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Wild Low-Dimensional Topology and Dynamics

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Abstract

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In this dissertation we discuss various results for spaces that are *wild*, i.e. not locally simply connected. We first discuss periodic properties of maps from a given space to itself, similar to Sharkovskiĭ's Theorem for interval maps. We study many non-locally connected spaces and show that some have periodic structure either identical or related to Sharkovskiĭ's result, while others have essentially no restrictions on the periodic structure. We next consider embeddings of solenoids together with their complements in three space. We differentiate solenoid complements via both algebraic and geometric means, and show that every solenoid has an unknotted embedding with Abelian fundamental group, as well as infinitely many inequivalent knotted embeddings with non-Abelian fundamental group. We end by discussing Peano continua, particularly considering subsets where the space is or is not locally simply connected. We present reduced forms for homotopy types of Peano continua, and provide a few applications of these results.

Keywords: Sharkovskii's Theorem, periodic points, solenoids, 3-manifolds, fundamental group, Peano continua, homotopy invariants.

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Chapter 1. Introduction

In this dissertation we discuss various results in wild low-dimensional topology and dynamics. There are three main topics we consider: periodic properties of maps from a given space to itself, solenoids together with their complements in S^3, and Peano continua. In each of these areas, we derive interesting results while considering spaces that are *wild*, i.e. not locally simply connected.

In Chapter 2, we study the periodic properties of maps on non-locally connected spaces. This work is motivated by the result of Sharkovskiĭ [44], which completely describes which orders of periodic points for a map of an interval to itself imply the existence of other orders. Sharkovskiĭ's result states that the possible period sets for interval maps are very structured, in that he defines a total order on the natural numbers that precisely determines when a certain period must imply the existence of succeeding periods.

Sharkovskiĭ's work has been generalized in various ways. A few authors have found other spaces that satisfy the same strong result as Sharkovskiĭ's Theorem. Others have considered what possible period sets and period implications may arise for spaces that definitely do not have such a rigidly structured result. Some of the spaces that have been considered are arbitrary linear continua, circles, and graphs. For more details and citations see Section 2.1.

Most spaces do not have a total order that describes the implications for the existence of certain periods for maps on the space. However, we are often able to describe the possible period sets, usually as the union of sets of a particular form. It is interesting to note that for many of the spaces previously studied, the form of period sets is often built from segments, or multiples of segments, of Sharkovskiĭ's order for the interval. There are also spaces that have essentially no structure in the form of their possible period sets, such as a 2 dimensional disk, which has no restrictions other than the existence of a fixed point.

We consider the periodic properties for spaces that are not locally connected, such as the topologist's sine curve and the Cantor set, as well as many others. We prove that the topologist's sine curve has the same periodic structure as the interval, while the Cantor set has essentially no periodic structure at all (except when restricting to maps with every point periodic). Additionally, we discuss many other examples with varying levels of periodic structure ranging from the strict total order of Sharkovskiĭ's Theorem to the flexibility of the Cantor set. We also investigate how the existence of aperiodic points for interval maps is related to the result of Sharkovskiĭ's Theorem, as this issue affects the result for the Cantor set. One last question we study in this chapter is what period implications are possible for any space, i.e. for m, n, is there a space where a period m point implies the existence of a period n point?

Chapter 3 discusses embeddings of solenoids in the three sphere. Solenoids are inverse limits of circles, where the bonding maps may be chosen to be any integer, corresponding to wrapping one circle around the next n times. There are different solenoids that result from different choices of bonding maps, although some different choices may lead to the same resulting inverse limit. In any case, the resulting topological space is a compact topological group, yet it is not locally path connected, nor are its path components.

Solenoids do in fact embed in three space, and in fact there are many inequivalent ways to embed any given solenoid, even if we restrict to a fixed choice of bonding maps. The complements of these embeddings provide interesting examples of open 3-manifolds with complicated structure at "infinity," as the boundary of the manifold in S^3 is the solenoid, which is not locally connected.

We can distinguish some of these embeddings by their fundamental groups. We show that each solenoid has embeddings with Abelian fundamental group, as well as embeddings with non-Abelian fundamental group. We calculate the fundamental group by decomposing the complement into pieces that are each a braid in a solid torus, and combining the resulting groups appropriately via the Seifert Van Kampen Theorem. The resulting presentation is infinite, and thus it is difficult to further distinguish the resulting groups, particularly as all of the groups for a given solenoid have a common Abelianization.

We can further distinguish the complements of certain embeddings by geometric means. As long as infinitely many of the pieces in our decomposition have hyperbolic structures, we may use an extension of the JSJ-decomposition to differentiate between manifolds with different hyperbolic pieces. This method provides infinitely many distinct complements for every solenoid, and in fact for every defining sequence of bonding maps except for one ending constantly with 2 for the dyadic solenoid.

In Chapter 4 we finish by discussing Peano continua, that is, spaces that are compact, connected, locally connected metric spaces. While Peano continua have these very nice properties, there are still many interesting examples that arise. In particular, it is useful to distinguish those points that have simply connected neighborhoods and those that do not. For example, the Hawaiian earring is a Peano continuum with a unique point where the space is not locally simply connected. On the other hand, the Sierpinski carpet is a space which is nowhere locally simply connected.

We first prove the existence of certain reduced forms for Peano continua up to homotopy equivalence. One of these reduced forms involves contracting all strongly contractible sets attached at cut points of the space. The second reduces the one-dimensional subspace to be a disjoint collection of arcs, together with the set of points where the space is not locally simply connected, which we prove are homotopically fixed. We give some applications of these reduced forms, including showing that adding arcs to a space preserves the homotopy groups of the original space as a subgroup.

We then state some special results that hold for one-dimensional Peano continua. In this setting, the first reduced form implies the existence of a minimal deformation retract. The second reduced form represents the continuum as a compactification of a null sequence of open arcs by some compact set of dimension 0 or 1. We also give an application of these results for one-dimensional continua with the shape of a graph.

We conclude by discussing homotopy invariants for one-dimensional Peano continua. Two of these are subsets of the continuum, consisting of points where the continuum is not locally simply connected, while the third is a number in $\mathbb{N} \cup \infty$. These invariant subsets are enough to determine the homotopy type of the continuum in many cases, and we conjecture that they are in fact complete homotopy invariants for all one-dimensional Peano continua.

Chapter 2. Periodic Properties of Maps on Non-Locally Connected Spaces

2.1 Introduction

A central topic of study in dynamical systems is that of periodic points of maps from a given space to itself. In this setting we may iterate the map, and then ask questions about when points of the space return to their original position, under repeated application of the function. The topology of certain spaces provides significant structure as to what periods, or sets of periods, may occur, while other spaces are essentially unrestricted in this regard.

Sharkovskiĭ [44] has proven a remarkable result about self-maps of an interval, and which periodic orders imply the existence of other orders. He introduces a total ordering of the positive integers that describes the periodic structure of interval maps. We first describe this total order, and then proceed with a few definitions before stating Sharkovskiĭ's Theorem (Theorem 2.1.2). Sharkovskiĭ's order on the positive integers ($\preceq$) starts with the odd integers in ascending order (excluding 1), followed by 2 times the odd integers, then 2^2 times the odds, and 2^i times the odds. The end of the ordering is the powers of 2 in descending order. It is interesting to note that Sharkovskiĭ's order is almost a well-ordering of the natural numbers, as there is only one Dedekind cut that has no least element in this ordering, namely the cut consisting of all powers of 2: $\{\ldots, 2^n, \ldots, 2^2, 2, 1\}$.

Sharkovskiĭ's Order:

$$3 \preceq 5 \preceq 7 \preceq 9 \preceq \cdots \preceq 2\cdot 3 \preceq 2\cdot 5 \preceq \cdots \preceq 2^2\cdot 3 \preceq 2^2\cdot 5 \preceq \cdots \preceq 2^n \preceq \cdots \preceq 2^2 \preceq 2 \preceq 1$$

Definitions 2.1.1. Let f be a map from a space to itself. A point x has *period* k if $f^k(x) = x$. If k is the smallest such positive integer, we say that x has *order* k, or *least period* k. The *period set* of f, $\mathrm{Per}(f)$, is the set of all least periods (orders) for the function f. A *tail* of the Sharkovskiĭ order is a set S of positive integers such that if $n \in S$, then $m \in S$ whenever $n \preceq m$.

We now have the definitions in place to state Sharkovskiĭ's Theorem:

Theorem 2.1.2 (Sharkovskiĭ [44])**.** *Let I denote an interval.*

A: For every continuous map $f : I \to I$, $\mathrm{Per}(f)$ is a tail of the Sharkovskiĭ order.

B: Every non-empty tail of the Sharkovskiĭ order occurs as $\mathrm{Per}(f)$ for some continuous map $f : I \to I$.

We show in Lemma A.1.1 that the maps f in Theorem 2.1.2B can be taken to fix the endpoints of the interval. This will be useful in some of our later results.

Although it is sometimes assumed that I is a closed interval, this is not necessary, and the theorem is true for open (or half open) intervals as well [14, 44]. In the case where I is not closed, I no longer has the fixed point property. However, Theorem 2.1.2A is still true, as any map without a fixed point has no periodic points. Theorem 2.1.2B is also true as stated, but can be strengthened by removing 'non-empty' from the statement: the map $f(x) = x+1$ on the real line has no periodic points, for example, and thus has $\mathrm{Per}(f) = \emptyset$. A proof of Sharkovskiĭ's Theorem can be found in many articles or books on dynamical systems, see for example [4, 12, 14, 24, 44].

About 11 years after Sharkovskiĭ proved his theorem, Li and Yorke [33] independently proved related results about interval maps, only finding out about Sharkovskiĭ's earlier work at a later date. One of the results in this paper was a partial result of Sharkovskiĭ's Theorem, in that Li and Yorke proved that a point of period 3 implies the existence of periodic points of all other periods. However, Li and Yorke also proved facts that were not a result of Sharkovskiĭ's Theorem; in particular they showed that period 3 implies the existence of an uncountable set of aperiodic points with certain properties.

Definition 2.1.3. A space X is called a *Sharkovskiĭ space* if Theorem 2.1.2A is true when I is replaced by X. That is, for maps $f : X \to X$, the period set $\mathrm{Per}(f)$ is some tail of Sharkovskiĭ's order.

Sharkovskiĭ's Theorem has been generalized in various ways. Schirmer proves that any connected linear order space is a Sharkovskiĭ space [43]. While Schirmer also proves that Theorem 2.1.2B also holds for connected linear spaces if they contain an arc, Baldwin shows that Theorem 2.1.2B does not hold for all connected linear spaces [8]. It is interesting to note that Baldwin shows that the only way Theorem 2.1.2B can fail for a connected linear space is if the space does not admit any map with a certain period. While chainable (or arc-like) continua are generally not Sharkovskiĭ spaces, Minc and Transue show that hereditarily decomposable chainable continua are Sharkovskiĭ spaces [37].

As most spaces are not Sharkovskiĭ spaces, many people have studied the possible period sets for maps on other spaces, and have classified what periods imply the existence of other periods, and more generally, what period sets are possible. We note that such work has been done for spaces such as n-ods, trees, circles, and others, see for example [2, 3, 5, 7, 9, 12, 23, 26, 31, 38, 45, 54, 56].

In this chapter, we prove that the topologist's sine curve and the Warsaw circle are both Sharkovskiĭ spaces, as are other examples of non-locally connected continua based on those spaces. These spaces also satisfy Theorem 2.1.2B, and most have the fixed point property. We also discuss examples that are not Sharkovskiĭ spaces, and we describe the possible period sets for functions on these spaces. For these spaces, the possible periods sets are usually based on combinations of multiples of tails of the Sharkovskiĭ order.

Further, we discuss period sets for maps of the Cantor set. The Cantor set is quite flexible, and has no restrictions on the possible period sets. However, if we require every point to be periodic then we do get a restricted result. Consequently, at the end of the chapter we consider the relationship between aperiodic points and Sharkovskiĭ's order for interval maps.

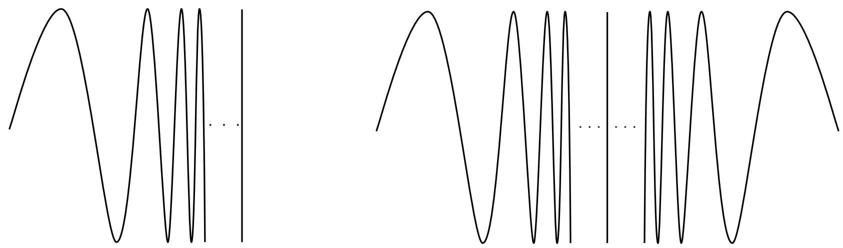

Figure 2.1: The topologist's sine curve (left) and the doubled sine curve (right).

We also briefly discuss period implications that may be possible for any space, which seem to be strongly related to Sharkovskiĭ's order.

2.2 Non-Locally Connected Spaces

Definition 2.2.1. A space X is *locally connected* if for every point $x \in X$, and for every neighborhood U containing x, there is a connected neighborhood V with $x \in V \subset U$.

Many common spaces are locally connected, for example: arcs, graphs, Euclidean n-space, and manifolds. While all of these examples are locally simply connected, there are also many examples of locally connected spaces that are not locally simply connected, including the Hawaiian earring, the Sierpinski curve, and the Menger curve.

We will discuss various non-locally connected spaces, in relation to Sharkovskiĭ's Theorem. Perhaps the simplest example of a space that is not locally connected is that of a convergent sequence, with its limit point, since no neighborhood of the limit point is connected. While this space is not connected, there are examples of connected, non-locally connected spaces such as the topologist's sine curve, and the Warsaw circle, which we discuss below.

Example 2.2.2 (Topologist's sine curve)**.** Let C be the graph of $\sin(1/x)$, for $x \in (0, 1]$, and let A be the limit arc $\{0\} \times [-1, 1]$. The topologist's sine curve is the space $X = C \cup A$. Note that X is compact and has two path components C and A which are intervals (C is

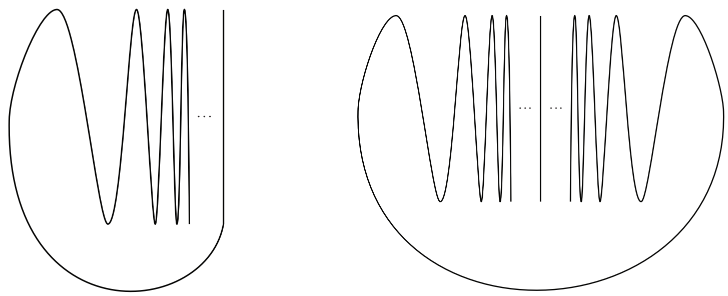

Figure 2.2: The Warsaw circle (left) and a doubled Warsaw circle (right).

half-open, while A is closed). See Figure 2.1.

Example 2.2.3 (Warsaw circle)**.** The Warsaw circle is a topologist's sine curve together with an arc connecting endpoints of the two path components. See Figure 2.2.

We note here that for the Warsaw circle it is important that the arc that connects the two path components of the topologist's sine curve connects to one of the endpoints of the limit arc A instead of an interior point of A; otherwise we get a different space that retracts onto a subspace triod, which clearly does not satisfy Sharkovskiĭ's theorem.

We will discuss various examples of spaces related to the topologist's sine curve and the Warsaw circle. We divide these examples into sections about arc-like, circle-like, and star-like continua. We recall the definition of $\mathcal{P}$-like (for more information see [41]): If $\mathcal{P}$ is a collection of continua, a continuum X is $\mathcal{P}$-like if for every $\varepsilon > 0$ there is a surjective ε-map $f : X \to P$, for some $P \in \mathcal{P}$. A map f is an ε-map if $\operatorname{diam}(f^{-1}(p)) < \varepsilon$ for every $p \in P$. Note that the collections of arc-like, circle-like, and star-like continua are not disjoint. For example, the buckethandle continuum, or Knaster continuum, is both arc-like and circle-like (see [41], especially 2.9 and 12.48); this continuum can be realized as a union of semicircles in the plane, or also as an inverse limit of arcs with the bonding maps folding the arc in half over itself. Another space that falls into two of these collections is Example 2.3.2, which is both arc-like and triod-like (star-like). We will also discuss a few non-compact examples, and include them in the section with similar spaces. We also include a section on spaces that we call archetypal, which is somewhat different than the other sections.

2.3 Arc-Like Continua

In this section we discuss certain arc-like continua as Sharkovskiĭ spaces. Arc-like continua are sometimes referred to as either snake-like or chainable continua. While the examples we present are hereditarily decomposable, and thus are considered in [37], we include them here because of our simple direct proof, which applies to other examples in later sections. Additionally, we discuss some non-compact examples where our methods apply (recall that continua are compact by definition).

Theorem 2.3.1. *The topologist's sine curve is a Sharkovskiĭ space.*

Proof. Denote the topologist's sine curve by $X = C \cup A$ as in Example 2.2.2, where C is the $\sin(1/x)$ curve, and A is the limit arc. Let $x \in X$ be a point of order n, and let $m \succeq n$. We will show that there is a point y of order m.

If $f(A) \subset C$, then by continuity f must also map C into C, and $f(X)$ is a compact connected subset of C, thus an interval I. Thus f maps I into I, and any periodic point of f must be in I; in particular $x \in I$. Therefore, by Sharkovskiĭ's Theorem, there must be a point $y \in I$ of order m.

If $f(C) \subset A$, then f also maps A into A, and any periodic point must be in A. Again Sharkovskiĭ's Theorem gives a point $y \in A$ of order m.

The remaining case is where f maps A into A and C into C. Since both A and C are intervals, Sharkovskiĭ applies, and the existence of the point x of order n implies the existence of a point y of order m (in the same path component as x).

□

We can now discuss many other examples of Sharkovskiĭ spaces based on this example.

For many of these examples, the proof is a fairly straightforward extension of the proof for the topologist's sine curve; the main idea is generally to consider where the path components of the space map to, and show that a periodic point must lie in a path component that maps to itself.

There are, however, a few more interesting examples that we will discuss. In particular, the following example shows some difficulties that also arise in examples in later sections.

Example 2.3.2 (A doubled topologist's sine curve)**.** Let $X = X_1 \cup X_2$ be the union of two topologist's sine curves, where $A_1 = A_2$. We can also write $X = C_1 \cup A \cup C_2$. See Figure 2.1.

Theorem 2.3.3. *The doubled topologist's sine curve is a Sharkovskiĭ space.*

Proof. Let $X = C_1 \cup A \cup C_2$ be the doubled sine curve as above. Similar to the case of the topologist's sine curve, by considering the images of path components, most of the cases reduce to the known cases of periodic points contained in either an interval or topologist's sine curve that maps to itself. The interesting case is where f maps A to A, but maps C_1 into C_2, and C_2 into C_1. If the periodic point x lies in A, then Sharkovskiĭ applies. If $x \in C_i$, then x must have even order $2n$.

Then x has order n as a periodic point for the map f^2, which maps each path component to itself. Then f^2 has a periodic point y of order m, for every $m \succeq n$ (in the same component C_i as x). Then y must have order $2m$ for f.

By considering Sharkovskiĭ's ordering, this proves everything except that a point of even order implies a fixed point. Note however, that in this case $f(A) = A$, and thus has a fixed point (so that this space has the fixed point property). □

Example 2.3.4 (A line of topologist's sine curves)**.** Let S be a consecutive sequence of integers (finite, infinite, or bi-infinite). For i in S, let $X_i = C_i \cup A_i$ be a topologist's sine curve, with C_i being the $\sin(1/x)$ curve, and A_i being the limit arc. Let $X = \bigcup X_i$, where $A_i \subset C_{i+1}$ for each i. We call the space X a line of topologist's sine curves. See Figure 2.3.

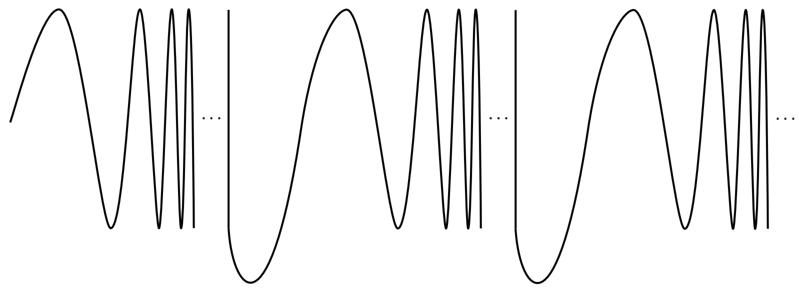

Figure 2.3: A line of topologist's sine curves.

Theorem 2.3.5. *Any line of topologist's sine curves is a Sharkovskiĭ space. It has the fixed point property if and only if it is a finite line of sine curves.*

Proof. This comes down to the fact that for any map $f : X \to X$, f must be weakly increasing on path components of X, i.e. if we define n_i so that $f(C_i) \subset C_{n_i}$, then for $i < j$ we have $n_i \leq n_j$. To see this, note that C_i limits on $A_i \subset C_{i+1}$, and that $f(A_i)$ is a compact subset of $f(C_{i+1}) = C_{n_{i+1}}$. Thus $f(C_i)$ limits on $C_{n_{i+1}}$, and therefore we can see that $n_i \in \{n_{i+1}, n_{i+1} - 1\}$.

With this property of being weakly increasing on path components, we see that the only way that a point can be periodic is if it lies in a path component that maps to itself. Each path component is an interval, so Sharkovskiĭ's Theorem applies.

Note that X has the fixed point property if and only if it is the union of finitely many sine curves. Even for an infinite line, Sharkovskiĭ's Theorem still holds, so that any map without a fixed point (such as a translation) cannot have any periodic points. Also note that the infinite line of topologist's sine curves is an infinite cover of the Warsaw circle. □

2.4 Archetypal

We now discuss a class of spaces that are bijective images of arcs, but not necessarily arc-like (chainable), or even Hausdorff. While these spaces may have certain strange properties, they are similar enough to arcs that they are still Sharkovskiĭ spaces (Theorem 2.4.4).

Definition 2.4.1. A space X is *archetypal* if X is a uniquely arcwise connected T_1 space and there is a continuous bijection from an arc to X. The arc may be open, closed, or half-open.

Conjecture 2.4.2. *The only archetypal spaces that are arc-like are arcs.*

An example of an archetypal space that is not an arc is the Warsaw circle, since it is the bijective image of a half-open interval. An example of an archetypal space that is the image of an open arc is shown in Figure 2.4, on the left. This example is the union of two Warsaw circles sharing a common limit arc. The circle S^1 is not archetypal: although there is a continuous bijection from $[0, 1)$ to S^1, it is not uniquely arcwise connected. Note that archetypal spaces need not be compact.

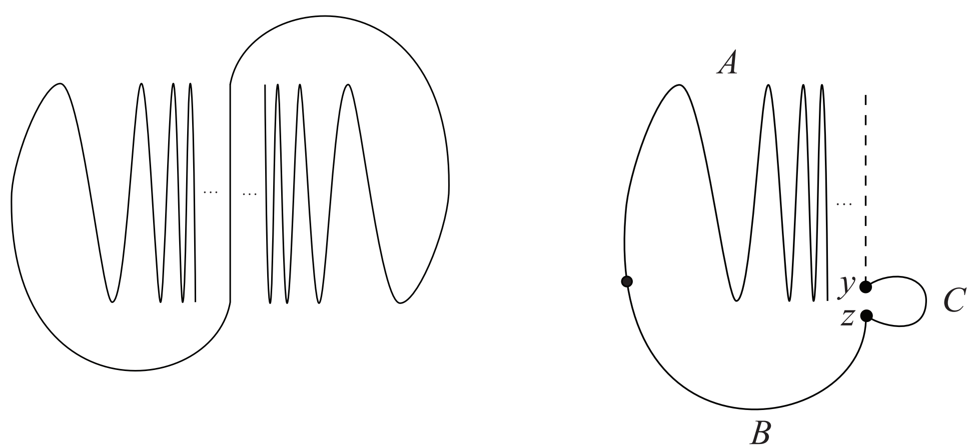

Figure 2.4: Two archetypal spaces. On the right is a schematic for Example 2.4.3, which is a non-Hausdorff archetypal space as described below.

Example 2.4.3. We construct an archetypal space X that is not Hausdorff. We will write X as a union of three arcs, and then describe the topology. The first arc A is a topologist's sine curve, without the limit arc. The second arc B is the arc in the Warsaw circle that connects A to its limit arc, although B does not include any of the limit arc. As it will be useful later in defining the topology, let x be the endpoint of B that lies in the limit arc of A. Note that $A \cup B$ is just an open arc, but $A \cup B \cup x$ is not locally connected at x. The final arc C is a closed arc with endpoints y, z. The space X as a set is $A \cup B \cup C$, where y, z are roughly identified with x, as described in the next paragraph. See Figure 2.4.

The topology of X at any point other than y, z is the standard topology of an arc; that is, a basis element is any open set in $X - \{y, z\} = A \cup B \cup \text{int}(C)$. Define a basis element containing y to be the union of an open set of $C - z$ containing y together with an open set of A that is a neighborhood of x in the plane. Define a basis element containing z to be the union of an open set of $C - y$ containing z together with an open set of $A \cup B$ that is a neighborhood of x.

There is a bijection from a half open arc onto X, that maps the endpoint to y, then proceeds along C to z, and then follows B and then A. The space X is T_1; the only possible difficulty lies in considering the points y, z, and there are open sets around each that miss the other point. The points y, z show that X is not Hausdorff, since neighborhoods of the two points must intersect in A (in a neighborhood of x).

The space X is clearly arc connected, being the bijective image of an arc. It remains to show that X is uniquely arc connected. First note that $X - \{y, z\}$ is the disjoint union of two open arcs, with the standard topology. Notice that in small neighborhoods of y, z, the path components containing these points are exactly the neighborhoods in the arc $B \cup C$ (which is homeomorphic to a standard arc). Thus any arc in the space X must correspond to the image under the bijection described in the previous paragraph, and we see that X is uniquely arcwise connected.

Thus X is an archetypal space that is not Hausdorff.

Theorem 2.4.4. *Archetypal spaces are Sharkovskiĭ spaces.*

This theorem generalizes the result in [54] that the Warsaw circle is a Sharkovskiĭ space. The proof relies on showing that a map of the Warsaw circle lifts to a map of the interval. Our definition of an archetypal space gives the necessary conditions to prove the generalized theorem.

Proof of Theorem 2.4.4. Let X be archetypal, with continuous bijection $p : I \to X$, and let $f : X \to X$ be a continuous map. Since p is a bijection, $\tilde{f} = p^{-1}fp : I \to I$ is a function, and we claim that $\tilde{f}$ is continuous. Then since p is a bijection, the periods of f and $\tilde{f}$ are the same, and since I is a Sharkovskiĭ space, so is X.

To see that $\tilde{f}$ is continuous, consider a point $t \in I$, and let U be a metric ball in I containing $\tilde{f}(t) = p^{-1}fp(t)$. Let A be the set of endpoints of U, i.e. $A = \partial U$ and $|A| \leq 2$. Since X is T_1, points are closed, and thus $X - p(A)$ is an open set in X containing $fp(t)$. Let $W \subset I$ be the path component of $(fp)^{-1}(X - p(A))$ containing t. The set W is open in I since fp is continuous, and I is locally path connected. Now consider $p\tilde{f}(W) = fp(W) \subset X - p(A)$. Since W is path connected, we see that $p\tilde{f}(W)$ is contained in the path component of $X - p(A)$ containing $fp(t)$.

This path component is just $p(U)$: clearly the path component contains $p(U)$, and it cannot contain any other point x since X is uniquely arcwise connected and p defines an arc from $p(t)$ to x that goes through $p(A)$. Thus $p\tilde{f}(W) \subset p(U)$, so $\tilde{f}(W) \subset U$, and therefore $\tilde{f}$ is a continuous map on the interval I. □

Corollary 2.4.5. *The Warsaw circle is a Sharkovskiĭ space.*

Note that while the Warsaw circle is the bijective image of a half-open arc, it does not retract to a half-open arc, but only to a closed arc. In fact, the Warsaw circle does satisfy Theorem 2.1.2B, and it has the fixed point property, even though the half-open arc does not.

Note that the proof that the Warsaw circle is a Sharkovskiĭ space is substantially different than the proof for the previous examples, such as the topologist's sine curve. The main difference is that the Warsaw circle only has one path component, which limits on itself, so that it is not an arc; on the other hand, in the previous examples, every path component was an arc.

2.5 Circle-Like Continua

We will now discuss various examples of non-locally connected circle-like continua. A primary example of these is the Warsaw circle, which was discussed in the last section. While some of the spaces we discuss are Sharkovskiĭ spaces, others are not, and for these we discuss the possible period sets for maps on these spaces. We note that the results for circle maps are quite different than Sharkovskiĭ's Theorem [5, 38, 45].

Example 2.5.1 (A doubled Warsaw circle)**.** Let X be the space obtained by taking the double topologist's sine curve from Example 2.3.2, and joining the endpoints of C_i by an arc. We can write $X = C \cup A$, where C is an open interval, with each 'end' limiting on the closed arc A as a topologist's sine curve. See Figure 2.2.

Theorem 2.5.2. *The doubled Warsaw circle is a Sharkovskiĭ space.*

Proof. This is actually much simpler than either the double topologist's sine curve or the Warsaw circle. Either A maps to C (in which case C also maps to C), or C maps to A (and A also maps to A), or each path component maps to itself. Any periodic point must then lie in a path component that maps to itself and Sharkovskiĭ's Theorem applies. Note that this space has the fixed point property. □

All of the above examples satisfy Theorem 2.1.2 as stated, and all have the fixed point property, except for infinite lines of topologist's sine curves. Also, these all retract onto an interval, and thus we can see that Theorem 2.1.2B is also satisfied for these spaces. Note that we can drop the requirement 'non-empty' for the infinite line of sine curves, as there is a shift map with no periodic points.

The examples that we will discuss for the remainder of this section, and in the following sections, are not Sharkovskiĭ spaces. Just as the earlier examples, these spaces retract to

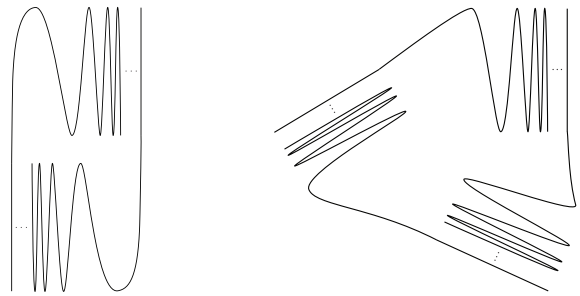

Figure 2.5: A double cover (left) and a 3-fold cover (right) of the Warsaw circle.

an arc, and thus Theorem 2.1.2B holds for these examples; additionally, some (but not all) have the fixed point property.

Example 2.5.3 (A double cover of the Warsaw circle)**.** This space is the union of two topologist's sine curves where $A_1 \subset C_2$ and $A_2 \subset C_1$. If each A_i is at the closed end of C_j $(i \neq j)$, then this connected space is a double cover of the Warsaw circle. See Figure 2.5.

The double cover is almost a Sharkovskiĭ space, but there is one set of implications that does not hold – an even period does not always imply a fixed point.

