delta + 1, \delta - 1\}$. $\square$

Remark 3.4. A tight example G for the lower bound $\lambda' - \delta$ is shown in Fig. 1, where A is a vertex independent set of order a, each vertex in A has degree a, B and C are two n-cliques, $F \subseteq [B, C]$ is a matching with $|F| = c$, $1 < a \le c \le n-1$ and $c \le 2a - 1$. It can be seen that $\delta = \lambda = a$, $\lambda' = c$. By Theorem 3.3, $m_\lambda \ge \min\{\lambda' - \delta, \delta - 1\} = c - a$. On the other hand, let S be a subset of F with $|S| = c - a + 1$. Then $\lambda(G - S) \le |F| - |S| = a - 1 < a = \delta(G - S)$, i.e., $G - S$ is not maximally-λ. Thus, $m_\lambda < c - a + 1$, which implies that $m_\lambda = c - a = \lambda' - \delta = \min\{\lambda' - \delta, \delta - 1\}$.

Note that when $c = 2a - 1$, the graph G also illustrates that the upper and lower bounds on $m_\lambda(G)$ for non-regular graphs in Theorem 3.3 are attainable.

4. The value $m_\lambda(G)$ under some conditions

In this section, we investigate the value $m_\lambda(G)$ under some conditions. We completely determine the exact value on $m_\lambda(G)$ for vertex transitive graphs.

Note that a graph G is super-λ if and only if either G is not λ'-connected or $\lambda' > \delta$ [14]. First, we determine $m_\lambda(G)$ for regular maximally-λ but not super-λ graphs G as follows.

Theorem 4.1. *Let G be a k-regular maximally-λ graph which is not super-λ. Then $m_\lambda(G) = 1$.*

Proof. Clearly, $k \ge 2$. When $k = 2$, $G \cong C_n$ and C_n is never super-λ except when $n = 3$. (See Example 2.2.) It follows from Proposition 2.3 that, $m_\lambda(G) = 1$ when $k = 2$. In the following, we assume that $k \ge 3$. Since G is λ'-connected but not super-λ, then $k = \lambda \le \lambda' \le \delta = k$ and $\lambda' = k$. Then by Theorem 3.3, $m_\lambda \ge \min\{\lambda' - k + 1, k - 1\} = 1$. To prove $m_\lambda = 1$, it suffices to show that G is not 2-maximally-λ.

Since G is not super-λ, there exists some minimum edge-cut F of G which is not the set of edges incident to a vertex of G. Then there exists a pair of non-adjacent edges, say e_1, e_2, in F. Set $S = \{e_1, e_2\}$ and let $G' = G - S$. Clearly, G' is connected and $\delta(G') = k - 1$. Let $S' = F - S$ and it follows that $G' - S'$ is not connected. Hence, $\lambda(G') \le |S'| = k - 2 < k - 1 = \delta(G')$ and G is not 2-maximally-λ. $\square$

By the observation that a super-λ graph is also maximally-λ, the following proposition is ready to see.

Proposition 4.2. *Let G be a super-λ graph. Then $S_\lambda(G) \le m_\lambda(G) \le \delta(G) - 1$.*

A natural question from Proposition 4.2 is that for super-λ graphs, whether or not there exist some family of graphs satisfying $S_\lambda(G) < m_\lambda(G)$ or $S_\lambda(G) = m_\lambda(G)$. We always call 3-regular graph *cubic graph*. The following result for cubic super-λ graphs explains the case $S_\lambda(G) < m_\lambda(G)$.

Theorem 4.3. *Let G be a cubic super-λ graph. Then $m_\lambda(G) = \delta(G) - 1 = 2$.*

Proof. Since G is super-λ and λ'-connected, then $\lambda' \ge \lambda + 1 = 4$. By Theorem 3.3, $2 = \delta - 1 \ge m_\lambda \ge \min\{\lambda' - \delta + 1, \delta - 1\} = 2$ and thus $m_\lambda = \delta - 1 = 2$. $\square$

Remark 4.4. The authors in [14] proved $S_\lambda(G) = \delta(G) - 2 = 1$ for cubic super-λ graphs G. Thus, Theorem 4.3 shows that cubic super-λ graphs are a family of graphs satisfying $S_\lambda(G) = 1 < 2 = m_\lambda(G)$. It is easy to see that the well-known Petersen graph is a cubic super-λ graph. In [4], Boesch and Wang showed that every connected circulant is super-λ unless it is $G(n; a)$ or $G(2i; 2, 4, \dots, i-1, i)$ for $i > 1$ is odd, implying that the connected circulant $G(2j; 1, j)$ for $j \ge 2$ is a cubic super-λ graph. Therefore, the Petersen graph and the circulant $G(2j; 1, j)$ ($j \ge 2$) are examples satisfying $S_\lambda(G) < m_\lambda(G)$.

