2015-06-01

English Learners' Participation in Mathematical Discourse

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Arts

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June 2015

ABSTRACT

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Due to the increasing diversity of mathematics classrooms today, teachers need guidance on how to support English Learners (ELs) in mathematics classes in a way that situates language learning within mathematical activity. Unfortunately, neither mathematics education research nor EL education research is sure how to navigate the complexity of teaching ELs mathematics while supporting both their language development and their mathematical development through their participation in mathematical activity. This study examined ELs' participation in mathematical Discourse, investigating both the mathematical purposes ELs accomplished by using multiple symbol systems, and the way ELs used non-English language (NEL) symbol systems to support their spoken English. The participants were college-aged ELs beginning their studies at the English Learning Center at an American university. The students all had fluency with basic conversational English, and had many different levels of mathematical experience. I identified five categories of purposes in which ELs engaged during mathematical Discourse. I also developed the Replace Augment Learn (RAL) framework that describes how ELs used NEL symbol systems to make up for their decreased English literacy and facilitate their participation in mathematical Discourse. Analysis of the data suggests ELs' use of NEL symbol systems (1) played a significant role in achieving many of the purposes associated with mathematical Discourse, and (2) opened up a space for effective language acquisition. These findings indicate that authentic mathematical activity can be a productive site for language development, and that ELs with basic conversational English and literacy with a variety of symbol systems can participate meaningfully in mathematical Discourse.

Keywords: mathematical Discourse, vocabulary acquisition, English Learner, symbol systems, mathematical activity, literacy

ACKNOWLEDGEMENTS

Thanks to Dan for investing so much in me. You helped me believe in myself and learn to push through my writing anxiety. Thanks for listening to many hours of my ramblings as I tried to formulate the ideas that became this thesis. Thanks also for spending so much time helping me improve my writing.

Thanks to Bryan for supporting this crazy idea I had to get a master's degree. Who knew that by the end we'd both be finishing our master's theses at the same time? Life has been crazy but you've been my biggest supporter. We made it! I love you!

Thanks to my parents for always motivating me to do my best and shoot for the stars.

Thanks to Kailie and Becca for being the best friends I never expected to find in graduate school.

Thanks to Lincoln for being the cutest reason ever for a thesis to take longer than planned.

Finally, they say it takes a village to raise a child, but it also takes a village to finish a thesis. Thanks to our family, our friends, Sesame Street, and especially my parents for helping take care of us and of Lincoln so that we could finish.

Now, on to the next adventure…

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CHAPTER ONE: RATIONALE

Due to the increasing diversity of mathematics classrooms today, mathematics teachers need guidance on how to support English Learners (ELs) in their classes. ELs need support for both general classroom participation and mathematical participation and growth. The two main sources for teachers to learn to provide this support are mathematics education research and EL education research. On the surface, it appears that these two groups both have the same goal—to support ELs in mathematics classes—but the groups attempt to do this in two different ways, neither of which is sufficient on its own. Mathematics education researchers focus on the mathematical activity that all students need to participate in to do and learn mathematics. EL education researchers' main focus is on supporting ELs' language use and development so they can participate in their classes. Goals to support students' mathematical activity and language use are valuable, but do not give teachers a complete picture of how to support ELs' mathematical activity.

Mathematics education researchers propose that for students to do and learn mathematics there are certain mathematical activities students should engage in (NCTM, 2000). The list of NCTM Practice Standards is a representative example of the many activities researchers claim are important for doing and learning mathematics. These are also considered a resource for teachers who are looking for ways to improve students' mathematical experience in the classroom. The five practice standards are *problem solving*, *reasoning and proof, communication*, making *connections*, and using *representations.* Mathematics education researchers propose that by participating in these activities, students are engaging in and learning mathematics. Unfortunately, the standards do not explain how to help all students engage in these activities, particularly students for whom English is their second language.

Mathematics education researchers value equity, but there are very few suggestions in the NCTM standards or in other literature (Schoenfeld, 2002) that specifically address teaching students who are still learning English. The discussion on equity in the NCTM standards explains that there must be high expectations for all students, and that reasonable and appropriate accommodations must be made when necessary. There are no specific suggestions regarding when to make those accommodations, what they might look like, or how to implement them. Similarly, there are very few suggestions within the practice standards regarding how a teacher might support a student who struggles with the language components of each practice. For example, the communication standard gives some insight into supporting students at different age levels (e.g., modeling age-appropriate discourse and writing or allowing younger students to communicate with pictures, gradually introducing more complex language as they mature), but does not explain how to support ELs. A teacher looking for suggestions for how to support ELs mathematical development would have to extrapolate from these few suggestions since none are given regarding ELs' participation in mathematical activity, particularly activities with language components. To deal with this lack of specificity, teachers may turn to EL education research to learn how to teach ELs mathematics.

EL education researchers propose that for ELs to learn mathematics they must have adequate language support. This is based on the belief that language (defined as spoken and written words) is the primary way students explain themselves, debate and discuss mathematics, and communicate knowledge (Gottlieb, Carnuccio, Ernst-Slavit, Katz, & Snow, 2006; Kersaint, Thompson, & Petkova, 2013; Khisty, 2002). The literature suggests that ELs have limited English language literacy, and teachers must help them develop that literacy so they can participate in the mathematics class. Research suggests doing this by focusing on vocabulary development, explicitly teaching culture-specific words, using technology and children's literature to support students' language development, and getting students to engage in different language domains (Gottlieb et al., 2006; Kersaint et al., 2013; Roberts & Truxaw, 2013). Teachers must also make sure the language they use to present content is comprehensible (Khisty, 2002) by speaking clearly and precisely. The lists of suggestions for supporting ELs in a mathematics classroom overwhelmingly focus on students' language development and pay little attention to engaging students in what mathematics educators might consider meaningful mathematics.

The suggestions in EL education research focus on how to engage students in a classroom and support their language development; these suggestions do not attend to how to engage ELs in the mathematics of a mathematics classroom. A particularly common suggestion may help to illustrate why this is problematic. Mathematics teachers are told that a major support for ELs is to use word walls or personal dictionaries because they can help students' language use. Unfortunately, the literature that offers this and similar suggestions does not show how to use these resources to enable students to engage in the ongoing mathematical activity in the classroom. The use of dictionaries, word walls, or even pre-teaching vocabulary does not help students know how to use language in the ways that will allow them to meaningfully engage in authentic mathematical activity. I highlight these particular suggestions not to say that dictionaries or word walls do not have a place in a mathematics class, but to show that the EL education research's focus on these supports does not attend to how ELs might use them to support their participation in mathematical activity. A focus on language use and development alone does not adequately address how to help ELs engage in mathematical activity, an important component of which is situation-specific language use.

It is apparent that no one group is sure how to best navigate the complexity of teaching ELs mathematics while supporting both their language development and their mathematical development through their participation in mathematical activities. The result is that teachers are not adequately prepared to teach mathematics to the ELs in their classes, and teachers are not adequately supported as they attempt to do so. Since neither mathematics education literature nor EL education literature offer comprehensive support to suggest otherwise, teachers may assume that until ELs have developed literacy with academic English the ELs cannot really participate in mathematical activity in the classroom.

Fortunately, the nature of mathematics and the mathematics classroom suggests that there are ways that ELs *can* participate in mathematical activity before they have such sophisticated English literacy. There are many communicative objects in mathematics like symbols and graphs that do not rely heavily on the use of English to communicate meaning. It is reasonable to believe that many ELs might be able to use graphs and mathematical symbols to engage in mathematical activity even if the ELs have lower levels of English literacy. We need to understand the kinds of mathematical activity ELs are able to engage in despite difficulties with the English language. We also need to understand how they are able to engage in that activity. Perhaps objects like symbols and graphs help them do so; it is important to understand if they do, and if so, how. Learning how ELs use these non-language components of mathematics to engage in mathematical activity will be a first step toward understanding what teachers can do to support ELs mathematically. Teachers want to provide support that works towards students' meaningful engagement in mathematics in a way that also adequately supports their language use and development. This study attempts to understand what kinds of mathematical activity ELs can engage in and understanding how they do so. This research will expand mathematics education researchers' and EL education researchers' understanding of how to support ELs in a mathematics classroom in a way that incorporates mathematics educators' goals to engage students in mathematical activity and EL educators' goals of supporting ELs' language use and development.

CHAPTER TWO: BACKGROUND

English Learner (EL) education aims to help ELs learn vocabulary and participate in academic discourse (reading, writing, speaking, and listening to English) to enable them to acquire knowledge (Gottlieb et al., 2006; Kersaint et al., 2013). EL educators' goals for teaching mathematics to ELs align with these same standards. They focus on helping ELs learn mathematical vocabulary and use it in discourse in order to acquire mathematical knowledge.