Theorem 2.5.4. *Let f be a map of the double cover of the Warsaw circle to itself. Suppose f has a point of order n, and $m \preceq n$. If either n is odd or if $m \neq 1$, then f has a point of order m.*

Proof. This is very similar to the doubled topologist's sine curve. The only difficult case is if f permutes the path components C_1, C_2. In this case, f^2 maps each path component to itself, and just as in the proof for the doubled sine curve, by looking at the map f^2 we get all of Sharkovskiĭ's Theorem, except for a point of even order implying the existence of a point of order 1. This part of Sharkovskiĭ's Theorem is not in fact true for this space, as a simple rotation gives every point order 2, with no fixed point. Otherwise, Sharkovskiĭ's Theorem holds as stated.

Note that any function that permutes the path components shows that this space does not have the fixed point property. However, we can prove that any map without a fixed point has a point of order 2.

If f maps X into one path component C_i, then $f(X)$ is a compact interval that maps to itself, giving a fixed point. If f maps into both path components, it can be seen that f must be surjective, and must satisfy $f(A_i) = A_j$. If f maps each C_i to itself, then we get fixed points in each A_i. If f permutes the path components, then $f^2(A_i) = A_i$, so that f^2 has a fixed point in each A_i, and thus f has a point of order 2 (in each A_i).

□

This example gives rise to more questions than the simple "Is X a Sharkovskiĭ space?" For any space X, we can define a partial order $\preceq_X$ on the positive integers by $n \preceq_X m$ if every map of X to itself that has a point of order n has a point of order m. We can then ask what this partial order $\preceq_X$ is. However, for many spaces this is not as informative as the standard Sharkovskiĭ Theorem. In general, it is more informative to ask: What are all possible $\mathrm{Per}(f)$ for a given space X? If we know all possible sets of least periods, we can reconstruct the partial order $\preceq_X$, however the converse is not true. We will discuss the partial order $\preceq_X$ further in Section 2.9.

We introduce some notation to deal with these questions. Write the Sharkovskiĭ order as a relation from $\mathbb{Z}^+$ to itself: $\mathcal{S} = \{(n, m) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid n \preceq m\}$. A multiple of the Sharkovskiĭ order is then $d \cdot \mathcal{S} = \{(dn, dm) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid n \preceq m\}$. This mutliple of Sharkovskiĭ's order only has multiples of d as initial points. As such, we can extend the relation to include all other integers as initial points, representing vacuous implications, e.g. we may add $k \preceq m$ when there are no maps with points of period k (for a certain multiple $d \cdot \mathcal{S}$). This will allow us to describe the periodic implications for various spaces, as they often involve certain combinations of tails of different multiples of Sharkovskiĭ's order. Thus we define $\overline{d \cdot \mathcal{S}} = d \cdot \mathcal{S} \cup \{(a, b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ \mid d \text{ does not divide } a\}$. We note that we could have extended $d \cdot \mathcal{S}$ to a total order, but that does not work as well in the following since we will want the maximal set of implications, even if many of them are vacuous in this case.

In the following, it will be useful to note that all possible tails of the Sharkovskiĭ order can be realized as the period set of an interval map that fixes the endpoints of the interval (see Lemma A.1.1). In other words, we may take the maps in Theorem 2.1.2B to be maps fixing the endpoints of the interval.

Example 2.5.5 (An n-fold cover of the Warsaw circle)**.** This space is the union of n topologist's sine curves $X_i = C_i \cup A_i$, where $A_i \subset C_{i+1}$ (indices taken mod n). If each A_i is at the closed end of C_{i+1}, then this connected space is an n-fold cover of the Warsaw circle. A 3-fold cover is shown in Figure 2.5.

Some dynamical properties of the n-fold cover of the Warsaw circle have been studied in [56]. They show that if a function has a fixed point, then it satisfies Theorem 2.1.2 for that function; however if there are no fixed points then it does not. While the n-fold cover is not a Sharkovskiĭ space, we prove which period sets are possible.

Theorem 2.5.6. *Let f be a map of the n-fold cover of the Warsaw circle to itself. Then* $\mathrm{Per}(f)$ *is a non-empty tail of $d \cdot \mathcal{S}$ for some $d|n$. Furthermore, every such tail is* $\mathrm{Per}(f)$ *for some f.*

Proof. As with the double cover, it can be seen that either f maps X into one component C_i, or f is surjective, in which case there is some fixed number k such that $f(C_i) = C_{i+k}$, and $f(A_i) = A_{i+k}$ for all i (indices taken mod n). Let d be the order of the induced map on components, so that $f^d(C_i) = C_i$; in other words, d is the order of k in $\mathbb{Z}_n$. We say that f has *type* d. Note that d divides n.

If $d = 1$, then we get the usual Sharkovskiĭ ordering, as any periodic point is in an interval C_i that maps to itself. Since $f(A_i) = A_i$, f has a fixed point as well. Now consider the case where $d \neq 1$. For any periodic point x of f, d must divide the order of x, call it $d \cdot n$. Then x is a point of order n for the map f^d. So there is a point y of order $m \succeq n$ for f^d, which will have order $d \cdot m$ for f.

Thus for any map f of type d, we see that Sharkovskiĭ's Theorem is true for the partial ordering on the integers which is d times the original Sharkovskiĭ order: $d \cdot \mathcal{S}$. Thus every map of type d satisfies Sharkovskiĭ's Theorem with the extended partial ordering $\overline{d \cdot \mathcal{S}}$. Then for the n-fold cover X of the Warsaw circle we get that the maximal partial ordering ($\preceq_X$) for Sharkovskiĭ's Theorem is defined by the relation $\mathcal{S}(n) = \bigcap_{d|n} \overline{d \cdot \mathcal{S}}$.

We note that this actually agrees with our result for the double cover of the Warsaw circle, although it may appear different at first. The number 2 is unique with respect to $\mathcal{S}$ in the sense that $2 \cdot \mathcal{S}$ is actually a subset of $\mathcal{S}$ (as a relation). In fact, except for 1, all of the numbers that do not show up in $2 \cdot \mathcal{S}$ (namely the odds) precede all the evens. Thus $\mathcal{S}(2) = \mathcal{S} \cap (\overline{2 \cdot \mathcal{S}}) = \mathcal{S} - \{(2n, 1) \mid n \in \mathbb{Z}^+\}$.

As with the double cover, we note that the n-fold cover does not have the fixed point property, but that every map of type d has a point of order d (since $f^d(A_i) = A_i$). This is related to the fact that all factors d of n are maximal elements in $\mathcal{S}(n)$ (or equivalently $\preceq_X$). So every self-map of the n-fold cover of the Warsaw circle has a point of period n (not necessarily least period).

Now we discuss the possible period sets $\mathrm{Per}(f)$ for the n-fold cover. We note that our partial order $\preceq_X$ defined by $\mathcal{S}(n) = \bigcap_{d|n} \overline{d \cdot \mathcal{S}}$ is not particularly informative here. It can easily be seen that for most integers this intersection will remove almost all information about periods of functions. For instance, we get no implications for a point of period $d|n$. However, the way we have written the partial order as an intersection of other partial orders is more informative, as will be seen.

If f maps into one path component, then the possible period sets are just tails of $\mathcal{S}$. Such maps must have a fixed point, since f is essentially the map from the compact set $\mathrm{im}(f)$ to itself. Each map f that does not map into one path component is surjective and has type d for some $d|n$, and $\mathrm{Per}(f)$ must be a tail of $d \cdot \mathcal{S}$. We show that we get all such non-empty tails.

Given any (non-empty) tail T of $\mathcal{S}$, by Lemma A.1.1 there is a map $h : I \to I$ fixing the endpoints of I with $\text{Per}(h) = T$. We will use h to construct a map f of type d with $\text{Per}(f) = d \cdot T$. First take homeomorphisms $f_i : C_i \to C_{i+1}$ (indices mod n) that respect the limit structure on A_i and that are coherent, i.e. so that F defined by $F|_{C_i} = f_i$ is not only a homeomorphism of type n, but also so that $F^n = \text{Id}_\text{X}$.

Now, choose intervals $I_i \subset C_i - A_{i-1}$ coherent with the homeomorphisms f_i, i.e. $f_i(I_i) = I_{i+1}$. Then define the map H_d to perform h on the first n/d intervals I_i,

$$H_d = \begin{cases} h & \text{on } I_i \quad \text{for } 1 \leq i \leq n/d \\ \text{Id} & \text{otherwise} \end{cases},$$

and then we can define f by

$$f = F^{(n/d)} \circ H_d.$$

Notice that $f^d = H_1$, which is a map of type 1 that performs h on each I_i, and is the identity otherwise. Thus $\text{Per}(f) = d \cdot \text{Per}(h) = d \cdot T$, which is a (non-empty) tail of $d \cdot \mathcal{S}$. Note that we only consider non-empty tails since every map of type d has a point of order d.

Thus for maps f of the n-fold cover of the Warsaw circle, $\text{Per}(f)$ is a non-empty tail of $d \cdot \mathcal{S}$ for some $d|n$, and every such tail is $\text{Per}(f)$ for some f.

□

2.6 Star-Like Continua

Our last set of examples based on the topologist's sine curve are star-like continua. An n-star, or n-od, is just a union of n arcs each sharing one endpoint, and disjoint otherwise. Baldwin discusses Sharkovskiĭ's Theorem for all n-ods, giving a complete characterization of the possible period sets [7]. We discuss non-locally connected n-ods, and note that our results are related to Baldwin's.

Example 2.6.1 (A topologist's n-od)**.** Let X_n be a union of n topologist's sine curves with a common limit arc, $X_n = A \cup \bigcup C_i$. Note that for $n = 2$ this is just the doubled sine curve of Example 2.3.2. Also, this space is planar for all n, which can be seen by letting C_i be the graph of $\sin(1/x) + ix$, for $x \in (0, 1]$.

Theorem 2.6.2. *Let f be a map of the topologist's n-od to itself. Then there is some partition K of n such that* $\mathrm{Per}(f)$ *is a union of non-empty tails of $d \cdot \mathcal{S}$, where d varies over all integers in $K \cup \{1\}$. Furthermore, every such union of tails occurs as* $\mathrm{Per}(f)$ *for some map f.*

Proof. First note that if A maps into any C_i, then X_n maps into a compact interval in C_i, and the standard Sharkovskiĭ Theorem applies. Suppose now that A maps into A. If $\mathrm{im}(f)$ contains any point in C_i, then by connectivity we can see that $\mathrm{im}(f)$ must contain a tail of the sine curve C_i, and thus limits on all of A. So if $f(A) \neq A$, then $\mathrm{im}(f) \subset A$, and the standard theorem applies.

We may then assume that $f(A) = A$. The map f induces a map g on the path components, which we think of as a map on $Y = \{0, 1, \ldots, n\}$ where we consider $A = 0$ and $C_i = i$. Thus we are assuming $g(0) = 0$. Some points may map to 0, which is fixed, others may not be in the image; we are concerned with those points that are always in the image (of any iterate of g), since any periodic point of f will correspond to some such path component.

Let N be the number of such points (not counting 0), i.e. $N = \lim_{m \to \infty} |g^m(Y)| - 1$. We note that equivalently, $N = |g^n(Y)| - 1$. By reordering, we may assume that for $1 \leq i \leq N$, $i \in g^m(Y)$ for all m (if $N = 0$ this statement is vacuous). Now if we restrict g to $Y(N) = \{1, \ldots, N\}$, we get a permutation in S_N. Let $K = \{k_1, \ldots, k_\ell\}$ be the cycle type of the permutation. We say that g (and also f) is of type K.

Suppose that f is of type K, and $k \in K$. Then there is a set of k of the path components C_i that f cyclicly permutes. Any periodic point in these path components must have period divisible by k. As in earlier examples, we get the implications on periodicity derived from the partial order $k \cdot \mathcal{S}$. Note that f^k need not have a fixed point in C_i, as the sine curve could be continually pushed toward the limit arc A. However, if this is the case then there are no periodic points in these path components, and we have an empty tail of $k \cdot \mathcal{S}$.

Then for a map f of type K, we see that $\mathrm{Per}(f)$ is a union of tails of $d \cdot \mathcal{S}$, where d ranges over $K \cup \{1\}$ (recall that $f(A) = A$). Note that the union of two tails of $d \cdot \mathcal{S}$ is still a tail of $d \cdot \mathcal{S}$. The possible types of f are all partitions of m, where $0 \leq m \leq n$. Since A is always fixed (as a set), we always include 1 as a possibility for d, whether or not 1 is in the type K. To simplify, we may then assume that K is a partition of n (instead of a partition of $m \leq n$).

Thus for each map f, there is some partition K of n such that $\mathrm{Per}(f)$ is a union of non-empty tails of $d \cdot \mathcal{S}$, where d varies over $K \cup \{1\}$. We may assume the tails are non-empty: since X_n has the fixed point property there is always a non-empty tail of $1 \cdot \mathcal{S}$, and if there were no non-empty tail for some $d \neq 1$ in the partition, we could consider a map of a different type K' where we replace d in the partition K by d copies of 1.

We will now show that all such $\mathrm{Per}(f)$ described in the last paragraph actually occur. For the most part, this is very similar to the last example of an n-fold cover of the Warsaw circle. Given a partition K of n, and an element $d \in K$, we can choose d path components of X_n to be cyclicly permuted by f, and design the map f to have the appropriate periods as before. For any other $d' \in K$, we repeat using distinct path components of X_n.

The only difficulty arises if $1 \notin K$, for example, if $K = \{n\}$. Then as before we can construct a map g that has any union of nonempty tails of $d \cdot \mathcal{S}$, for $d \in K$. The trick is to still get a tail T of $1 \cdot \mathcal{S}$ corresponding to the limit arc A. To do this, first note that our construction of g will give a neighborhood U of the limit arc A where g only permutes the path components by the coherent homeomorphisms f_{ij}. In other words, we want U to be disjoint from all the sets $I_i \subset C_i$ used to create the tails of $k \cdot \mathcal{S}$ for $k \in K$.

Now choose an interval $J \subset A$. Consider the preimage of J under a (suitably nice) projection map from X_n to A. For instance, with the standard embedding of the topologist's sine curve in the plane, take horizontal projection. Let J_i be the components of the preimage of J that are contained in U. By Lemma A.1.1 there is a map $h : J \to J$ that fixes the endpoints of J, and has $\mathrm{Per}(h) = T$. To define f, precompose the function g with the map h on J and each J_i, where the map h must be performed coherently on each J_i. This will define a continuous map f, that has $T \subset \mathrm{Per}(f)$.

Unfortunately this may also give periodic points that were not desired in the $\sin(1/x)$ curves C_i. This can be corrected however, by pushing all of the sine curves C_i toward A, in the portion $C_i \cap U$. This is still continuous, and will avoid the creation of undesired periodic points. Thus all of the possible sets $\mathrm{Per}(f)$ are exactly as described above. □

We note here the relationship between our results for the topologist's n-od and Baldwin's results for the standard n-od [7]. Baldwin defines partial orders $\preceq_d$ for all integers d, and shows that for a map f of the n-od, $\mathrm{Per}(f)$ is a union of tails of $\preceq_d$, where d is allowed to vary over all positive integers less than n. We note that $d \cdot \mathcal{S}$ is a terminal segment of $\preceq_d$, so that all period sets for the topologist's n-od can be achieved as a period set for the standard n-od. However, the partial order $\preceq_d$ is a nontrivial extension of $d \cdot \mathcal{S}$, and allows for different periods. Additionally, there is no restriction in the case of the standard n-od that all the values of d used sum to n (or more precisely, some value $\leq n+1$, since we are allowed a partition of n, union 1). In fact, Baldwin shows that you can achieve any possible period set fixing a neighborhood of the basepoint. This extra freedom is possible since the arcs of the n-od can map to more that just one other arc.

Example 2.6.3 (A topologist's ∞-od)**.** $X = A \cup \bigcup C_i$, for $i = 1, \ldots, \infty$. This is similar to Example 2.6.1, but with infinitely many sine curves. This space is still planar; for instance take C_i to be the graph of $\sin(1/x) + x/i$.

This example can also be made compact by adding C_0 as the graph of $\sin(1/x)$.

Theorem 2.6.4. *For a map f of the topologist's ∞-od to itself,* $\mathrm{Per}(f)$ *can be any set of* *positive integers that contains 1.*

Proof. First note that X has the fixed point property. If $f(A) \subset A$, then there is clearly a fixed point, and if $f(A) \subset C_k$, then f maps the topologist's sine curve $A \cup C_k$ to itself, which also has the fixed point property.

Let S be a subset of $\mathbb{N}$ that contains 1. First note that $\mathbb{N}$ can be partitioned into sets N_i where each N_i has s_i elements, for every $s_i \in S$ (and N_0 will be infinite if S is finite). Then define the map f to be the identity on the limit arc A together with the sine curves C_k for $k \in N_0$, and for $i \neq 0$ define f to cyclicly permute the s_i sine curves C_k for $k \in N_i$. If the cyclic permutation is defined nicely, i.e. so that f restricts to coherent homeomorphisms that compose to the identity, then this gives points of all orders s_i, and fixed points, with no other periodic points. For an example of coherent homeomorphisms, consider the planar embedding described above, and use vertical projection between the corresponding sine curves. Note that if the ∞-od is taken to be compact, then the curve C_0 should be fixed by f.

While it may not be obvious at first, this result is similar to the n-od case. The answer for that case deals with $K \cup 1$ for partitions of n. Here, we consider $S \cup 1$, for any subset $S \subset \mathbb{N}$; and any set of positive integers could be considered a partition of infinity, in some sense, as we allow repetitions in our partitions. □

Note that the above result can also hold for path connected, and even locally connected spaces. For instance, consider a 'standard' ∞-od as a cone over $\{1/n\}$, with or without the limit point 0 included. For an example of a locally connected space, simply make each arc of the ∞-od get smaller. Another way of expressing this last space is as the one-point compactification of a sequence of half-open arcs. Note that this space is planar and contractible, being a dendrite. The proof for these spaces is essentially the same as for the topologist's ∞-od, where the main difference is showing the fixed point property.

We also note that the same result also holds for a 2-dimensional disk. It is easy to construct examples for the disk if for any n you can construct a map of the disk that fixes the boundary with $\mathrm{Per}(f) = \{1, n\}$. We leave the details to the reader.

A similar result holds for the Hawaiian earring H, which is the one-point compactification of a sequence of open arcs. However, there is a difference here, as the Hawaiian earring does not have the fixed point property since H retracts to a circle, which we can rotate without a fixed point. So while we can have any period set containing 1, we can also have any period set of a circle map of degree one (without a fixed point) – these are described by Misiurewicz in [38]. We note that any finite set of periods can be added to any period set of a degree one map of a circle, and there are possibly many more complicated things that can happen on the Hawaiian earring:

Question 2.6.5. *What are the possible period sets for maps of the Hawaiian earring that do not contain 1?*

2.7 Cantor Set Maps

The last few examples in the preceding section had maps with any period set desired, assuming a fixed point. This mainly relied on being able to describe the space in a symmetric fashion, and then permuting the different pieces while limiting on a fixed point to ensure that the map is continous. In a similar manner, we can find self-maps of the Cantor set with any period set that contains 1. There are also many Cantor set maps with no fixed point that have various period sets, and after considering the question of possible period sets for the Cantor set, we came to believe and eventually prove:

Theorem 2.7.1. *Given any subset S of the natural numbers, there is a continuous map f from the Cantor set to itself with* $\mathrm{Per}(f) = S$.

Why should this result be true? While the interval is very restrictive as far as what period sets are possible as given by Theorem 2.1.2, the restriction is mainly due to the fact that the interval is path connected. The proof of Sharkovskiĭ's Theorem relies on the fact that the interval between points in a periodic orbit must map onto the interval between the images of those points. When applied to an appropriate interval and iterate of the map, this fact together with a version of the fixed point theorem (due to the intermediate value theorem) shows the existence of other periodic points.

The Cantor set, on the other hand, is totally disconnected, and it seems that you should always be able to arrange for the periodic points that arise for the interval maps to lie in the complementary open sets of the Cantor set, thus avoiding any undesired periods. Additionally, it is a well known fact that the Cantor set maps onto any compact metric space. This would seem to imply that any flexibility available in the period sets for any compact metric space should also exist for the Cantor set.

While any period set is possible for a Cantor set map, it is not simple to construct maps with arbitrary infinite period sets that do not contain 1. There are also interesting restrictions that arise when considering maps where every point is periodic. In the remainder of this section, we will discuss these restrictions, as well as maps achieving any desired set of periods.

2.7.1 The Cantor Set. The standard construction of the Cantor set is as an intersection of nested subsets of the unit interval. Let $X_0 = [0, 1]$. Remove the middle third of this interval to get $X_1 = [0, 1/3] \cup [2/3, 1]$. Thus X_1 is the disjoint union of 2^1 intervals of length $1/3^1$. The set X_2 is obtained by removing the middle third of both intervals in X_1: $X_2 = \big([0, 1/9] \cup [2/9, 1/3]\big) \cup \big([2/3, 7/9] \cup [8/9, 1]\big)$. The subsequent sets X_i are define similarly: X_i consists of 2^i intervals of length $1/3^i$, and X_{i+1} is obtained by deleting the middle third of each interval. The Cantor set C is then the intersection of the sets X_i: $C = \bigcap X_i$.

The Cantor set can also be represented in other ways which will be useful in constructing various maps. First, the Cantor set is a product of countably many two point sets $\{0, 1\}$. Equivalently, a point in the Cantor set is a sequence of 0's and 1's, where two sequences are close if they agree on initial segments. The first digit of a point corresponds to which interval in X_1 the point lies, 0 for the left, and 1 for the right. The next digit corresponds to which of the subintervals in the next level X_i the point is in.

$$C = \prod_{i=1}^{\infty} \{0, 1\}$$

Another useful representation for the Cantor set is that it is equal to the union of copies of itself. Since the construction of the Cantor set is the same independent of the level i, the points of C that lie in any number of the intervals of a given X_i will also form a Cantor set.

$$C = \bigcup_{i=1}^{n} C$$

Similarly, the Cantor set can be written as an infinite union of copies of itself, together with a limit point. This can be seen by taking the right half of X_1 as a Cantor set $C_1 = [2/3, 1]$, then the right half of what is left of X_2 as a Cantor set $C_2 = [2/9, 1/3]$, and so on, which limits on the point 0.

$$C = \{p\} \cup \bigcup_{i=1}^{\infty} C$$

More generally, we have the following lemma:

Lemma 2.7.2. *If X is a compact, totally-disconnected metric space and D is a Cantor set, then $C = X \times D$ is also a Cantor set.*

Corollary 2.7.3. *If $f : X \to X$, then for the map $g = f \times \mathrm{Id}_D : C \to C$ we have $\mathrm{Per}(g) = \mathrm{Per}(f)$.*

This corollary is the main reason we will use this lemma, so that we may achieve a desired period set on some simpler totally disconnected space, and then apply the corollary to get that period set for a map of the Cantor set.

Definitions 2.7.4. Recall that given a map f, a point x is periodic if $f^n(x) = x$ for some

n. A point x is *pre-periodic* if some iterate of x is periodic, i.e. $f^m(x)$ is periodic for some m. The *pre-periodic length* of x is the minimal such m. Note that periodic points are pre-periodic, with pre-periodic length 0. A point is *aperiodic* if it is not pre-periodic. Aperiodic points have infinite orbits.

2.7.2 Maps with every point periodic. We now construct examples of maps where every point is periodic. In this section, all of the maps will be homeomorphisms of the Cantor set.

First consider any set $P \subset \mathbb{N}$ containing 1. Enumerate $P = \{1, p_1, p_2, \dots\}$. Let X be a convergent sequence $x_n \to x_0$. Define $f : X \to X$ as follows.

- Fix the limit point x_0.
- Permute the first p_1 points $x_1, \dots, x_{p_1}$ cyclicly.
- Permute the next p_2 points cyclicly, the next p_3, and so on.

Then $\mathrm{Per}(f) = P$, with x_0 being a fixed point, and the first p_1 points having period p_1, and the next p_i points having period p_i. Since X is totally disconnected, compact, and metric, we can apply Lemma 2.7.2 and Corollary 2.7.3 to get a map g from the Cantor set to itself with $\mathrm{Per}(g) = P$.

Note that if P is finite, we may fix most of the points of X after permuting the initial points of X to get the desired periods.

Given a function $f : C \to C$ and a number $n \in \mathbb{N}$ we now show how to construct a map $g : C \to C$ with $\mathrm{Per}(g) = n \cdot \mathrm{Per}(f) := \{np \mid p \in \mathrm{Per}(f)\}$. Write $C = \bigcup_{i=1}^{n} C_i$, and define g as follows.

- $g|_{C_i} = \mathrm{id} : C_i \to C_{i+1}$ for $i = 1, \dots, n-1$.
- $g|_{C_n} = f : C_n \to C_1$.

Note that each C_i only is mapped to itself after n iterates of g. Then we can see that the map g^n performs f on each C_i, so that $\mathrm{Per}(g^n) = \mathrm{Per}(f)$, and therefore $\mathrm{Per}(g) = n \cdot \mathrm{Per}(f)$ as desired.

Now, given $f_i : C \to C$ for $i = 1, \ldots, n$, we construct a map f whose period set is the union of the period sets for f_i. Write $C = \bigcup_{i=1}^{n} C_i$, and define $f|_{C_i} = f_i : C_i \to C_i$, where each f_i acts on C_i independently of the other maps. Then it is easy to see that $\mathrm{Per}(f) = \bigcup_{i=1}^{n} \mathrm{Per}(f_i)$.

Combining these three previous results, we see that we can construct a map f with $\mathrm{Per}(f) = P$ for any set P that is a finite union of sets with a common divisor in the set. Equivalently,

$$P = \bigcup_{i=1}^{k} n_i \cdot (P_i \cup \{1\}).$$

Recall that all of these maps can be constructed as homeomorphisms, with every point periodic.

Cannon et al [19] prove that any Cantor set map with every point periodic has this form:

Theorem 2.7.5. *If $f : C \to C$ is a map of the Cantor set C such that every point $x \in C$ is periodic, then there is a finite set $B = \{p_1, \ldots, p_k\}$ of periods of f such that every period $p(x)$ is divisible by at least one of the elements of B. We call the set B a basis for the period set $P = P(f)$. Every subset P of $[1, \infty)$ that has such a finite basis can be realized as the period set of a Cantor set homeomorphism f, with every point periodic.*

A proof of Theorem 2.7.5 is given in [19], but we give a simple corollary here.

Corollary 2.7.6. *The same result holds if every point of C is pre-periodic, with uniformly bounded pre-periodic length.*

Proof. Let m be a bound on the pre-periodic length of points in C for a map f, i.e. $f^m(x)$ is periodic for every $x \in C$. Since every periodic point is in the image of every iterate of f, we see that $X = f^m(C)$ is a retract of C containing exactly the periodic points of f. Let

$\tilde{f} = f|_X$. Since X is totally disconnected, compact and metric, we may apply Corollary 2.7.3 to get a map $g : C \to C$ with every point periodic, and $\text{Per}(g) = \text{Per}(\tilde{f}) = \text{Per}(f)$. We may now apply the theorem to see that the period set of f must be of the same form, having a finite basis. □

2.7.3 Maps with any period set.

Cannon et al [19] have also proved the following result for Cantor set maps when the restriction that every point be periodic is removed. This is essentially Theorem 2.7.1 as stated above, but with the additional result that the maps can be homeomorphisms or totally pre-periodic.

Theorem 2.7.7. *For any set $P \subset \mathbb{N}$, there is a Cantor set map whose period set is P. The map f may be taken to be either a homeomorphism, or a map with every point pre-periodic, but not both.*

Note that for f to be a homeomorphism will usually require there to be aperiodic points, unless the set P has a finite basis as in Theorem 2.7.5. We will now discuss examples of maps with various infinite period sets, leading up to examples of arbitrary period sets.

We now construct a map s with all periods, that is, $\text{Per}(s) = \mathbb{N}$. Recall that $C = \prod^{\infty}\{0, 1\}$, where elements are sequences in $\{0, 1\}$. The shift map s is defined by shifting all digits to the left once. Some examples of this definition are given in Table 2.1. The shift map has all periods: $\text{Per}(s) = \mathbb{N}$. To see this, notice that for every $n \in \mathbb{N}$, the repeating sequence $c_n = \overline{00\ldots01}$ has period $(n + 1)$ if there are n 0's in the repeating sequence. Also, the shift map has aperiodic points, for example $c = 101001000100001\ldots$ is aperiodic, where the number of 0's between consecutive 1's continually increases. Additionally, the shift map has pre-periodic points of unbounded pre-periodic length, such as $11\ldots1000\ldots$ which has pre-periodic length n, where n is the number of 1's.

Notice that s has precisely two fixed points, $\overline{0}$ and $\overline{1}$. We can modify s to have no fixed points, but retain all other periods. To do so, take neighborhoods of the fixed points where

$$
\begin{aligned}
x_1x_2x_3x_4x_5x_6\ldots &\mapsto x_2x_3x_4x_5x_6\ldots\\
11111111111\ldots &\mapsto 1111111111\ldots\\
00000000000\ldots &\mapsto 0000000000\ldots\\
10110010111\ldots &\mapsto 0110010111\ldots\\
&\mapsto 110010111\ldots\\
&\mapsto 10010111\ldots\\
&\mapsto 0010111\ldots
\end{aligned}
$$

Table 2.1: The shift map s on the Cantor set.

Period	point
2	01
3	110
4	0110
5	11010
6	011010
7	1101010

Table 2.2: Periodic points in the altered shift map on the Cantor set.

the first three digits agree, i.e. 000. . . and 111. . . , and change s to map these neighborhoods to a chosen aperiodic point a. The point a needs to be chosen so that there are no strings of three consecutive 0's or 1's, as such a point would then become periodic. For example, a can be any non-repeating word composed from the strings 01 and 001.

To see that we retain all other periods n, consider repeating sequences of length n. In constructing such a periodic point, you must be careful to make sure that it is not repeating of a smaller period (e.g. 0101 has period 2, not period 4). Also, you need to ensure that there are never three adjacent 0's, nor three adjacent 1's, as such points would eventually be mapped to the aperiodic point a. One way to achieve this is to consider repeating strings in 01,10, and 110. In particular, to find a point of period $2m + 2$, start with 01 followed by m copies of 10; to find a point of period $2m + 3$, start with 110 followed by m copies of 10. This construction will give you any period $p \geq 2$ desired. See Table 2.2 for examples of

$$
\begin{aligned}
11111111111\ldots &\mapsto 00000000000\ldots \\
&\mapsto 10000000000\ldots \\
10110010111\ldots &\mapsto 01110010111\ldots \\
&\mapsto 11110010111\ldots \\
&\mapsto 00001010111\ldots \\
&\mapsto 10001010111\ldots
\end{aligned}
$$

Table 2.3: The odometer map on the Cantor set.

points with periods 2 through 7.

In a similar manner, we may alter the shift map s to remove any finite set of periods we desire. There are only finitely many points of any period, as there are only finitely many strings in $\{0, 1\}$ of length n. By taking small neighborhoods of these points and mapping these neighborhoods to a specified aperiodic point, we can remove those periods. If the neighborhoods are taken sufficiently small ($2^n + 1$ digits should work), and if the aperiodic point is chosen to miss all of those neighborhoods, then the period set of the altered shift map will be precisely those periods we did not remove.

This works to give an altered shift map with any co-finite period set. Unfortunately, this method doesn't quite work to obtain an arbitrary infinite period set. In removing infinitely many different periods, there are continuity issues because of limit points of the altered neighborhoods.