Since edge transitive graphs are always super-λ except for C_n [27], $S_\lambda(G) = \delta(G) - 1$ for edge transitive graphs G with $\delta(G) \ge 5$ [14], and $m_\lambda(G) = \delta(G) - 1$ [7], it follows that $S_\lambda(G) = m_\lambda(G) = \delta(G) - 1$ for $\delta(G) \ge 5$. Thus, edge transitive graphs with $\delta(G) \ge 5$ are a family of graphs satisfying $S_\lambda(G) = m_\lambda(G)$. For example, the hypercube Q_n ($n \ge 5$) and the star graph $ST(n)$ ($n \ge 6$) [1] are such graphs.

Let $\xi(G) = \min\{d(x) + d(y) - 2 : xy \in E(G)\}$. It is proved in [9] that, for a λ'-connected graph G, $\lambda'(G) \le \xi(G)$ holds. Thus, we call a graph G with $\lambda'(G) = \xi(G)$ *λ'-optimal*. It is clear that super-λ' graphs are λ'-optimal.

Lemma 4.5. *([12]). If G is a λ'-optimal graph, then G is maximally-λ.*

Theorem 4.6. *Let G be a λ'-optimal graph. Then*

(1) *If $\delta = 1$, then $m_\lambda = 0$, otherwise, $\delta - 2 \le m_\lambda \le \delta - 1$;*

(2) *$m_\lambda = \delta - 1$ if G is a regular or a biregular graph.*

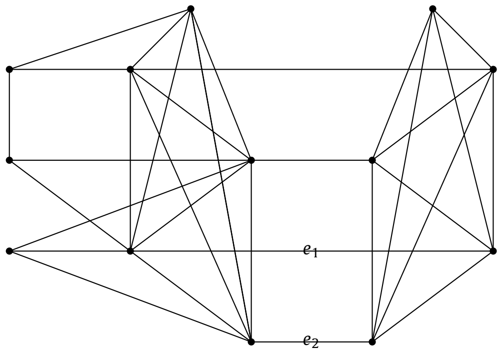

Fig. 2. The graph G satisfies $m_\lambda(G) = \delta(G) - 2$.

Proof. (1) If $\delta = 1$, clearly, $m_\lambda = 0$. Next, assume that $\delta \geq 2$. Since G is λ'-optimal, $\lambda' = \xi = \min\{d(x) + d(y) - 2 : xy \in E(G)\} \geq 2\delta - 2$ and $\lambda' - \delta \geq \delta - 2$. It follows from Theorem 3.3 that $\delta - 1 \geq m_\lambda \geq \min\{\lambda' - \delta, \delta - 1\} \geq \delta - 2$.

(2) Now suppose that G is a regular or a biregular graph, then

$$\lambda' = \xi = \begin{cases} 2\delta - 2, & G \text{ is regular}, \\ \delta + \Delta - 2, & G \text{ is biregular with degrees } \delta \text{ and } \Delta, \text{ respectively}, \end{cases}$$

where $\Delta > \delta$. By Theorem 3.3, it follows that

$$\delta - 1 \geq m_\lambda \geq \min\{\lambda' - \delta + 1, \delta - 1\} = \begin{cases} \delta - 1, \\ \min\{\Delta - 1, \delta - 1\} = \delta - 1. \end{cases}$$

The result follows. □

Now, we give an example, see Fig. 2, to illustrate that the lower bound $\delta - 2$ in Theorem 4.6(1) is attainable. It is easy to check that $\lambda'(G) = \xi(G) = 4$. Let $S = \{e_1, e_2\}$. Then $\lambda(G - S) \leq 2 < 3 = \delta(G - S)$, which implies that $m_\lambda(G) \leq 1$. Combining with Theorem 4.6(1), we have $m_\lambda(G) = 1 = \delta(G) - 2$.