In this chapter I argue that the suggestions given by EL education research for how to teach mathematics to ELs are potentially useful but inadequate. These suggestions reflect the importance of vocabulary as a part of doing mathematics, but focus mainly on artificial vocabulary acquisition. This is inadequate for two reasons. First, vocabulary learning must take place in authentic mathematical activity such as when students are engaged in mathematical practices. Second, there is more to doing and learning mathematics than word use in classroom discourse; students must behave and fluently communicate in mathematically authentic ways using the many representations used in mathematics. In other words, students need to engage in authentic mathematical practices and use language in appropriate ways as they do so. Mathematical practices have language components that can make participation difficult for ELs. Fortunately, authentic mathematical activity affords the use of symbol systems other than spoken and written words (Siebert & Draper, 2012) that can support EL's participation, mathematical development, and language use and development.

The Importance of Mathematical Vocabulary

Knowing mathematical vocabulary is an important part of learning and doing mathematics. Language is a major form of communication and classroom mathematical conversations are full of vocabulary that students will need to understand and use. Students will encounter many problems, assignments, and assessments that rely on the use of words to present ideas. Students are also expected to use vocabulary-laden textbooks as a resource to learn definitions and concepts. Mathematics teachers and other experts use language to communicate precise meanings and ideas to each other and to students. Students need to know important mathematical vocabulary in order to fully participate in learning and doing mathematics.

EL education research recognizes the importance of vocabulary; much of it focuses on vocabulary instruction as an essential way to support ELs in learning and doing mathematics (Kersaint et al., 2013; Khisty, 2002). The recommendations for how students should learn vocabulary include using word walls, personal glossaries, children's books, illustrations and clipart, and technology. The literature suggests that these tools will effectively teach students mathematical vocabulary. It suggests that these resources can also support students' participation in the class because the resources can make assessments more equitable by giving students a resource to help them communicate mathematical knowledge. Other recommendations include supporting students' vocabulary by giving tests and assignments with simplified language, and providing translations. These are all suggestions that EL research suggests teachers incorporate into their instruction to support ELs' mathematical language development.

At first glance, the suggestions given by EL education research seem to offer a useful list of supports for teachers to provide for ELs in their mathematics classes. However, overemphasizing a focus on vocabulary acquisition does not support ELs' full participation in mathematical activity. While these suggestions have the potential to be useful for teachers and may have a role in mathematical activity, they do not provide all of the language support students need to do and learn mathematics. Much of the meaning in language comes from the situations in which it is produced; supports provided to ELs must attend to language use within mathematical activity.

Authentic Mathematical Situations

The conventional suggestions given by EL education research that focus on vocabulary acquisition fall short in supporting students in learning and doing mathematics because they do not attend to the situated nature of language. Language is “inextricably a product of the activity and situations in which [it is] produced” (Brown, Collings, & Duguid, 1989, p. 33). As a result, the full meaning of a word can never be fully captured by a definition. Students cannot effectively or meaningfully learn mathematical words outside of the authentic mathematical situations and activities that give the words meaning (Brown, Collings, & Duguid, 1989). An *authentic mathematical situation* is a situation in which the activities, contexts, goals, and behaviors are “ordinary practices of the culture” (p. 34) of doing mathematics. These authentic situations are those in which students are engaged in meaningful mathematical activity such as mathematical practices. *Authentic*, in this case, does not necessarily imply that students must learn language within “real-world” contexts but rather that they must learn language as they engage in mathematical practices that are authentic to the activity and goals within a productive mathematics classroom.

NCTM’s Practice Standards (NCTM, 2000) offer general examples of the kinds of activity that occur during the authentic mathematical situations in which vocabulary learning must be situated. The first NCTM standard is *problem solving*. Activities in this standard include solving problems in mathematical and nonmathematical contexts, using and adapting multiple strategies, and monitoring and reflecting on problem solving attempts. The second standard is *reasoning and proof*. Activities in this standard include valuing reasoning and proof, making and investigating conjectures, creating and evaluating mathematical arguments, and choosing the most useful methods of proof. The third standard is *communication*. Communicating mathematically involves organizing ideas, communicating mathematical thinking clearly, analyzing others' thinking, and expressing ideas. The fourth standard is *connections*. This involves recognizing and using connections between different mathematical concepts, understanding these connections and how they fit in a bigger picture, and seeing and using mathematics in non-mathematical contexts. The fifth standard is *representation*. Activities in this standard include creating multiple representations to communicate ideas, using and switching between multiple representations, and using representations to interpret mathematical phenomena. Vocabulary learning should be situated within authentic mathematical situations because they involve these kinds of activity.

Learning vocabulary in authentic mathematical situations is necessary and solves or sidesteps many of the problems caused by vocabulary instruction in contrived situations. First, when students learn vocabulary in authentic situations, they can learn to determine which word meanings and properties are applicable in a situation. Mathematical words have associated meanings and properties that are used to accomplish particular purposes. For example, a student who has not learned the word "variable" in an authentic mathematical situation would not fully understand the many different uses of letters and variables in mathematics (Arcavi, 1994). On the other hand, if students learn about variables as they use letters to communicate an arithmetic generalization, they will understand how letters can be used to represent arbitrary numbers. If students use variables as they represent and explore a linear relation, they experience how letters are used to represent covarying values and parameters. As students learn and use words while they engage in mathematical activity, they will learn how to draw on word meanings and properties to successfully engage in those activities.

Second, learning vocabulary in authentic mathematical situations helps students understand when a concept can be used even if it is not explicitly mentioned in a problem. For instance, if a student is given the graph of a linear equation and instructed to draw a line that is parallel to the given line, a student who has only been taught the definition of *slope* using a word wall, a graphic organizer, or flash cards would not know that the concept of slope is essential to solve this problem. A student who has learned the concept of *slope* by connecting many representations of linear relationships like graphs and equations is likely prepared to use the concept of *slope* to solve the problem effectively. Students who learn mathematical words and concepts as they engage in mathematical practices are prepared to use those concepts even if they are not mentioned explicitly in a problem.

Finally, as students learn vocabulary in authentic mathematical situations they can develop informal and empirical foundations for definitions of words. Artificial situations do not give students opportunities to have meaningful experiences that motivate a word's definition, making it difficult to build an understanding of a formal definition (Moschkovich, Schoenfeld, & Arcavi, 1993). For example, students who spend time solving problems about linear functions, justifying claims about linear equations based on a graph, or using graphs to compare rates of change will have meaningful experiences that prepare them to make sense of the formal definition of *slope.* In fact, these informal experiences are essential for making sense of and using the formal definition (Lobato & Thanheiser, 2002). Students who learn mathematical words by engaging in mathematical practices develop strong informal foundations for the formal definitions they must eventually understand and use (van Lier, 2004).

An example of students successfully learning words in authentic mathematical situations takes place in the bilingual classroom described by Chval and Khisty (2009). The students in this classroom regularly engaged in authentic mathematical activity; they spoke and wrote explanations of thinking, produced justifications, critiqued each others' thinking, and negotiated mathematical meanings. In one lesson, students were learning about right triangles. The teacher and students used authentic mathematical situations to develop meaning for words like *right* and *congruent.* Because important words were introduced, learned, and used within authentic mathematical situations, students built important foundational ideas that supported word meanings and learned how to use the words in mathematically appropriate ways.

A lot of the literature on teaching mathematics to ELs gives suggestions regarding vocabulary support without describing how to apply those suggestions in authentic mathematical situations. For example, Roberts and Truxaw (2013) suggest that teachers should use word walls and graphic organizers to help ELs learn vocabulary, but they do not explain how to use those tools as a part of authentic mathematical activity. Word walls are large posters with words and definitions that students can use as a reference for unfamiliar vocabulary. Graphic organizers are visual tools (such as charts or concept maps) that organize ideas and are often used to help students remember many different components or examples of a definition. Robert and Truxaw explain that a teacher should use word walls to pre-teach vocabulary before a unit. Teachers can then assess students' understanding by having them fill out a graphic organizer, perhaps by sorting the words from the word wall into categories. During the unit that follows, the authors suggest that the same vocabulary will need to be retaught to reinforce and clarify definitions. The authors suggest that teachers should also refine definitions on the word wall to include more formal definitions and applicable "real life" situations as they are encountered in a unit. They then suggest that students can refine their graphic organizers in a similar way and refer to them when they need to remember particular vocabulary.