We now give an example of a completely aperiodic map, that is, a map whose period set is empty. Again we will write elements of C as sequences in $\{0, 1\}$. The odometer map, or the '+1' map, is defined by adding 1 (base 2) to the first digit, and carrying to the right, as in the odometer of a car. Thus we are considering elements of C as 'infinite binary integers'. Every point is aperiodic under the odometer map, as adding 1 finitely many times will never return a point to itself.

Figure 2.6: Constructing an 'irrational rotation' on the Cantor set.

2.7.4 An irrational rotation. We now give another example of a totally aperiodic map. This map was presented by Cannon et al [19], and was used to construct homeomorphisms with arbitrary period sets. This map is based on an irrational rotation of a circle, and is essentially the same as the example due to Denjoy, which Denjoy showed can in fact be made a C^1 diffeomorphism (see for example [24, 29]).

Let α be any irrational number. Take a Cantor set $D \subset S^1$. Let $A \subset D$ be the accessible points of the Cantor set. Collapsing complementary intervals of D to single points gives a map $\pi : D \twoheadrightarrow S^1$, where $\pi(A)$ is a countable dense set $X \subset S^1$. Since any two countable dense sets in S^1 are equivalent up to an ambient homeomorphism of S^1, we may take X to be invariant (as a set) under the irrational rotation by α. For instance, given any other countable dense set Y, let $X = \bigcup_{n\in\mathbb{Z}} \alpha^n(Y)$.

Then as shown in the diagrams in Figure 2.6, we may lift the rotation by α on X to a map $\hat{\alpha}$ on D as follows. The map π is 2-1 on the accessible points A, and 1-1 on $D - A$. As the rotation α is orientation preserving and takes the set of images of accessible points X to itself, we may define $\hat{\alpha}$ on A by requiring that it be orientation preserving. The lift α is well defined on the inaccessible points $D - A$. Thus we get the 'irrational rotation' $\hat{\alpha}$ on the Cantor set, which is a homeomorphism. Every point is aperiodic under $\hat{\alpha}$ as the rotation α is totally aperiodic on S^1.

Cannon et al [19] then use this irrational rotation to construct a homeomorphism of C with an arbitrary period set. The idea is based on the example of a sequence of circles limiting on a circle, i.e. $S^1 \times X$, where $X = \{x_0, x_1, x_2, \dots\}$ is a convergent sequence $x_i \to x_0$. Define a map f on this space by rotating $S^1 \times \{x_0\}$ by an irrational amount α, and then rotating each circle $S^1 \times \{x_i\}$ by a rational amount r_i. If the rational numbers r_i converge to α, we get a continuous map. If the rational numbers $r_i = p_i/q_i$ in lowest terms, then we get $\mathrm{Per}(f) = \{q_i\}$. It is a number theoretic result that for any irrational number α, and any set of positive integers $\{q_i\}$, there exist rational numbers $r_i = p_i/q_i$ in lowest terms with $r_i \to \alpha$.

This map we constructed was on the space $S^1 \times X$. We will use this same idea to get a map on the Cantor set. Write $C = D \times X$, where C, D are Cantor sets, and X is a convergent sequence as before. We have already discussed how to get an 'irrational rotation' $\hat{\alpha}$ on $D \times \{x_0\}$. It remains to approximate the rational rotations by $r_i = p_i/q_i$ in a manner that converges to $\hat{\alpha}$. In particular, we need to construct maps f_i on $D \times \{x_i\}$ such that $f_i^{q_i} = \mathrm{Id}$ and such that f_i agrees with $\hat{\alpha}$ on accessible points adjacent to large complementary intervals. This process is discussed in more detail in [19].

Thus we can produce a homeomorphism on the Cantor set with any desired period set, as long as we are allowed aperiodic points.

For our final examples, we discuss related examples by Zastrow and Cannon. Zastrow's example has aperiodic points, while Cannon's example has no aperiodic points, but has pre-periodic points of unbounded pre-periodic length. Both examples are based on the following idea. Given a set $P \subset \mathbb{N}$, take some element $k \in P$, and partition P into equivalence classes modulo k: $P = \bigcup_{r=0}^{k-1} P_r$, where $P_r = \{p \in P \mid p = qk + r\}$. Then, as we may take unions of period sets, it suffices to construct maps with period sets $\{k\} \cup P_r$, or equivalently, period sets $\{k\} \cup \{q_i k + r\}$ for fixed k, r, where the q_i may vary arbitrarily.

In both of these examples, we will take a space that is a union of sequences, and then apply Corollary 2.7.3 to get a map on the Cantor set. We give an overview of the idea in this paragraph, and then discuss the examples in more details in the following paragraphs. Start with the space S, which is a convergent sequence of sequences, i.e. if X_i is a convergent sequence of points, then S is the one point compactification of $\bigcup_{i=1}^{\infty} X_i$. Also let the space

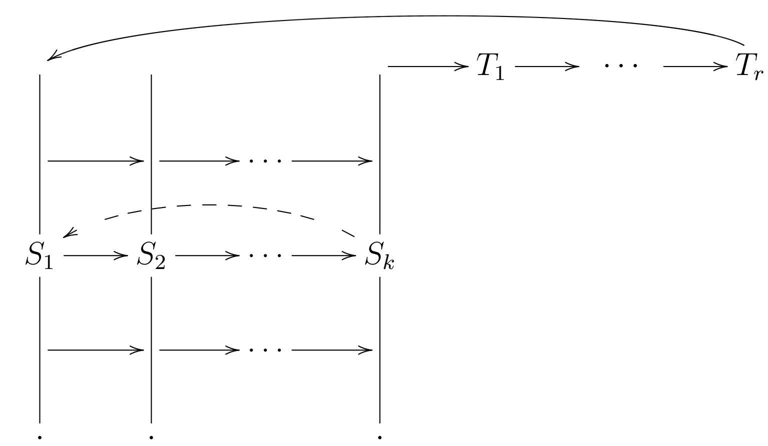

Figure 2.7: Schematic for examples of Cannon and Zastrow.

T be just a convergent sequence of points. Then our space Y will be the union of k copies of S_i, and r copies of T_j. A schematic for the space Y is shown in Figure 2.7. In both examples, the map f sends S_i to S_{i+1} by the identity, for $i < k$, and then f maps S_k to S_1 in a manner that permutes the levels of the sequences, in a periodic fashion according to the numbers q_i. This gives periods $q_i k$; to get periods $q_i k + r$, each periodic point runs through the sets T_i once before completing the periodic cycle.

Zastrow's Example. As stated above, we want a map with period set $\{k\} \cup \{q_i k + r\}$ for fixed k, r, where the q_i may vary arbitrarily. For ease in defining the function for this example, we will write each S_i as a union of two sequences S_i^+, S_i^- with a common limit point s_i. As S_i^* is a sequence of sequences, we will write $S_i^* = \bigcup_{m=0}^{\infty} S_i^*(m)$, where each $S_i^*(m)$ is a convergent sequence, and the sets $S_i^*(m)$ converge to s_i. Thus we can write our space as

$$Y = \left[\bigcup_{i=1}^{k} \left(\{s_i\} \cup \bigcup_{m=0}^{\infty} \left(S_i^+(m) \cup S_i^-(m) \right) \right) \right] \cup \left(\bigcup_{j=1}^{r} T_j \right).$$

First we will define a map $g : Y \to Y$, and then modify the map g to get a map $f : Y \to Y$ with the desired periods. Define the map g in the following manner, where sequences are mapped to other sequences, with the limit points being mapped to limit points.

$$
g : \begin{cases}
S_i^+(m) \mapsto S_{i+1}^+(m) & \text{if } \; i \neq k \\
S_i^-(m) \mapsto S_{i+1}^-(m) & \text{if } \; i \neq k \\
\hline
S_k^+(m) \mapsto S_1^+(m+1) & \\
S_k^-(m) \mapsto S_1^-(m-1) & \text{if } \; m \neq 0 \\
\hline
S_k^-(0) \mapsto T_1^n & \\
T_j^n \mapsto T_{j+1}^n & \text{if } \; j \neq r \\
T_r^n \mapsto S_1^+(0) &
\end{cases} \tag{2.1}
$$

Note that the map g permutes the sets S_i cyclically, and the subsets $S_i^+(m)$ are shifted toward the limit point s_i (with the index $m \mapsto m+1$), while the subsets $S_i^-(m)$ are shifted away from the limit point s_i (with the index $m \mapsto m-1$). Additionally, the sequence $S_k^-(0)$ takes a tour through the sets T_j. The only periodic points for g are the limit points s_i of the sets S_i, and their period is q. All other points are aperiodic.

We will modify the map g to get a map f with the desired periods $q_n k + r$. We will redefine the map only on a sequence of points of Y. To this end we need to write the sequence $S_i^*(m) = \{s_i^*(m)\} \cup \bigcup_{n=0}^{\infty} S_i^*(m,n)$, where each $S_i^*(m,n)$ is a single point, and the sequence $S_i^*(m,n) \to s_i^*(m)$ as $n \to \infty$.

Let $m_n = \lfloor q_n/2 \rfloor$ and $\phi(m_n) = \lfloor (q_n - 1)/2 \rfloor$. Thus if q_n is odd, then $q_n = 2m_n + 1$ and $\phi(m_n) = m_n$. If q_n is even, then we have $q_n = 2m_n$, and $\phi(m_n) = m_n - 1$. Now we define the change from g to f by the following formula:

$$
S_k^+(m_n, n) \mapsto S_1^-(\phi(m_n), n)
$$

On the remainder of Y, the map f agrees with g. In other words, for each n, we just change the map on the n^{th} point of the sequence $S_k^+(m_n)$, and now instead of mapping to S_1^+, the point maps to S_1^- in the appropriate m-level. The 'correct' m-level is either m_n or $(m_n - 1)$ depending on whether q_n is odd or even. For these formulas to work, it is important that we start indexing at $m = 0$.

This map is still continuous, as we only redefined the map g on a sequence of points that limit on s_k, and their images limit on s_1, which is the image of s_k. The point $S_1^+(1, n)$ has period $q_n k + r$, as the point moves down the m-levels on the sets S_i^+, then back up the m-levels of S_i^- for a total of q_n circuits through the k sets S_i, and then finally through the r sets T_j. One can check that our modifications give no other periodic points. Note that the intermediate limit points $s_i^*(m)$ are all aperiodic. The points $s_i^-(m)$ cycle through i and m decreases until these points traverse the sets T_j and end at $s_1^+(0)$. The points $s_i^+(m)$ also cycle through the subscripts i, while m continually increases, thus giving aperiodic points.

Thus $\text{Per}(f) = \{k\} \cup \{q_n k + r\}$. We note that if there are any points with $q_n = 0, 1$ then the construction above does not quite work as stated, but there are only finitely many such points, and we can add any finite period set as discussed in section 2.7.2.

Cannon's Example. Cannon gives a similar example with no aperiodic points, but with pre-periodic points. The basic form of the map is the same as Zastrow's example, as shown in Figure 2.7. However, the map from S_k to S_1 is quite different, so as to avoid aperiodic points. We do not present this map here, but a complete description is given in [19].

2.8 Aperiodic and pre-periodic points for interval maps

While the main result from the previous section simply states that any period set is attainable for Cantor set maps, the complete result is more interesting than just that, in that the existence of aperiodic or pre-periodic points changes the result. In particular, if every point is periodic on the Cantor set, then there is a restriction that the period set have a 'finite basis' (see Theorem 2.7.5).

This raises the question of how the presence of aperiodic or pre-periodic points is related to the possible period sets for other spaces. The period sets for most spaces discussed in this chapter (other than the Cantor set) are built upon pieces of Sharkovskiĭ's order for period sets of the interval. Thus we focus on the following questions.

Question 2.8.1. *For maps on an interval, what period sets can occur in the absence/presence of aperiodic points? of pre-periodic points?*

First recall that for a non-compact interval there are maps with empty period set, i.e. every point aperiodic, such as a translation of the real line. For $P = \{1\}$, the identity map has every point periodic, and for $P = \{1, 2\}$, the map $f(x) = 1 - x$ on the interval $[0, 1]$ also has every point periodic. It seems that these are the only possibilities if there are no pre-periodic points.

Conjecture 2.8.2. *A point of period 4 implies the existence of pre-periodic points.*

Now we consider the case of aperiodic points. As mentioned above, the identity is a map with $P = \{1\}$ and with no aperiodic points. We now describe another map that also has $P = \{1\}$, but that also has aperiodic points. Consider a closed interval as the extended real line, that is, $\mathbb{R}$ taken with the endpoints $-\infty, \infty$. The desired map $a : I \to I$ shifts $\mathbb{R}$ by 1 and fixes $\pm\infty$, thus having fixed points and aperiodic points. Note that this can also be done on a half open interval simply by deleting one endpoint and using the same map, and on an open interval by taking two such half open intervals together and identifying the two endpoints as in $(x, y) = (x, z] \cup [z, y)$.

As noted above, the map $f(x) = 1 - x$ has $P = \{1, 2\}$ with no aperiodic points. Using the map a from the preceding paragraph together with the map $b = \hat{T}_{h(2)}$ from Lemma A.1.1, we will construct a map with $P = \{1, 2\}$ having aperiodic points. Recall the map $b : [0, 2] \to [0, 2]$ fixes the endpoints $0, 2$, and has $\text{Per}(b) = \{1, 2\}$. Define the map $c : [-1, 2] \to [-1, 2]$ by the map a on $[-1, 0]$ and the map b on $[0, 2]$. Thus $\text{Per}(c) = \{1, 2\}$ and c has aperiodic points. In a similar manner, using maps $\hat{T}_h$ for different values of h, we can get maps with any period set that also has aperiodic points.

If we want to consider ∞ as the period of an aperiodic point (as the orbit has infinitely many points), we see that ∞ does not fit in nicely to Sharkovskiĭ's ordering, as neither 2 nor ∞ imply the other. Only in the case of a compact interval does ∞ imply 1, as there is always a fixed point. While ∞ does not imply any other period, it does seem that certain periods imply the existence of aperiodic points.

Conjecture 2.8.3. *If* $\mathrm{Per}(f)$ *is an infinite set, then* f *has an aperiodic point. For every* n*, there are maps with finite period set* $\{2^n, \ldots, 2, 1\}$ *without aperiodic points.*

By Sharkovskiĭ's Theorem (Theorem 2.1.2) we know that a non-empty period set P of an interval map falls into one of three categories:

- P contains some number that is not a power of 2
- P contains all powers of 2: $P = \{\ldots, 2^n, \ldots, 2, 1\}$
- P contains only finitely many powers of 2: $P = \{2^n, \ldots, 2, 1\}$

Conjecture 2.8.3 states that only in the first two cases is there necessarily an aperiodic point.

Li and Yorke [33] prove a special case of Conjecture 2.8.3, when $3 \in P$. Not only do they prove the existence of an aperiodic point, but they show that there is an uncountable set of aperiodic points with certain properties that warrant the title of their paper: "Period Three Implies Chaos." We extend their methods to prove the conjecture in the case where there is a period not a power of 2.

We also will use a few techniques from the proof of Sharkovskiĭ's Theorem, which we briefly discuss here. The basic idea is that the image of an interval must cover at least the interval between the image of its endpoints. Then a periodic orbit of a point gives various intervals that must cover each other in a particular way, based on the order of the orbit. A particular type of orbit, called a Štefan cycle, is then shown to correspond directly to the minimal number in the Sharkovskiĭ order that lies in the period set. For more details, consult a proof of Sharkovskiĭ's Theorem, for instance [14] or [4].

Now, suppose that the period set P contains a period that is not a power of 2. Let p be the minimal element of the period set, according to Sharkovskiĭ's order. Note that p will not be a power of 2. We first prove the result in the case where p is odd, and the case when p has factors of 2 follows easily.

Case A: p is odd. Let x be a point of period p. The orbit of x must be a Štefan cycle, by Proposition 5.3 in [14]. The orbit of x creates $(p-1)$ intervals I_i that satisfy the following covering properites, where we write $I \to J$ if $f(I) \supset J$.

$$\overbrace{I_1 \to I_1 \to \cdots \to I_1}^{k} \to I_2 \to I_3 \to \cdots \to I_{m-1} \to I_1$$

We may concatenate copies of this finite cyclic sequence to get an infinite sequence of intervals M_n such that $f(M_n) \supset M_{n+1}$, and where each M_n is some I_i, $i = 1, \ldots, m-1$. As we want to avoid periodicity, we change the number k of copies of I_1 used in each repetition of our sequence; for example it suffices to set $k_j = j$. It is also important to note that no point in the original orbit of x may follow this sequence of intervals M_n, as these points may only stay in I_1 for two consecutive iterations, while the sequence has arbitrarily large strings of copies of I_1. This is then essentially the same setup as in Li and Yorke [33], and we then get intervals Q_n such that $Q_{n+1} \subset Q_n$ and $f^n(Q_n) \subset M_n$. Then any point z in $Q = \bigcap Q_n$ will satisfy $f^n(z) \in M_n$, which we have chosen to be aperiodic.

We note that the existence of intervals such as Q_n is a standard lemma used in proving Sharkovskiĭ's Theorem, and is also given in Li and Yorke [33]. Also, the other results of Li and Yorke involving properties of these aperiodic points should also follow from this similar setup we have produced, but we do not desire to get into that discussion here.

Case B: $p = q2^k$ with q odd. We use the following proposition from [14]. We note that we have defined Sharkovskiĭ's order in the opposite direction as they have, and thus we have switched 'maximal' for 'minimal.'

Proposition 2.8.4 (Proposition 2.3 from [14]). *If $m = 2n$ is Sharkovskiĭ minimal for f then n is Sharkovskiĭ minimal for f^2.*

As we have chosen p to be Sharkovskiĭ minimal, by Proposition 2.8.4 we see that f^{2^k} has odd minimal Sharkovskiĭ period q. Thus from case A, f^{2^k} has an aperiodic point. This point must then also be aperiodic for f, as the orbit of a point under f^{2^k} is a subset of the orbit under f.

Thus we have shown that if an interval map has a point with period not a power of 2, then there must be an aperiodic point.

The second case, where P consists of all powers of 2 is the least clear of all. Unlike the previous case, there is no one periodic orbit which guarantees infinitely many others, and the methods akin to Li and Yorke fail here. We will discuss an example of this case after the next case, as it is constructed from the examples there.

For the final case where $P = \{2^n, \dots, 2, 1\}$, we construct examples of maps with no aperiodic points, that is, with every point pre-periodic. In this discussion, we will often use the idea of the double of a map. This idea is presented in [24] and is used to give examples of maps with certain period sets. Given a map $f : [0, 1] \to [0, 1]$, a new map $F(x)$ called the double of f is defined as follows.

$$F(x) = \begin{cases} 2/3 + f(3x)/3 & \text{if } 0 \le x \le 1/3 \\ -(2 + f(1))(x - 2/3) & \text{if } 1/3 \le x \le 2/3 \\ x - 2/3 & \text{if } 2/3 \le x \le 1 \end{cases}$$

This map 'shrinks' the original map f and performs it on an altered domain and range: $\tilde{f} : [0, 1/3] \to [2/3, 1]$. The rest of the map is piecewise linear, passing through the points $\big(1/3, (2 + f(1))/3\big)$, $(2/3, 0)$, and $(1, 1/3)$.

Note that the interval [0,1/3] maps to [2/3,1], which in turn is mapped to [0,1/3]. Thus any periodic point in these intervals has even period. Now consider F^2; on both of these intervals this map is just an adjusted version of the original map f, thus the periods of points for F^2 are exactly the periods of points of f, which gives periodic points for F of periods $2 \cdot \text{Per}(f)$. In the interval $(1/3, 2/3)$, there is a unique fixed point $x_0 = \dfrac{2(2+f(1))}{3(3+f(1))}$, and any other point x in this interval is not periodic as $F^n(x)$ eventually lies outside of $(1/3, 2/3)$. This last statement is related to the fact that the fixed point x_0 is repelling, as $|F'(x_0)| = 2 + f(1) > 1$. Thus $\text{Per}(F) = 2 \cdot \text{Per}(f) \cup \{1\}$.

We now use the process of doubling to find functions with any possible finite period, having no aperiodic points. Start with $f_0(x) = x$, the identity map, with $\text{Per}(f_0) = \{1\}$. As before, note that f_0 has no aperiodic points. Let f_1 be the double of f_0. Then $\text{Per}(f_1) = \{2, 1\}$. As mentioned in the last paragraph, every point in $(1/3, 2/3)$ other than the fixed point is eventually sent outside of $(1/3, 2/3)$. The map f_1 has slope 1 on $[0, 1/3] \cup [2/3, 1]$, and so ${f_1}^2$ is the identity on this set. Therefore every point is pre-periodic for f_1.

The maps f_i are defined similarly: f_i is the double of f_{i+1}, so that $\text{Per}(f_i) = \{2^i, \ldots, 2, 1\}$. Since every point is pre-periodic under f_{i-1}, we also get that every point is pre-periodic under f_i: the middle third interval again gets sent to the outer intervals (other than the fixed point), and the map ${f_i}^2$ is essentially f_{i-1} on these outer intervals, so that every point is pre-periodic.

This proves Conjecture 2.8.3 in the case where P is finite and non-empty.

The only case where Conjecture 2.8.3 is not proven is the second case, where P consists of all powers of 2. We will consider a limit of the maps f_i above as an example of this case, which is also discussed in [24], where they call this example the Adding Machine, due to Misiurewicz. We note that our example may seem slightly different as we start with a different function f_0, but the resulting limit function is the same.

Consider the function f defined as the pointwise limit of the functions f_i described above: $f(x) = \lim_{i \to \infty} f_i(x)$. In fact, for every $x \in (0, 1]$, the sequence $f_i(x)$ is eventually constant, and thus the limit exists. The sequence $f_i(0)$ is not eventually constant, but $f_i(0) = 1 - 1/3^i$ so that $f(0) = 1$. This map does in fact have period set being exactly all the powers of 2: $\text{Per}(f) = \{\ldots, 2^n, \ldots, 2, 1\}$. However, the property that each f_i only has pre-periodic points

i	Orbit of $x = 0$ under f_i		Orbit of $x = 1$ under f_i
0	0		1
1	0, $\frac{2}{3}$		1, $\frac{1}{3}$
2	0, $\frac{8}{9}$, $\frac{2}{9}$, $\frac{2}{3}$		1, $\frac{1}{3}$, $\frac{7}{9}$, $\frac{1}{9}$
3	0, $\frac{26}{27}$, $\frac{8}{27}$, $\frac{20}{27}$, $\frac{2}{27}$, $\frac{8}{9}$, $\frac{2}{9}$, $\frac{2}{3}$		1, $\frac{1}{3}$, $\frac{7}{9}$, $\frac{1}{9}$, $\frac{25}{27}$, $\frac{7}{27}$, $\frac{19}{27}$, $\frac{1}{27}$

Table 2.4: Orbits of 0 and 1 under the maps f_i, in order of the orbit.

is not preserved in the limit. The points 0,1, for example, are now aperiodic. Consider the orbits of 0 and 1 under the various maps f_i.

The orbits are shown in Table 2.4, ordered by the orbit structure, i.e. where $f(x_i) = x_{i+1}$. Note that the orbit under f_i is a subset of the orbit for f_{i+1}, and that the structure remains intact from one level to the next, as the additional points in the orbit are all inserted in between one pair of points, with the rest remaining in the same order in the orbit. It also becomes instructive for this discussion if we order the points of the orbits in the standard order from [0,1], as shown in Table 2.5, where the top row shows all of the points in the orbits for both 0 and 1 combined, which is followed by the different orbits of 0 under the maps f_i, followed by the orbits of 1 under f_i, with all of the points positioned consistently. Notice from Table 2.5 how the orbits of 0 and 1 under the map f_i each pick out half of the endpoints of the intervals in stage i of the construction of the middle thirds Cantor set. Then for the limit function $f(x)$, these become aperiodic points whose orbits lie in the accessible points of the Cantor set. Notice that as $f(0) = 1$, these orbits are joined, and the points in the orbit of 0 under f_i will still eventually map to 0, then 1, and continue on in that orbit. Thus considering f^{-1} as well as f on these points, we get a bi-infinite orbit.

Thus this map f has $\text{Per}(f) = \{\ldots, 2^n, \ldots, 2, 1\}$, and although f is a limit of maps f_i, each with no aperiodic points, the map f does have aperiodic points.

In summary, Conjecture 2.8.3 is true if P has a period not a power or 2, or if P is a finite set, and is undecided in the case that P is the set of all powers of 2: $P = \{\ldots, 2^n, \ldots, 2, 1\}$. We state these results from this section in the following theorem.

	0,	$\frac{1}{27}$,	$\frac{2}{27}$,	$\frac{1}{9}$,	$\frac{2}{9}$,	$\frac{7}{27}$,	$\frac{8}{27}$,	$\frac{1}{3}$,	$\frac{2}{3}$,	$\frac{19}{27}$,	$\frac{20}{27}$,	$\frac{7}{9}$,	$\frac{8}{9}$,	$\frac{25}{27}$,	$\frac{26}{27}$,	1
i	Orbit of $x = 0$ under f_i															
0	0,															
1	0,								$\frac{2}{3}$,							
2	0,				$\frac{2}{9}$,				$\frac{2}{3}$,				$\frac{8}{9}$			
3	0,		$\frac{2}{27}$,		$\frac{2}{9}$,		$\frac{8}{27}$,		$\frac{2}{3}$,		$\frac{20}{27}$,		$\frac{8}{9}$,		$\frac{26}{27}$	
i	Orbit of $x = 1$ under f_i															
0																1
1								$\frac{1}{3}$,								1
2				$\frac{1}{9}$,				$\frac{1}{3}$,				$\frac{7}{9}$,				1
3		$\frac{1}{27}$,		$\frac{1}{9}$,		$\frac{7}{27}$,		$\frac{1}{3}$,		$\frac{19}{27}$,		$\frac{7}{9}$,		$\frac{25}{27}$,		1

Table 2.5: Orbits of 0 and 1 under the maps f_i, in order of the real line.

Theorem 2.8.5. *Let P be the period set of an interval map.*

- *If P has a period that is not a power of 2, then there must be an aperiodic point.*
- *If P is finite and non-empty, there need not be an aperiodic point.*

2.9 Period implications

In the final section of this chapter, we return to the topic of the partial order of period implications for a given space X, as discussed in Section 2.5 prior to Example 2.5.5. Recall that for a space X, we define the partial order $n \preceq_X m$ to mean that every map of X to itself having a point of order n also has a point of order m. As noted earlier, this partial order is generally not very informative, as few spaces have as much structure in the period sets as the interval does. However, it is interesting to ask what period implications are possible for any space. In this section we will want to avoid vacuous implications, so we make the following definition:

Definition 2.9.1. Given a space X, $n \trianglelefteq_X m$ if there exists a map $f : X \to X$ with a point of order n, and if every such map also has a point of order m.

For most of the examples discussed in this chapter, the structure of possible period sets is built around unions of multiples of Sharkovskiĭ's order. In fact, these spaces retract onto an interval, and so there are maps on these spaces with every period set possible for the interval (as given by Theorem 2.1.2B). Thus any implication $n \trianglelefteq_X m$ must be contained in Sharkovskiĭ's order. For other spaces such as the Cantor set or a compact disk B^2 there are no implications other than maps on the disk must have a fixed point, i.e. $n \trianglelefteq_B 1$ for all n. This leads us to make the following conjecture.

Conjecture 2.9.2. *For any space X, any non-vacuous period implication is a multiple of a Sharkovskiĭ implication. That is, if $n \trianglelefteq_X m$, then there is some number a such that $n = an_0, m = am_0$, and $n_0 \preceq m_0$ in Sharkovskiĭ's order.*

While all of the examples considered above have every implication being precisely a Sharkovskiĭ implication, the fact that some period sets are based on multiples of Sharkovskiĭ's order hints that taking multiples might be necessary, and we will present some examples that show this is indeed the case.

In addition to considering the examples discussed previously in this chapter, we have begun to look at finite topological spaces. The finite spaces of interest will not be Hausdorff, as a finite Hausdorff space is discrete, and has no period implications other than the existence of periodic points of periods $n \leq |X|$. While some may think it uninteresting to consider non-Hausdorff spaces in and of themselves, it is interesting to note that the homotopy type of any finite simplicial complex can be encoded in a finite non-Hausdorff space, as well as the information of simplicial maps.

2.9.1 Computer aided search.

We wish to describe these finite spaces and continuous maps in a manner that can be checked by a computer program. First note that a finite topology has a basis consisting of minimal open sets: for every point $x \in X$, define U_x to be the minimal open set containing x, $U_x = \bigcap_{U \ni x} U$, where U is open. Since X is finite, the intersection is finite and U_x is open. We then use this basis of minimal open sets to define a partial pre-order, or equivalently a directed graph on X, that is equivalent to the topology. Define $x \leq y$, or $x \to y$, if $y \in U_x$, i.e. every open set containing x also contains y. Note that this relation is clearly transitive and reflexive. It may not be anti-symmetric as the existence of distinct points x, y such that $x \to y$ and $y \to x$ is equivalent to having a partial pre-order instead of a partial order, which is also equivalent to the directed graph having a directed cycle, and to the existence of a nontrivial indiscrete subset of X.

Thus we may encode a topology on a finite space as a matrix M with entries in $\{0, 1\}$, where entry $M_{ij} = 1$ means that $i \to j$, while $M_{ij} = 0$ means that $i \not\to j$. A continuous map on this space is simply a map that preserves the partial pre-order: f is continuous if and only if whenever $x \to y$ then $f(x) \to f(y)$. Thus given a function, which can be represented as an n-tuple with entries in $\{1, \ldots, n\}$, we may check the corresponding entries in the topology matrix to check if the continuity condition is satisfied.

Note that to answer the question of which period implications are possible, for any given space we must consider *all* continuous functions. We may sample different spaces, and see what implications do or do not arise for that space, but to find these implications we must consider all continuous functions on each space considered. This makes it difficult to compute all implications for all spaces of a certain size, as the most obvious method of constructing spaces of size n and maps on spaces gives approximately $2^{(n^2)}$ spaces ($n \times n$ matrices with entries in $\{0, 1\}$), with n^n maps on each space (n-tuples with entries in $1, \ldots, n$).

Fortunately, many of these $2^{(n^2)}$ matrices do not give topologies, and there are many homeomorphic repetitions among those that are. Also, many of the n^n maps are not continuous. We have taken some preliminary measures to cut down the number of spaces and functions to check. To ensure that the $n \times n$ matrix gives a topology, we must have the following conditions. First, $a_{ii} = 1$ as a point is in every open set containing it. Second, the relation $x \to y$ must be transitive; this can be checked by taking M^n and checking that the 0 pattern matches that of M. This is due to the fact that a non-zero entry in M^n means that there is a directed path from i to j of length at most n. Thirdly, the intersection property for open sets must be satisfied, which can be checked by taking intersections of rows of M. One last simplification of the space we have implemented is that we require the row sums of the matrix to be increasing; this can be done by a simple renumbering of the points.

While we must still consider all continuous functions on the space, there is a simplification so that we do not need to enumerate all possible functions and then check if they are continuous. The basic idea is to partially define the function (on some subset of the points), and check if that is continuous so far by checking the preservation of the partial pre-order. If some partial function is already discontinuous, then we do not need to enumerate the remaining functions that agree with this partial function. When defining the function on one additional point, we only need to compare the new point to each of the previously defined points for order preservation, so that for a continuous function this method has the same number of comparisons as if we had defined the entire function and then checked for continuity.