Corollary 4.7. *Let G be a k-regular super-λ' graph. Then $m_\lambda(G) = k - 1$.*

Theorem 4.8. *([34]). Every vertex transitive graph, either containing no 3-cliques or having odd order, is λ'-optimal.*

Hence by Theorems 4.6 and 4.8, for vertex transitive graphs, we obtain the following result immediately.

Corollary 4.9. *Let G be a k-regular-connected vertex transitive graph. If G either contains no 3-cliques or has odd order, then $m_\lambda(G) = k - 1$.*

Next, we will completely determine the exact value of m_λ for vertex transitive graphs. For subset $X \subseteq V(G)$, if $\omega(X)$ is a minimum restricted edge-cut of G, then X is called a λ'-*fragment* of G. We call a λ'-fragment with minimum cardinality λ'-*atom*. A λ'-fragment X is called *strict* if $3 \leq |X| \leq |V(G)| - 3$. If G contains strict λ'-fragments, then those ones with the smallest cardinality are called λ'-*superatoms*. With the help of the following lemma showed in [23], we can completely determine m_λ for vertex transitive graphs.

Lemma 4.10. *([23]). Let G be a k-regular-connected vertex transitive graph which is neither a complete graph nor a cycle. If G is not λ'-optimal, then*

(1) *any two distinct λ'-atoms of G are disjoint;*

(2) *the subgraph Y induced by a λ'-atom is a $(k-1)$-regular graph and $k \leq |V(Y)| \leq 2k - 3$.*

Theorem 4.11. *Let G be a k-regular-connected vertex transitive graph with $k \geq 3$. Then $m_\lambda = \lambda' - k + 1$.*

Proof. If G is λ'-optimal, then $\lambda' = 2k - 2$. And by Theorem 4.6(2), we have $m_\lambda = k - 1 = \lambda' - k + 1$. In the following we assume that G is not λ'-optimal. Clearly, K_{k+1} is λ'-optimal, then $G \ncong K_{k+1}$. Since G is not λ'-optimal, we have $\lambda' \leq 2k - 3$ and so $\lambda' - k + 1 \leq k - 2$. By Theorem 3.3, $m_\lambda \geq \min\{\lambda' - k + 1, k - 1\} = \lambda' - k + 1$. To prove $m_\lambda = \lambda' - k + 1$, it suffices to show that there exists some edge subset S of G with $|S| = \lambda' - k + 2$ such that $G - S$ is not maximally-λ.

Since G is vertex transitive, every vertex of G lies in a λ'-atom. Let $A \subseteq V(G)$ be a λ'-atom of G. Let $y \in \overline{A}$. Then there exists $\phi \in Aut(G)$ such that $\phi(x) = y$ for a fixed vertex $x \in A$. Let $A_y = \phi(A)$. Then $A_y \cap A = \emptyset$ by Lemma 4.10(1) and the fact that A_y is also a λ'-atom and $y \notin A$. It follows that, for each vertex y in G, there exists a λ'-atom A_y containing y, such that $A_y \cap A_z = \emptyset$ or $A_y = A_z$ for $y \neq z \in V(G)$. Thus, we see that $V(G)$ is a disjoint union of distinct λ'-atoms.

By the definition of λ'-atom, both $G[A]$ and $G[\overline{A}]$ are connected. For each $uv \in \omega(A)$, without loss of generality, assume $u \in A$. It follows that v belongs to another λ'-atom, say B. By Lemma 4.10(2), both $G[A]$ and $G[B]$ are $(k-1)$-regular. Thus, $d_{\bar{A}}(u) = d_{\bar{B}}(v) = 1$. Then it can be seen that the edges of $\omega(A)$ are independent. By $k \leq |A| \leq 2k-3$, we have $|\omega(A)| = |A| \geq k > \lambda' - k + 2$. Take an edge subset $S \subseteq \omega(A)$ such that $|S| = \lambda' - k + 2$. Since $|S| < k$, $G - S$ is connected and

$$\lambda(G-S) \leq |\omega(A)| - |S| = \lambda' - (\lambda' - k + 2) = k - 2 < k - 1 = \delta(G-S),$$

where $\delta(G-S) = k-1$ since S is an independent set of edges. Thus, $G-S$ is not maximally-λ and $m_\lambda \leq |S| - 1 = \lambda' - k + 1$. The proof is complete. $\square$

The *Cartesian product* of graphs G_1 and G_2 is the graph $G_1 \Box G_2$ with vertex set $V(G_1) \times V(G_2)$, vertices (x_1, y_1) and (x_2, y_2) are adjacent in $G_1 \Box G_2$ if and only if either $x_1 = x_2$ and $y_1 y_2 \in E(G_2)$, or $x_1 x_2 \in E(G_1)$ and $y_1 = y_2$.