Creating or referring to a word wall or graphic organizer is not an activity that is inherently mathematical. It may require organizing mathematical words and definitions in an accessible way, but does not involve engaging in mathematics. I am not necessarily suggesting that word walls or graphic organizers could not be used as a part of authentic mathematical activity, but Roberts and Truxaw describe their use separate from authentic mathematical activity and provide no insight into how students might create and use word walls and graphic organizers as part of authentic mathematical activity. Simply referring to one of these resources to see what a word means would not fully support students in using that word and concept to do mathematics; for example, it would not help them know when to use particular mathematical concepts, and it would not show them how to use those words to accomplish mathematical goals. This may cause students to have only superficial or artificial experiences with mathematical words instead of experiences with words that are embedded in authentic mathematical activity.

Summary

Recommendations given by EL educators often have a focus on vocabulary support and use word-acquisition strategies in artificial situations. Some of these strategies like using clip art and children's books to learn words seem trivial. Yet others like having personal glossaries are superficial since no definition can fully capture how to use a word to engage in mathematical practices. These strategies may make vocabulary instruction more memorable, but memorability does not teach students how to use those words during mathematical activity or to understand how those words express and are related to important concepts. Authentic mathematical activity for secondary students rarely consists of clipart and children's books. Other recommendations are to use word walls, personal glossaries, or technology to support vocabulary acquisition. Individually, these practices, while not authentically mathematical, could possibly be used to support students during mathematical activity. Perhaps many of the strategies *could* be used during mathematical practices, but the literature does not explain how to do so. Advice as to how to incorporate these suggestions into authentic mathematical practices would make those tools far more valuable to teachers with ELs in their classrooms.

Participation in Mathematical Discourse

The previous section discussed the necessity of learning vocabulary in authentic mathematical situations. Fortunately, authentic mathematical situations are not only ideal settings in which to learn language, but are also productive settings in which to learn and do mathematics. Authentic mathematical situations involve the use of practices like those described in the NCTM practice standards (NCTM, 2000) or the Common Core Standards for School Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Mathematics educators believe that participation in these authentic mathematical practices is the best way for students to learn and do mathematics. Of interest to mathematics teachers of ELs, then, is not only how ELs learn vocabulary best in authentic situations, but how their language development can be addressed in authentic mathematical situations in the classroom. This section describes students' language use during the discourse in the mathematics classroom. However, focusing on ELs' language use and development in a mathematics classroom does not sufficiently address their mathematical participation and development; there is far more to learning and doing mathematics than learning and using language. This section will then discuss why it is important to focus on ELs' overall participation in a mathematics classroom by discussing their participation in *mathematical "big D" (Gee, 1996) Discourse*.

Language development and classroom discourse

EL education research and standards (Gottlieb et al., 2006; Kersaint et al., 2013) focus on how teachers should support students' use of academic vocabulary in the four language domains (listening, speaking, reading, and writing). The literature that stems from this research consists of many recommendations for teachers. One recommendation is for teachers to make their instruction comprehensible by minimizing how much students must learn solely by listening (Khisty, 2002). They recommend that teachers use drawings and gestures as resources to clarify instruction, write words as they say them, and model how they want their students to talk (e.g., Khisty, 2002). Another strategy is to give students extra time to prepare and practice talking by engaging in a "think-pair-share" activity. Yet another common strategy is to have ELs begin to participate by reading (aloud) another student's work, alleviating the pressure of presenting their own thinking. Teachers may also choose to give ELs the chance to participate by answering simple questions that they are unlikely to struggle with linguistically or mathematically to help them be more comfortable using language in the class.

A focus on ELs' word use in the four language domains *is* valuable but is still not enough to adequately support students' mathematical development. Word use in these four domains in the classroom is often referred to as classroom *discourse* (Gee, 1996). Getting students to engage in activities that support discourse is a good goal, but it only prepares students to use and receive predigested content, consistent with the activities in a traditional, monolingual mathematics class. It does not fully prepare students to engage in mathematical activity or fully support construction of their own understanding of mathematical concepts. For example the suggestions for how teachers can make their talk comprehensible do not give insight into supporting students' talk during small-group mathematical activity. Similarly, the suggestion to use "think-pair-shares" gives students opportunities to practice speaking. It is not clear, however, how to give ELs adequate mathematical support to use "think-pair-shares" as authentic mathematical conversation. Practicing what to say beforehand or sharing someone else's thinking will help students' language development and is a good pedagogical practice. Unfortunately, it gives minimal attention to the mathematical activity students are (or are not) engaged in because the strategy is focused solely on language use and not the content of the language or the role of that language use within the mathematical activity. Using these strategies as the sole way to support ELs in a mathematics class is inequitable because the strategies limit the mathematical activities in which ELs can engage and limits their mathematical development by solely supporting ELs' language use with little or no support to the ELs' participation in the mathematical activity. This result is undesirable and should be considered unacceptable by both mathematics educators and EL educators.

Mathematical Discourse

Because a focus solely on ELs' language use (i.e., discourse) limits their mathematical activity and is not enough to support their mathematical development, researchers and educators must shift their focus and learn how to help students become fluent in *mathematical Discourse* (Gee, 1996). Being fluent in a "big D" Discourse (Gee, 1996) means behaving, thinking, believing, communicating, and using language in ways that identify a person as a particular type of person. For a student to become fluent in mathematical Discourse then, they must learn to behave, think, believe, communicate, and use language in the same ways as expert participants in mathematical activity. Researchers' and teachers' goal should be to support ELs in participating in mathematical Discourse by helping them behave, think, and communicate appropriately during authentic mathematical activity. This broader view of what ELs should do in a mathematics class includes language use as well as mathematical development, and should be attractive to EL educators and mathematics educators alike.

An example of mathematical Discourse may help demonstrate the type of mathematical activity in which ELs need to engage. In a lesson about *slope*, a teacher might begin by orchestrating a discussion about real world examples of slope. The teacher may then transition into talking about the idea of slope in mathematical contexts such as when representing linear relationships. The students may then work on a task that requires them to make sense of the concept of slope by using graphs, tables, and equations. Students may then need to decide if a claim about the comparative slope of two lines is true, write a justification for their answer, and explain their ideas to other students. The class might conclude by talking about a more formal definition of slope and critiquing each group's justification. These activities, behaviors, and examples of language use are crucial to mathematical Discourse when learning about slope, and are precisely the kinds of activities in which ELs should engage. Focusing on *discourse* alone ignores all that should go on in a mathematics class along with and in addition to language use.

For this study, I am particularly interested in the kind of mathematical Discourse that is consistent with the practice standards found in the Common Core State Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) because these are perhaps the most widely adopted practice standards in the US. To be consistent with these standards, ELs need to learn to engage in the following types of activity in a mathematics class: making sense of problems, persevering in solving them, reasoning abstractly and quantitatively, constructing arguments, critiquing others' reasoning, modeling real-world situations with mathematics, using mathematical tools and ways of communication appropriately, attending to precision, making use of the structure of mathematics, and generalizing repeated reasoning. If ELs can engage in mathematical Discourse in which these practices are valued and practiced, they will be participating in mathematics classes in such a way that aligns with the new standards for mathematics classes in this country. They will also be engaging in mathematical activity that will support their mathematical development and foster understanding.

Using a Discourse perspective of doing and learning mathematics highlights why focusing on word use in the language domains is not enough to fully support students' language use and development with regards to the language they will need to use to engage in mathematical practices. The students engaged in the slope task discussed above would need to understand the changing meaning of the word *slope* depending on whether it is being used to talk about the steepness of a road in the task launch or the rate of change of a linear relationship in the task itself. They would need to learn appropriate ways to discuss and represent certain ideas about slope. They would need to learn to create justifications, know when they had adequately justified a claim (for instance, that one line has a steeper slope than another), and learn how to convince others of their claims. Students would also need to coordinate word use as well as the use of graphs, charts, and equations as they discussed, represented, and justified their ideas about *slope*. All of these activities students must engage in and ways to use English that students must understand to fully learn and do mathematics are components of mathematical Discourse. Discourse attends to students' mathematical activity as well as the language use necessary to engage in those activities. Therefore, a Discourse perspective is necessary for fully supporting ELs in a mathematics class.

Moschkovich (2007a) agrees that teachers must help ELs participate in mathematical Discourse and explains why a Discourse view of mathematics learning is necessary. She claims that vocabulary-acquisition or word-meaning perspectives limit the mathematics students can learn to computations and traditional word problems because they, in effect, define mathematics that way. She explains that vocabulary acquisition is not adequate for the language needs of mathematical practices that are valued today, like those practices in the CCSSM (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) She does not give any explanations or research to back up this statement, which highlights the need for further research using a Discourse perspective.