With these simplifications in place, we have written a program for Mathematica that runs through spaces and checks for non-vacuous period implications that hold for every continuous map on that space. We enumerate the spaces by considering the matrix M as a binary number with n^2 digits, and then check that it satisfies all of the properties in the preceding paragraphs before considering the continuous functions on that space. The code for this program is given in Appendix A.2.

Over a period of weeks, this program has calculated period implications for over one billion matrices. This includes all matrices up to 5×5, while the vast majority of the matrices considered are 6×6. As there are $2^{36} \approx 68$ billion 6×6 matrices, we will need to find a more efficient algorithm for enumerating topologies, or resort to some sort of Monte Carlo approach. Of the spaces considered, the program has only found implications of $n \trianglelefteq n$ and $n \trianglelefteq 1$, which are Sharkovskiĭ implications.

$$X_{2,1}: \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad X_{3,1}: \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \quad X_{6,1}: \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$X_{6,2}: \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & & & & \\ 1 & 1 & 0 & 0 & & & & \\ 1 & 0 & 1 & 0 & & & & \\ 1 & 0 & 0 & 1 & & & & \\ \hline & & & & 1 & 0 & 0 & 0 \\ & & & & 1 & 1 & 0 & 0 \\ & & & & 1 & 0 & 1 & 0 \\ & & & & 1 & 0 & 0 & 1 \end{array}\right), \quad X_{6,3}: \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & 0 & & & & & & \\ 1 & 1 & 0 & & & & & & \\ 1 & 0 & 1 & & & & & & \\ \hline & & & 1 & 0 & 0 & & & \\ & & & 1 & 1 & 0 & & & \\ & & & 1 & 0 & 1 & & & \\ \hline & & & & & & 1 & 0 & 0 \\ & & & & & & 1 & 1 & 0 \\ & & & & & & 1 & 0 & 1 \end{array}\right)$$

Table 2.6: Spaces $X_{mn,n}$ with period implications $mn \trianglelefteq n$.

2.9.2 Non-trivial period implications. Independent of the computer program discussed in the previous section, we have discovered spaces $X_{mn,n}$ with implications $mn \trianglelefteq n$ for all m, n. These are non-trivial multiples of the Sharkovskiĭ implications $m \preceq 1$. The space $X_{mn,n}$ is constructed as the disjoint union of n copies of the space $X_{m,1}$, which has $(m+1)$ points. The space $X_{m,1}$ has one special point 1 that is in every nonempty open set, while the minimal open set for any other point i is just $\{1, i\}$. Examples of the matrices are shown in Table 2.6, where the empty blocks are all zeroes.

Theorem 2.9.3. *The space $X_{mn,n}$ has non-vacuous period implication $mn \trianglelefteq n$.*

Proof. We first consider the spaces $X_{m,1}$, i.e. where $n = 1$. It is easy to see that there is a map with periods m and 1, as the space is symmetric with respect to the points $2, \ldots, m+1$, and we may permute them cyclically, while fixing 1. To see that m does imply 1, recall that continuous maps must preserve the partial pre-order. Thus if the special point 1 maps to any point $i \neq 1$, then the whole space must map to i, as all points precede 1 but none precede i. Thus if there is a point of period m, then the point 1 must be fixed.

Now consider the space $X_{mn,n}$, with $n \geq 2$. Again, it is fairly simple to construct a map with periods mn and n. Permute the n blocks cyclically, via the identity map on the first $(n-1)$ blocks, and on the final block, permute the m undistinguished points cyclically as above. Thus the special points of each block form a cycle of order n, while the remaining points form a cycle of order mn.

Now we show the implications $mn \trianglelefteq n$. First consider the case where $n < m$. Note that $X_{mn,n}$ has $n(m+1) = mn + n$ points, and that $mn + n - m < mn$. Then if any of the special points of one of the blocks maps somewhere other than another special point, then all of the other m points of that block must also map to the same point, resulting in at most $mn + n - m < mn$ points in the image, so there cannot be an mn cycle in this case. Thus any mn cycle cannot contain any of the n special points, but must consist of all mn of the remaining points of the space. A similar argument shows that the n special points must be permuted bijectively, otherwise there are not enough points in the image of the map. If the permutation is not an n cycle, then the map partitions the blocks according to the permutation on the special points, and there cannot be a mn cycle on all mn of the non-distinguished points.

A slight alteration of the above argument works for the case $m < n$, where $m \neq 1$. Again, if the mn cycle contained a special point of any block, the the remainder of that block could not be in the cycle, as the entire block eventually maps to the same point as the special point does. Then consider the quotient collapsing all such blocks to their special points; the resulting space will have some blocks together with isolated points. The special points for the blocks that remain cannot be part of our mn cycle, as we collapsed all such points. Then there no longer remain enough points to create an mn cycle, as we have deleted m points for each block collapsed, an can only use the 1 special point from that block. Thus any mn cycle cannot contain any of the special points, and as before the special points must themselves form an n cycle in order to create an mn cycle.

If $m = 1$, then we are looking for the implication n implies n, which is always true, but may be vacuous if there are no possible maps with an n cycle. However, our space $X_{n,n}$ is just the discrete topology on n points, which clearly admits an n cycle. $\square$

Chapter 3. Solenoids and their Complements

3.1 Introduction

A solenoid is a topological space that is an inverse limit of circles. Let $\{n_i\}$ be a sequence of positive integers, and let $f_i : S^1 \to S^1$ be defined by $f_i(z) = z^{n_i}$, where S^1 is thought of as the unit circle in the complex plane. Then we define the solenoid

$$\Sigma(n_i) = \varprojlim(S^1, f_i).$$

If the tail of the sequence is $1, 1, 1, \ldots$, then the solenoid is degenerate, being simply a circle. If the sequence ends in $2, 2, 2, \ldots$, then we have what is called the *dyadic solenoid*, Σ_2. We will use the dyadic solenoid for specific examples throughout this chapter.

We note that multiple sequences $\{n_i\}$ can determine the same solenoid, up to homeomorphism. For instance, we may assume each n_i is prime by replacing any composite number by the sequence of its prime factors. We may also remove any finite initial segment of the sequence, and we may reorder the sequence (infinitely). Bing notes that if you remove a finite number of elements from two sequences so that in the remainders, every prime occurs the same number of times, then the solenoids are topologically equivalent; he also says that perhaps the converse is true [11]. The converse is confirmed by McCord [35]. A few other references discussing solenoids are [28, 32, 49, 50].

As solenoids are obtained via an inverse limit construction of compact topological groups S^1, we get the standard result that solenoids are also compact topological groups. Additionally, it is standard that a solenoid has uncountably many path components, each of which is dense in the solenoid, and also that solenoids are not locally connected, nor are its path components. However, the path components are fairly nice in that they are archetypal, as defined in Section 2.4. In particular, there is a continuous bijection from the real line onto each path component. This bijection however is not a homeomorphism, as small neighborhoods in the solenoid path component are not locally connected. A lift of a small neighborhood to the real line contains infinitely many small disjoint neighborhoods centered at a collection of points unbounded on the line.

While these standard facts together with the inverse limit construction give some nice properties of solenoids, they do not make it apparent that all solenoids embed in S^3. To see this, we will construct the solenoid $\Sigma(n_i)$ as a nested intersection of solid tori. Take a solid torus T_0 with cross-sectional diameter d_0 in S^3, using the standard metric from S^4. Embed a solid torus T_1 with cross-sectional diameter $d_1 < d_0/2$ inside of T_0 that wraps around T_0 n_1 times. Continue this process, embedding a solid torus T_i with cross-sectional diameter $d_i < d_{i-1}/2$ inside of T_{i-1}, which wraps around T_{i-1} n_i times. The nested intersection $\bigcap T_i$ is an embedding of $\Sigma(n_i)$ in S^3. See Figure 3.1 for an example with the dyadic solenoid (where $n_i \equiv 2$).

We note that while this nested intersection construction may seem canonical, there are in fact many ways to embed each T_i inside of T_{i-1}, even if we require that T_i never 'folds back' on itself (i.e. T_i is embedded in a monotone fashion inside T_{i+1}). In the simple case where $n_i \equiv 2$, T_i can have any odd number of half twists with itself; when $n_i > 2$, there can be much more complicated braiding. While this does not change the topology of the solenoid itself, this does change its complement significantly. This is analogous to knot theory: while every knot is itself a circle, knot complements are quite different. Thus, we could consider the study of solenoid embeddings and their complements as solenoid knot theory. This is also quite related to braid groups, as each (nice) embedding of T_i into T_{i-1} can be represented by a braid on n_i strands that gives a transitive permutation of the strands (otherwise the closed braid will result in a link with multiple components). This issue will be discussed further in the following sections, and some diagrams are given in Figure 3.3.

All of the embeddings of solenoids that we will consider here will be obtained as nested intersections of solid tori, where each torus is a closed braid in the previous torus. We note that similar work has been done in [32], where they discuss what they call *tame* embeddings,

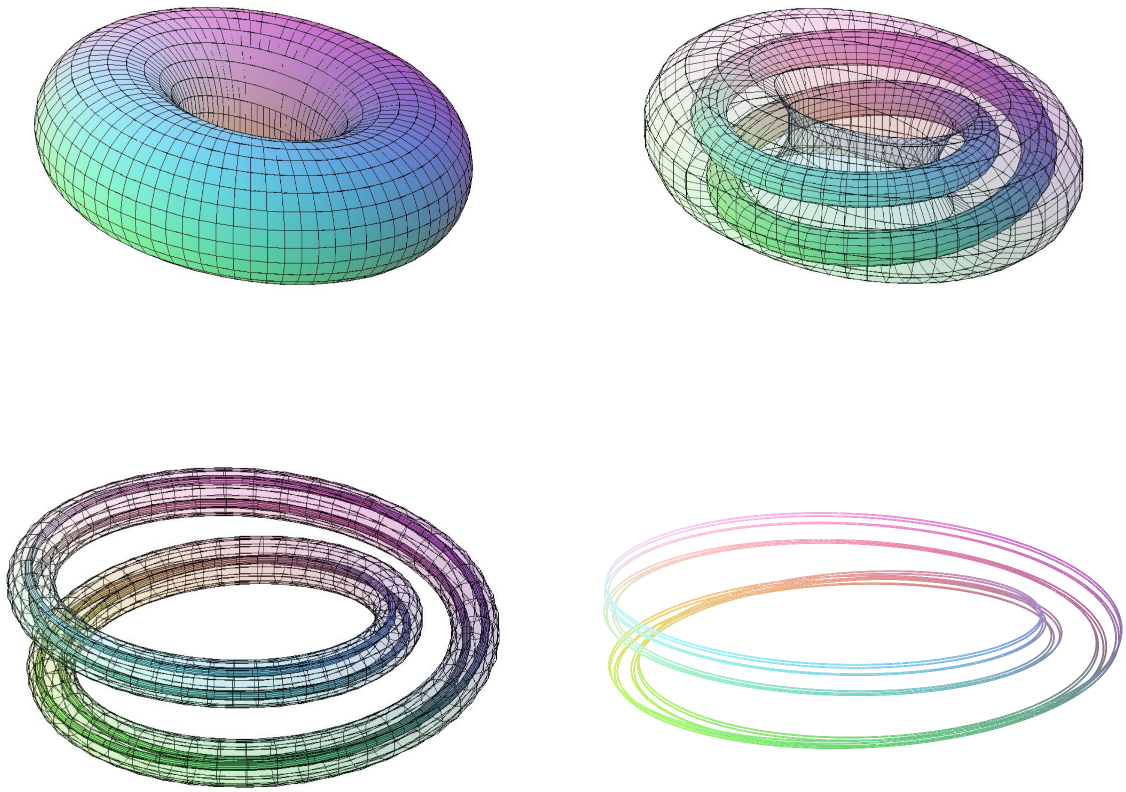

Figure 3.1: Embedding the dyadic solenoid in S^3. Begin with a standard unknotted solid torus T_0 (top left). Then embed a second torus T_1 inside T_0, wrapping around the longitude of T_0 twice (top right). A third torus T_2 is shown wrapping twice inside T_1 (bottom left). The solenoid is the infinite intersection of such nested tori (bottom right).

similar to our braided embeddings.

It is also interesting to note that solenoids arise in the theory of dynamical systems. In the case where the sequence n_i is constant, the solenoid can be a hyperbolic attractor of a dynamical system. These solenoids as attractors were first studied by Smale, and are sometimes called Smale attractors. A discussion of solenoids as hyperbolic attractors can be found in many books on dynamics, see for instance [29]. A recent result of Brown [13] shows that generalized solenoids (classified by Williams [53]) are the only 1-dimensional topologically mixing hyperbolic attractors in 3-manifolds.

3.2 Fundamental Groups

When a solenoid Σ is embedded in S^3, the complement $\Sigma^c = S^3 - \Sigma$ is an open 3-manifold. As these manifolds are the complement of a non-locally connected space, they have a complicated structure "at infinity," and are not the interior of a compact manifold with boundary. We will discuss the fundamental groups of such manifolds, which will depend on the particular embedding chosen for the solenoid. Recall that we are starting with an embedding of the solenoid as a nested intersection of solid tori:

$$T_0 \supset T_1 \supset T_2 \supset \ldots; \qquad \Sigma = \bigcap T_i.$$

This gives us that the solenoid complement is an increasing union of torus complements:

$$(S^3 - T_0) \subset (S^3 - T_1) \subset (S^3 - T_2) \subset \ldots; \qquad \Sigma^c = \bigcup (S^3 - T_i).$$

These torus complements are in fact knot complements, where the knots will generally be satellite knots, assuming there is some knotting in the embedding (see the following sections).

The fundamental group of the solenoid complement is then the direct limit of the fundamental groups of the knot complements. This direct limit is in fact injective, i.e. each group injects into the final direct limit, so that it is in fact a union of knot groups, as given by the following lemmas. Note that our embeddings of solenoids as nested closed braids ensure that the core curve of each torus links the meridional curve of the previous solid torus with linking number $n_i \neq 0$.

Lemma 3.2.1. *Suppose that T_1, T_2 are solid tori in $\mathbb{R}^3$ with $T_2 \subset \text{int}(T_1)$ and such that the core curve J of T_2 links the meridional curve K of ∂T_1 having linking number $lk(J, K) \neq 0$. Then the map $\pi_1(\mathbb{R}^3 - T_1) \to \pi_1(\mathbb{R}^3 - T_2)$ is injective.*

Proof. Suppose to the contrary that there is a loop ℓ in $\mathbb{R}^3 - T_1$ that is not nulhomotopic in $\mathbb{R}^3 - T_1$ but is nulhomotopic in $\mathbb{R}^3 - T_2$. Let $D : \mathbb{B}^2 \to \mathbb{R}^3 - T_2$ be a singular disk in $\mathbb{R}^3 - T_2$ bounded by ℓ.

Put D in general position with respect to ∂T_1. By cut and paste, remove all curves of intersection with ∂T_1 that are nulhomotopic in ∂T_1. Since the core curve J is not nulhomotopic in $\mathbb{R}^3 - T_1$, at least one curve of intersection must remain.

Take such a curve whose preimage is innermost in the domain $\mathbb{B}^2$ of D. This curve is essential in ∂T_1 but trivial either in $\mathbb{R}^3 - \text{int}(T_1)$ or in $T_1 - T_2$. The loop theorem thus supplies a nonsingular disk D' whose boundary is nontrivial in ∂T_1 but whose interior either lies in $\mathbb{R}^3 - T_1$ or in $T_1 - T_2$.

In the latter case, $\partial D'$ must be the meridian of ∂T_1, hence must link the core curve J of T_2, and D' must intersect J, a contradiction. Hence $D' \subset \mathbb{R}^3 - \text{int}(T_1)$, $\partial D'$ must be the longitude of T_1, and T_1 must be unknotted.

But that implies that ℓ is a multiple $m \cdot K$ of the meridional curve K of ∂T_1, hence must have linking number $m \cdot lk(J, K) \neq 0$ with J, hence cannot be nulhomotopic missing T_2, a contradiction. □

Lemma 3.2.2. *Let $\Sigma = \bigcap T_i$ be the intersection of solid tori T_i in S^3, such that for each i, the core curve J of T_{i+1} links the meridional curve K of ∂T_i having linking number $lk(J, K) \neq 0$.*

Then for every i, the map $\iota_* : \pi_1(S^3 - T_i) \to \pi_1(S^3 - \Sigma)$ induced by inclusion is injective, and $\pi_1(S^3 - \Sigma) = \varinjlim_i \pi_1(S^3 - T_i) = \bigcup_i \pi_1(S^3 - T_i)$.

Proof. Let γ be a nulhomotopic loop in $S^3 - \Sigma$, and let H be a nulhomotopy of γ in $S^3 - \Sigma$. As Σ and the images of γ, H are compact, we see that there must be indices i, k such that γ lies in $S^3 - T_i$, and the image of H lies in $S^3 - T_{i+k}$. Since the core curve of each T_{i+k} links with the meridional curve of ∂T_i with linking number $\prod_{j=i+1}^{i+k} n_j \neq 0$, we may use Lemma 3.2.1 to see that γ is in fact nulhomotopic in $S^3 - T_i$. Thus each $\pi_1(S^3 - T_i)$ injects into $\pi_1(S^3 - \Sigma)$, and the lemma is proven. □

Recall that S^3 is the union of two solid tori; we will embed a solenoid into one of these. In order to calculate the fundamental group of the solenoid complement, we will cut the space along the tori $\{T_i\}$, to get pieces $T_i - T_{i+1}$ that are each a solid torus minus a braid, together with one piece that is simply a solid torus (the initial complementary solid torus in S^3). We will calculate the fundamental group of each piece, and then use the Seifert Van Kampen Theorem to get relations between the pieces, as the outer torus of one piece is the inner torus, or braid, in the previous piece. The union of all of these groups and the Van Kampen relations will give a presentation for the fundamental group by Lemma 3.2.2.

The fundamental group $\pi_1(T_i - T_{i+1})$ can be calculated by considering the space $T_i - T_{i+1}$ as a mapping cylinder over an n_i-punctured disk. Thus the group has the form

$$\pi_1(T_i - T_{i+1}) = \left\langle t, x_1, \ldots, x_{n_i} \;\middle|\; t^{-1} x_k t = w_k(x_1, \ldots, x_{n_i}) \right\rangle$$

The x_i's represent free generators of the fundamental group of a punctured disk, and t represents the longitude of T_i. Here w_k is some word in the x_j's, depending on the embedding (braiding) of one solid torus inside the previous. We note that the for each k, if strand k attaches to strand m in the closed braid, then the word w_k is a conjugate of x_m. See Figure 3.2.

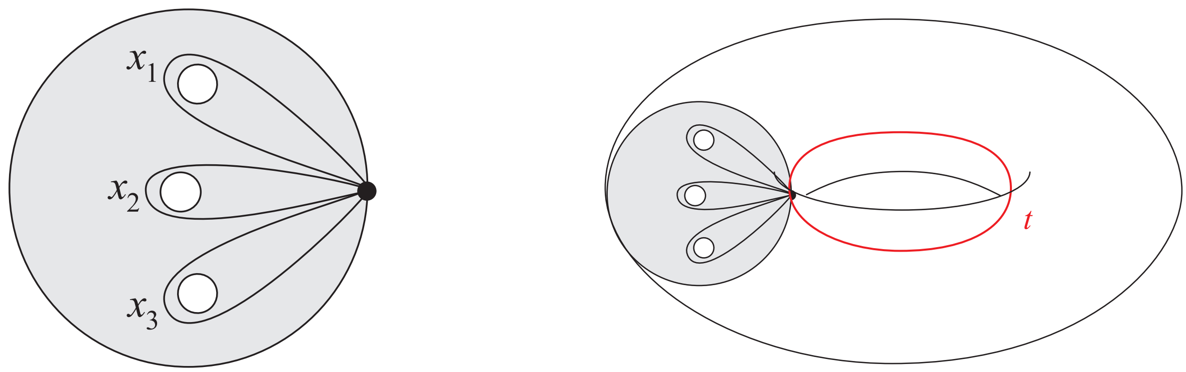

Figure 3.2: Generators for $\pi_1(T_i - T_{i+1})$.

Now apply Seifert Van Kampen to get the relations connecting the various pieces. Using the variables $x_{(i)k}, t_{(i)}$ to denote the generators of the fundamental group of the piece $(T_i - T_{i+1})$, we get relations such as

$$x_{(i)1} = \prod_{k=1}^{n_{i+1}} x_{(i+1)k}, \qquad t_{(i+1)} = t_{(i)}^{n_i} v_{(i)}(x_{(i)1}, \ldots, x_{(i)n_i}).$$

The word $v_{(i)}\big(\{x_{(i)k}\}\big)$ is determined by the embedding, relating the longitudes t_i, t_{i+1} of the tori T_i, T_{i+1}.

Putting all of this together, we get an infinite presentation for $\pi_1(\Sigma^c)$. The generators are $t_{(i)}, x_{(i)k}$ from each level i, with $k = 1, \ldots, n_i$. The relations come from each level and Van Kampen's theorem. Recall that the words $w_{(i)k}, v_{(i)}$ are dependent on the braided embedding of one torus in the previous. Also note that $t_{(0)} = e$, since the longitude of T_0 is trivial in S^3, as its complement is simply a solid torus.

$$\pi_1(\Sigma^c) = \Big\langle t_{(i)}, x_{(i)k} \;\Big|\; t_{(i)}^{-1} x_{(i)k} t_{(i)} = w_{(i)k}(\{x_{(i)k}\}), t_{(0)} = e$$

$$x_{(i)1} = \prod_{k=1}^{n_{i+1}} x_{(i+1)k}, t_{(i+1)} = t_{(i)}^{n_i} v_{(i)}(\{x_{(i)k}\})\Big\rangle$$

Example 3.2.3 (Dyadic Solenoid)**.** In the case of the dyadic solenoid with defining sequence $n_i \equiv 2$, our presentation for π_1 simplifies. There are only two $x_{(i)k}$'s at each level i, and since

$x_{(i)1} = x_{(i+1)1}x_{(i+1)2}$, we do not actually need any of the generators $x_{(i+1)2}$. If we let $z_i = x_{(i)1}$ be the meridian of T_i, and $s_i = t_{(i)}$ the longitude of T_i, we then get a simplified presentation, where R represents relations dependent on the braiding:

$$\pi_1 = \left\langle s_i, z_i \;\middle|\; [s_i, z_i] = e,\ R,\ s_0 = e \right\rangle$$

3.3 Unknotted Solenoids

We define an unknotted embedding of any solenoid in S^3, and discuss the fundamental group of its complement. We will discuss knotted embeddings in the next section.

Definition 3.3.1. An embedding of a solenoid as a nested intersection of solid tori T_i is *unknotted* if each T_i is unknotted (in S^3).

We will show that every solenoid has an unknotted embedding, and that the complement of an unknotted embedding has Abelian fundamental group.

While there are many braids on n strands that give the unknot, the simplest is probably $b(n) = \prod \sigma_i = \sigma_1\sigma_2 \ldots \sigma_{n-1}$, in terms of the standard braid generators σ_i. Note that we could just as easily have reversed the order, or used inverses (σ_i^{-1}). These closed braids just wrap around $(n-1)$ times without any crossings, and then take the first (or last) strand over (or under) all of the other strands.

There is an obvious way to try to embed the next level in this one: thicken each strand to a tube, draw n_{i+1} parallel strands in each tube (crossing all of the strands in one tube over all of the strands in another when the tubes cross), and put the braid for the next level in one tube in some portion where there are no crossings of the tubes. Unfortunately, this obvious way to iterate this process does not produce an unknot. This is due to the fact that there is some inherent twisting in each stage that will show up in the following stages, if not dealt with carefully.

As an example, consider just two levels, where both $n_1, n_2 = 2$, as shown in Figure 3.3. On the first level, we have two strands, and we will use $b(1) = \sigma_1$ as our braid (if we had chosen to use inverses for $b(n)$ the following works out similarly). On the next level, we have four strands. If we start with σ_1, and then just follow the previous stage with the strands parallel to each other, the resulting knot is actually a trefoil, rather than the unknot. However, if you instead start with σ_1^{-1} (or even σ_1^{-3}), you do get the unknot. It is more enlightening to say that if you begin with $\sigma_1^{\epsilon}\sigma_1^{-2}$ you get the unknot, if $\epsilon = \pm 1$. This is true because unwinding the doubled structure from the first level cancels out the σ_1^{-2}, leaving σ_1^{ϵ}, which is the unknot. We leave it to the reader to verify that the given braids yield the specified knots. These braids and the resulting knots are shown in Figure 3.3.

Even though the obvious method does not work, it is possible to keep track of the twists in such a way to get an unknotted embedding of the solenoid. This basically amounts to adding some amount of extra full twists (of all the strands) to correct for the twisting from the previous level. In the case of the braids $b(n)$ which we have chosen above, this ends up being precisely $(n-1)$ full twists. The case of the dyadic solenoid with $n_i = 2$ amounts to adding one full twist, and three levels of this embedding are shown in Figure 3.4. This twisting will also become apparent as we discuss the algebraic structure later, particularly in the example of the dyadic solenoid (see Example 3.3.2).

We note here that this process of constructing unknotted braids can be continued indefinitely, thus providing an embedding of the solenoid. At first there may seem to be a difficulty due to the fact that our embedding requires nested tori, while our braid construction here does not obviously satisfy that condition. However, one can check that each level of our braid construction does nicely embed in the previous. For example, in Figure 3.4, taking a tubular neighborhood of the four strands on the left and the four strands on the right gives a 2-braid with one crossing, just as in the top left single crossing in the diagram. Also, taking a neighborhood of two strands at a time gives a 4-braid that is the same as the top left portion of the diagram (above the full twist on four strands).

We briefly describe one other way to see that this always works, even for more complicated

Figure 3.3: Two levels of the dyadic solenoid embedded as the trefoil (top row), the unknot (middle row), and another version of the unknot (bottom row). The diagrams on the right show the corresponding braids.

braids that may represent the unknot. Start with one level embedded in S^3 as a torus, which has an ambient isotopy h to the standard unknotted torus. Embed the next desired level in the interior of the standard unknotted torus, and then composing with h^{-1} gives the desired embedding of the next level. While this process works for any braid representation for the unknot, our chosen simple braids $b(n)$ admit a formulaic description.

To compute the fundamental group of the solenoid complement, we first compute the fundamental group of a solid torus minus the chosen closed braid $b(n) = \prod \sigma_i$. As in the previous section, we can present the fundamental group of this piece as $G(n) = \langle t, x_1, x_2, \ldots, x_n \mid R \rangle$, where x_i represents the loop going around the i^{th} puncture once, t represents the longitude of the solid torus. The relators R are determined by the braid $b(n)$ as follows: for $i > 1$, we have $t^{-1}x_i t = x_{i-1}$, together with the relation $t^{-1}x_1 t = x_1 x_2 \ldots x_{n-1} x_n x_{n-1}^{-1} x_{n-2}^{-1} \ldots x_1^{-1}$. Note that if we kill t (i.e. set $t = e$), then these relators become $x_i = x_{i-1}$, and thus the quotient $G(n)/ \ll t \gg = \langle x_1 \rangle = \mathbb{Z}$. This should be expected, as this is equivalent to gluing in a solid torus to get S^3 minus the braid $b(n)$, which was the unknot. In the following, it will be convenient to set $x_0 = \prod x_i$, which satisfies the relation $t^{-1}x_0 t = x_0$.

Now we consider the Seifert Van Kampen relations. If we denote the elements of the inner piece with 'primes,' (as in x_i' compared to x_i for the outer piece), then the relations determined by the meridian and longitude of the intersection torus are $x_1 = x_0' = \prod x_i'$, and $t' = t^{n_1} w(x_i)$, where w is some word in the x_i's.

This is where the issue of twisting comes into play. By considering a diagram, one can see that $w = x_0$ does work (it's useful to remember that x_0 commutes with t). While not every other word in x_i will work, we can match this longitude of the intersection torus with any longitude $t'(x_0')^k$ of the inner torus, perhaps wrapping around more (or fewer) times than we think we should. As $x_0' = x_1$, we see that we can append any number of x_1's at the end of w. In order to get the unknot at this second level, we choose $w = x_0 x_1^{-n_1}$. Thus when we set $t = e$, we get that $x_i = x_j$, so that $x_0 = x_1^{n_i}$ and $w = e$. Then $t' = e$, and we similarly have $x_i' = x_j'$. Also, $x_1 = x_0' = \prod x_i' = (x_1')^{n_2}$. Thus the fundamental group is generated

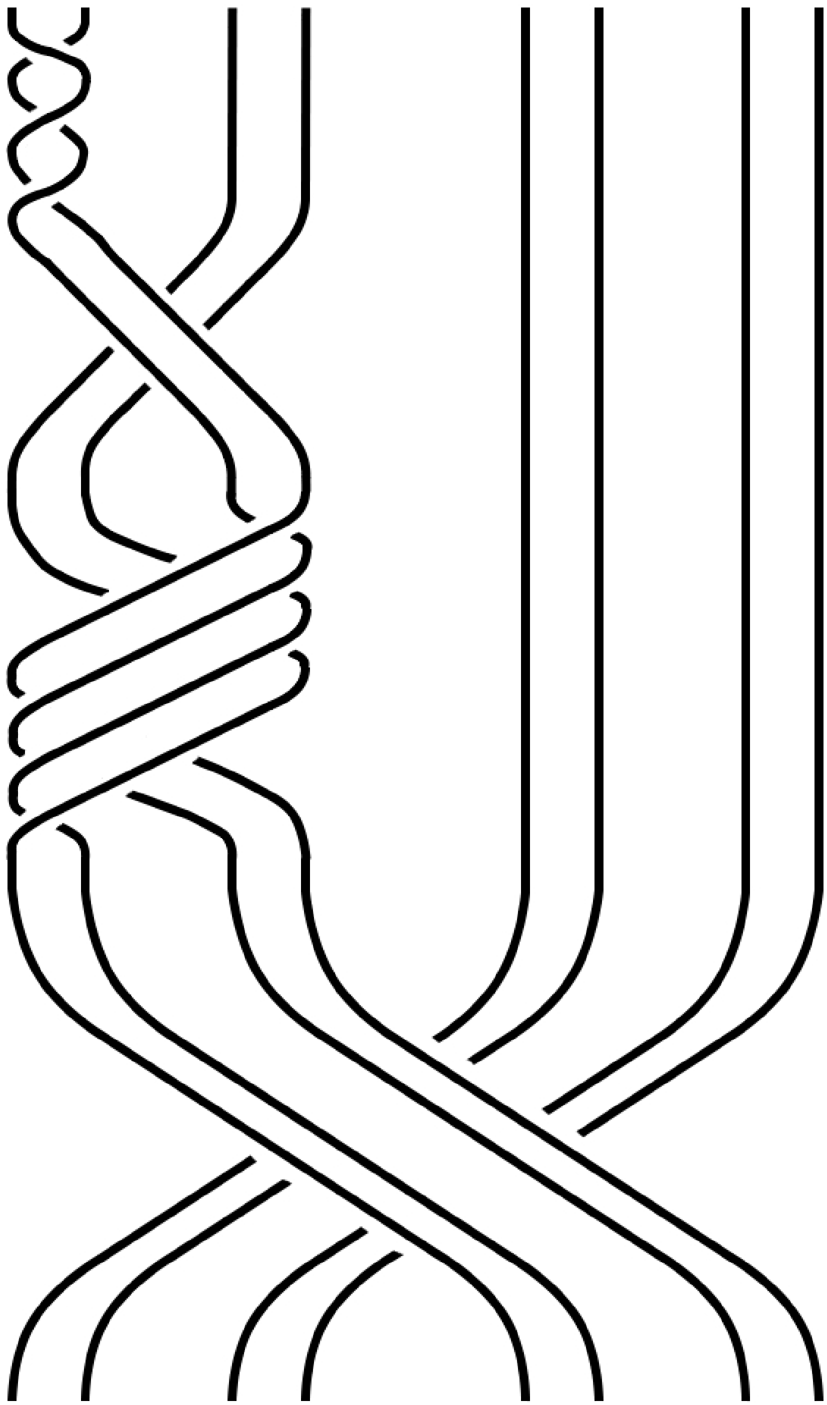

Figure 3.4: Multiple levels of unknotted braids.

by x_1', where the generator x_1 from the previous step satisfies $x_1 = (x_1')^{n_2}$. Therefore the second stage is unknotted, being a knot with fundamental group $\mathbb{Z}$, and the fundamental group from the first stage embeds via the map $\mathbb{Z} \to \mathbb{Z} : 1 \mapsto n_2$.