By Theorem 3.3, for a k-regular super-λ graph with $k \geq 4$, we have $k - 1 \geq m_\lambda \geq \min\{\lambda' - k + 1, k - 1\} \geq 2$. For k-regular super-λ graphs with $k \geq 4$, whether or not there exist some graphs G such that $m_\lambda(G)$ attains every integer between 2 and $k-1$ is quite a natural question. We deal with this question by using Theorem 4.11.

Lemma 4.12. *([21]). For any given integers k and s with $k \geq 3$ and $0 \leq s \leq k-3$, there is a k-regular-connected vertex transitive graph $G = K_2 \Box G_0$, where G_0 is a circulant graph satisfying*

$$G_0 = \begin{cases} G(k+s; 1, 2, \ldots, t), & \text{if } k = 2t+1; \\ G(k+s; 1, 2, \ldots, t-1, t+\frac{1}{2}s), & \text{if } k = 2t \text{ and } s \text{ is even}; \end{cases}$$

and $t \geq 1$, such that $\lambda'(G) = k + s$ if and only if either k is odd or s is even.

Theorem 4.13. *For any given integers k and l with $k \geq 4$ and $2 \leq l \leq k-1$, there exists a k-regular-connected super-λ vertex transitive graph G such that $m_\lambda = l$.*

Proof. Let $G = K_2 \Box G_0$ be the graph defined in Lemma 4.12. If $1 \leq s \leq k-3$, then $\lambda'(G) = k + s > k = \delta(G)$ and G is super-λ. Besides, by Theorem 4.11, we have $m_\lambda(G) = \lambda'(G) - k + 1 = s + 1$. Thus, the Cartesian product graph G satisfies $2 \leq m_\lambda(G) \leq k-2$. Besides, since $\lambda'(G) = 2k - 2 > k = \delta(G)$ for k-regular λ'-optimal graph G with $k \geq 4$, then λ'-optimal graphs with $k \geq 4$ are always super-λ. By Corollary 4.9, the hypercube Q_k and the star graph $ST(k+1)$ are super-λ graphs satisfying $m_\lambda = k - 1$. $\square$

5. $S_\lambda(G)$ for vertex transitive graphs

In view of Proposition 4.2 and Theorem 4.11, we see that $S_\lambda(G) \leq m_\lambda(G) = \lambda'(G) - \delta(G) + 1$ for super-λ vertex transitive graphs. In this section, we will completely determine the exact value of $S_\lambda(G)$ for vertex transitive graphs. In [14], Hong et al. proved the following results about $S_\lambda(G)$.

Lemma 5.1. *([14]). A graph G is super-λ if and only if $|\omega(A)| > \delta(G)$ holds for any vertex subset $A \subseteq V(G)$ with $2 \leq |A| \leq \lfloor \frac{|V(G)|}{2} \rfloor$ and $G[A]$, $G[\overline{A}]$ being connected.*

Theorem 5.2. *([14]). Let G be a k-regular super-λ graph which is λ'-connected. Then*
(1) $\min\{\lambda' - \delta, \delta - 1\} \leq S_\lambda(G) \leq \delta - 1$;
(2) If $k = 3$, $S_\lambda(G) = 1$;
(3) If G is a k-regular super-λ' graph, then $S_\lambda(G) = k - 1$.

In [23], Meng characterized super-λ vertex transitive graphs as the following.

Theorem 5.3. *([23]). Let G be a k-regular-connected vertex transitive graph which is neither a complete graph nor a cycle. Then G is super-λ if and only if it contains no k-cliques.*

Let G be a k-regular-connected vertex transitive graph of order n. For $k = 1$, $G \cong K_2$, G is super-λ, and $S_\lambda(G) = 0$. For $k = 2$, $G \cong C_n$. If $n \geq 4$, then G is not super-λ, thus $S_\lambda(G)$ does not exist. If $n = 3$, then $S_\lambda(G) = 1$. For $k = 3$, by Theorem 5.3, G is super-λ if and only if G contains no 3-cliques. Hence, for $k = 3$, if G contains no 3-cliques, by Theorem 5.2(2), $S_\lambda(G) =$