Moschkovich also explains that vocabulary-acquisition or word-meaning perspectives are inequitable. They cause a focus on students' obstacles (instead of competencies), positioning students as mathematical learners with major deficiencies; they could cause a focus on less important, non-mathematical things like the pronunciation of words; they ignore the situational meanings and resources that ELs can use to show and gain mathematical competence; and they can decrease a teacher's expectations of EL students. A Discourse perspective, on the other hand, positions students as capable mathematics learners who use a variety of resources to show competency, maintains a focus on important mathematics, supports a focus on the situational meanings of words and resources ELs may use (Moschkovich, 2007a), and maintains high expectations for all learners.

Mathematical Acts

Since teachers must support ELs' participation in mathematical Discourse, it is important to know how people behave in mathematical Discourse. Mathematical acts are the ways that people behave and engage in mathematical Discourse. A mathematical act is made up of a communicative act, a meaning for the utterance or act, and a goal or purpose (Moschkovich, 2007b). For this study, I define a *mathematical act* to be made up of the use of a symbol system, a meaning associated with the symbol system use, and a goal or purpose. The symbol use may be the use of a single symbol system (like speech) or may consist of the use of multiple systems (such as *gesturing* to certain parts of a *graph* while *saying* a sentence). The following example illustrates the components of a student's mathematical act. A student uses a fraction bar that has been divided into twelfths by marking it to divide it into fourths and says, "two fourths would be the same thing [gestures across 6- 1/12 pieces represented by the fraction bar]". The symbol systems that were used were a fraction bar (a drawing/diagram), gestures, annotations, and verbal English. The meaning of the sign use is, "2/4 is the same as 6/12 because it takes up the same amount of the whole fraction bar." The purposes are to compare the size of two fractions and use the fraction bar representation to justify a claim. In this situation we are able to articulate all three components of the mathematical act. There are situations, however, when we cannot articulate one or more of the components. This may be due to limitations with the data collection methods. It may also occur when parts of the mathematical act are not made public, like when a student draws and annotates a diagram but does not explain her thinking and intentions as she does so.

To understand how ELs engage in mathematical Discourse, one should avoid just studying ELs' mathematical practices; the mathematical practices typically studied and described are too general to shed much insight into how ELs are using symbol systems to engage in particular purposes during mathematical Discourse. The term "mathematical practice" is often used colloquially and is usually not well defined when used. Common lists of practices like the NCTM Practice Standards (NCTM, 2000) or the standards found in the Common Core State

Standards for Mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) describe the goals and purposes of an activity and mention general sign use (like using multiple representations with the goal of communicating mathematics), but are not specific to particular mathematical situations or concepts. This lack of specificity is problematic since teachers need to understand how to support ELs' mathematical acts in specific mathematical situations. None of the general mathematical practices shed much insight into how a teacher could support an EL in the fraction comparison situation described above. For example, telling teachers to get students to use multiple representations does not specify how the students should use them or what they should be trying to accomplish as they do. A suggestion like this based on these general mathematical practices does not describe what this should look like when ELs are involved. It also does not make clear what an EL's role should be, which makes it difficult for the teacher to know what they should be supporting.

Studying ELs' mathematical acts is a far more effective way to study their participation in mathematical Discourse. Mathematical acts are ways of reasoning, arguing, symbolizing, and communicating, *that are specific to particular mathematical ideas* (Bowers, Cobb, & McClain, 1999; Cobb, Stephan, McClain, & Gravemeijer, 2011; Moschkovich, 2007b). Examining ELs' mathematical Discourse by looking at their mathematical acts reveals what symbol systems students use, what meaning their symbol uses have, and the purposes of the symbol use. Viewing ELs' engagement in mathematical Discourse by examining their mathematical acts has the potential to shed insight into how to help ELs use a variety of symbol systems to engage in mathematical Discourse. It has the potential to shed insight into what ELs can engage in and accomplish as they participate in mathematical Discourse. Analyzing their mathematical acts also may shed insight into *how* ELs use various symbol systems to facilitate their participation in mathematical Discourse even with lower levels of English literacy.

A focus on mathematical Discourse does not disregard the fact that language is an important component of doing mathematics; language remains an important, powerful component of mathematical Discourse. This can be illustrated by examining the NCTM Practice Standards (NCTM, 2000). The problem solving standard says that students should read and understand written problems, ask the right questions in order to understand problems presented orally, and develop a common language with other students to use to compare strategies and results. The reasoning and proof standard says that students should produce convincing arguments to justify conjectures. The communication standard clearly involves language, requiring reflecting on and discussing ideas, making ideas public, communicating clearly and convincingly both orally and in writing, understanding others' explanations, communicating mathematics in order to learn mathematics, and building on others' thinking. The connections standard requires that students experience contextualized mathematics—contexts which may be extremely language-laden. Finally, the representation standard says that students should use many representations (those that are based on language, e.g., spoken and written words, as well as those that are not) to reason and communicate mathematically.

Discourse Participation Paradox

The goal for ELs to engage in mathematical Discourse seems like a worthy one, but this expectation may seem paradoxical. ELs really should learn vocabulary within authentic mathematical activity. ELs should engage in meaningful mathematics along with the four language domains during mathematical Discourse by doing mathematical acts. But, how are ELs supposed to fully engage in authentic mathematical activity when the use of the English language is typically such a central component of those activities? If we claim that engaging in mathematical Discourse is how ELs will learn mathematics and the only way ELs will be able to learn the kind of English they need, how are they to participate in the Discourse in order to learn the language when they likely lack sufficient language to participate in the Discourse? Is Discourse participation a paradoxical and inequitable expectation?

Symbol Systems in Mathematical Discourse

A potential solution to this paradox comes from understanding the symbol systems used to communicate meaning in mathematics. A symbol is a type of sign. A *sign* is something that stands for something, to someone, in some capacity (Chandler, 2007). A sign is made up of the signifier, the sense made of the signifier, and the thing to which the signifier refers. For this study we are interested in a particular type of sign called a *symbol*. A *symbol* is a sign whose signifier does not necessarily innately resemble that which is being signified; the connection between the two is conventional and must be learned (Chandler, 2007). A *symbol system* is a set of symbols that can be used to create and convey meaning along with the conventions and norms that are necessary for appropriately interpreting and using the symbols. Examples of symbol systems used to communicate meaning in mathematics are *graphs, drawings/diagrams,* and *spoken English*. To understand a solution to the Discourse participation paradox, one must understand how symbol systems are used to communicate meaning in mathematical Discourse.

To understand how symbol systems are used to communicate meaning in mathematical Discourse, one must understand the reconceptualization of mathematical literacy by Siebert and Draper (2012). Traditionally, the word *text* has referred solely to words, and *reading* and *writing* those texts has meant interpreting and creating those words. Siebert and Draper explain that in mathematics many things besides words are used to communicate meaning. These include particular graphs, charts, tables, gestures, drawings, and mathematical symbols (e.g., ÷), in addition to words. In their reconceptualized view of mathematical literacy, they use *text* to “include all objects created or interpreted for the purpose of constructing, sharing, and negotiating meaning” (p. 182). *Reading* and *writing* means interpreting and creating those objects, respectively. These texts facilitate how one communicates, reasons, problem solves and justifies in mathematical Discourse. These texts are created using the symbol systems that are used to communicate meaning in mathematical Discourse.

A possible solution to the Discourse participation paradox may be in ELs’ use of many symbol systems other than spoken English to engage in mathematical acts. It is reasonable to expect that ELs could participate in mathematical acts using a variety of symbol systems that do not depend heavily on language competency (English or another language). For example, a student with limited English language competency may be able to justify the claim that $x = 3$ and $x = 2$ are the roots of $f(x) = x^2 - 5x - 6$ by symbolic manipulation or by creating a graph and indicating the x-values where $f(x) = 0$. It follows that a teacher could support an EL who is engaging in a mathematical act by supporting and building on their use of symbol systems that do not rely heavily on English language use.

Symbol systems and representations in EL education literature. There are some examples in the literature that support the idea that ELs can use many different types of symbol systems to participate in mathematical Discourse even with limited English literacy. Moschkovich (2007a) described a situation where using a Discourse perspective revealed a student’s mathematical competency as the student described a pattern. Students were given a task to find all rectangles with area 36, calculate the perimeter for each rectangle, and then describe the pattern. Alicia, an EL student in the class, was asked to describe the pattern. She was unable to use the word rectangle in her description but provided a good explanation using gestures and a drawing of a rectangle. Clearly Alicia's mathematical activity relied on the use of symbol systems other than spoken or written English.