We can continue this process of inserting unknotted solid tori T_i, and we get that each fundamental group $\pi_1(S^3 - T_i)$ is cyclic. If we call the generators from two consecutive stages a, a', then we have $a = (a')^{n_k}$. Thus we see that the fundamental group of the complement of the unknotted solenoid $\Sigma(n_i)$ is the direct limit $G(n_i) = \varinjlim(\mathbb{Z}, f_i : 1 \mapsto n_i)$. This group can be described more directly as follows, since we are allowed to divide by any of the n_i:

$$G\big(\{n_i\}\big) = \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; q = \prod_{i=1}^{k} n_i \text{ for some } k \right\}.$$

The element 1 in this group represents the meridian loop of the first torus in the construction, and $1/n_1$ represents the meridian loop of the second level, or going around one strand of the braid in the first level. At each stage we can divide by n_i, and in general $1/\left(\Pi^k n_i\right)$ represents a loop going around a strand of the braid on the k^{th} level, or equivalently, a meridian of the $(k+1)^{st}$ level. Since any loop can only come to within a finite (non-zero) distance of the solenoid, this gives us all loops in the fundamental group.

Example 3.3.2 (Dyadic Solenoid)**.** If Σ is the dyadic solenoid with defining sequence $n_i \equiv 2$, then this tells us that the fundamental group is the direct limit $\varinjlim(\mathbb{Z}, 2)$, which is just the dyadic rationals $G = \{p/2^k\}$.

This can also be seen from the presentation as given in Example 3.2.3. For each level being unknotted we get the following presentation, where we have filled in the relations R from the presentation earlier.

$$\pi_1 = \left\langle s_i, z_i \;\middle|\; [s_i, z_i] = e,\ {s_i}^{-1} z_{i+1} s_i = {z_{i+1}}^{-1} z_i,\ s_{i+1} = {s_i}^2 z_i {z_{i+1}}^{-2},\ s_0 = e \right\rangle$$

Notice that on any level, if $s_i = e$, then ${z_{i+1}}^2 = z_i$, and then $s_{i+1} = s_i^2 = e$. Thus this group becomes $\langle z_i \mid z_{i+1}^2 = z_i \rangle = \varinjlim(\mathbb{Z}, 2)$.

Similarly, for an n-adic solenoid, where $n_i \equiv n$, we get the (non-complete) n-adic rationals $\{p/q \mid q = n^k\}$. In general, the group G can be any non-trivial subgroup of $\mathbb{Q}$. We characterize the subgroups of $\mathbb{Q}$ in Lemma A.1; we restate the lemma here for convenience, and give a proof in the appendix. We then describe how to achieve those as the fundamental groups of specific solenoid complements.

Note that for additive subgroups of $\mathbb{Q}$, multiplication by a nonzero constant is an isomorphism, so that we may assume that any non-trivial subgroup contains 1. In the lemma, the number k_i represents the number of times (plus 1) that the prime p_i is allowed to appear in the denominators of the subgroup elements.

Lemma A.1. *Let $\{k_i\}$ be a sequence in $\mathbb{N} \cup \infty$. Define*

$$Q\big(\{k_i\}\big) = \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; q = \prod_{i=1}^{m} p_i^{n_i} \quad \text{for some } n_i < k_i \text{ and some } m \right\}$$

where p_i denotes the i^{th} prime number.

Then $Q\big(\{k_i\}\big)$ is a subgroup of $\mathbb{Q}$ containing 1. Furthermore, every subgroup $G \leq \mathbb{Q}$ containing 1 is equal to $Q\big(\{k_i\}\big)$ for some sequence $\{k_i\}$.

For a solenoid with defining sequence $\{n_i\}$, the fundamental group is $G\big(\{ni\}\big)$ as mentioned above, which can also be described as the subgroup $Q\big(\{k_j\}\big)$ from Lemma A.1 by setting k_j to be one more than the cumulative number of times the j^{th} prime occurs as a factor in the sequence $\{n_i\}$ (where k_j might be infinite). For example, if the sequence $\{n_i\}$ begins with $2, 4, 6, 8, 5, \ldots$, where the tail of the sequence consists of odd numbers, then for $i = 1$, we have $p_i = 2$, and $k_i = 1 + (1 + 2 + 1 + 3 + 0) = 8$, as we add one more than the sum of the powers of 2 that appear in the n_i.

From this, it is now easy to see that given any subgroup $Q(k_j) \leq \mathbb{Q}$, there is a solenoid Σ

and an unknotted embedding into S^3 such that $\pi_1(S^3 - \Sigma) = Q(k_j)$. The defining sequence $\{n_i\}$ can be chosen in various ways, but the homeomorphism type of the solenoid described is uniquely determined. One construction that will always work is as follows:

$$n_i = \prod_{j=1}^{i} p_j^{m_{ij}}, \quad \text{where } m_{ij} = 1 \text{ if } i - j < k_j - 1 \text{ and } 0 \text{ otherwise.}$$

This construction may have some $n_i = 1$, and these may be removed if the tail of the sequence n_i is not identically 1. In the case where the n_i are eventually 1, the subgroup $Q\big(\{k_j\}\big)$ is cyclic ($\cong \mathbb{Z}$), and the required solenoid is the circle S^1 (with $n_i \equiv 1$). The circle is not always considered a solenoid, being degenerate. If not considering S^1 to be a solenoid, then we may reformulate the result to say that any subgroup of $\mathbb{Q}$ that is neither $\{0\}$ nor $\mathbb{Z}$ may be obtained as the fundamental group of a solenoid complement.

As mentioned earlier, any finite segment of $\{n_i\}$ does not change the solenoid $\Sigma(n_i)$. It also does not change the fundamental group of the unknotted complement. If $G\big(\{n_i\}\big)$ is the group for the sequence $\{n_i\}$, and $G\big(\{n_i\}, k\big)$ is the group where we start the sequence at $i = k$, then we have the isomorphism $\varphi : G\big(\{n_i\}\big) \to G\big(\{n_i\}, k\big)$ defined by $\varphi(x) = x \prod_{i=1}^{k-1} n_i$.

Also notice that any reordering of $\{n_i\}$, or replacing a term n_j by a sequence of its prime factorization, will also not change the group (here the isomorphism is the identity map).

In the previous discussion, we considered a particular unknotted embedding, based on a choice of braids $b(n)$ that give the unknot. There are obviously many other choices of braids that give the unknot; for example, the combined braid from the first and second stages described above is an unknot on $n_1 n_2$ strands, which differs from our chosen braid $b(n_1 n_2)$. However, the results stated above still hold. Given any unknotted embedding, we have $\pi_1(S^3 - \Sigma) = \varinjlim \pi_1(S^3 - T_i) = \varinjlim \mathbb{Z}$, where the bonding maps are still $f_i : 1 \mapsto n_i$.

While, for a given solenoid, the fundamental group of the complement of any unknotted embedding is the same, one may ask the following question.

Question 3.3.3. *Are all unknotted embeddings of a given solenoid equivalent?*

Here we might take equivalent to mean that there is an ambient isotopy, or perhaps ambient homeomorphism (possibly orientation preserving) between the two embeddings, or perhaps just requiring that the complements in S^3 be homeomorphic.

As noted above, changing any finite segment of the n_i's does not change the fundamental group of the complements. Additionally, if we use the same braids $b(n_i)$ in each embedding, then in this case the complements will be homeomorphic, as we may 'unwind' the first k levels of unknotted tori, with an ambient isotopy. Similarly, any finite reordering of the n_i or replacement by factorizations or products will also give ambient isotopic embeddings. If there are infinitely many of these changes made, it is no longer clear whether this changes the homeomorphism type of the complement.

It seems likely that infinitely many changes will result in different complements, or at least embeddings that are not ambient isotopic, and that there should be uncountably many inequivalent unknotted embeddings for any solenoid.

Conjecture 3.3.4. *For every solenoid, there are uncountably many inequivalent unknotted embeddings in S^3.*

We summarize the results of this section in the following theorem:

Theorem 3.3.5. *For any solenoid Σ, there exists an embedding $\Sigma \subset S^3$ such that $\pi_1(S^3 - \Sigma)$ is Abelian, and in fact a subgroup of $\mathbb{Q}$.*

Furthermore, for every nontrivial subgroup $G \leq (\mathbb{Q}, +)$, there exists a solenoid Σ and an embedding $\Sigma \subset S^3$ such that $\pi_1(S^3 - \Sigma) \cong G$.

3.4 Knotted Solenoids

In the previous section, we took care to ensure that each torus in the nested intersection construction was unknotted in S^3. First, we used a braid $b(n)$ that represents the unknot, and then we took care how we glued in the next stage, with respect to twisting. Relaxing these conditions, we will consider any braid b on n strands that is transitive on the strands; transitivity gives us a knot instead of a link.

Again, the fundamental group of a solid torus minus this closed braid will have the form $G(b) = \langle t, x_1, \ldots, x_n \mid R \rangle$. The relators in R are of the form $t^{-1}x_i t = w_i$, where the word w_i can be determined directly from the braid. We only mention here that if the braid b sends strand i to strand j, then the corresponding relator has the form $t^{-1}x_i t = g^{-1}x_j g$, where g is some word depending on the braiding. Then due to the transitivity, we see that after Abelianization, the relators give $x_i = x_j$ for all i, j.

To connect two such tori, we need the extra relations $x_1 = x'_0 = \prod x'_i$, and $t' = t^{n_1} w(x_i)$. By careful consideration of a braid diagram, one can determine a suitable word w_b for a given braid. Again, we may allow $w = w_b x_1^k$ for any k (since we are not worried about extra twisting anymore).

After Abelianization, these relations become $x_1 = (x'_1)^{n_2}$, and $t' = t^{n_1} w(x_i)$. At each level, we get a $\mathbb{Z}$ generated by x'_1, and while t' might not equal zero, it can be written as a word in x'_1 as the previous t could be written as a word in x_1. We note here that we can always take the first solid torus T_0 to be standardly embedded, so that the longitude $t_0 = e$. This follows from a theorem of Alexander [1], which states that every knot (or link) can be represented as a closed braid. Then the maps from $\mathbb{Z} \to \mathbb{Z}$ are again multiplication by n_i. Thus the Abelianization of all these groups depends only on the solenoid, not the embedding.

The preceding fact is actually a simple consequence of Alexander duality:

Theorem 3.4.1 (Alexander Duality)**.** *For a compact set $K \subset S^n$, $H_i(S^n - K) \cong \check{H}^{n-i-1}(K)$.*

In our setting, this tells us that the first homology, or the Abelianization of the fundamental group, of the complement of an embedded solenoid is equal to the first Čech cohomology of the solenoid, which is independent of the embedding: $(\pi_1)_{\mathrm{Ab}} = H_1(S^3 - \Sigma) = \check{H}^1(\Sigma)$.

Of course the Čech cohomology of $\Sigma(n_i)$ must then be the group $G(n_i)$ as discussed in the previous section, as in that case the fundamental group is the first homology group, being Abelian. That this group is in fact the Čech cohomology is shown/discussed in [35].

Example 3.4.2 (Dyadic Solenoid)**.** Again, consider the dyadic solenoid with $n_i \equiv 2$. On each level we will use the braid ${\sigma_1}^3$, which gives the trefoil knot. In this case the presentation for the fundamental group becomes:

$$\pi_1 = \left\langle s_i, z_i \;\middle|\; [s_i, z_i] = e,\ {s_i}^{-1} z_{i+1} s_i = {z_i}^{-1} {z_{i+1}}^{-1} {z_i}^2,\ s_{i+1} = {s_i}^2 {z_i}^3 {z_{i+1}}^{-6},\ s_0 = e \right\rangle$$

Note that if we Abelianize, then $z_{i+1}^2 = z_i$, and then $s_{i+1} = s_i^2 = e$ as before. This gives us that $H_1 = (\pi_1)_{\mathrm{Ab}}$ is the dyadic rationals.

However, this fundamental group is non-Abelian. This follows from Lemma 3.2.2 and the fact that the trefoil group is non-Abelian. This can also be seen more directly, as the fundamental group maps onto the infinite alternating group A_∞. To see this, map each generator z_i to the 3-cycle $\big(i\ (i+1)\ (i+2)\big)$. As each s_i is just a word in the z_i's and the previous s_j's, this defines the homomorphism.

While the homology of a solenoid complement only depends on the solenoid, the fundamental groups can be quite different. However, it is still difficult to tell them apart. We have given a way to present these groups, but our presentations are infinite, which makes it difficult to determine when two groups are isomorphic; in fact it is difficult to tell when two finite presentations give isomorphic groups. For instance, if we take a dyadic solenoid with $n_i \equiv 2$, at any level we may either use the unknotted embedding from Example 3.3.2, or the trefoil embedding from Example 3.4.2. The presentation will look similar to those in the examples, using the relations from one or the other at different levels i depending on which embedding was chosen. While it seems likely that these give different fundamental groups, it is hard to prove that for these given infinite presentations, especially as they have isomorphic Abelianizations (see Theorem 3.4.1).

However, despite these difficulties, we can tell some of these embeddings apart via the fundamental group. Lemma 3.2.2 tells us that the fundamental groups of the various stages inject into the fundamental group of the entire complement. A standard result from knot theory states that the fundamental group of the complement of any knot that is not the unknot is non-Abelian. Thus if there is any knotting in our embedding of the solenoid, $\pi_1(S^3 - \Sigma)$ will be non-Abelian, in contrast to the unknotted embeddings which always have Abelian fundamental groups.

There are many knotted embeddings of any solenoid, which seemingly should all be different. As fundamental groups determine knots (up to chirality), it seems that if there is any substantial difference in the knottings, the fundamental groups should differ. Unfortunately, it is hard to show this given our infinite presentations.

We summarize the results of this section in the following theorem and conjecture.

Theorem 3.4.3. *For every solenoid Σ, there are knotted embeddings $\Sigma \subset S^3$, and such embeddings have $\pi_1(S^3 - \Sigma)$ non-Abelian. These embeddings are inequivalent to unknotted embeddings, whose complements have Abelian fundamental groups.*

Conjecture 3.4.4. *If a solenoid is embedded in two 'different' knotted ways, the fundamental groups of the complements are different.*

3.5 Distinguishing Non-Abelian Complements

As discussed in the previous sections, for any solenoid there is an embedding with a non-Abelian fundamental group, which is clearly not equivalent to the Abelian embeddings. As knots are essentially determined by the fundamental group of their complements (up to an issue of chirality), it seems that unknotted embeddings of a solenoid that are knotted in different ways should give different fundamental groups. Unfortunately, the result for knots does not easily carry over to solenoids, as the fundamental groups are now ascending unions of knot groups, and it is not clear whether two ascending unions could be equal in the end, yet differ at every finite stage.

In order to distinguish non-Abelian embeddings of a given solenoid, we consider the geometry of the complements. A standard tool we will use is the JSJ-decomposition, cutting the manifold along incompressible tori. As the JSJ-decomposition only applies to compact manifolds, we will generalize it to apply to a certain class of embeddings of solenoids. The following statement is taken from Hatcher's notes on 3-manifolds [30], under the section on Torus Decomposition.

Theorem 3.5.1 (JSJ-Decomposition). *A compact irreducible orientable 3-manifold has a minimal collection of disjoint incompressible tori such that each component of the complement of the tori is either atoroidal or Seifert fibered. This minimal collection is unique up to isotopy.*

To generalize this result for solenoid embeddings, we need to consider embeddings such that infinitely many of the 'solid torus minus a braid' pieces are hyperbolic. As long as the braid has at least 3 strands, this should generically be the case. If there are only 2 strands, the piece will always be Seifert fibered.

Proposition 3.5.2. *Given $n \geq 3$, there exist (at least) two n-braids $B(n,i)$ in a solid torus T such that the complements $T - B(n,i)$ have distinct hyperbolic structures for $i = 1, 2$.*

Proof. An n-braid in a solid torus is the mapping torus of an n-punctured disk B^2. Thurston [47, 48] proves that such manifolds are hyperbolic precisely when the monodromy is pseudo-Anosov, and states that this is in fact the generic case (see Theorem 0.1 in [48]). □

The proof above using Thurston's results only shows that an n-braid in a solid torus will generically give a hyperbolic 3-manifold with 2 cusps, without constructing specific examples. For a fixed choice of n, we can construct specific examples with different hyperbolic structures quite easily, and in Table 3.1 we present a few specific braids for $n = 3, 4, 5$ in terms of the

n	Braid	Hyperbolic Volume
3	$\sigma_1^{-1}\sigma_2$	4.05
	$\sigma_1^{-3}\sigma_2$	5.97
4	$\sigma_1^{-1}\sigma_2\sigma_3$	4.85
	$\sigma_1^{-1}\sigma_2\sigma_3^{-1}$	7.51
5	$\sigma_1^{-1}\sigma_2\sigma_3\sigma_4$	5.08
	$\sigma_1^{-1}\sigma_2^{-1}\sigma_3\sigma_4$	5.90
	$\sigma_1^{-1}\sigma_2\sigma_3^{-1}\sigma_4$	11.2

Table 3.1: Braids in a solid torus with distinct hyperbolic volumes.

standard braid generators σ_i. In general, it seems that the braids $\prod_{i=1}^{n-1} \sigma_i^{e_i}$, where $e_i = \pm 1$, each give different volumes, unless there is either some obvious symmetry (i.e. $-++$ gives the same as $++-$, $+--$ and $--+$), or if it is Seifert fibered (i.e. $---$ or $+++$). Of course, for $n = 3$ we must add extra twisting, since there are only 2 generators σ_i, which only gives one hyperbolic 3-braid knot with two crossings, up to symmetry. The hyperbolic volumes given in Table 3.1 were calculated using SnapPea [51].

Recall that hyperbolic structures on 3-manifolds are in fact topological invariants, as given by Mostow-Prasad rigidity [40, 42]:

Theorem 3.5.3 (Mostow-Prasad Rigidity)**.** *If a 3-manifold admits a complete hyperbolic structure with finite volume, then that structure is unique up to isometry.*

Using Mostow-Prasad rigidity and Proposition 3.5.2, we are able to prove the existence of inequivalent non-Abelian embeddings for any given solenoid.

Theorem 3.5.4. *For any solenoid, there exist uncountably many inequivalent non-Abelian embeddings, i.e. such that the complements are different manifolds.*

Proof. Choose a defining sequence n_i for the solenoid Σ, with the condition that $n_i \neq 2$. If necessary, we may take the product of consecutive terms n_i to ensure that $n_i \neq 2$.

We will construct different non-Abelian embeddings of Σ. Let T_0 be a knotted solid torus with cross-sectional diameter 1 in S^3. To the complement of T_0, glue in either $T - B(n_1, 1)$

or $T - B(n_1, 2)$, one of the hyperbolic manifolds from Proposition 3.5.2. Continue attaching either $T - B(n_i, 1)$ or $T - B(n_i, 2)$. As we fill in the braids, make sure that the cross-sectional diameter of each braid is less than half the diameter of the previous level. This will embed the solenoid $\Sigma(n_i)$. As we have two choices at each stage, there are uncountably many ways of doing this. It remains to show that these each give different complements.

We will use the JSJ-decomposition. Take any incompressible torus T^* in $S^3 - \Sigma$. This cuts S^3 into a compact piece and a noncompact piece, because Σ is connected. There is a small torus T_k in our construction that lies inside the non-compact piece, as T^* is bounded away from Σ, and we ensured that the tori T_i had cross-sectional diameter less than 2^{-i}. This torus T_k then cuts S^3 into two new pieces, again one compact and one not, with the originally chosen incompressible torus T^* in the compact piece. Now apply the JSJ-decomposition (Theorem 3.5.1) to the compact piece. As the pieces $T^2 - B(n, i)$ in our construction were chosen to be hyperbolic they are atoroidal, and thus the torus T^* must be isotopic to one of our defining tori T_i.

Thus we get a canonical JSJ-decomposition of our solenoid complement, with every incompressible torus in the complement being isotopic to one of the defining tori. In particular, the incompressible tori cut $S^3 - \Sigma$ into pieces, one of which has one cusp (the original knot complement), and all the rest having 2 cusps. These pieces may be ordered by taking the piece with one cusp as the first, and then considering which other pieces share a common boundary. So we have a canonical way of cutting up the solenoid complement into these ordered pieces. If any of the pieces are different at any spot in the sequence, the resulting manifolds are distinct, which proves the theorem. □

Corollary 3.5.5. *Let $\{n_i\}$ be any defining sequence of a solenoid, other than a sequence that is eventually 2 for the dyadic solenoid. Then there are uncountably many inequivalent embeddings of the solenoid using the sequence n_i.*

Proof. Proceed with the construction as in the proof of the theorem, except when $n_i = 2$, fill in with any Seifert fibered 2-braid. In fact, all we need is that infinitely many of the pieces are hyperbolic. Then to get the generalized JSJ-decomposition, when given an incompressible torus T^*, choose the small torus T_k such that T_k represents the inner braid in one of the hyperbolic pieces. Again we may apply the standard JSJ-decomposition to the compact complementary component of T_k. This gives us that T^* is either one of our defining tori T_i, or that T^* lies in one of the Seifert fibered pieces.

Again, we get a canonical JSJ-decomposition, where on each compact piece we take the minimal collection of tori guaranteed by the standard JSJ-decomposition. As we have infinitely many hyperbolic pieces, and since we can choose to fill in with non-isometric pieces, we get uncountably many distinct complements. □

Note that this argument cannot extend to the defining sequence $n_i \equiv 2$, as the homeomorphism type of a solid torus minus any 2-braid is only dependent on the number of components of the closed braid (i.e. either 1 or 2). As we have only been considering knots, we will always have 1 component, giving only one homeomorphism type of a solid torus minus a 2-braid.

Chapter 4. Peano Continua

4.1 Introduction

This chapter is dedicated to studying Peano continua, which are compact, connected, locally path connected metric spaces. We first concentrate on simplifying Peano continua by finding certain reduced forms, up to homotopy equivalence. In the remainder of this section we briefly discuss these results together with some informal definitions, and also state the main theorems. The precise definitions, theorems and proofs are given in the subsequent sections.

The first reduced form we consider deals with finding deformation retractions which are in certain senses minimal and canonical. We *deforest* Peano continua by contracting subcontinua attached at single points, in a maximal way (see Definition 4.2.5).

Theorem 4.3.1. *Every non-contractible Peano continuum has a strong deformation retraction to a deforested continuum.*

Such a *deforested* Peano continuum is a strong deformation retract, and is minimal in the sense that it admits no proper deforestation. However, a deforested Peano continuum may not be a minimal deformation retract, or even admit a minimal deformation retract, as we show in Examples 4.1 and 4.2.

Some types of relatively simple Peano continua, such as simplicial complexes, always admit minimal deformation retracts. If such a minimal deformation retract of a space exists, it is called a *core* of the space (Definition 4.2.6). It is evident that for a given space all cores must be homotopy equivalent, but cores need not be homeomorphic, nor unique as subspaces.

Theorem 4.3.2. *In a deforested Peano continuum, the set of points with simply connected one-dimensional neighborhoods forms a locally finite graph.*

While this graph admits many homotopies, it will be shown in Theorem 4.3.3 that the complement of this graph in the one-dimensional subspace is homotopically rigid, and is exactly the set of points with one-dimensional neighborhoods that are fixed by all self-homotopies of the space.

We proceed to show that there is a homotopy equivalence to another reduced form, which we call *arc-reduced*, where the graph mentioned above is the disjoint union of a countable family of arcs (Definition 4.2.8). If the entire space is one-dimensional, we see that such a space is in fact a compactification of a null sequence of arcs by a particular homotopically fixed subspace, as mentioned in the previous paragraph. We now briefly explain the notation in the following theorem, and note that the corresponding definitions are given in Definition 4.2.7. The *one-dimensional set* $I(X)$ is the set of points in X with one-dimensional neighborhoods, the *bad set* $B(X)$ is the set of points in $I(X)$ where X is not locally simply connected, and the *good set* $G(X)$ is the complement of $B(X)$ in $I(X)$.

Theorem 4.3.4. *Every Peano continuum X is homotopy equivalent to an arc-reduced continuum Y, i.e. where $G(Y) = I(Y) - B(Y)$ is a disjoint union of arcs.*

As an example, consider a compact, zero or one-dimensional metric space B together with an infinite binary tree limiting on a Cantor set. We can form a one-dimensional Peano continuum by mapping the Cantor set at the end of the tree onto the set B and taking the quotient space. If the original space B is locally path connected, then the set of points with simply connected neighborhoods is just the tree, so Theorem 4.3.4 tells us that this space is homotopy equivalent to a null sequence of arcs attached to the set B on a countable dense set. Figure 4.1 shows the case where B is an interval.

While Theorem 4.3.4 tells us that the graph from Theorem 4.3.2 can be made very nice, Theorem 4.3.3 tells us that the complement of this locally finite graph is rigid, being exactly

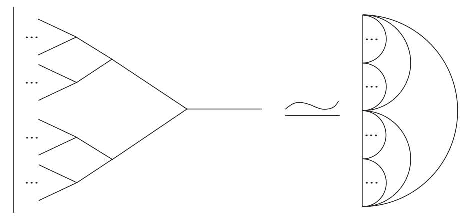

Figure 4.1: Example of Theorem 4.3.4.

the set of points in $I(X)$ that are fixed by all self-homotopies of X.

These reduced forms give stronger results when restricting to one-dimensional Peano continua. In fact, they lead to homotopy invariants that together form a complete invariant of homotopy type. We have proven the following theorem for continua where $B(X)$ is *nice* in various ways, and conjecture the result to hold for all continua regardless of the complexity of $B(X)$. In the theorem, $Q(X)$ and $\operatorname{rank}(X)$ are certain homotopy invariants, defined in Definition 4.6.2 and Definition 4.6.10. Briefly, $Q(X)$ is a subset of $B(X)$, containing certain limit points of arcs in $G(X)$, while $\operatorname{rank}(X)$ is the number of non-separating arcs in $G(X)$, and is related to the rank of free groups that are free factors of $\pi_1(X)$.

Theorem 4.6.12. *The triple* $(B(\cdot), Q(\cdot), \operatorname{rank}(\cdot))$ *is a complete invariant of the homotopy type of one-dimensional Peano continua that are sufficiently nice.*

We now outline the structure of the chapter. In Section 4.2, we give definitions and results which will be used throughout the chapter. In Section 4.3, we discuss the existence of the deforested and arc-reduced forms for all Peano continua. In Section 4.4, we study the special case of one-dimensional Peano continua, where we can prove some stronger results; for instance, all such continua have a unique core. In Section 4.5, we present an application discussing one-dimensional Peano continua with the shape of a graph. Finally, in Section 4.6, we discuss topologically defined homotopy invariants for one-dimensional Peano continua, and show that these often determine the homotopy type of the continuum.

We note that many of the results in this chapter are generalizations of work from [36], which only considered one-dimensional Peano continua. Herein we generalize many of the results to continua of arbitrary dimension, considering subspaces that are one-dimensional, as well as adding new results.

4.2 Preliminaries and Definitions

We begin by recalling a few standard definitions that will be used throughout the rest of the chapter. *Dimension* will mean covering dimension. We will denote the identity map on X by Id_X. A *Peano continuum* is a compact, connected, locally path connected separable metric space. A *dendrite* is a one-dimensional Peano continuum containing no simple closed curves. A *loop* in a space X is a continuous map from S^1 to X.

Theorem 4.2.1 (Hahn-Mazurkiewicz)**.** *A space is a Peano continuum if and only if it is a metric space that is the continuous image of a closed arc.*

Many of our arguments in this paper will use the notion of reduced loops:

Definition 4.2.2. A loop $f : S^1 \to X$ is *reduced* if whenever $f|_I$ is a closed curve for some interval $I \subset S^1$, then $f|_I$ is either essential or constant.

The following lemma about reduced loops will be important in many of our proofs. For a proof of Lemma 4.2.3 and more information on reduced loops, see the work of Cannon and Conner [16].

Lemma 4.2.3. *In a one-dimensional Peano continuum, every loop f is homotopic to a reduced loop $\widetilde{f}$, which is unique up to reparametrization. Furthermore, the image of $\widetilde{f}$ is contained in the image of f. We call $\widetilde{f}$ a* reduced representative *for f.*

Definition 4.2.4. A connected open set D in X is an *attached strongly contractible subset* if

(i) the boundary of D in X consists of one point, and

(ii) the closure $\overline{D}$ has a strong deformation retraction to its boundary point.

Note that the boundary point of an attached strongly contractible subset is a cut-point of the space, where one complementary component of the cut point can be contracted fixing the cut-point.

Definition 4.2.5. A space is *deforested* if it contains no attached strongly contractible subsets.

The name for the previous definition is motivated by the one-dimensional case, where the closure of strongly contractible subsets are dendrites, which are tree-like.

Definition 4.2.6. A *core* of a continuum is a minimal strong deformation retract. That is, a strong deformation retract that admits no proper strong deformation retraction.

Definition 4.2.7. The one-dimensional set $I(X)$ of a space is the set of points with one-dimensional neighborhoods. A point $x \in I(X)$ in a Peano continuum is *bad* if it has no simply connected neighborhood, or equivalently if every neighborhood of x contains a simple closed curve. Notice that by Lemma 4.2.3 every simple closed curve in a one-dimensional Peano continuum is essential. We denote the set of all bad points of X by *B(X)*, and its complement in $I(X)$ is the good set, denoted by $G(X) = I(X) - B(X)$.

These subsets of the space will be useful in the proofs of the main theorems, and have certain nice properties that will be discussed in the remainder of the paper. In fact, the set $B(X)$ has been studied before in slightly different contexts. Cannon and Conner [17] define the set $B(X)$ for connected planar sets, and prove that every self-homotopy must fix $B(X)$ pointwise. This agrees with our definition in the case of planar one-dimensional Peano continua, and we prove a similar theorem in Theorem 4.3.3 and a stronger version for one–dimensional spaces in Theorem 4.4.3. While not explicitly defined, these same notions are also used in Theorem 5.2 of [18]. Conner and Eda also discuss the set $B(X)$ in [22, 25], although there they use the notation X^w for the set $B(X)$, and call these points *wild*. While they define the set as those points where the space is not semi-locally simply connected, by Lemma 4.2.3 we see that for one-dimensional spaces this is the same as points that are not locally simply connected, or points where every neighborhood contains a simple closed curve, which is our definition of $B(X)$.