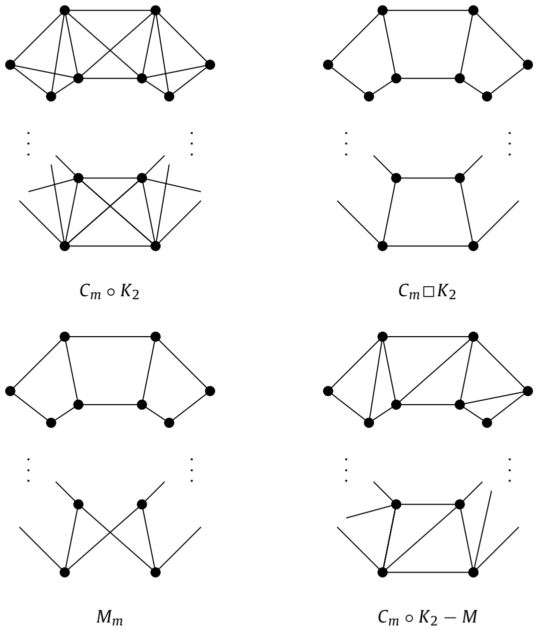

Fig. 3. Graphs in $\mathcal{F}$.

1; otherwise, $S_\lambda(G)$ does not exist. For $k \geq 4$, if furthermore, G is super-λ', by Theorem 5.2(3), $S_\lambda(G) = k - 1$. Therefore, we only need to determine $S_\lambda(G)$ for k-regular-connected super-λ vertex transitive graphs G which are not super-λ' with $k \geq 4$.

The *lexicographic product* of G_1 and G_2 is the graph $G_1 \circ G_2$ with vertex set $V(G_1) \times V(G_2)$. In $G_1 \circ G_2$, two vertices (x_1, y_1) and (x_2, y_2) are adjacent if and only if either $x_1x_2 \in E(G_1)$, or $x_1 = x_2$ and $y_1y_2 \in E(G_2)$. Let M_m be the Möbius ladder with m rungs, $C_m \circ K_2 - M$ be the vertex transitive graph obtained from $C_m \circ K_2$ by removing a perfect matching M (see Fig. 3), and D_t be the family of 4-regular vertex transitive graph every vertex of which is contained in exactly two 3-cliques. In fact, we can find some graphs, such as the line graph of any cubic graph that contains no 3-cliques, in D_t. Let $\mathcal{F} = \{C_m \circ K_2, C_m \square K_2, C_m \circ K_2 - M, M_m, C_m\} \cup D_t$. Then every graph in $\mathcal{F}$ is vertex transitive [36]. Thus, by Theorem 5.3, we can see that every graph in $\mathcal{F}$ except for C_m is super-λ. It can be seen in [36] that every graph in $\mathcal{F}$ is λ'-optimal but not super-λ'. Yang et al. [36] gave a characterization of super-λ' vertex transitive graphs which are λ'-optimal as follows.

Theorem 5.4. *([36]). Let G be a λ'-optimal k-regular vertex transitive graph which is not a graph in $\mathcal{F}$. Then*

(1) If G is not super-λ', then any two distinct λ'-superatoms of G are disjoint;

(2) G is not super-λ' if and only if either G contains a $(k-1)$-regular subgraph with $2k-2$ vertices and $|V(G)| \geq 2k+1$, or G contains a $(k-1)$-clique and G is not isomorphic to a $(k+1)$-clique.

Let G be a vertex transitive graph which is not super-λ'. Then each vertex lies in a λ'-atom or λ'-superatom. Hence, by an argument similar to the proof of Theorem 4.11, combining Lemma 4.10(1) with Theorem 5.4(1), the next lemma follows.

Lemma 5.5. *Let G be a k-regular-connected vertex transitive graph with $k \geq 4$ which is not a graph in $\mathcal{F}$. If G is not super-λ', then $V(G)$ is a disjoint union of distinct λ'-atoms or λ'-superatoms.*

For convenience, we use $e_1 \sim e_2$ to indicate that two edges e_1 and e_2 are adjacent, and $e_1 \nsim e_2$ otherwise. A matching M *covers* a vertex v, if some edge of M is incident with v.