Moschkovich (2007a) described another situation where two students used many symbol systems as resources to clarify a mathematical description. In this instance, two EL students (Marcela and Giselda) worked together to decide whether the line $y = -0.6x$ is steeper or less steep than $y = x$. As they explained their ideas to each other, they used their first language (Spanish), the axes and graphed lines on their papers, gestures, and descriptions of real world objects to communicate and justify ideas. They were able to use many symbol systems to participate in mathematical Discourse even though their English language competency was limited.

In the bilingual classroom described by Chval and Khisty (2009), students regularly used many symbol systems to communicate mathematically. When students were asked to find the perimeter of a three-quarter circle that has the area 100 cm^2, a student named Violetta used multiple symbol systems in her work. Violetta wrote an explanation of her answer, drew arrows to parts on a drawing of the three-quarter circle to clarify what she meant by phrases like "the two straight lines," and documented her calculator keystrokes to show her work. The text she created included written English, a drawing, arrows, and written representations of calculator keystrokes. While we don't know the specifics of Violetta's level of English literacy, it is still clear that her mathematical activity involved the use of many symbol systems including some that did not rely heavily on English use. This suggests that regardless of ELs' English proficiency, they may rely heavily upon a variety of symbol systems as they participate in mathematical Discourse.

EL education research does not explain how to support ELs' use of the many symbol systems used in mathematical Discourse or how to help students leverage the use of these symbols in a way that allows them to engage in mathematical acts in the class. Students *leverage* symbol systems by using them to fulfill particular goals in specific mathematical situations, perhaps using literacy with some symbol systems to compensate for a lack of literacy with others. I suspect that not only are ELs able to use symbol systems that are not heavily language laden to accomplish meaningful goals in mathematical Discourse, but that they also leverage non-language symbol systems (e.g., equations, diagrams, gestures) to make up for decreased English literacy. In the EL education literature, there are articles and books that claim that ELs are able or should be permitted to use many symbol systems in mathematics classes (e.g., Kersaint, Thompson, and Petkova, 2013, p. 55). The authors of such statements do not discuss how these symbol systems might be crucial for EL's participation in mathematical Discourse, what ELs are accomplishing in mathematical Discourse, or how ELs leverage symbol systems to participate in mathematical Discourse. The general tendency is that EL education scholars focus on ELs' use of words to acquire knowledge but not their use of multiple symbol systems to engage in mathematical acts in mathematical Discourse.

Symbol systems and representations in mathematics education literature. The mathematics education field values students' use of many symbol systems, but like EL education research, does not adequately explain how ELs can leverage these symbol systems to participate in mathematical Discourse. In the NCTM standards (NCTM, 2000), a clear goal is to get students to use and make connections across many representations. Five representations that are particularly valued and discussed in mathematics education are verbal descriptions, graphs, equations, tables, and drawings/diagrams (Lesh, Post, & Behr, 1987). Some researchers recognize that other symbol systems like gestures or annotations support the use of these five types of representations. For example, gestures can support and clarify a students' use of graphs (Radford, 2003). CCSSM (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) says that ELs should use a variety of representations as resources to communicate mathematically as they learn English. Unfortunately, neither the NCTM standards nor the CCSSM standards explain how ELs can leverage these symbol systems to make up for decreased English literacy to successfully participate in mathematical Discourse.

It is necessary to establish how I will use the term *representations* for the duration of this study. Typically in mathematics education literature the word *representation* is used to describe objects used in mathematics; unfortunately *representation* is not well defined. For this study, I use *representation* to refer a particular instance of symbol use. Representations are the objects created using particular symbol systems to represent something in a specific situation. For example, if a student uses a graph that represents a bird's distance from a tree over time, that graph is a *representation* that was created using the *symbol system* of graphs.

Examining what ELs accomplish and how they leverage the use of different symbol systems in their mathematical acts can shed insight into the kind of support teachers should provide to help ELs participate in mathematical Discourse. Participation in Discourse is crucial for learning and doing mathematics, and using a variety of symbol systems is an integral part of mathematical Discourse; supporting ELs' use of these symbol systems is a productive way to support ELs as competent doers and learners of mathematics. This study focuses on the kinds of mathematical acts ELs are able to engage in as they use a variety of symbol systems in mathematical Discourse. Once we know what kinds of purposes ELs are accomplishing and engaging in during Discourse, the next step is to understand how the various symbol systems facilitate their participation in mathematical Discourse. In particular, since this study is about ELs, we need to understand how they leverage non-English language (NEL) symbol systems to make up for any lack of literacy with English as they participate in mathematical Discourse. NEL symbols systems include symbol systems that do not rely heavily on the use of language (e.g., graphs, gestures, diagrams) as well as spoken and written language besides English.

There is very little in mathematics education research about the use of mathematical symbol systems but there is a large body of related research about the representations that are used to build and express meaning in mathematics. Since representations are particular instances of symbol use, research on representations may contribute to our understanding of how ELs use symbols in mathematical acts. Most of the literature does not use the words *symbol system*, even if they are talking about a concept similar to my definition of *symbol systems.* As I discuss the mathematics education literature below, I use their terms, but note when the concepts they discuss are consistent with the idea of a symbol system.

The main representations (symbol systems) used in mathematics are mathematical symbols (like the ones used in equations), verbal descriptions, graphs, tables, and drawings/diagrams (Lesh et al., 1987). Radford (2003) also suggests that gestures and body language are representations (symbol systems) that communicate meaning in mathematics. Three main sources of meaning making in mathematics are using, translating between, and linking representations (Kaput, 1989; Moschkovich et al., 1993). Students often use many representations to make sense of a problem, particularly if the problem is a realistic problem (Lesh et al., 1987). This research suggests that ELs need to draw upon multiple symbol systems to engage in mathematical activity, to learn mathematics, and to show competency. Most of this research uses a primarily cognitive view, describing the importance of using multiple representations to learn mathematics, but does not explain how students use multiple symbol systems to engage in mathematical practices. In particular, none of this research describes how ELs can leverage their literacy with multiple symbol systems to engage in mathematical acts in the classroom.

Roth and Bowen (2001) take a more social perspective and discuss how representations (symbol systems) are used to *do* things like engage in mathematical practices. The uses of representations are socially decided and often based on conventions, so different people in different situations use representations (symbol systems) differently. Roth and Bowen focus in particular on the use of graphs and other inscriptions and explain that different people use them differently, in different contexts. For example, scientists produce and read graphs differently based on their knowledge and the context in which they work. Most of Roth and Bowen's (2001) work is not based in classrooms, but it seems reasonable to assume that students will use graphs and other symbol systems differently based on their knowledge and the contexts in which they are working. EL students may use symbol systems differently than non-EL students in part because of their differing levels of English language competency, and their differences in cultural and mathematical backgrounds. While Roth and Bowen's research suggest that ELs may use symbol systems differently than non-ELs, their research does not describe these differences, particularly how ELs might support their use of the English language by using other symbol systems.

Moschkovich (2007a) and Chval and Khisty (2009), as described earlier in this chapter, have begun to contribute to research that shows what types of activity ELs are engaging in as they use symbol systems in mathematical Discourse (e.g., justifying a claim and describing a shape) but further research is needed. These authors have shown that in order to equitably teach

ELs, teachers cannot focus solely on vocabulary or academic discourse because that promotes a deficiency view of EL students. The authors have also shown that ELs are able to use a variety of symbol systems to participate in mathematical Discourse. However, they do not provide a robust description of the kinds mathematical acts ELs are able to engage in and accomplish through the use of a variety of symbol systems. They also do not explain how NEL symbol systems make up for decreased literacy with English to allow ELs to participate meaningfully in mathematical Discourse.

Hypothesis about ELs' use of symbol systems. I hypothesize that if ELs are using NEL symbol systems to support their use of English, then their use of NEL symbol systems will contribute to their increased English-language proficiency. An example from the bilingual class described by Chval and Khisty (2009) provides evidence for this hypothesis. Sara (the teacher) valued writing, and the practices in the classroom reflected that. She believed that language development involves more than learning words (a claim supported by the framework described in this chapter), and so she shifted her focus from a more general focus on English development to developing students' ability to express mathematical ideas. As students worked to improve and clarify their communication, often through writing, Sara encouraged them to clarify their explanations by using drawings and other symbol systems that were appropriate in the context. As they engaged in these mathematical acts, their ability to use English language and NEL symbol systems to accomplish particular mathematical goals increased. The teacher saw that the students' descriptions and explanations were improving both mathematically and in relation to their use of English. This suggests that students' use of NEL symbol systems to clarify their written English may have supported, at least in part, their increased English proficiency. Unfortunately Chval and Khisty's work does not describe how that might have happened. The work in Sara's classroom also focused on ELs use of symbol systems to support written English but did not describe how the symbol systems might be used to support spoken English. This study will contribute to understanding the latter.