Definition 4.2.8. We say that X is *arc-reduced* if $G(X)$ is either a disjoint union of a null sequence of (open) arcs or a finite bouquet of circles.

The Hawaiian earring (the one point compactification of a null sequence of arcs) is a simple example of an arc-reduced continuum that is not just a finite bouquet of circles. In fact, Theorem 4.3.4 shows that every one-dimensional arc-reduced continuum (with $B(X) \neq \emptyset$) is a compactification of a null sequence of arcs by the set $B(X)$, which is homotopically fixed. We note that the set $B(X)$ can be any zero or one-dimensional compact metric space.

4.3 Reduced Forms for Peano Continua

We will first prove the existence of the deforested reduced form for Peano continua. Recall that a space is deforested if it has no subsets attached at a point that are strongly contractible to that point. It is important here that we require *strongly* contractible subsets in the definition of deforested, even if we only want to require a weak deformation retraction to the reduced form. To see this, consider taking two copies of a cone over the Hawaiian earring joined at their base points. While both of the cones are contractible, the wedge is a space with uncountable fundamental group [15, 16, 27, 39, 55].

It is also important that each strongly contractible subset is only attached at a single point. In Example 4.2 we present a space with infinitely many disks attached along arcs where it is not possible to contract all of the disks.

Also note that while a contractible continuum has a contraction to a point, which is a deforested continuum, it is unclear whether every contractible Peano continuum has a strong deformation retraction to a point. This issue comes up in the construction of a partial order in the proof.

Theorem 4.3.1. *Every non-contractible Peano continuum has a strong deformation retraction to a deforested continuum.*

Proof. Let X be a non-contractible Peano continuum. Let D be the union of all attached strongly contractible subsets in X. Recall that attached strongly contractible subsets are open, thus D is also open. Let E be the set of points in X that are the boundary of some attached strongly contractible subset.

Now, consider the points in $E - D$, those points that are the boundary of some attached strongly contractible subset, but are not in the interior of any strongly contractible subset. Each such point has at least one complementary component whose closure is an attached strongly contractible subset, and we will call such strongly contractible subsets *maximal*. Maximal strongly contractible subsets must be disjoint: the boundary point of one cannot lie in another by definition of maximal, and if any point were in the intersection of two maximal strongly contractible subsets, then there would be a path in each strongly contractible subset to its boundary point, and some point along that path would give a second boundary point for the other dendrite. Thus the collection of all maximal strongly contractible subsets must form a null sequence by Lemma A.4.1. Since X is locally connected, any point that is a limit of the collection of maximal strongly contractible subsets cannot be contained in the interior of any maximal strongly contractible subset, but could possibly be the boundary point of some strongly contractible subset. Because the strong deformation retractions fix the boundary points of the strongly contractible subsets, then by Lemma A.4.2 we can paste all of the deformation retractions of the maximal strongly contractible subsets together. Thus there is a strong deformation retraction of X that retracts all of the maximal strongly contractible subsets simultaneously.

We will show that every point of D is contained in a maximal strongly contractible subset, so that the deformation retraction above maps to $X-D$. Partially order the set E by defining $x < y$ if x is contained in a strongly contractible subset with boundary point y. Note that if $x_1 < x_2$ and $x_2 < x_1$, then we have a strongly contractible subset D_1 with boundary x_1, and some other strongly contractible subset D_2 with boundary x_2, with $x_1 \in D_2$ and $x_2 \in D_1$. If this is the case, then X contractible: if $r_i(x,t)$ are strong deformation retractions from X to $X - D_i$, then the map r defined by

$$r(x,t) = \begin{cases} r_1(x, 2t) & t \in [0, \frac{1}{2}] \\ r_1(r_2(x, 2t-1), 1) & t \in [\frac{1}{2}, 1] \end{cases}$$

gives a contraction of X to x_1. Note that this contraction may not fix x_1.

Thus if X is not contractible, the relation $x < y$ is antisymmetric and we now show that it is also transitive. Let $x < y$ and $y < z$, so that $x \in D_y$ that strongly contracts to y, and $y \in D_z$ that contracts to z. If $z \in D_y$, then we have the case discussed in the previous paragraph, which can only happen if X is contractible. Thus $z \notin D_y$ and so since y separates x from z we can see that x lies in the same component of $X - z$ as y does, namely D_z, and therefore $x < z$. So the relation $x < y$ is indeed a partial order if X is not contractible.

We will prove that every chain in E has an upper bound, so that by Zorn's lemma there exist maximal points in E. To see that the maximal points in E do not lie in D, note that given strongly contractible subsets D_i with boundary x_i as above, if x_1 lies in D_2, then $D_1 \cup D_2$ is an attached strongly contractible subset with boundary x_2 (since x_2 cannot lie in D_1 if X is not strongly contractible). These facts together will show that every point in D is contained in a maximal strongly contractible subset, as defined above.

Let C be a chain in E, and suppose that C has no maximum element (otherwise that element is an upper bound for C). Since X is compact, there is a point z in the nested intersection of the closure of tails of the chain C; i.e., there is a point z that is a limit of every tail of C. We will show that z is unique, and that z is an upper bound for the chain C.

Suppose there are two such limits z_1, z_2. Then since X is first countable, we can find a countable subchain $\{c_i\}$ in C so that c_{2i+1} converges to z_1 and c_{2i} converges to z_2. For each c_i, some number of the complementary components are attached strongly contractible subsets with boundary c_i. Lemma A.4.1 tells us that they must form a null sequence. Denote the union of this null sequence of attached strongly contractible subsets as D_i; then by Lemma A.4.2 we get a strong deformation retraction h_i from X to $X - D_i$. Note that the sets $(D_{i+1} - D_i)$ have nonempty interior, connected closures, and boundary consisting of at most two points (namely c_i, c_{i+1}). Thus Lemma A.4.1 applies again, and the sets $(D_{i+1} - D_i)$ form a null sequence, which must converge to z_1 since $c_{2i+1} \to z_1$, but they must also converge to z_2, since $c_{2i} \to z_2$. Thus there can only be one such limit point z.

There is one complementary component of z containing the entire chain C. If not, then there is some k such that c_k, c_{k+1} are in different complementary components. Every path from c_k to c_{k+1} then passes through z, and since there is a deformation retraction that takes c_k to c_{k+1}, this deformation also takes z to c_{k+1}. But then z would not be a limit of every tail of C. So there is one complementary component of z containing all of the chain C, call it A. Notice that A is open, with boundary equal to z.

We claim that A is an attached strongly contractible subset, and thus z is an upper bound for C. First take a sequence c_i in C that limits on z. As before, define the set D_i to be the union of all dendrites attached at c_i. For each i there is a strong deformation retraction h_i that contracts D_i. It will be useful to name the retraction $f_i(x) = h_i(x, 1)$, and define $f_0 = \mathrm{Id}_X$.

To deformation retract X to $X - A$, first perform $h_1 = f_0 \circ h_1$ for $t \in [0, \frac{1}{2}]$, followed by $f_1 \circ h_2$ for $t \in [\frac{1}{2}, \frac{2}{3}]$, and $f_{i-1} \circ h_i$ for $t \in [\frac{i-1}{i}, \frac{i}{i+1}]$. Composing with f_i ensures that we don't backtrack too much, in particular that the image of our deformation retraction h misses D_i

for times $t > \frac{i}{i+1}$. More precisely, the deformation retraction is defined by

$$h(x,t) = \begin{cases} f_{i-1} \circ h_i(x, i(i+1)(t - \frac{i-1}{i})) & \text{if } t \in [\frac{i-1}{i}, \frac{i}{i+1}], \text{ for } i \geq 1, \\ z & \text{if } (x,t) \in A \times \{1\}, \\ x & \text{if } (x,t) \in (X - A) \times \{1\}. \end{cases}$$

This function is clearly continuous on $X \times [0,1)$ by the standard pasting lemma. To see that h is continuous when $t = 1$, first recall that the sets $\overline{(D_{i+1} - D_i)}$ form a null sequence, which must converge to z since $c_i \to z$. Note that $h(z,t) = z$ for all t. Also, the image of $\overline{(D_{i+1} - D_i)} \times [0,1]$ is contained in $\overline{(A - D_i)} = \{z\} \cup \bigcup_{k \geq i} \overline{(D_{k+1} - D_k)}$, which also forms a null sequence converging to z because the sets $\overline{(D_{k+1} - D_k)}$ converge to z. Since $X = (X - A) \cup \bigcup_i \overline{(D_{i+1} - D_i)}$, and since h fixes every point $x \in X - A$ for all t, we can apply Lemma A.4.2 (which is an infinite pasting lemma), to see that h is continuous on all of $X \times [0,1]$.

Thus A is a strongly contractible subset attached at z, so by Zorn's lemma maximal elements of E exist, so that every attached strongly contractible subset is contained in a maximal one, and there is a strong deformation retraction from X to $X - D$. It is easy to see that $X - D$ contains no attached strongly contractible subsets, since the preimage of a strongly contractible subset of $X - D$ under the deformation retraction would be an attached strongly contractible subset of X.

Therefore X has a strong deformation retraction to $X - D$, which has no attached strongly contractible subsets. $\square$

We will now discuss a nice property of deforested continua: the set $G(X)$ is in fact a locally finite graph, i.e. a one-dimensional CW-complex such that each 0-cell intersects the closure of only finitely many 1-cells. In the proof we will use Lemma A.4.3, which gives an equivalent definition for a locally finite graph, and also the technical results Lemma A.4.4

and Lemma A.4.5.

Theorem 4.3.2. *In a deforested Peano continuum, the set of points with simply connected one-dimensional neighborhoods forms a locally finite graph. If this graph is nonempty and finite then it is the entire continuum.*

Proof. Assume that $G(X) \neq \emptyset$. By Lemma A.4.3, it suffices to show that each point in $G(X)$ has a deleted neighborhood that is a finite disjoint union of arcs.

Fix $x \in G(X)$, and let W be a simply connected one-dimensional neighborhood of x. Fix a path connected neighborhood U of x such that $\overline{U} \subset W$. Then for any $w \in U$, there is a unique arc $p(w)$ from x to w. For points $w_1, w_2 \in U$, if neither w_1 nor w_2 lies on the arc from x to the other, we will call the last point in $p(w_1) \cap p(w_2)$ a *y-point* (relative to x). So each y-point y is the endpoint of at least three arcs. By Lemma A.4.5, we see that these arcs can be extended to join y to ∂U, and these extended arcs intersect only at y since $\overline{U} \subset W$ is simply connected. For a set $A \subset U$ define $Y(A)$ to be the set of all y-points in U formed by the paths from x to the points of A. In other words, $Y(A)$ consists of the first points of intersection of paths from A to x. Note that if A is finite, then so is $Y(A)$.

We claim that $Y(U)$ is finite. To see this, consider an open set V such that $\overline{U} \subset V \subset \overline{V} \subset W$. Suppose that $Y(U)$ is infinite. Then we may choose a sequence $\{y_n\}$ in $Y(U) - \{x\}$ such that $y_i \notin Y(\{y_1, \ldots, y_{i-1}\})$. Choose the unique arc p from x to y_1 in U. Since y_1 is a y-point in U, there are three arcs emanating from y_1. Choose one of these arcs that is disjoint from p, and call it $\widetilde{p}$. By Lemma A.4.5, $\widetilde{p}$ can be extended to an arc p' which joins y_1 to ∂U to ∂V. Let p_1 be $p\,p'$. Continuing by induction, since y_i is a y-point, but not for any two of the previous y_k's, there are three arcs emanating from y_i, of which at most two intersect the previous p_k's. Thus there is an arc p_i joining x to y_i to ∂U to ∂V, with the segment of p_i from y_i to ∂U to ∂V not intersecting the previous p_k's. Let r_i be the segment of p_i from ∂U to ∂V. Then $\{r_i\}$ is infinite, which contradicts Lemma A.4.4, so $Y(U)$ must be finite.

Then there is a path connected neighborhood $A \subset W$ of x with no y-points (other than possibly x). Each point in A lies on an arc with x as one endpoint, and which extends to ∂A by Lemma A.4.5. Then a path connected neighborhood of each point contained in $A - \{x\}$ is exactly a portion of that arc, since if there were any other point in the neighborhood it would need to be connected by a path to that arc, which would give a y-point, but there are no y-points in $A - \{x\}$. Again by Lemma A.4.4, there can only be finitely many of these arcs from x to ∂A. So $A - \{x\}$ is a disjoint union of finitely many arcs, hence $G(X)$ is a locally finite graph.

If $X - G(X) = \emptyset$, then $X = G(X)$ is a compact graph, which must be finite. If neither $G(X)$ nor its complement is empty, then there exists a path from some point in $G(X)$ to some point in $X - G(X)$. This path gives a ray that must hit infinitely many vertices of the graph $G(X)$, otherwise it could not limit on $X - G(X)$. Thus if $G(X)$ is a finite graph then $X = G(X)$. □

This now allows us to characterize the points $B(X)$ as those points in the one-dimensional subspace $I(X)$ which must be fixed by all self homotopies of the space.

Theorem 4.3.3. *A point in $I(X)$ is contained in $B(X)$ if and only if it is fixed by every self-homotopy of X, that is, a map $f\colon X \to X$ where $f \simeq \mathrm{Id}_X$.*

Proof. Suppose $x \in G(X) = I(X) - B(X)$. We may assume that X has no attached strongly contractible subsets by Theorem 4.3.1, and if x is not in the image of the deformation retraction given there, then that is a self-homotopy that does not fix x. Then by Theorem 4.3.2, we know that x has a neighborhood that is a locally finite graph, and can clearly be moved by a self-homotopy of X.

Let $x \in B(X)$ and suppose by way of contradiction that $f\colon X \to X$ is homotopic to Id_X with $f(x) \neq x$. Let $H : X \times I \to X$ be a homotopy between f and Id_X. Choose a one-dimensional neighborhood U of x and a value $t \in I$ such that $H(U \times [0,t]) \subset I(X)$ and $H(x,t) \neq x$. Take a smaller neighborhood $V \subset U$ so that $V \cap H(V,t) = \emptyset$. Then any simple closed curve ℓ in V is disjoint from its image $H(\ell,t) \subset H(V,t)$. But ℓ and $H(\ell,t)$ are freely homotopic in the one-dimensional subspace $I(X)$, and since a simple closed curve is reduced, $H(\ell,t)$ must contain its reduced representative ℓ by Lemma 4.2.3, which is a contradiction. Thus f fixes x for every $x \in B(X)$. $\square$

This says that $B(X)$ is fixed pointwise by all homotopies of X within X, and is exactly the set of such points in $I(X)$, although there may be other points in $X - I(X)$ that must also be fixed. For example, consider a sphere wedged with a Hawaiian earring. The base point of the Hawaiian earring will be fixed by every self-homotopy of X, but this point is not in $I(X)$, and therefore not in $B(X)$.

We now prove the existence of the arc-reduced form for all Peano continua.

Theorem 4.3.4. *Every Peano continuum X is homotopy equivalent to an arc-reduced continuum Y, i.e. where $G(Y) = I(Y) - B(Y)$ is a disjoint union of arcs.*

Proof. We first prove the theorem in the case where $X - G(X) \neq \emptyset$, and the case when $X = G(X)$ will follow easily. The theorem is obviously true if $G(X) = \emptyset$, so we assume that $G(X) \neq \emptyset$.

First note that by Theorem 4.3.2 we may assume that $G(X) = I(X) - B(X)$ is a locally finite graph. Since X is separable, $G(X)$ must have a countable number of edges and vertices. Recall that instead of allowing a half-open (or open) edge in our graph, we choose a sequence of points on the edge approaching the endpoint(s) (X is compact) and consider those as vertices of valence 2.

To define the continuum Y and the homotopy equivalence between X and Y, we will choose paths in $G(X)$ along which we will retract the vertices of $G(X)$ into $X - G(X)$. Essentially, we want a maximal forest in $G(X)$ that strongly deformation retracts to $X - G(X)$.

In order to do this, define a sequence of covers $\mathcal{U}_i$ of $X - G(X)$ by path connected open sets in X of diameter ϵ_i, where $\mathcal{U}_0 = \{X\}$, $\epsilon_0 = \text{diam}(X)$, and $\epsilon_{i+1} < \epsilon_i/2$. Since $X - G(X)$ is compact, we can also require that $\epsilon_{i+1} < \lambda_i$, a Lebesgue number for the cover $\mathcal{U}_i$, so that each $U \in \mathcal{U}_{i+1}$ is contained in some $U' \in \mathcal{U}_i$. For convenience, we define A_i to be the union of all $U \in \mathcal{U}_i$. By our choice of ϵ_i, we have the following inclusion: $A_i \supset A_{i+1} \supset X - G(X)$. Note that there are only finitely many vertices in each $A_i - A_{i+1}$, since each A_{i+1} contains a λ_{i+1}-neighborhood of $X - G(X)$, and so an infinite set of vertices in $A_i - A_{i+1}$ would have a limit point in $G(X)$, which is a locally finite graph.

We now begin to choose the paths by which we will retract the vertices of $G(X)$ to $X - G(X)$. Consider the vertices in $A_i - A_{i+1}$. Each such vertex x is contained in some $U \in \mathcal{U}_i$, and there is an arc $a(x)$ from x to $X - G(X)$ contained in U. Since our graph contains no open edges, this arc $a(x)$ must hit vertices as it approaches $X - G(X)$. Let $q(x)$ be the initial segment of $a(x)$ until it first hits a vertex in A_{i+1}. Denote the end vertex of $q(x)$ by $v(x)$. Continue choosing such paths with the added condition that for vertices $x, x' \in A_i$, the paths $q(x)$ and $q(x')$ are either disjoint, or they coincide after their first intersection. Note that for vertices $x \in A_i$, we have $d(x, y) \leq \epsilon_i$ for all $y \in q(x)$, since $q(x) \subset a(x) \subset U \in \mathcal{U}_i$. Iterate this process for the remaining vertices in the sets A_j, $j > i$.

After all the paths $q(x)$ have been chosen, set $r(x)$ to be the ray defined by the concatenation of the paths $q(x), q(v(x)), q(v^2(x)), \ldots$ (i.e. $r(x)$ follows the path $q(x)$ from x, then the path from the endpoint of that path, and so on). The sequence $\{v^k(x)\}$ is Cauchy since $\epsilon_{k+1} \leq \epsilon_k/2$, and thus limits on a unique point $b(x) \in X - G(X)$. Thus we can define the path $p(x)$ to equal $r(x)$ up to time 1, and then map to the limit point $b(x)$. It will be important to note that if $x \in U \in \mathcal{U}_i$, then the path $p(x)$ has diameter at most $\sum_{k=i}^{\infty} \epsilon_k \leq \sum_{k=0}^{\infty} \epsilon_i 2^{-k} = 2\epsilon_i$, and thus the set of paths $\{p(x)\}$ forms a null sequence, since there are only finitely many vertices in each of the levels $(A_i - A_{i+1})$.

Denote the union of all the paths $p(x)$ as $F = \bigcup_x p(x)$. Note that F is 'mostly' contained in $G(X)$, except for a countable number of endpoints $\bigcup_x b(x) \subset X - G(X)$. This set F is our desired maximal forest. To see that F deformation retracts to $X - G(X)$, first note that each path $p(x)$ deforms to its endpoint $b(x)$, and does so in its image. Then since any limit of the sets $p(x)$ cannot be contained in the locally finite graph $G(X)$, all such limit points must be in $X - G(X)$ and thus fixed by all the deformations of the paths $p(x)$. These conditions are sufficient to apply Lemma A.4.2, which states that we can paste all these maps together to get a continuous deformation of F.

We now define the space Y and the maps $f\colon X \to Y$ and $g\colon Y \to X$, and show that they are homotopy inverses. The space Y is a quotient space of X, where each path $p(x)$ constructed above is identified with its endpoint $b(x)$ in $X - G(X)$, and f is the corresponding quotient map. This is well defined since if any of the paths intersect, they coincide the rest of the way to $X - G(X)$. Thus Y is a Peano continuum with $G(Y)$ a disjoint union of arcs, corresponding to those edges in $G(X)$ that are not part of any ray.

To define the map g, first label all of the edges in $G(X)$. Then subdivide each arc a in $G(Y)$ into a bi-infinite sequence of subarcs. Label the middle third of a with the label for $b = f^{-1}(a)$, the corresponding arc in X. Then for an end third of a, let v_i be the corresponding endpoint of the arc b. Label the infinite sequence of subarcs with the labels of the edges in the path $p(v_i)$, noting orientation (i.e. which direction is going to toward $X - G(X)$). The map g is also a quotient, identifying all closed subarcs in $G(Y)$ that have the same oriented label.

The composition $f \circ g$ simply takes the arcs of $G(Y)$ and slides the end thirds to their endpoints in $Y - G(Y)$, and stretches the middle third over the whole arc. This is clearly homotopic to the identity map on each arc, and is the identity on $Y - G(Y)$. Since the arcs in $G(Y)$ form a null sequence, and any limit point of the arcs is in $Y - G(Y)$ which is fixed by each homotopy, we can paste these homotopies together by Lemma A.4.2 to see that $f \circ g$ is homotopic to Id_Y.

We now show that $g \circ f$ is homotopic to Id_X. Let $h\colon F \times I \to F$ be a strong deformation retraction of F onto $\bigcup_x b(x)$, as described above.

For each arc a in $G(X)-F$ with endpoints v_j and v_k, define $C(a)$ to be the concatenation of paths $\overline{p_j}\ a\ p_k$. The homotopy $H\colon X \times I \to X$ fixes $X - G(X)$ for all times, retracts F by h, and stretches each arc a in $G(X)-F$ over the arc $C(a)$. Explicitly, for $x \in C(a)$ the homotopy looks like

$$H(x,t) = \begin{cases} h(x,t) & \text{if } x \in p_j, \\ h(a(0), t - s(2t+1)) & \text{if } x = a(s) \text{ for } s \in [0, t/(2t+1)], \\ a(s(2t+1)-t) & \text{if } x = a(s) \text{ for } s \in [t/(2t+1), 1-t/(2t+1)], \\ h(a(1), t-(1-s)(2t+1)) & \text{if } x = a(s) \text{ for } s \in [1-t/(2t+1), 1], \\ h(x,t) & \text{if } x \in p_k, \end{cases}$$

Again we use Lemma A.4.2 to paste the homotopies on each $C(a)$ together to get a homotopy on all of X. This follows since the collection $\{C(a)\}$ forms a null sequence, with each $C(a)$ mapping into itself continuously for all time, and as before, any limit point x_0 of the $C(a)$'s will be in $X - G(X)$, which is fixed for all time. Thus $g \circ f$ is homotopic to Id_X.

Thus $f\colon X \to Y$ is a homotopy equivalence of Peano continua X and Y, where $G(Y)$ is a disjoint union of arcs and f maps $X - G(X)$ homeomorphically onto $Y - G(Y)$.

In the case where $X = G(X) = I(X) - B(X)$, by Corollary 4.4.2 and Theorem 4.3.2 X is homotopy equivalent to a locally finite graph. Pick any vertex of X to play the role of $X - G(X)$ above. The rest of the proof follows as above. The bouquet must be finite since X is compact, and has no bad points ($B(X) = \emptyset$) by hypothesis. □

The following lemma gives an application, discussing the homotopy groups of such arc-reduced spaces. We note that a version of this lemma for $n = 1$ is given and used in [20], but the proof there does not extend to higher dimensions.

Lemma 4.3.5. *Let X be a metric space. Let $\Xi \subset X$ be a disjoint union of arcs, that is, open sets in X, each of which is homeomorphic to an open arc. Let $Y = X - \Xi$. For $n \geq 1$, if $f : B^{n+1} \to X$ with $f(S^n) \subset Y$, then there is a map $f' : B^{n+1} \to Y$ with $f'|_{S^n} = f|_{S^n}$.*

Alternatively, we could say that for $n \geq 1$, the map $i_* : \pi_n(Y) \to \pi_n(X)$, induced by the inclusion $i : Y \to X$, is injective.

Proof. Let $X_0 = f(B^{n+1})$. Then X_0 is a Peano continuum contained in X, since it is the image of a Peano continuum B^{n+1}. Note that since X_0 is a Peano continuum, there can only be countably many of the arcs in Ξ that intersect X_0, and that these must form a null sequence. For every arc c_i in Ξ, let a_{2i-1}, a_{2i} be two points in the interior of c_i.

We will define a sequence of maps $f_i : B^{n+1} \to X$ such that f_i agrees with f on the boundary, S^n, and that the image of f_{2i} misses the open interval $(a_{2i-1}, a_{2i}) \subset c_i$. Then we will show that the limit of these maps is continuous, and that since it misses some point in each arc in Ξ, we can retract the image off of all of the arcs in Ξ.

To define f_i, start with f_{i-1}, where $f_0 = f$. Let $A_i = f_{i-1}^{-1}(a_i)$. Since $A_i \cap S^n = \emptyset$, there is only one component of $B^{n+1} - A_i$ that intersects (and contains) S^n. Define the map f_i to agree with f_{i-1} on this component containing S^n, and define $f_i = a_i$ elsewhere. Thus $f_i^{-1}(a_i)$ does not separate. Note that if A_i does not separate B^{n+1}, then this construction gives $f_i = f_{i-1}$.

Additionally, it will be useful to note that $f_j^{-1}(a_i)$ does not separate, for $j \geq i$, since in subsequent steps, all we might do is delete components of $f_i^{-1}(a_i)$ to get $f_j^{-1}(a_i)$.[*]

It is fairly easy to see that each f_i is continuous, since all we have done is take a continuous function and redefine it to be a constant on certain set whose boundary originally mapped to that particular constant. In particular, using the $\epsilon - \delta$ definition of continuity, we see that for any ϵ, any value of δ that shows that f_{i-1} is continuous will also work to show that f_i is continuous.

We also claim that the image of f_{2i} does not hit the interval $(a_{2i-1}, a_{2i}) \subset c_i$. Suppose that it did. Let $\widehat{A}_{2i-1}, \widehat{A}_{2i}$ be the preimages of a_{2i-1}, a_{2i} under the map f_{2i}. Note that $\widehat{A}_* \subset A_*$ (recall that A_* was the preimage under the map f_{*-1}), but might not be equal if we redefined the function on part of that set. Since the set $\{a_{2i-1}, a_{2i}\}$ separates the image of f_{2i}, then $\widehat{A}_{2i-1} \cup \widehat{A}_{2i}$ separates B^n. We now note a certain property of Euclidean space, that is related to unicoherence: if F_k is a countable collection of disjoint closed sets whose union separates, then one of the sets F_k separates. We will actually only use this property for the case of the two closed sets $\widehat{A}_{2i-1}$ and $\widehat{A}_{2i}$, to see that at least one of these sets must separate. But we redefined the maps so that these sets would not separate (see [*] above).

We now define the limit map f_∞ to be the pointwise limit of the maps f_i. We first show that this is well defined. Let x be a point in B^{n+1}, and consider the sequence $\{f_i(x)\}$. Note that by the way we defined f_i, the only repetitions in this sequence must be consecutive, and that the only interesting case is when the sequence is not eventually constant. Let p be a path in B^{n+1} from x to S^n. We claim that there is an increasing sequence of points y_i along p (actually non-decreasing) that satisfy $f(y_i) = f_i(x)$. Thus the sequence $\{y_i\}$ converges to some point y, and by continuity of f, we see that $f_i(x)$ converges to $f(y)$.

To show that such a sequence of y_i's exists, let $y_0 = x$. Assume that y_{i-1} has been defined. If $f_i(x) = f_{i-1}(x)$, then let $y_i = y_{i-1}$. Otherwise, we may assume that $f_i(x) = a_i$ and that $A_i = f_{i-1}^{-1}(a_i)$ separates x from S^n. Define y_i to be the first point along the tail of the path p, from y_{i-1} to S^n, with the property that $f_{i-1}(y_i) = f_i(x) = a_i$.

The fact that f_∞ is continuous will follow easily from the fact that we can use the same δ for all the f_i's. Given ϵ, choose a δ that works for $\epsilon/2$ (for all f_i). Then f_∞ will map the δ-ball into a closed $\epsilon/2$-ball, which is contained in an open ϵ-ball. Thus f_∞ is continuous.

It is now easy to modify f_∞ into the desired map f'. Since the image of f_∞ does not contain any arc in Ξ, there is a deformation retraction h that retracts the image of f_∞ of off all of the arcs in Ξ. That we can do all of these retractions simultaneously follows from

Lemma A.4.2. Then the composition $f' = h \circ f_\infty$ is the desired map.

$\square$

4.4 Results for One-Dimensional Peano Continua

In the previous section, we made no assumption about the dimension of a Peano continuum in our theorems, although in some theorems we were able to prove results about the one-dimensional portion of the continuum. In this section, we restrict to one-dimensional Peano continua, and prove stronger results that hold in this special case. Note that in a one-dimensional Peano continuum, strongly contractible subsets are exactly dendrites, that is, simply connected one-dimensional Peano continua. Recall that a dendrite is strongly contractible to any of its points.

We will prove the existence of minimal deformation retracts (core continua) for all one-dimensional Peano continua. It is not apparent that every continuum has a minimal deformation retract, and in fact this is not true for all continua, not even for planar Peano continua.

Example 4.1. We give an example of a planar Peano continuum X that has no minimal deformation retract, but is deforested. To construct X, start by embedding a Warsaw circle W into the inaccessible points of a Sierpinski curve S in the plane. This can be done by embedding the Warsaw circle W into the interior of a closed disc, and then removing the interiors of a null sequence of closed discs that are disjoint from W. By Whyburn's characterization theorem [52], if these smaller closed discs are dense in the original disc, then we have embedded W into the Sierpinski curve, and in fact into the inaccessible points.

Next, fill in the interior of the embedded Warsaw circle in the plane. Finally, remove the interior of a small closed disc from the interior of the filled-in Warsaw circle. The resulting space is X. See Figure 4.2.

The space X is a planar Peano continuum with no minimal deformation retract. To see

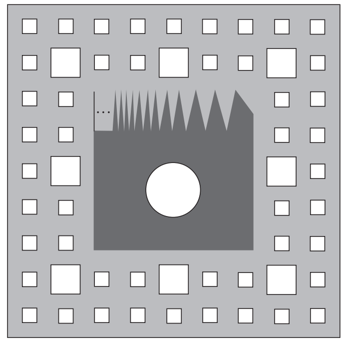

Figure 4.2: A planar Peano continuum with no minimal deformation retract.

this, it is first important to note that the Warsaw circle is contained in $B(X)$, since W is contained in the inaccessible points of S, and therefore no point of W will have a simply connected neighborhood (not even after filling in the interior of W). As such, it must be fixed by every homotopy of X within X (as proved in [17]), in particular, by any deformation retraction of X.

It is possible to deformation retract the interior of the punctured Warsaw disc into any neighborhood of the Warsaw circle, so the only possible minimal deformation retract would be the set $B(X)$. However, the punctured Warsaw disk does have a deformation retraction to the boundary circle of the puncture, so it can not deform to the simply connected Warsaw circle.

Thus it is not possible to deformation retract the space X to the set $B(X)$ which is the Warsaw circle and the exterior portion of the Sierpinski curve containing it. Therefore X has no minimal deformation retract. Clearly X is deforested, having no cut-points to allow attached strongly contractible subsets.