Lemma 5.6. *Let G be a k-regular-connected vertex transitive graph with $k \geq 4$. If G is super-λ but not super-λ', then $S_\lambda(G) = \lambda' - k$.*

Proof. Since $\lambda' \leq 2k - 2$, we have $\lambda' - k < k - 1$. Then combining Proposition 4.2, Theorems 4.11 and 5.2(1), it follows that $\lambda' - k + 1 = m_\lambda(G) \geq S_\lambda(G) \geq \min\{\lambda' - k, k - 1\} = \lambda' - k$. To show $S_\lambda(G) = \lambda' - k$, it suffices to prove that there exists some $S \subseteq E(G)$ with $|S| = \lambda' - k + 1$, such that $G' = G - S$ is not super-λ. In view of Lemma 5.1, we will show how to take an edge subset S such that there is a vertex subset $A \subseteq V(G)$ with $2 \leq |A| \leq \frac{|V(G)|}{2}$, $G'[A]$ and $G'[\overline{A}]$ being connected, and $|\omega_{G-S}(A)| \leq \delta(G - S)$. Suppose that $A \subseteq V(G)$ is a λ'-superatom of G when G is λ'-optimal but not super-λ', and A is a λ'-atom of G when G is not λ'-optimal. Then both $G[A]$ and $G[\overline{A}]$ are connected. We consider the following cases.

Case 1. G is not λ'-optimal.

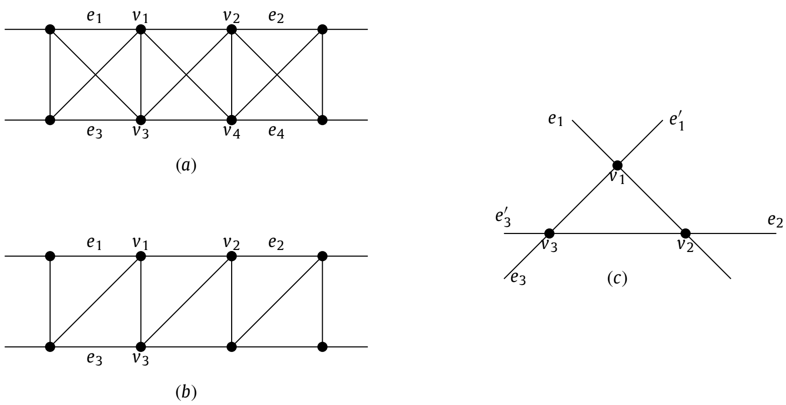

Fig. 4. An illustration about the proof of Case 3 in Lemma 5.6.

By the proof of Theorem 4.11, we see that edges of $\omega(A)$ are independent, and $|\omega(A)| = |A| \geq k > \lambda' - k + 1$. Take an edge subset $S \subseteq \omega(A)$ such that $|S| = \lambda' - k + 1$. Then

$$|\omega_{G-S}(A)| = |\omega_G(A)| - |S| = \lambda' - (\lambda' - k + 1) = k - 1 = \delta(G - S).$$

Hence, $G - S$ is not super-λ for $|S| = \lambda' - k + 1$.

Case 2. G is λ'-optimal but not super-λ' and $G \notin \mathcal{F}$.

By Theorem 5.4(2), either $G[A]$ is a $(k-1)$-regular subgraph with $2k-2$ vertices, or $G[A] \cong K_{k-1}$. For the former case, it can be proved by a similar argument used in Lemma 4.12 that edges of $\omega(A)$ are independent. Furthermore, $|\omega(A)| = |A| = 2k - 2$. Take $S \subseteq \omega(A)$ such that $|S| = k - 1$. Then

$$|\omega_{G-S}(A)| = 2k - 2 - (k - 1) = k - 1 = \delta(G - S).$$

For the case that $G[A] \cong K_{k-1}$, let uv be an edge of $\omega(A)$ with $u \in A$. Suppose v is in another λ'-superatom, say B. Then $d_A(v) \leq d_{\overline{B}}(v) = 2$. Let H be the subgraph of G induced by the edges between A and $N_G(A)$. For any vertex subset $U \subseteq A$,

$$2|U| = \sum_{u \in U} d_H(u) = \sum_{v \in N_G(U)} d_U(v) \leq \sum_{v \in N_G(U)} d_A(v) \leq \sum_{v \in N_G(U)} d_{\overline{B}}(v) = 2|N_G(U)|.$$

So, we have $|N_G(U)| \geq |U|$ for any $U \subseteq A$ in H. By Hall's Theorem, there is a matching $S \subseteq \omega(A)$ which covers every vertex in A. Such an edge subset S satisfies $|S| = |A| = k - 1$ and

$$|\omega_{G-S}(A)| = 2|A| - (k - 1) = k - 1 = \delta(G - S).$$

In any case, $G - S$ is not super-λ for $|S| = k - 1 = \lambda' - k + 1$.