Research Questions and Summary

The examples used in this framework show that at least some EL students *can* leverage their literacy in multiple symbol systems to perform mathematical acts while engaged in mathematical Discourse. The examples give some idea as to what kinds of mathematical acts students are accomplishing as they use these symbol systems, but more work needs to be done to provide more detailed descriptions of the purposes ELs can engage in. I will address this need by answering the question, *What mathematical purposes do ELs accomplish when using literacy with various symbol systems while engaging in mathematical acts during mathematical Discourse?* The literature also suggests that ELs may leverage their use of symbol systems that are not heavily dependent on the use of the English language to support a lack of literacy with the English language, but the literature does not describe how ELs do this. I address this need by answering the question, *How do ELs use literacy with NEL symbol systems to support their spoken English?* As ELs participate in mathematical Discourse, they will improve a specific type of English literacy. The literacy they will develop is the kind of English literacy needed to participate meaningfully in authentic mathematical activity. As this study addresses the two research questions, it will provide evidence that this type of ELs' English literacy will improve as they participate in mathematical Discourse. Answering these questions is an important step in more effectively supporting and teaching ELs.

CHAPTER THREE: METHODOLOGY

In this section I explain the methodology for my study. I describe the setting and participants, the data collection, and the data analysis.

Participants and Setting

Even though teachers need a great deal of support teaching mathematics to secondary-age ELs, studying college-age ELs will provide insight into how to help both age groups. As evidenced by the literature review in this same study, secondary teachers are perhaps most at need for support for teaching ELs mathematics. This study focuses on college students, however, because it is likely that they have more experience with mathematics and therefore more literacies to draw upon to engage in mathematical acts. Both adolescent and post-adolescent young adults are likely to have the ability to use a variety of symbol systems to do mathematics. Studying these ELs in particular will allow me to learn about how ELs' literacy with a variety of symbol systems facilitates their participation in mathematical Discourse while doing secondary-level mathematical tasks. This knowledge may suggest how secondary students could potentially use symbol systems similarly to engage in secondary-level mathematics, and will then help both secondary mathematics teachers and college mathematics teachers understand how to help ELs in their classrooms successfully learn and do mathematics.

Eight of the ten participants in this study were students from the English Learning Center (ELC) at a large US university, where college-age students come to take classes specifically designed to improve their English. Many students come to the ELC after high school to prepare for college. Others come after completing a college degree in their own country so they can improve their English and prepare for graduate school in America. Still others take a break during their work toward an undergraduate degree in their own country to spend time improving their English before they go back and complete their degree. These eight participants all spoke English as a second language. Since they were volunteers, there were a variety of levels of English proficiency among the participants. The ELs also had a wide variety of first languages, varying levels of mathematical experience, and varying levels of perceived mathematical ability. One of the ten participants was a Mathematics Education native-English speaking masters student. The tenth participant was a native-English speaking undergraduate student who had no background in mathematics education. More detailed descriptions of the individual participants are included later.

The participants were selected on a volunteer basis. I posted an advertisement for students learning English to participate in a study about how English language learners participate in mathematics classes. Some people may have taken fliers, but initial interest was minimal. I contacted a department secretary at the ELC, and she emailed students in the program with the highest levels of English proficiency (those in the "Academic Prep" classes) and sent them information about the study. After she promoted the study I had more than enough volunteers. I chose students who met my criteria based on their first language (I wanted to include students with a variety of first languages in the study) and their availability. I recruited students from only the top few English-proficiency levels so that interviews could be successfully conducted without the use of a translator. I asked for students who had a basic understanding of linear functions and right-triangle trigonometry, so I assumed that any students who contacted me fit these criteria. All of the EL volunteers were on the Academic Track at the ELC, which is the track on which students with the highest English fluency were placed. Students on this track take classes in reading, writing, speaking, listening, and grammar. They also have a *directed studies* course in which they meet with an advisor to plan language and academic goals.

Once I had enough students volunteer for the study, I selected participants and began pairing them up and scheduling their meetings. Each pair was designed to model a different type of interaction that would occur in an English-speaking mathematics classroom. Two pairs modeled a pairing with two ELs with the same first language (which I refer to as the *SFL pairing*, for *same first language* pairing). One pair modeled a pairing with two ELs with different first languages (*DFL pairing*). One EL modeled a pairing of an English-speaking student who does not speak the EL's first language, and an EL (*ELE pairing*). The last pairing modeled the pairing of an EL with a mathematics teacher who does not speak the EL's first language (*ELT pairing*). The following section describes the participants in each pairing.

Session Pairings

To easily identify and refer to specific study participants as I write about them, I give them each a label. The first part of the label (before the dash) identifies what pairing the student was a part of, and the second part identifies the particular student in that pairing. For example, SS-1 is the Spanish-Spanish pairing, student 1. CK-C is the Chinese-Korean pairing, and the Chinese student. MP-P means the Mathematics Teacher-Portuguese pairing, the Portuguese-speaking student. The Mathematics Teacher (MP-M) and the English student (SE-E) were both native English speakers.

Spanish-Spanish (Pairing 1- SFL Pairing). Pairing 1 consisted of a pair of students whose first language is Spanish. This pair piloted the first three mathematical tasks. The time they committed to participate ran out before they were able to do task 4. Because minimal changes were made to the first three tasks after piloting them, the data for Pairing 1 is included in the analysis.

The first student **(SS-1)** was 17 years old and had come to the ELC right after his high school graduation. He began learning English in elementary school in Mexico from native-Spanish speakers. He did not like his experience learning English from those teachers. At the time of the interview he had been in America learning English for two months. Until he came to America he had never taken any content classes in English. He had taken mathematics classes up through Calculus 1 in Mexico (none of which were in English).

The second student **(SS-2)** was 37 years old and was at the ELC to prepare for graduate school in America. He began learning English in elementary school in Chile. He said that he only learned a little bit of English during that time. He began studying English more seriously when he served a mission for the Church of Jesus Christ of Latter-day Saints in southern Chile, since foreign missionaries are encouraged to learn English when given the opportunity (e.g., when they are paired with an American missionary). He had been in America studying English for two months at the time of the interview. He has a degree in computer engineering, so he had taken all the mathematics classes necessary for that profession. Those mathematics classes were all in Spanish. When he came to America (two months prior) he enrolled in a G-MAT prep course to prepare for graduate school applications, so he had done a small amount of mathematics in English.

Chinese-Korean (Pairing 2- DFL Pairing). Pairing 2 consisted of a pair of students with different non-English first languages. The first was a student whose native language is Chinese, and the second was a student whose native language is Korean.

The first student (**CK-C**) was 36 years old and is from Taiwan. Her native language is Mandarin Chinese. She began learning English when her parents hired a tutor to teach English to a large group of elementary-aged kids. The tutor was a native-Mandarin speaker. In Junior high and high school she had to take English as part of the requirements to graduate. Those teachers were also native-Mandarin speakers. Their studies focused on what she called "life English" and did not include any mathematics in English. At the time of the interview she had been in America for six months learning English. She said her current English classes focused on listening, writing, speaking, and reading. In senior high school she chose the language-history track and did not take higher-level mathematics classes like calculus.

The second student (**CK-K**) is from South Korea and speaks Korean. I do not know his age but based on our conversation I would guess that he was in his early twenties. He began learning English in Korea in middle school. He told me that the English education he received mainly focused on grammar. He did not study reading, writing, listening, or speaking in English in Korea. The teachers who taught him English were not native-English speakers. At the time of the study he had been in America learning English for approximately a year and a half. He told me he was surprised how different the English he learned in Korea was from the English he experienced and was learning here in America. He took mathematics classes up through calculus in Korean in high school. Here in America he had sat in on a mathematics class at the local university to get a feel for how mathematics is taught in America. The class he observed was the extent of his mathematical experience in English.