Example 4.2. We define a specific deformation retract Y of the space X from Example 4.1. Expand the central puncture so that the upper boundary of the puncture touches each of the valleys of the sine curve in the Warsaw circle. Essentially, Y is the union of a portion of the Sierpinski curve together with countably many disks D_i, each of which is attached along an arc of Y in the sine curve.

Since Y is a deformation retract of X, and X has no core, neither does Y. Then it is not possible to deformation retract all of the disks D_i. Thus Y demonstrates why the existence of deforested continua cannot be generalized to a similar reduced form where strongly deformation retractible subsets are attached along arcs (or other sets) instead of only being attached at single points.

While the examples above show that not every Peano continuum has a minimal deformation retract, we will prove that every one-dimensional Peano continuum does have a unique minimal deformation retract, or core continuum. We first prove that core continua are precisely those with no attached dendrites and state a few other convenient equivalent characterizations.

Theorem 4.4.1. *If X is a non-contractible one-dimensional Peano continuum, then the following conditions are equivalent:*

(i) X is a core continuum.

(ii) X admits no proper strong deformation retraction.

(iii) X has no attached dendrites.

(iv) $\forall\, x \in X$ and $\forall$ path component p of $X - \{x\}$, $p \cup \{x\}$ is not simply connected.

(v) $\forall\, x \in X$, $\exists$ an essential loop ℓ such that ℓ cannot be homotoped off x.

(vi) Every point of X lies on an essential reduced loop.

Proof. We prove the equivalences as follows: (1)⇔(2), (3)⇔(4), (5)⇒(6)⇒(2)⇒(3)⇒(5).

(1)⇔(2) By definition.

(3)⇔(4) A path component p of $X - \{x\}$ with $p \cup \{x\}$ simply connected is an attached dendrite.

(5)⇒(6) Follows from the existence of reduced loops (Lemma 4.2.3).

(6)⇒(2) If X admits a proper strong deformation retraction $r \colon X \to Y$, then any loop through a point in $X - Y$ can be homotoped into Y (by r), and thus cannot be an essential reduced loop.

(2)⇒(3) An attached dendrite can be contracted, giving a proper strong deformation retract.

(3)⇒(5) We consider a few cases.

Case 1: No neighborhood of x is simply connected. Then there is a sequence of essential loops converging to x. Concatenating these loops together with small paths connecting them to x gives an essential curve that passes through x, and cannot be homotoped off x.

Case 2: x separates X. Then by (4) each component (together with x) must contain an essential loop. Concatenating two of these loops, together with paths to the point x, gives an essential loop that cannot be homotoped off of x.

Case 3: x does not separate X, and has a simply connected neighborhood U that x separates. Since $X - \{x\}$ is locally path connected and connected, it is path connected. Thus any two points $y_1, y_2 \in U$ separated by x are connected by a path in $X - \{x\}$ (that does not stay in U). Since there is a path from x to y_i in U, we get an essential loop passing through x, that cannot be homotoped off x.

Case 4: x does not separate any simply connected neighborhood U. Since X has no attached dendrites, x is in a locally finite graph by Theorem 4.3.2. Then the above conditions require that x must be a pendant vertex (i.e. a vertex with valence 1), but this would be an attached dendrite, which is a contradiction.

□

We can now easily show the existence of core continua for all one-dimensional Peano continua. Note that for any space that has a core, all cores must be homotopy equivalent, but they need not be a uniquely defined subspace or even homeomorphic. For example, consider a twice punctured disk, which deformation retracts to both a theta and a figure

8. However, in the case of one-dimensional Peano continua we do get the result that cores always exist as a unique subspace as long as the space is not contractible.

Corollary 4.4.2. *Every one-dimensional Peano continuum has a minimal strong deformation retract, or* core. *If the continuum is not contractible, then the core is a unique subspace.*

Proof. That core continua exist for all one-dimensional Peano continua is a simple combination of Theorem 4.3.1 and Theorem 4.4.1 (3)$\Rightarrow$(1). If the continuum is contractible, the core will not be unique, as it may contract to different points. On the other hand, if the continuum is not contractible, then by Theorem 4.4.1 (6) every point must lie on an essential reduced loop, and all such points must be contained in every deformation retract (cf. Lemma 4.2.3). Thus the core of a one-dimensional Peano continuum may also be defined as the union of all essential reduced loops in the space, which is a uniquely defined subspace. □

When restricting to one-dimensional continua, we get the following result, which is related to Theorem 4.3.3, but stronger.

Theorem 4.4.3. *In a one-dimensional Peano continuum X, a point is bad if and only if it is fixed by every self-homotopy of X, that is, a map $f\colon X \to X$ where $f \simeq \mathrm{Id}_X$. Moreover, if $h\colon X \to Y$ is a homotopy equivalence, then $h|_{B(X)}$ is a homeomorphism onto $B(Y)$.*

Proof. Since X is one-dimensional, $X = I(X)$, so by Theorem 4.3.3 we see that $B(X)$ is precisely those points in X that are fixed by every self-homotopy of X.

Now let $h\colon X \to Y$ and $g\colon Y \to X$ be homotopy inverses. For $b \in B(X)$, if $h(b) \notin B(Y)$, then there is a simply connected neighborhood of $h(b)$, and thus h_* cannot be injective since h must map small loops to small loops. So h maps $B(X)$ into $B(Y)$, and similarly g maps $B(Y)$ into $B(X)$.

Then $g \circ h \cong \mathrm{Id}_X$, so by Theorem 4.3.3 $g \circ h|_{B(X)} = \mathrm{Id}_{B(X)}$, and thus $h|_{B(X)}$ is injective. Also, $h \circ g \cong \mathrm{Id}_Y$, so $h \circ g|_{B(Y)} = \mathrm{Id}_{B(Y)}$, which implies that $h|_{B(X)}$ is surjective (onto $B(Y)$) since g maps $B(Y)$ into $B(X)$. Since both h and g are continuous, it follows that h maps $B(X)$ homeomorphically onto $B(Y)$. □

While the set $B(X)$ is homotopically rigid, being fixed pointwise by every self-homotopy of X, its complement $G(X)$ can be homotoped to be a disjoint union of arcs. Note also that the result that homotopy equivalences restrict to homeomorphisms on $B(X)$ does not generalize to higher dimensional continua. To see this, consider a Hawaiian earring where each arc is thickened everywhere except at the base point, i.e. each arc becomes a pinched annulus. Another way of viewing this space is to fill in alternating complementary components of the Hawaiian earring in the plane. This new space X is clearly homotopy equivalent to the Hawaiian earring, but $I(X) = \emptyset$.

As an application of Theorem 4.3.4, we will discuss the following example for one-dimensional Peano continua X:

If $B(X)$ is finite (and nonempty), then X is homotopy equivalent to a finite number of Hawaiian earrings connected in either a line or a circle.

If $B(X)$ is non-empty and finite, then X is homotopy equivalent to an arc-reduced continuum Y with the finite set $B(X) = B(Y)$, and the remainder of Y is a null sequence of arcs. Each point of $B(Y)$ must have a Hawaiian earring attached to it, as these points are in the bad set. Thus we have finitely many Hawaiian earrings connected by arcs (with only vertices of valence 2). Note that there can only be finitely many arcs connecting distinct points of $B(Y)$. So we have some finite connected graph, with a Hawaiian earring attached at each vertex. By a homotopy equivalence, we may assume that the graph is a circle; any extra arcs can be absorbed by one of the Hawaiian earrings, and if we need an extra arc (i.e. to make the circle from a line) we can take it from a Hawaiian earring.

Thus X is homotopy equivalent to an n-fold cover of the Hawaiian earring, where $n = |B(X)|$.

We can then use this form to see that any homomorphism from the fundamental group of the Hawaiian earring, $\pi_1(\mathbb{H})$, to the fundamental group of an n-fold cover of the Hawaiian earring (as in the arc-reduced form above) is conjugate to a homomorphism induced by a continuous map. This is an extension of the result by Summers [46] which proved the case where $n = 2$, and is related to the result of Eda [25] which states that every homomorphism from $\pi_1(\mathbb{H})$ to itself is conjugate to one induced by a continuous map. It is convenient to note that if we replace the Hawaiian earring by a homotopy equivalent space obtained by joining one arc to the basepoint of the Hawaiian earring, we get that all homomorphisms are in fact continuous, removing the possibility of being conjugate to a continuous map. We can extend this idea to prove the following theorem.

Theorem 4.4.4. *If $m < n$, then there is no surjection $\pi_1(m\mathbb{H}) \twoheadrightarrow \pi_1(n\mathbb{H})$.*

Proof. First note that $n\mathbb{H}$ is a covering space of the Hawaiian earring with covering map p. Thus if we have a homomorphism $\phi : \pi_1(\mathbb{H}) \to \pi_1(n\mathbb{H})$, then $p_* \circ \phi$ is a homomorphism from the Hawaiian earring to itself, and is therefore continuous by Eda's result [25].

Now suppose $\phi : \pi_1(m\mathbb{H}) \to \pi_1(n\mathbb{H})$ is a homomorphism. Writing $\pi_1(m\mathbb{H})$ as a free product of $\pi_1(\mathbb{H})$, we can see that each of the m factors are mapped continuously to $\pi_1(n\mathbb{H})$. As $m < n$, there is at least one basepoint b in $n\mathbb{H}$ that is not in the image of the basepoints of $m\mathbb{H}$. Let r be a retraction of $n\mathbb{H}$ to a Hawaiian earring based at b that does not contain the images of the basepoints of $m\mathbb{H}$. Now consider the composition $r_* \circ \phi$. As ϕ is induced by a continuous map, and each of the basepoints of $m\mathbb{H}$ do not map to b, the image of $r_* \circ \phi$ can only be countable. If ϕ were surjective then this image would be uncountable, since $\pi_1(\mathbb{H})$ is uncountable. Therefore ϕ cannot be a surjection. □

4.5 Shapes of One-Dimensional Continua

We would like to thank Craig Guilbault for telling us about the following problem: Does a one-dimensional Peano continuum with the shape of a circle contain a circle? We prove a somewhat stronger result that answers this question in the affirmative, and is used in Maggie May's dissertation [34].

We recall the definition of shape for compact metric spaces: Two compact metric spaces have the same shape if when embedded in the Hilbert cube, their complements are homeomorphic. The *shape group of* X is the inverse limit of fundamental groups of nerves of covers of X, where the mesh of the covers goes to zero. Recall that the shape group is an invariant of the shape of the space. Note that for CW-complexes, there is a cover where X is homotopy equivalent to its nerve, and thus the shape group is isomorphic to the fundamental group.

Theorem 4.5.1. *Every one-dimensional Peano continuum* X *with the shape of a graph* G *has a core which is homotopy equivalent to* G*. Thus such a space has a strong deformation retraction to a finite subgraph.*

Proof. Let H be the core of X assured by Corollary 4.4.2, which is a strong deformation retract of X. Note that X, G and H all have the same shape, and thus the same shape group. Since G is a graph, its shape group is isomorphic to its fundamental group, which is countable. If H were not locally simply connected, then its fundamental group would be uncountable by a result from Cannon and Conner [16]. Also by [16] we know that one-dimensional Peano continua are shape injective (the natural map from the fundamental group into the shape group is injective). Thus the fundamental group of H must embed in that of G, and so must be countable. Thus H is locally simply connected [16, 21]. Then by Theorem 4.3.2 we see that H is a (locally finite) graph. Since H is compact, it must be a finite graph. Since H, G are shape equivalent, they have the same shape group, which is their common fundamental group since they are CW-complexes. Therefore they are both homotopy equivalent to a bouquet of n circles, where n is the rank of their fundamental group, which is a free group. Thus we see that $H \subset X$ is homotopy equivalent to G. □

Corollary 4.5.2. *Every one-dimensional Peano continuum that has the shape of a circle admits a strong deformation retraction to a circle.*

Proof. By the theorem, the space has a core graph Γ that is homotopy equivalent to a circle. It remains to show that Γ is in fact a circle.

Since Γ is not simply connected, it contains a subgraph C which is a simple closed curve. Suppose $\Gamma \neq C$, and let $x \in \Gamma - C$. Let y be an interior point on a path from x to C in Γ. If y does not separate x from C, then there is another path from x to C, and thus $\pi_1(\Gamma)$ has rank at least two, which is a contradiction. Thus y separates x from C, and by part 4 of Theorem 4.4.1 we see that the path component of $\Gamma - y$ containing x is not simply connected (when taken together with y). This also implies that $\pi_1(\Gamma)$ has rank at least two, which is a contradiction.

Thus the only core graph which is homotopy equivalent to a circle is a circle, so we immediately obtain that a one-dimensional Peano continuum having the shape of a circle has a circle as its core. □

The following examples show that the hypothesis of being one-dimensional is necessary.

Example 4.3. Let X be the wedge product of a circle with the cone over the Hawaiian earring, where the wedge point is the bad point of the base Hawaiian earring. The cone over the Hawaiian earring is contractible, hence has the shape of a point. Thus X has the shape of a circle. However, X does not have a strong deformation retraction to a circle, since the base point of the cone over the Hawaiian earring must move over the cone point in any contraction. Note that X does have a weak deformation retraction to a circle.

Let Y be the wedge product of a circle with two cones over the Hawaiian earring, again with the wedge point as the bad point of the base Hawaiian earrings. Similar to X, the space Y has the shape of a circle, but the fundamental group of Y is uncountable, since the fundamental group of the doubled cone is uncountable [16]. Thus Y cannot deformation retract to a circle.

4.6 Homotopy Invariants for One-Dimensional Peano Continua

The reduced forms given in Section 4.3 lead to homotopy invariants for one-dimensional Peano continua. As mentioned in Theorem 4.4.3, the set $B(X)$ of non-locally simply connected points in a one-dimensional Peano continua is invariant, in that any homotopy equivalence restricts to a homeomorphism on the set $B(X)$. However, the set $B(X)$ does not by itself determine the homotopy type of the continuum. In this section, we define two more homotopy invariants, and prove for some cases that these do determine the homotopy type.

A number of times in this section we will consider the limit set L of a collection of sets $C = \{S_\alpha\}_\alpha$ (usually the sets will be arcs). By this we mean that a point x is in L if every neighborhood of x contains a set S_α that is disjoint from x. Note that it is not sufficient for S_α to intersect the neighborhood; we will make that distinction when we want to consider that case.

Lemma 4.6.1. *Let Y_1, Y_2 be Peano continua each containing the Peano continuum X as a subspace such that $Y_i - X = A_i$ is a disjoint union of infinitely many open sets, each of which is homeomorphic to an open arc. If the limit set (in X) of the arcs of A_1 is the same as the limit set of the arcs of A_2, then Y_1 is homotopy equivalent to Y_2.*

Proof. Let $Q \subset X$ denote the common limit set of A_1, A_2. Partition A_1 and A_2 into disjoint sequences iteratively as follows:

To get the sequence d^{2i}, take the lowest indexed arc in A_1 (which has not yet been chosen) to be d_1^{2i}. Choose q_{2i} to be a closest point in Q to d_1^{2i} (note this may not be unique), and choose a sequence in A_1 (of arcs not yet chosen) that converges monotonically to q_{2i}, starting with d_1^{2i}. Take every other element of this sequence to form the sequence d^{2i}, so that the remaining arcs in $A_1 - \bigcup^{2i} d^k$ still limit on all of $Q(X)$.

Then choose a matching monotone sequence in A_2 (of arcs not yet chosen) converging to q_{2i}, starting with an arc c_1^{2i} in A_2 that is within 2^{-2i} of q_{2i}. Again, take every other arc in the sequence to make up the sequence c^{2i}.

Now for $2i+1$, do the same as above, this time starting with the lowest indexed arc in A_2 not used yet, to first get a sequence c^{2i+1} in A_2, and a matching sequence d^{2i+1} in A_1, where both sequences limit on the same point $q_{2i+1} \in Q$.

Repeat.

This now gives us $A_2 = \bigcup_i c^i$, and $A_1 = \bigcup_i d^i$, with a pairing $d^i_j \leftrightarrow c^i_j$. First note that $\min_q\{\text{diam}(d^i_j, q)\}$ forms a sequence converging to zero; if not, then there would be a sequence of arcs with no point in Q nearby, but since the space is compact we must have a limit point, which by hypothesis is in Q. Now we can see that the diameters of the sets $d^i_j \cup c^i_j$ form a null sequence: $\text{diam}(d^i_j \cup c^i_j) \leq \text{diam}(d^i_j \cup q_i) + \text{diam}(q_i \cup c^i_j) \leq \text{diam}(d^i_1 \cup q_i) + \text{diam}(q_i \cup c^i_1)$. The one term above is chosen to be less than 2^{-i}, and the comment above shows that the other term is small if i is chosen large enough since we pick q_i to be as close as possible.

Now since X is compact and locally path connected, for each i, j we can choose small diameter paths p^i_j, r^i_j in X connecting the pairs of endpoints of d^i_j and c^i_j. The homotopy equivalence $f : Y_1 \to Y_2$ will send d^i_j to $p^i_j\, c^i_j\, \bar{r}^i_j$ while fixing X, and its homotopy inverse g will send c^i_j to $\bar{p}^i_j\, d^i_j\, r^i_j$. Since the images of the arcs d^i_j form a null sequence, Lemma A.4.2 shows that f (and similarly g) is continuous. Since both f, g fix X, then $g \circ f$ maps d^i_j to $p^i_j\, \bar{p}^i_j\, d^i_j\, r^i_j\, \bar{r}^i_j$, which is homotopic to the identity. Again we use the continuity lemma Lemma A.4.2 to see that the compositions are homotopic to the identity.

Thus Y_1 is homotopy equivalent to Y_2. □

We will define a subset $Q(X)$, which will be the limit set of the arcs A_i in the lemma. This set $Q(X)$ will be another homotopy invariant for X, which together with $B(X)$ will essentially determine the homotopy type of X in many cases. We will also develop some equivalent definitions and properties of the set $Q(X)$ so that we will be able to apply Lemma 4.6.1 by getting collections of arcs that limit on $Q(X)$.

Definition 4.6.2. The subset $Q(X) \subset B(X)$ is the set of points of X such that every neighborhood contains a simple closed curve intersecting $G(X)$.

We first prove the analog of Theorem 4.3.3 for $Q(X)$.

Theorem 4.6.3. *If $f\colon X \to Y$ is a homotopy equivalence of one-dimensional Peano continua, then $f|_{Q(X)}$ is a homeomorphism onto $Q(Y)$. Furthermore, if f is a self-homotopy of X, then f fixes $Q(X)$ pointwise.*

Proof. Let $x \in Q(X)$, and let ℓ be a simple closed curve near x that intersects $G(X)$. Since f is a homotopy equivalence, $f(\ell)$ cannot be nulhomotopic, and thus must contain a simple closed curve. Suppose that this simple closed curve is contained in $B(Y)$. Then since $g|_{B(Y)}$ is a homeomorphism by Theorem 4.3.3, $g \circ f(\ell)$ must contain a simple closed curve in $B(X)$, but this cannot be homotopic to ℓ, which is a simple closed curve not contained in $B(X)$, since simple closed curves are reduced (see Lemma 4.2.3). Thus $f(\ell)$ contains a simple closed curve intersecting $G(Y)$, so that f maps $Q(X)$ into $Q(Y)$. We note that it can also be shown that the reduced representative for $f(\ell)$ is a simple closed curve (which intersects $G(Y)$).

Then since $g \circ f|_{Q(X)} = Id_{Q(X)}$, $f|_{Q(X)}$ must be injective, and since $f \circ g|_{Q(Y)} = Id_{Q(Y)}$, $f|_{Q(X)}$ must be surjective because g maps $Q(Y)$ into $Q(X)$. Since f and g are continuous, we see that f maps $Q(X)$ homeomorphically onto $Q(Y)$.

If f is a self-homotopy of X, then by Theorem 4.3.3 f must fix $Q(X)$ pointwise since $Q(X) \subset B(X)$. □

In order to characterize the points of $Q(X)$ as limits of certain arcs in $G(X)$, we first need the following lemma.

Lemma 4.6.4. *Let X be an arc-reduced one-dimensional Peano continuum. Then there is a subcollection T of the arcs in $G(X)$ such that $B(X) \cup T$ is connected and every arc in T separates $B(X) \cup T$.*

Proof. Enumerate the null sequence of arcs in $G(X)$ as $T_0 = \{a_i \mid i \in \mathbb{N}\}$. Given T_i, define T_{i+1} iteratively by sequentially throwing out all non-separating arcs. To be more precise, if

a_{i+1} does not separate $B(X) \cup T_i$, then let $T_{i+1} = T_i - \{a_{i+1}\}$, otherwise $T_{i+1} = T_i$. Then set $T = \bigcap T_i$.

Suppose that $B(X) \cup T$ is not connected. First note that $B(X) \cup T$ is compact, being closed (its complement is a union of open arcs), so given any two distinct components, there is a positive distance between them. So there are only finitely many arcs in $G(X)$ which can connect (in X) these components of $B(X) \cup T$, but none of the arcs we pulled out separated the space. This is a contradiction, hence $B(X) \cup T$ must be connected. It is then clear by the construction that every arc in T separates. □

We now give two equivalent definitions of $Q(X)$.

Lemma 4.6.5. *Let X be an arc-reduced one-dimensional Peano continuum.*

- *With T as in Lemma 4.6.4, $Q(X)$ is the limit set of the collection of arcs not in T (i.e. $G(X) - T$).*
- *A point is in $Q(X)$ if it is the limit of some collection of arcs $C \subset G(X)$ that do not collectively separate, i.e. $X - C$ is connected.*

Proof. Let R be the limit set of $G(X) - T$, and let S be the union of limit sets of collections C, where the union is take over collections $C \subset G(X)$ with $X - C$ connected. We will show that $Q(X) \subset R \subset S \subset Q(X)$.

Every neighborhood of a point in $Q(X)$ contains a simple closed curve intersecting $G(X)$. Since every arc of T separates $B \cup T$, T cannot contain all of the arcs of $G(X)$ in this simple closed curve. So $Q \subset R$, and clearly $R \subset S$ since $X - (G - T) = B \cup T$ is connected.

It remains to show that $S \subset Q$. Suppose that x is a limit of a collection of arcs C with $X - C$ connected. First, if $X - C$ is locally path connected at x, then in any neighborhood of x, there is a small arc a in $C \subset G(X)$ whose endpoints are connected in $X - C$ by a small path. This gives a closed curve that traverses a exactly once, and can be modified to give a simple closed curve traversing a. Thus $x \in Q$.

Now suppose that $X - C$ is not locally path connected at x. Consider any small neighborhood U of x such that U contains no path connected neighborhood of x in $X - C$. Since $X - C$ is connected, we must have a non-null sequence of path components P_i in $V \cap (X - C)$ approaching the path component P containing x, and thus these path components must also limit on another point $x' \neq x$. Let $V, W \subset U$ be disjoint path connected neighborhoods of x, x' in X. Then there are two path components P_i, P_j that intersect both V and W. Since V and W are both path connected, we can construct a closed curve by connecting P_i to P_j by a path a_V in V, traversing P_j, then connecting to P_i by a path a_W in W, and then traversing P_i. Since P_i, P_j were different path components of $U - C$, we see that our closed curve must intersect $C \subset G(X)$. Again we get a simple closed curve intersecting $G(X)$ from this, showing that $x \in Q$. □

Notice that Lemma 4.6.5 implies that while the choice of arcs in T from Lemma 4.6.4 may not be unique, any choice of T will work to define $Q(X)$ as the limit of arcs not in T. Thus every point in $Q(X)$ is a limit of arcs in $G(X)$, but the arcs in $T \subset G(X)$ may limit on points not in $Q(X)$. Because of this, we have the following definition and lemma, which characterize all limit points of the collection of arcs $G(X)$.

Definition 4.6.6. In a one-dimensional Peano continuum X, define $P(X)$ to be the set of all points that are limits of components of $B(X)$. In other words, a point x is in $P(X)$ if every neighborhood of x contains a component of $B(X)$ disjoint from x.

Lemma 4.6.7. *In a one-dimensional Peano continuum X, the limit set of the arcs in $G(X)$ is exactly $P(X) \cup Q(X)$.*

Proof. From Lemma 4.6.5 we know that the limit set of the arcs in $G(X) - T$ is exactly $Q(X)$.

Suppose that x is a limit point of the arcs in T. Every neighborhood of x must intersect infinitely many components of $B(X)$, since otherwise there could not be infinitely many arcs of T in that neighborhood (recall that every arc of T separates $B(X) \cup T$). Either every neighborhood contains a component of $B(X)$, in which case $x \in P(X)$, or there is some neighborhood U of x intersecting infinitely many components of $B(X)$, but containing none of them. This then will also be true for all neighborhoods contained in this one.

For such a neighborhood U, choose a sequence of points $x_i \in U$ that converge to x, where the points x_i come from distinct components C_i of $B(X)$. These components C_i cannot form a null sequence (otherwise they would eventually be contained in a neighborhood of x), so by passing to a subsequence we may assume that $\operatorname{diam}(C_i) \geq \delta$. Let $\varepsilon < \delta/2$ be such that $B(x, 2\varepsilon) \subset U$. Take a sequence of points $y_i \in C_i$ such that $d(x_i, y_i) = \varepsilon$. Then the points y_i converge to some point $y \in U$ with $d(x, y) = \delta$. Then for small disjoint path connected neighborhoods of x and y, we can find two distinct components C_j, C_k that intersect both small neighborhoods. In these neighborhoods, there are paths connecting C_j, C_k, and each of these paths must contain an arc of $G(X)$. At least one of these arcs must not be in T, since otherwise they would not separate $B(X) \cup T$. Therefore any small neighborhood U of x contains an arc of $G(X) - T$, and thus $x \in Q(X)$.

It remains to show that every $x \in P(X) - Q(X)$ is in the limit set of T. Since $x \notin Q(X)$, there is a neighborhood V of x that does not contain any arc of $G(X) - T$. Let U be an arbitrary neighborhood of x contained in V. Because X is locally path connected, we may assume that U is path connected. There are infinitely many components of $B(X)$ in U because $x \in P(X)$, and since U is connected there must be an arc of $G(X)$ in U. This arc must then be in T, hence x is in the limit set of T. □

Lemma 4.6.8. *Let X be an arc-reduced one-dimensional Peano continuum with $Q(X) \neq \emptyset$. Then there is a partition of the arcs in $G(X)$ as $G(X) = T \cup C \cup D$ with*

(i) $B(X) \cup T$ is connected and every arc in T separates $B(X) \cup T$,

(ii) $B(X) \cup T \cup C = X - D$ is a Peano continuum, and

(iii) The limit sets of both C, D are $Q(X)$.

Proof. Part 1 is just Lemma 4.6.4. Now we will partition the remaining arcs in $G(X) - T$ to achieve conditions 2 and 3. To do this, we will use a sequence of small finite covers of $Q(X)$, that limits on all of $Q(X)$, and pull out arcs to constitute D (only from every other step to assure C also limits on $Q(X)$), while being careful that the remaining arcs (in C) are sufficient to locally path connect the subspace $X' = B(X) \cup T \cup C$. Lemma 4.6.5 tells us that C, D can only limit on a subset of $Q(X)$, so constructing them to limit on all of $Q(X)$ is sufficient to prove part 3.

Start with some $\varepsilon_0 > 0$. Having ε_i, cover $Q(X)$ by finitely many path connected neighborhoods S_k in X of diameter $< \varepsilon_i$. Throwing out some of the sets S_k, if necessary, we may assume that each S_k intersects $Q(X)$. For every set S_k, choose a point $q_k \in S_k \cap Q(X)$. Let r be a small radius such that $r < \frac{1}{2}\min_{j,\ell}\{d(q_j, q_\ell)\}$ and also such that for all k we have $R_k = B(q_k, r) \subset S_k$.

Since $q_k \in Q(X)$, there is a simple closed curve $\ell \subset R_k$ that intersects $G(X)$. So either ℓ contains an arc in $G(X)$ that connects a component of $B(X)$ to itself, or it contains multiple arcs in $G(X)$ connecting different components of $B(X)$. Since every arc in T separates $B(X) \cup T$, ℓ must contain at least one arc in $G(X) - T$. Choose one such arc from each R_k. To iterate, choose $\varepsilon_{i+1} < \varepsilon_i/2$ such that no chosen arc is contained in $B(Q(X), \varepsilon_{i+1})$.

We define D as the set of arcs chosen when i is even, and then C consists of the remaining arcs: $G(X) - (T \cup D)$. Thus both C and D limit on all of $Q(X)$. It remains to show that $X' = B(X) \cup T \cup C$ is a Peano continuum, in particular that it is locally path connected.

Since D only limits on $Q(X)$, the only place $X' = X - D$ cannot be locally path connected is at points of $Q(X)$. Given $q \in Q(X)$ and $\epsilon > 0$, there is some neighborhood U containing q of diameter $< \epsilon/2$, where U was one of the S_k's chosen in the above construction, say at step i_0. Now while U is path connected, $U - D$ may not be, since an arc removed at some step after i_0 may disconnect U. To fix this, set $U_{i_0} = U$, and for $i > i_0$ let $U_{i+1} = U_i \bigcup \{$all S_k from step $i+1$ that intersect $U_i\}$. Thus $\{U_i\}$ is an ascending chain of path connected neighborhoods, so that $V = \bigcup U_i$ is a path connected neighborhood (in X) of q. Note that since $\varepsilon_{i+1} < \varepsilon_i/2$ and since the diameter of U is less than $\epsilon/2$, we see that the diameter of V is less than ϵ.

It remains to show that $V - D$ is still path connected. Let $a, b \in V - D$. Then there is a path p_0 in V from a to b. Given p_i, define p_{i+1} by replacing any arc $\ell \subset R_k \subset S_k$ in p_i that was chosen in step i (to go to D) by a complimentary path $\ell' \subset R_k$ (Recall that ℓ was chosen on a simple closed curve in R_k). We claim that the paths p_i converge to a path p from a to b.

Well, by our construction, we know that $\forall x \in [0,1]$, $d(p_i(x), p_{i+1}(x)) < \varepsilon_{i+1}$, so that p_i converges pointwise to a function p. It is easy to verify that the image of p lies in $V - D$. To check that p is continuous, let $x \in [0,1]$ and $\rho > 0$. Choose n such that $\sum_{i=n}^{\infty} \varepsilon_i < \rho/3$, and choose δ such that if $d(x,y) < \delta$ then $d(p_n(x), p_n(y)) < \rho/3$. Then for such y we have $d(p(x), p(y)) \leq d(p(x), p_n(x)) + d(p_n(x), p_n(y)) + d(p_n(y), p(y)) < \rho$. [Note that $d(p(x), p_n(x)) \leq \sum_{k=n}^{\infty} d(p_k(x), p_{k+1}(x)) < \sum_n \varepsilon_k < \rho/3$.]

Thus p is continuous and $V - D$ is path connected. Hence $X - D$ is locally path connected.