Case 3. G is a graph of $\mathcal{F}$ with $k \geq 4$.

Thus, $G \cong C_m \circ K_2, C_m \circ K_2 - M$ or D_t, and G is λ'-optimal.

If $G \cong C_m \circ K_2$, then G is 5-regular and $\lambda' = 8$. Refer to Fig. 3(a) for an illustration. Let $S = \{e_1, e_2, e_3, e_4\}$ and $A = \{v_1, v_2, v_3, v_4\}$, then $|\omega_{G-S}(A)| = 4 = \delta(G - S)$.

If $G \cong C_m \circ K_2 - M$, then G is 4-regular and $\lambda' = 6$. Refer to Fig. 3(b) for an illustration. Let $S = \{e_1, e_2, e_3\}$ and $A = \{v_1, v_2, v_3\}$, then $|\omega_{G-S}(A)| = 3 = \delta(G - S)$.

If $G \cong D_t$, then G is 4-regular and $\lambda' = 6$. Refer to Fig. 4(c) for an illustration. Let $A = \{v_1, v_2, v_3\}$ such that $G[A] \cong K_3$. We can find an independent edge set $S \subseteq \omega(A)$ with $|S| = 3$. In fact, because every vertex of G is contained in exactly two K_3, we can find $e_1 \in [\{v_1\}, \overline{A}]$ and $e_2 \in [\{v_2\}, \overline{A}]$ such that $e_1 \nsim e_2$. If there is an edge $e_3 \in [\{v_3\}, \overline{A}]$ which is adjacent to neither e_1 nor e_2, then we can take an independent edge set $S = \{e_1, e_2, e_3\}$. If not, without loss of generality, assume $e_3 \sim e_1$ and $e'_3 \sim e_2$, where $e'_3 \in [\{v_3\}, \overline{A}]$, and $e'_3 \neq e_3$. Then $S = \{e'_1, e_2, e_3\}$ is an independent edge subset, and $|\omega_{G-S}(A)| = 3 = \delta(G - S)$.

In any case $|\omega_{G-S}(A)| = k - 1 = \delta(G - S)$ and $|S| = \lambda' - k + 1$. $\square$

Therefore, we can sum up to determine the exact value on $S_\lambda(G)$ for vertex transitive graphs.

Theorem 5.7. *Let G be a k-regular-connected vertex transitive graph with order n. Then*

(1) *For $k = 2$, if $n = 3$, then $S_\lambda(G) = 1$. If $n \geq 4$, then $S_\lambda(G)$ does not exist;*

(2) *For $k = 3$, if G contains no 3-cliques, then $S_\lambda(G) = 1$, otherwise $S_\lambda(G)$ does not exist;*

(3) *For $k \geq 4$, if G is super-λ', then $S_\lambda(G) = k - 1$. If G is super-λ but not super-λ', then $S_\lambda(G) = \lambda' - k$.*

6. Conclusions

Based on the definition proposed by Esfahanian [10], a network is fault tolerant if it can remain functional in the presence of failures. The parameter $m_{\mathcal{P}}(G)$ considers a network functional if the network still has the property $\mathcal{P}$ in the presence of certain failures. Clearly, the larger this parameter, the more reliable the network with respect to the property $\mathcal{P}$ [31]. In this paper, we investigate maximally-λ tolerance $m_\lambda(G)$ and super-λ tolerance $S_\lambda(G)$ to edge-faults of graphs. We present upper and lower bounds on $m_\lambda(G)$ for general graphs and regular graphs. More refined bounds are obtained under some conditions. Exact values on $m_\lambda(G)$ and $S_\lambda(G)$ for vertex transitive graphs are obtained. These results can be used in the reliability analysis of the network. Our further work is to study the exact values of $m_\lambda(G)$ and $S_\lambda(G)$ for the graphs in general. Besides, we will also investigate $m_{\mathcal{P}}(G)$ in the presence of node/link and node failures.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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