Chinese-Chinese (Pairing 3- SFL Pairing). Pairing 3 consisted of a pair of students whose native-language is Mandarin Chinese. This pairing also represented a situation where a teacher pairs two ELs with the same first language to work together. I included a second SFL pairing (after Pairing 1) because while Spanish may be the dominant minority language in many areas of the country, it is certainly not the only possibility. I wanted to make sure any generalizations I made considered the activity of as wide a variety of first languages as possible, so I included a second SFL pairing. I also wanted to have a same-language pairing complete all four tasks, and since Pairing 1 was the pilot and only did the first three tasks it seemed reasonable to include this second SFL pairing.

The first student (**CC-1**) was 21 years old. She began learning English in third grade in China. These English lessons focused on reading, writing, and grammar, but did not focus at all on listening or speaking. At the time of the interview she had been in America for three months. She studied some calculus in high school. All of her mathematics instruction had been in Chinese.

The second student (**CC-2**) was about the same age as CC-1. (I do not know her actual age but got the impression she was at about the same age and grade level as CC-1). She began learning English in fourth grade in China. She said the classes were "for fun"; they only took them once or twice a week and just learned simple things. At the time of the interview she had been in America for seven months and had been at the ELC learning English for the past two months. She had finished her first year at a Chinese University and had taken a higher-level mathematics class (she did not know how to explain what class it was in English). She said she took mathematics all the way through senior high school but did not say what the highest class was that she took. During her work on the task, however, she correctly used differential calculus, which suggests she had at least taken the equivalent of Calculus 1. She had never taken any mathematics classes in English.

MathEd-Portuguese (Pairing 4- ELT Pairing). Pairing 4 consisted of a mathematics educator whose native language was English and a student from Brazil whose native language was Portuguese.

The first participant (**MP-M**) was the mathematics teacher. She was 23 years old and at the time of this study was finishing a master's degree in mathematics education. She had less than one year of experience in public school classrooms but had two years of experience teaching university classes. Her native language is English and she also spoke Spanish fluently. I paired her with someone whose first language was not Spanish to represent a situation where a teacher is working with an English Learner whose first language is not spoken by the teacher.

The second participant (**MP-P**) was an 18-year old from Brazil who spoke Portuguese as her first language. She started studying English in high school when she took it as a second language class much like Americans might learn Spanish or French, but said that she never learned it well. She said her real experience learning English started when she came to America (specifically the ELC) to study English. At the time of the interview she had been here in America for approximately three months. She said she really liked math and had been studying it seriously starting when she was 13 years old until the end of high school (she hadn't yet started college). She reported having taken Algebra, Geometry, Arithmetic, Financial Math, and Trigonometry. She also said she studied about matrices, combinatorics, probability, and some statistics. She had not taken Calculus. Her mathematics classes were all in Portuguese. She did not speak Spanish and Spanish was not explicitly used at any time by this pairing. At one point MP-P was trying to think of how to say a word in English. MP-M asked her to say it in Portuguese in case she recognized a similar word in Spanish, but she did not recognize the word.

Spanish-English (Session 5- ELE Pairing). Session 5 consisted of a student whose native language is Spanish and an undergraduate student whose native language is English.

The first participant (**SE-S**) was from Columbia and his native language is Spanish. I do not know his age. I would guess he was in his late twenties. He had been studying English for six months at the ELC. He had the lowest level of English literacy of any of the participants. He had only taken basic mathematics in secondary school, and had some statistics classes in college but had never taken college algebra, trigonometry, or calculus. None of his mathematics classes were in English. He was also the student with the most limited mathematics experience of the participants in this study.

The second participant (**SE-E**) was a 21-year old, undergraduate, English-speaking college student. She was not training to be a mathematics teacher. She spoke some Spanish, something I did not know beforehand. I still use this data to represent the pairing of an English-speaking student and a EL student where the English-speaker does not speak the EL's first language; SE-S did not know that SE-E spoke some Spanish, and Spanish was never used by either individual during their work on the mathematical tasks. SE-E took up through (and including) "AB Calculus" (Calculus 1) in high school and had not taken any other mathematics classes during her undergraduate work.

Not only were the participants at varying levels of mathematical proficiency, but also had varying levels of perceived mathematical ability. Some participants did not think they were very good at mathematics. They seemed to base this opinion on the classes they took, their prior success, or how long it had been since they had taken a mathematics class. For example, CC-1 said she was not good at mathematics. I told her that she did really well on the tasks and she said, "Well, I'm Asian". The variety of tasks posed to students attempted to ensure that each of these students would be able to at least engage in some parts of the tasks so I could observe their activity.

Mathematics Sessions and Tasks

Each pair met together to work on some mathematical tasks. The full tasks as the students received them are found in Appendix A. Refer to Figure 1 for a summary of each task. These tasks were chosen and designed to lend themselves well to the use of multiple symbol systems and encourage interaction and collaboration between participants. These tasks vary according to content and level so that hopefully no student would struggle with all four tasks. The levels of the tasks vary, but are all considered appropriate for secondary students (i.e., they do not consist of mathematics that one would have to have completed secondary mathematics to understand.) Pairing 1 piloted the first three tasks and the other four pairings completed all four tasks during their sessions.

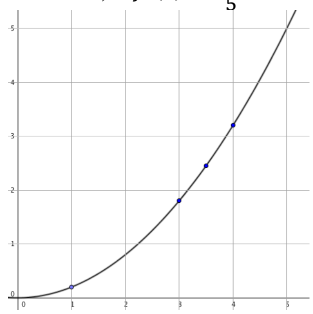
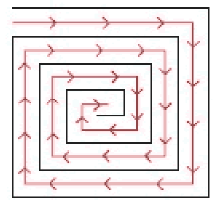
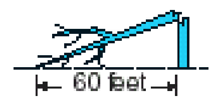
	Task 1	Task 2	Task 3	Task 4
Title	The “handshake” task	The “average speed” task	The “pathway” task	The “broken tree” task
Graphics	*No graphics*	This graph shows a bird’s distance (in meters) from a tree over time (in seconds). $f(t) = \frac{1}{5}t^2$ 		
Question(s)	There are 5 people at a party. Everybody shakes everybody else’s hand. How many handshakes happen at this party? What if there are 10 people at the party? How many handshakes happen? What if there are *n* people at the party? How many handshakes will happen?	What is the bird’s average speed from t=1 to t=4? From t=3 to t=4? From t=3.5 to t=4? Use average speeds to approximate the bird’s speed at exactly 4 seconds.	Jon walks along the spiral path on the red line. The line is in the middle of the path. If the path is 2 meters wide, how far does Jon walk?	A tree broke ¼ of the distance up the trunk (from the ground). The tree fell so that the top landed 60 feet away from the base. How tall was the tree before it fell?

Figure 1. Overview of tasks

Students met together in their pairing, and after signing disclosure documents they began their work on task 1. I gave them each their own copy of the task but suggested that they work together to solve the problems. After each person in a pair decided they were done with the task, I gave them task 2, and so on until they had completed all four tasks or until their time had expired. The structure of each session allowed me to observe students’ mathematical acts as they engaged in mathematical Discourse. I took the role of a silent observer unless students asked me clarification questions when trying to understand the questions or what was given via symbol systems such as drawings/diagrams. One exception to this occurred during the session with pairing 4 (MP). I interacted with MP-P more during their work on task 2 than I did with other students; we discussed what “average speed” means more than I did in the other sessions. This was deemed appropriate since MP-M represented the role of a teacher working with an EL; more “teacher interaction” did not detract from the EL’s work on the tasks.

Data Collection

I collected video data by filming the meetings. I arranged students on one half of a hexagonal table and placed a tripod with a video camera on the other half of the table. The camera pointed down at the students and their work. It recorded their gestures, their work on the task, and their interaction. I also collected their written work on the tasks to help with my articulation of the three components of their mathematical acts.

I also took field notes as I observed the students working during each session. I made note of trends that I observed and interesting moments that I wanted to look at in further depth. My field notes about trends I observed assisted my analysis as I categorized purposes and explored ways ELs used symbol systems to facilitate their participation in the mathematical Discourse. An example of the interesting moments I looked for were times students use NEL literacies where a native-English speaker might have spoken English to communicate something. During the sessions I also made note of any questions that came to my mind to ask that student in their interview. For example, I noted when a student struggled to identify examples to discuss during the interview regarding whether the student felt like she was struggling with the mathematics or the language.

I held the first session and piloted the first three tasks. I added a “2 meter” label to task 3 and labeled the axes on the graph in task 2 (I had forgotten to include those before that session). I determined that as long as I manned the camera adequately (e.g., zooming in on hard-to-read work) one camera was adequate. I used the data from the first session to complete a small portion of the first pass of my analysis (described below) in which I marked the ELs' turns, and articulated the symbol use (SU), the meaning, and began to articulate their purposes. I did enough coding to determine that my data collection procedures would be adequate, and I scheduled the rest of the sessions.