□

Corollary 4.6.9. *Let X be a one-dimensional Peano continuum with $Q(X) \neq \emptyset$, and let A be null sequence of arcs (possibly a finite collection) with endpoints in X. If the limit set of A is contained in $Q(X)$, then $X \cup A$ is homotopy equivalent to X.*

Proof. By Lemma 4.6.8 we have $X = (X - D) \cup D$ where $X - D$ is a Peano continuum, and where D limits on all of $Q(X)$. Then since $A \cup D$ also limits on $Q(X)$, by Lemma 4.6.1 we have

$$X = (X - D) \cup D \simeq (X - D) \cup (A \cup D) = X \cup A.$$

It is fairly easy to check that $X \cup A$ is a one-dimensional Peano continuum, so that

Lemma 4.6.1 actually applies. □

We now proceed to define one last invariant. In an arc-reduced continuum X, some of the arcs in $G(X)$ may form a loop attached at a single point of $B(X)$ instead of having two distinct endpoints. In this case, we can consider X as a pointed union, or wedge, of a subspace with a circle: $X = X' \vee S^1$. If multiple loops form at the same point, we have $X = X' \vee kS^1$. Note that since X is path connected, X' must be also, so that by a homotopy equivalence we can move this circle to any point of X'. Also note that X' will be a Peano continuum, since removing finitely many loops will leave the space locally path connected. This gives rise to the following definition.

Definition 4.6.10. The *rank* of a one-dimensional Peano continuum X is the number $\mathrm{rank}(X) = \sup\{k \mid X \simeq X' \vee kS^1\}$.

Note that $\mathrm{rank}(X)$ is either a non-negative integer or infinite, and is equal to the number of arcs in $G(X) - T$. Since $Q(X)$ is the limit set of $G(X) - T$ (Lemma 4.6.5), we see that $\mathrm{rank}(X)$ is infinite if and only if $Q(X) \neq \emptyset$. This number is clearly a homotopy invariant from its definition.

This new invariant $\mathrm{rank}(X)$ together with the invariant sets $B(X)$ and $Q(X)$ should form a complete homotopy invariant for one-dimensional Peano continua, i.e. these invariants together determine the homotopy type of the Peano continuum. We have proven this result in many cases, and conjecture it to always hold.

Theorem 4.6.11. *If X and Y are one-dimensional Peano continua that are* sufficiently nice*, then X and Y are homotopy equivalent if and only if one of the following hold:*

(i) $Q(X) \neq \emptyset$ *and there is a homeomorphism of pairs* $h : (B(X), Q(X)) \to (B(Y), Q(Y))$.

(ii) $Q(X) = \emptyset$, *there is a homeomorphism* $h : B(X) \to B(Y)$, *and* $\mathrm{rank}(X) = \mathrm{rank}(Y)$.

A more compact way of stating this result is the following:

Theorem 4.6.12. *The triple* $(B(\cdot), Q(\cdot), \mathrm{rank}(\cdot))$ *is a complete invariant of the homotopy type of one-dimensional Peano continua that are* sufficiently nice*.*

The phrase "sufficiently nice" is determined by the cases at the end of the proof, and only depends on the structure of $B(X)$ and $Q(X)$.

Proof. We will prove this in various cases, where we put various restrictions on the set $B(X)$. The beginning of the proof is the same for the different cases, so we first present that, and then show how to complete the proof in the separate cases.

By Theorem 4.3.3 and Theorem 4.6.3, if X and Y are homotopy equivalent, then there is a homeomorphism of pairs $h : (B(X), Q(X)) \to (B(Y), Q(Y))$, and $\mathrm{rank}(X) = \mathrm{rank}(Y)$.

To prove the converse, first we assume that $Q(X) \neq \emptyset$ and that there is a homeomorphism of pairs $h : (B(X), Q(X)) \to (B(Y), Q(Y))$. By Theorem 4.3.4 we may assume that X and Y are both arc-reduced. We will extend the homeomorphism $h : B(X) \to B(Y)$ to a homotopy equivalence of X and Y.

Using Lemma 4.6.8, we can write each of the spaces X and Y as the disjoint unions $X = B(X) \cup T(X) \cup C(X) \cup D(X)$ and $Y = B(X) \cup T(Y) \cup C(Y) \cup D(Y)$ (recall that $B(Y) = B(X)$) with the properties from the lemma. In particular, we know that the subspaces $\big(B(X) \cup T(X) \cup C(X)\big)$ and $\big(B(Y) \cup T(Y) \cup C(Y)\big)$ are both Peano continua and that the collections $C(X), D(X), C(Y), D(Y)$ all limit on $Q(X) \subset B(X) = B(Y)$. Thus we may apply Lemma 4.6.1 to get the following homotopy equivalences.

$$X = \big(B(X) \cup T(X) \cup C(X)\big) \cup D(X) \simeq \big(B(X) \cup T(X) \cup C(X)\big) \cup C(Y)$$

$$Y = \big(B(X) \cup T(Y) \cup C(Y)\big) \cup D(Y) \simeq \big(B(X) \cup T(Y) \cup C(Y)\big) \cup C(X)$$

To consolidate, call $C(X) \cup C(Y) = A$ and $B(X) = B$. Now it remains to show that

$B \cup A \cup T(X) \simeq B \cup A \cup T(Y)$. We will pause with the case $Q \neq \emptyset$ for a moment and consider the finite rank case (when $Q = \emptyset$), and then continue with the individual cases.

Suppose that $Q(X) = \emptyset$, and let $k = \text{rank}(X) = \text{rank}(Y) < \infty$. Then we have a space X' containing $B(X)$ such that $X \simeq X' \vee kS^1$. Recall that X' is a Peano continuum. We may assume that X' is arc-reduced. Consider any arc a in $G(X')$. If a does not separate X', then there is a path in $X' - a$ between the endpoints of a, and we could homotop a to an additional loop making $X \simeq X'' \vee (k+1)S^1$, contradicting the fact that $k = \text{rank}(X)$. Thus $\text{rank}(X') = 0$, and we may write $X' = B(X) \cup T(X)$, which is connected (in fact path connected) and every arc in T separates.

The same results hold for Y, and since X' and Y' are path connected, we can homotop the kS^1 in either space to any point of $B(X)$ we choose, so that they match up for both X and Y. Denoting the arcs in the loops kS^1 as the set A, we get $X \simeq B \cup A \cup T(X)$, and $Y \simeq B \cup A \cup T(Y)$. Note that this is almost the exact same situation as in the infinite rank case, but here we have the added nicety that each component of B is itself a Peano continuum; in particular the components are locally path connected.

Thus to prove the finite rank case, it suffices to prove the rank zero case.

So in either case, we need to show that $B \cup A \cup T(X) \simeq B \cup A \cup T(Y)$. We now proceed with various cases, depending on the structure of the set B.

(i) There are finitely many components of B.

Here T consists of only finitely many arcs, and is easily homotoped to match.

(ii) $B = Q$.

Here T limits on a subset of Q, and so by Corollary 4.6.9 we can add the arcs of T to each space. Thus $X \simeq B \cup A \cup T(X) \simeq B \cup A \cup T(X) \cup T(Y) \simeq Y$.

Sometimes the structure of the set B itself requires that $B = Q$, as in the following three cases.

(iii) There are no arbitrarily small essential curves in B.

Here $B(B(X)) = \emptyset$, or in other words, $B = Q$, since arbitrarily small loops are not contained in B.

(iv) Some iterate $B^{(k)}(X) = B(B(\ldots(X)\ldots)) = \emptyset$.

As in the previous case, it can be shown that B must equal Q (with a few more steps).

(v) Each component of B is a simply connected Peano continuum.

This fits the previous cases, but we present a more direct approach, which will be useful in further cases.

Then $B \cup T$ is simply connected. We can map each arc in $T(X)$ to a reduced path in $B \cup T(Y)$, and vice versa. These maps are continuous since the collection of the images of the arcs (the reduced paths) form a null sequence by local path connectivity, and so Lemma A.4.2 applies.

The composition of these maps sends an arc in T to a path in $B \cup T$ with the same endpoints. Since $B \cup T$ is simply connected, this is homotopic to the identity on each arc. Applying Lemma A.4.2 again, we see that this is a homotopy equivalence.

The remaining two cases we consider here are with the space having rank 0.

(vi) $Q = \emptyset$ and $|P \cap C| < \infty$ for each component C of B.

Since $Q = \emptyset$, each component of B is a Peano continuum (Lemma A.4.6). Then we can homotop the arcs with endpoints in C so that all the endpoints are in P. Choose a tree (simply connected graph) in C connecting the points of P. If C has no points of

P, then choose a tree connecting the endpoints of arcs in C, or alternatively, homotop the arcs so all share the same endpoint in C. Since the components of B form a null sequence (Lemma A.4.7), we can perform all of these homotopies on all components at the same time, and get a continuous map by Lemma A.4.2.

Then the subspace consisting of the union of T with the tree in each component is simply connected, and just as above, we get a homotopy equivalence from X to Y.

(vii) $Q = \emptyset$ and $|P \cap C| \neq \emptyset$ for finitely many components C of B.

Take disjoint neighborhoods of such components C. Homotop the arcs in each of these neighborhoods so that they have one endpoint in C. Recall that since $Q = \emptyset$, C is a Peano continuum. So then we can homotop the endpoints of the arcs in C to match up between X and Y. Note that there will be finitely many arcs not contained in any of these neighborhoods, which are easily dealt with.

□

We close by noting that Theorem 4.6.12 should be true for all one-dimensional Peano continua. The difficulty lies in proving that the set of arcs $T(X)$ may be homotoped to the arcs $T(Y)$ and vice versa in a consistent manner – that is, so that the composition is in fact homotopic to the identity. In Theorem 4.6.12 we have proven that this is possible in many specific cases, and we conjecture that the theorem holds for all one-dimensional Peano continua, independent of the structure of the set $B(X)$.

Conjecture 4.6.13. *The triple* $(B(\cdot), Q(\cdot), \text{rank}(\cdot))$ *is a complete invariant of the homotopy type of all one-dimensional Peano continua.*

Appendix A.

A.1 Interval Maps Fixing Endpoints

Lemma A.1.1. *For a closed interval I, there are maps $f : I \to I$ fixing both endpoints of I with all possible period sets* $\mathrm{Per}(f)$ *as described in Theorem 2.1.2.*

Proof. Consider the one-parameter family of truncated tent maps $T_h : [0,1] \to [0,1]$, for $0 \le h \le 1$, as discussed in [4].

$$T_h(x) = \min(h, 1 - 2|x - 1/2|) = \begin{cases} 2x & \text{if } x \in [0, h/2] \\ h & \text{if } x \in [h/2, 1 - h/2] \\ 2(1-x) & \text{if } x \in [1 - h/2, 1] \end{cases}$$

They note that given m there is a way to determine h so that $\mathrm{Per}(T_h)$ is exactly the tail of the Sharkovskiĭ order beginning with m. While they do not give a precise formula, they describe h in terms of the orbits of size m of T_1 (of which there are less than 2^m): $h(m) = \min\{\max P \mid P \text{ is an } m\text{-cycle of } T_1\}$. They also give a value of h corresponding to 2^∞, so that $\mathrm{Per}(T_h) = \{2^n \mid n \in \mathbb{N}\}$: $h(2^\infty) = \sup\{h(2^n) \mid n \in \mathbb{N}\}$. The family of maps T_h provide examples of functions that achieve every possible period set, although the endpoints are not fixed, as $T_h(1) = 0$ for all h.

We extend these to maps $\widehat{T}_h : [0,2] \to [0,2]$ that fix both endpoints of the interval $[0,2]$.

$$\widehat{T}_h(x) = \begin{cases} T_h(x) & \text{if } x \in [0,1] \\ 2x - 2 & \text{if } x \in [1,2] \end{cases}$$

Clearly $\widehat{T}_h$ fixes 0 and 2, and 1 is not periodic since $(\widehat{T}_h)^n(1) = 0$. Any other periodic point must lie in $(0,1)$, since for $x \in (1,2)$ we have $\widehat{T}_h(x) < x$, which together with $\widehat{T}_h([0,1]) \subset [0,1]$

implies that $f^n(x) \neq x$. Thus the maps $\widehat{T}_h$ have the same periodic properties as T_h, but they also fix both endpoints of the interval.

□

A.2 Mathematica Code for Period Implications on Finite Spaces

We include here the Mathematica code for computing period implications on finite topologies. This code was run on Mathematica version 7.

```
Topologize[A_] := (
  n = Length[A];
  AA = A + IdentityMatrix[n];
  B = ConstantArray[1, {n, n}];
  For[i = 0, i < n, i++;
   For[ii = 0, ii < n, ii++;
    If[AA[[ii]][[i]] == 0, , B[[i]] = B[[i]]*AA[[ii]]]
    ]
   ];

  BB = MatrixPower[B, n];
  For[i = 0, i < n, i++;
   For[ii = 0, ii < n, ii++;
    If[BB[[ii]][[i]] == 0, , BB[[ii]][[i]] = 1]
    ]
   ];
  BB
  )
```

```
RandomTopology[n_] :=
 Module[{A = RandomInteger[1, {n, n}] + IdentityMatrix[n],
   B = ConstantArray[1, {n, n}], BB = B},
  For[i = 0, i < n, i++;
   For[ii = 0, ii < n, ii++;
    If[A[[ii]][[i]] == 0, , B[[i]] = B[[i]]*A[[ii]]]
    ]
   ];

  BB = MatrixPower[B, n];
  For[i = 0, i < n, i++;
   For[ii = 0, ii < n, ii++;
    If[BB[[ii]][[i]] == 0, , BB[[ii]][[i]] = 1]
    ]
   ];
  BB
  ]

isLegal := (value = True;
  For[i = 0, i < k - 1, i++;
   If[A[[i]][[k]] == 1 && A[[array[[i]]]][[array[[k]]]] == 0,
    value = False];
   If[A[[k]][[i]] == 1 && A[[array[[k]]]][[array[[i]]]] == 0,
    value = False];
   ];
  value
  )
```

```
increment := (
  While[(array[[k]] == n && k != 0), array[[k]] = 1; k--]
   If[k != 0, array[[k]] = array[[k]] + 1, k--]
  )

FindImps := (
  array = ConstantArray[1, n];
  SpacePeriods = {};
  SpaceImps = ConstantArray[1, {n, n}];

  For[k = n, k != 0,
   If[(isLegal && k == n),
    (*Print[array]*)
    per = PeriodSet;
    perComp = Complement[Array[# &, n], per];
    SpacePeriods = Union[SpacePeriods, per];

    For[q = 0, q < Length[per], q++;
     For[t = 0, t < Length[perComp], t++;
      SpaceImps[[per[[q]]]][[perComp[[t]]]] = 0;
      ]
     ]

    ];
   If[(isLegal && k != n), k++, increment; w++];
   ];
```

```
  (* now ignore periods that never show up *)
  SpacePeriods2 = ConstantArray[0, n];
  For[i = 0, i < Length[SpacePeriods], i++;
   SpacePeriods2[[SpacePeriods[[i]]]] = 1;
   ];
  SpaceImps = SpaceImps*SpacePeriods2;

  (*  Print[MatrixForm[A]];
      Print[SpacePeriods];
      Print[MatrixForm[SpaceImps]];
  *)
  (* update master imp list *)

  For[i = 0, i < n, i++;
   For[j = 0, j < n, j++;
    If[SpaceImps[[i]][[j]] == 1 && MasterImps[[i]][[j]] == 0,
     MasterImps[[i]][[j]] = SpaceNumber]
    ]
   ];
  )

Space2Num[A_] := (
  m = Length[A];
  For[Num = 0; i = 0, i < m, i++;
   For[j = 0, j < m, j++;
    Num += 2^(j - 1 + (i - 1) m)*A[[i]][[j]];
```

```
    ]
   ];
  Num
  )

Num2Space[num_] := (
  number = num;
  m = Ceiling[Sqrt[Log[2, num]]];
  Aa = ConstantArray[0, {m, m}];
  For[i = 0, i < m, i++;
   For[j = 0, j < m, j++;
    Aa[[i]][[j]] += Mod[number, 2];
    number = Quotient[number, 2];
    ];
   ];
  Aa
  )

PeriodSet := (
  fn = Array[# &, n];(* f^0=id *)
  For[i = 0, i < n, i++;  (* each point *)
   For[j = 0, j < n, j++; fn[[i]] = array[[fn[[i]]]]
     ] (*find f^n(i)*)
   ];

  periods = {};
  points2consider = ConstantArray[0, n];
```

```
  For[i = 0, i < n, i++; points2consider[[fn[[i]]]] = 1
   ];
  For[i = 0, i < n, i++; p = 1;
   If[points2consider[[i]] == 1,
    For[j = array[[i]], j != i, p++; j = array[[j]]
     ];
    AppendTo[periods, p];
    ]
   ];
  periods = Union[periods]
  )


(*************** main program starts here ****************)

MasterImps = ConstantArray[0, {30, 30}];
lowersize = 2;
uppersize = 6;

For[space = lowersize - 1, space < 2^(uppersize^2), space++;
  If[Mod[space, 100000] == 0, Print[space];];
  A = Topologize[Num2Space[space]];
  SpaceNumber = Space2Num[A];
  (*Print[MatrixForm[A]];
  Print[space];*)
  If[SpaceNumber == space,
   rowsums = A.ConstantArray[1, Length[A]];
```

```
   For[index =
     1, (index < Length[A] &&
      rowsums[[index]] >= rowsums[[index + 1]]), index++
    ];
   If[index == Length[A], FindImps];
   ];
  ];
```

```
(*** to construct random spaces and check ***)

trials = 100;
For[index = 0, index < trials, index++;
 n = RandomInteger[{6, 10}];
 A = RandomTopology[n];
 FindImps;
 (*If[Mod[index,10]==0,Print[index]];*)
 Print[index];
 ]
```

A.3 Subgroups of $\mathbb{Q}$

This lemma characterizes the additive subgroups of the rational numbers. We note that these subgroups were previously discussed and characterized in [6, 10], but we give our own proof here. Note that for additive subgroups of $\mathbb{Q}$, multiplication by a constant is an isomorphism, so that we may assume that the subgroup contains 1. In the lemma, the numbers k_i represent the number of times (plus 1) that the prime p_i is allowed to appear in the denominators of the subgroup elements.

Lemma A.1. *Let $\{k_i\}$ be a sequence in $\mathbb{N} \cup \infty$. Define*

$$Q\big(\{k_i\}\big) = \left\{ \frac{p}{q} \in \mathbb{Q} \;\middle|\; q = \prod_{i=1}^{m} p_i^{n_i} \quad \textit{for some } n_i < k_i \textit{ and some } m \right\}$$

where p_i denotes the i^{th} prime number.

Then $Q\big(\{k_i\}\big)$ is a subgroup of $\mathbb{Q}$ containing 1. Furthermore, every subgroup $G \leq \mathbb{Q}$ containing 1 is equal to $Q\big(\{k_i\}\big)$ for some sequence $\{k_i\}$.

Proof. Since the definition does not require the fraction p/q to be in lowest terms, $Q\big(\{k_i\}\big)$ is clearly closed under addition and inverses, and is thus a subgroup containing 1.

Let Q be any subgroup of $\mathbb{Q}$ containing 1. Let D be the set of denominators of elements of Q when written in lowest terms, i.e. $D = \{q \mid p/q \in Q \text{ in lowest terms}\}$. Note that for every $q \in D$, we must have $1/q \in Q$, since $p/q \in Q$ with $(p, q) = 1$, so that if we multiple p/q by the multiplicative inverse of $m \mod q$ we get $mp/q = M + 1/q$. Since $1 \in Q$, then $1/q \in Q$. Then also $a/q \in Q$ for every $a \in \mathbb{Z}$ and $q \in D$, and in fact $Q = \{a/q\}$ as every has a reduced fraction with denominator $q \in D$.

Define the number $k_i \in \mathbb{N} \cup \infty$ to be one more than the maximum number of times the prime p_i appears in an element of D; $k_i = \sup\big\{1 + k \bigm| {p_i}^k \text{ divides } q \text{ for some } q \in D\big\}$. We first show that $Q \subset Q\big(\{k_i\}\big)$. Let $a/q \in Q$, where $q \in D$. Consider the prime factorization $q = \prod_{i=1}^{m} p_i^{n_i}$, where $n_i < k_i$ by the definition of k_i. Thus $a/q \in Q\big(\{k_i\}\big)$ for every $a/q \in Q$.

It remains to show $Q\big(\{k_i\}\big) \subset Q$. Note that $Q\big(\{k_i\}\big)$ is generated by elements of the form $1\big/\prod^m p_i^{n_i}$. In fact, we can take elements of the form $1/p_i^{n_i}$ as our generating set: since the $p_i^{n_i}$ are relatively prime, we may choose a_i so that $\sum(a_i/p_i^{n_i}) = 1\big/\prod^m p_i^{n_i}$. Thus it suffices to show that $1/p_i^{n_i} \in Q$ if $n_i < k_i$. By the definition of k_i, we know that there is an element $a/(bp_i^{n_i}) \in Q$ in reduced form. As before, since a is relatively prime to the denominator q, we may multiply by the inverse of $a \mod q$ and thus assume that $a = 1$. Then multiplying by b gives $1/p_i^{n_i} \in Q$.

Therefore every subgroup of $\mathbb{Q}$ is of the form $Q\big(\{k_i\}\big)$ for some sequence $\{k_i\}$. $\square$

We note that while different sequences $\{k_i\}$ give distinct subsets of $\mathbb{Q}$, they do not always give non-isomorphic subgroups. This is due to the fact that multiplication gives isomorphisms of subgroups of $\mathbb{Q}$. Thus if two sequences $\{k_i\}, \{k_i'\}$ differ in only finitely many spots by a finite amount (i.e. if $k_i \neq k_i'$ then both are finite), then the subgroups are isomorphic by multiplication/division by $\prod p_i^{k_i - k_i'}$. This is in fact the only way differing sequences can give isomorphic groups.

A.4 Peano Continua Lemmas

Lemma A.4.1. *Let $\mathcal{U} = \{U_\alpha\}$ be a collection of disjoint sets in a Peano continuum X, such that each U_α has nonempty interior, and also so that each $\overline{U_\alpha}$ is connected. If there is a uniform finite bound n on the number of points in the boundary of each U_α, then $\mathcal{U}$ forms a null sequence.*

Proof. Since X is second countable, $\mathcal{U}$ must be countable since the sets in $\mathcal{U}$ are disjoint and each has nonempty interior. Suppose that $\mathcal{U} = \{U_i\}$ is not a null sequence. Then by passing to a subsequence, we may assume that $\operatorname{diam}(U_i) > \varepsilon$ for all i. Since there are at most n boundary points of U_i, we claim there is a point $x_i \in U_i$ such that $d(x_i, \partial U_i) \geq \varepsilon/(2n)$. If not, then the union of balls B_k of radius $\varepsilon/(2n)$ centered at the boundary points of U_i would cover $\overline{U_i}$. In the connected set $\overline{U_i}$, there would then be a chain of the sets B_k between any two points $x, y \in U_i$. Since each B_k has diameter less than or equal to ε/n, we see that $d(x, y) \leq n \cdot \varepsilon/n = \varepsilon$, which contradicts the fact that $\operatorname{diam}(U_i) > \varepsilon$. So there exist points $x_i \in U_i$ with $d(x_i, \partial U_i) \geq \varepsilon/(2n)$.

Then since X is compact, there is a limit x of the sequence x_i, and by passing to a subsequence we may assume that $x_i \to x$. Since X is locally path connected, there is a path connected neighborhood V about x of diameter less than $\varepsilon/(2n)$. Eventually, the points x_i are contained in V, and so there are paths in V that join U_j and U_k, for j, k large enough. But this gives a boundary point in U_j that is within $\varepsilon/(2n)$ of the point x_j, which is a contradiction. Thus $\mathcal{U}$ is a null sequence. $\square$

Lemma A.4.2. *Let H be a function from the metric space $X \times Y$ into a metric space Z. Let $\{C_i\}$ be a null sequence of closed sets whose union is X. Suppose that H is continuous on each $C_i \times Y$, and that the images $D_i = H(C_i \times Y)$ form a null sequence of sets in Z. If for every subsequence C_{i_k} converging to x_0 there exists a point $z_0 \in Z$ such that $D_{i_k} \to z_0$ and $H(\{x_0\} \times Y) = \{z_0\}$, then H is continuous on all of $X \times Y$.*

Proof. Let $(x_n, y_n) \to (x_0, y_0)$. We need to show that $H(x_n, y_n) \to H(x_0, y_0)$. For each n, choose $i(n)$ such that $x_n \in C_{i(n)}$. If $\{C_{i(n)}\}$ is finite, then by restricting H to $\bigcup_n C_{i(n)} \times Y$ we have $H(x_n, y_n) \to H(x_0, y_0)$ by an application of the finite pasting lemma.

If the collection $\{C_{i(n)}\}$ is infinite, then by ignoring repetitions, we see that the sets $C_{i(n)}$ converge to x_0 since the C_i's form a null sequence and the points $x_n \in C_{i(n)}$ converge to x_0. Thus by our hypothesis, the images $D_{i(n)}$ converge to $z_0 = H(x_0 \times Y)$. Then for any neighborhood U of z_0, only finitely many $D_{i(n)}$ are not contained in U. Again, by a finite application of the pasting lemma, we see that the points $H(x_n, y_n)$ that correspond to these finitely many $D_{i(n)}$ must converge to $H(x_0, y_0) = z_0$, and so are eventually contained in U. The remainder of the points $H(x_n, y_n)$ are all contained in U, since each corresponding $D_{i(n)}$ is contained in U. Thus the sequence $H(x_n, y_n)$ converges to $z_0 = H(x_0, y_0)$, and therefore H is continuous. $\square$

Lemma A.4.3. *If X is a second countable metric space such that each $x \in X$ has a deleted neighborhood that is a 1-manifold with finitely many components, then X is homeomorphic to a locally finite graph.*

Proof. For $x \in X$, we can choose a small deleted neighborhood of x that is a 1-manifold with finitely many components, with each component limiting on x 'exactly once,' i.e., if any component together with x forms a circle, then delete one point of the circle, as well as deleting any components that do not limit on x. Define the valence of a point, $v(x)$, to be the number of components of such a deleted neighborhood. Let $V = \{x \in X \,|\, v(x) \neq 2\}$, which will be a subset of our set of 0-cells. Then $X - V$ is just a disjoint union of open arcs. Suppose there is one of these arcs, a, without compact closure. Let a be parameterized by $(0, 1)$, and consider the sequences $\{a(1/n)\}$ and $\{a(1 - 1/n)\}$ $(n \geq 2)$. If either sequence does not converge in X, then include that sequence in V as well. It can be seen that V is discrete and that $X - V$ is a collection of open arcs with boundary in V, each having compact closure. Also, each 0-cell intersects only finitely many closed 1-cells, thus X is a locally finite graph. □

Lemma A.4.4. *Let C_1, C_2 be disjoint closed subsets of a simply connected one-dimensional neighborhood W of x in a Peano continuum. Then any collection of paths $\{p_i\}$ in W from C_1 to C_2 that are pairwise disjoint on their interiors is finite.*

Proof. Suppose that $\{p_i\}$ is infinite. Let δ be the distance from C_1 to C_2. Let y_i be a point on p_i that is a distance $\delta/3$ away from C_1. Then there is a limit point q_1 of $\{y_i\}$. Let $U \subset W$ be a path connected neighborhood of q_1 with diameter $< \delta/6$. Then in U there must be infinitely many of the paths p_i, which then have points z_i a distance $\delta/3$ from C_2. These points z_i have a limit q_2. Let $V \subset W$ be a path connected neighborhood of q_2 with diameter $< \delta/6$. Then there are distinct p_j, p_k that are joined by a path in V as well as one in U. Since the paths p_i are disjoint, as are the sets U and V, we get an essential curve contained in U, V, p_j and p_k. This contradicts the fact that W is simply connected and one-dimensional. Thus $\{p_i\}$ must be finite. □

Lemma A.4.5. *Let K be a simply connected closed set contained in the one-dimensional set $I(X)$ of a non-degenerate Peano continuum X with no attached strongly contractible subsets. Then any closed arc p in K can be extended to an arc from either of its endpoints to ∂K.*

Proof. First note that $\partial K \neq \emptyset$, since otherwise K would be both open and closed, hence $K = X$ would be a simply connected one-dimensional continuum with no attached strongly contractible subsets, and must therefore be degenerate, which it is not. Now let y be an endpoint of an arc p in K. We show that there is an arc from y to ∂K that does not intersect the interior of p.

We first show that the result follows if there is a component C of $K - \{y\}$ that does not contain the interior of p. If $C \cap \partial K = \emptyset$, then C is also a component of $\text{int}(K) - \{y\}$ hence of $X - \{y\}$, but $C \cup \{y\} \subset K$ is simply connected and contracts to y, contradicting the fact that X has no strongly contractible subsets. So $C \cap \partial K \neq \emptyset$, and since K is arc connected $C \cup \{y\}$ is also arc connected, and must contain an arc from y to ∂K.

Now suppose by way of contradiction that the result is false, i.e. every arc from y to ∂K intersects the interior of p. By the result of the last paragraph, there can only be one component of $K - \{y\}$, namely the one containing p. Consider a point z in the interior of p. Since K is simply connected and one-dimensional, z must separate the path p into distinct components of $K - \{y_i\}$, in particular, there is a component A that does not contain the portion of p from z to y, and a component B that does contain that portion of p from z to y. Then using the point z and the component A as C in the paragraph above, we see that there is an arc a_z from z to ∂K that does not intersect the interior of A. The arc a_z cannot pass through y due to our assumption. So a_z leaves the arc p at some point x_z in B between y and z, possibly at z.

Consider a sequence of points z_i in the interior of the arc p that converge to y, and the associated arcs a_i as discussed in the previous paragraph. We may choose z_i such that z_{i+1} separates y from x_i. Thus we get a sequence of arcs from points x_i on p to ∂K. These arcs must be disjoint since K is simply connected and one-dimensional. Then in a small neighborhood of y, this gives infinitely many disjoint arcs connecting p to ∂K which contradicts Lemma A.4.4.

Thus the result of the lemma is true: there is an arc from y to ∂K that does not intersect the interior of p. □

Lemma A.4.6. *In a rank 0 one-dimensional Peano continuum, every component C of B is a Peano continuum.*

Proof. Components are always connected and closed. Since B is compact, then C is compact also. It remains to show that C is locally path connected. Assume that X is arc-reduced (recall that homotopy equivalences fix B homeomorphically).

Suppose that C is not locally path connected at x. Then there is some open set $U \subset C$ containing x such that no open set V with $x \in V \subset U$ is path connected. Now U corresponds to some open set U' in X (i.e. $U = U' \cap C$). Since X is locally path connected, x has a neighborhood $V' \subset U'$ that is path connected. However, $V = V' \cap C$ is not path connected, since $x \in V \subset U$. Let a, b be points in different path components of V. Then we get a path $\alpha : I \to V' \subset X$ connecting a, b in X, and α must hit some arc ℓ in $G(X)$, since V is not path connected, and C is a component of B. Then this arc ℓ does not separate X, which contradicts the fact that X is rank 0. ($X - \ell$ is still connected and locally path connected, so we can homotop the endpoints of ℓ to a single point in X.) □

Lemma A.4.7. *In a rank 0 one-dimensional Peano continuum, the non-degenerate components of B form a null sequence.*

Proof. Suppose there is a sequence of components C_i of B, each of diameter $\geq \delta$. Choose $x_i, y_i \in C_i$ such that $d(x_i, y_i) \geq \delta$. By passing to subsequences, we may assume that $x_i \to x$ and $y_i \to y$ in X, and then $d(x, y) \geq \delta$. There are path connected neighborhoods of x, y of radius $< \delta/2$, and for $j, k \gg 0$ we have paths connecting x_j, x_k and y_j, y_k in the respective neighborhoods. These paths must contain some arc in G, since they connect different components C_j, C_k of B. But then we have an arc (at least one, but actually lots) that doesn't separate, contradicting rank 0. □

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