After each session, I coded the ELs' turns by marking them on a StudioCode timeline. I marked any places where I had questions or was unsure what the EL was saying or writing. Then I scheduled an interview with each EL participant.

I interviewed each EL that participated in the study. Interview questions can be found in Appendix B. During each interview I checked my interpretations of what they were saying or doing during their work on the mathematical tasks by briefly discussing assumptions I made regarding their strategies. I also asked them questions about their experience doing the tasks and about both their mathematical and English background. At the end of each interview I gave students the opportunity to tell me anything they wanted about how they best learned mathematics and how they thought mathematics teachers should help ELs. ELs' responses during this part of the interview helped focus my decision about what was most important to focus on during my data analysis. For example, many students discussed that when teachers gave them graphs or drawings/diagrams to use, they had an easier time understanding the questions and working on the problems they were given in English. Hearing this over and over emphasized the importance of investigating *how* the use of various NEL systems might help ELs do mathematics.

Data Analysis

To analyze the data collected from the sessions, I primarily used StudioCode to code the videos of the sessions. StudioCode connects a coding timeline to the video and audio so that codes are connected to the timestamps of the video, enabling coding to be done while simultaneously running the video. Coding within StudioCode also allowed for instant access to the video segment that corresponded to a particular code. StudioCode allows for as many lines of code as necessary, which allowed codes to overlap (like if two ELs had turns that overlapped) or for any one moment to have multiple codes attached. For the instance of any particular code, the code extended from the beginning to the end of the moment so there was no ambiguity about what part of the video the code is referring. For each box of code connected to a portion of video, there is a textbox feature. In this textbox I annotated each turn with information regarding each component of the mathematical act that took place in that turn.

Coding Turns for Components of a Mathematical Act

The first step in coding the data consisted of identifying each turn that each person took as they worked on the mathematical tasks. A turn is a symbol use that is made up of the use of at least one mathematical symbol system; one turn may have the use of many symbol systems. Within each turn I articulated (in a text box) the components of a mathematical act, or in other words 1) the symbol system use, 2) the meaning, and 3) the purpose(s). When there was a clear place where the person shifted to a new sign use with a new meaning and purpose (e.g., there was a pause, nobody else took a turn, and then the person started talking about another aspect of the problem), I split this turn into two turns. Therefore, there could be consecutive turns taken by the same person that still count as separate turns. A student may have been trying to accomplish multiple purposes simultaneously through a sign use; this did not merit dividing a turn into multiple turns. *If, however, that student tried to accomplish one purpose during the first part of their turn, and then stopped trying to accomplish that purpose and began trying to accomplish another, that turn was separated into two turns.* Likewise, if a student tried one particular symbol use (using one combination of symbol systems) to accomplish a purpose, and then stopped and began using another set of symbol systems to accomplish the same purpose, I coded the segment as two turns.

The meaning and the purpose were determined based on my best interpretation of the participants' actions in the immediate context of the problem such as their interaction with the other people in the pairings, their written work, their word use, their actions, and their voice inflections. There were times where the turn was ambiguous or there were multiple equally likely meanings. In those cases, the meaning was coded as "cannot articulate." On these turns I made note of what I thought they meant or the multiple possibilities for the meaning so that I could ask for clarification during the interview. The purposes I coded for each turn were careful articulations of the purposes they were likely trying to accomplish based on the evidence I had from their words, actions, and the context. I used any further information I gathered during the interviews to refine my articulations of the components of a mathematical act for each turn. Each turn consisted of only one mathematical act. If the students stopped engaging in a particular set of purposes and began engaging in a new set of purposes, the turn ended and a new turn and mathematical act began.

I was able to articulate all three components of a mathematical act for most turns. There were situations, however, when I could not articulate one or more of the components. Sometimes this was due to limitations with the data collection methods. These limitations included occasional problems with the camera angle (like when a student put their hand between the camera and their work); times when a student's voice was not loud enough to articulate something being said; and times when I could not figure out a student's meaning and when I asked, neither could the student. Sometimes turns also occurred in such a way that parts of the turns were not made public, like when a student drew and annotated a diagram but did not explain her thinking and intentions as she did so. In these situations I still used in my analysis the parts of the components of a mathematical act that I *was* able to articulate; just because parts of a turn were not made public did not mean the turn was not a mathematical act.

Symbol Use. For each turn I articulated the student's symbol use. This was, essentially, a transcript of what they said and did, with each use of a symbol system articulated. For example, if a person wrote an equation as she was describing something, I articulated in brackets what she wrote in relation to what she was saying. I referred carefully to students' written work as I watched the video to be as precise as possible with my articulations of participants' symbol use. As I articulated the symbol use, I kept track of the various symbol systems that ELs used as they worked on the tasks in this study and report that list in the first results chapter.

Meaning. I articulated my best interpretation of the students' meaning for the symbol use in each turn based on careful analysis of the evidence I had from their words, actions, and the context. The meaning for each turn often took a statement (symbol use) with some ambiguity and clarified it (within reason according to the information available) and removed habits of speech. Meanings often had sentences and phrases reordered slightly to communicate more effectively what I thought the student was trying to communicate. The following example will illustrate the relationship between a symbol use and the corresponding meaning. The student's symbol use and the corresponding meaning were as follows:

Symbol use: "We can do that, do you remember the form-- formula for average? It's equal to [writes "$\bar{x} =$"], to summatory divided by the [writes "$\frac{\Sigma}{n}$"], the number of the [circles n], uh, whole numbers. So this is a summatory [gestures to $(9 + 8 + 7 + \ldots + 1 + 0)$]."

Meaning: The formula for the average is the summation divided by the number of things, which is n. So, $\bar{x} = \frac{\Sigma}{n}$. $(9 + 8 + 7 + \ldots + 1 + 0)$ is a summation.

The articulated "meanings" were only used for purposes of convenience as the data was analyzed so that the sometimes-difficult task of figuring out a student's utterances and strategies did not have to be done over and over again each time the data was coded. When analyzing how ELs used symbol systems to support spoken English, I analyzed their symbol use—not the "meaning"—to remain loyal to their work.

Purposes. To code purposes, I used an open coding scheme. I began to articulate purposes in writing in StudioCode text boxes associated with each EL turn by describing the immediate mathematical activities they seemed to be engaged in at each point during the turn. I considered what mathematical activity they were engaged in at times like for each sentence, for each incomplete thought, for each related cluster of non-language symbol usages, and for the overall turn. It quickly became apparent that ELs engaged in many different purposes. I decided that to prevent the purposes from becoming too unwieldy I needed to create a standard list of the purposes students were engaging in so I could be consistent with the language I used while coding. I coded and recoded small selections of data from all of the different sessions until I felt I had a complete list of the purposes ELs in this study engaged in as they did mathematics. I then coded all turns ELs took, determining which purposes applied to each turn using the same guidelines that helped me during the first attempts to articulate purposes. As I coded I refined the definitions of each purpose code. I considered a purpose a viable purpose code if it seemed to adequately describe with a high level of specificity similar activities ELs seemed to be engaged in across all five sessions. When I could adequately describe the purposes for all turns the ELs took during the study with the list of purpose codes I had created, I considered the list complete for the purposes of the analysis. These purpose codes will be discussed in Chapter Four.

Upon completion of this analysis I sorted the purpose codes into categories. I originally grouped them in ten different categories. Further work comparing the purposes I found to the descriptions of the CCSSM practice standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) led me to establish the five categories I discuss in the first results chapter. It is important to note that the CCSSM did not guide me as I analyzed the data and observed the purposes of the ELs' turns. I analyzed and compiled the purposes first, and then in an effort to sort them into meaningful categories I compared them to the CCSSM practice standards and found that many of them aligned well. Because of that alignment I used those practice standards to guide the categorizing of the purposes, but not the creation and description of the purposes themselves. A complete list of purposes and purpose codes is found in Appendix C. This analysis led to answers for the research question *What mathematical purposes do ELs accomplish when using literacy with various symbol systems while engaging in mathematical acts during mathematical Discourse?*

Coding how ELs use NEL Symbol Systems to Support their Spoken English

The analysis I now describe was done to answer the question, *How do ELs use literacy with NEL symbol systems to support their spoken English?*

Developing the Replace-Augment-Learn Framework. To determine how ELs were using NEL symbol systems to support their spoken English, I analyzed turns that had already been coded for components of a mathematical act using an open coding scheme. I began by going through the data and flagging turns where it seemed ELs were relying on NEL symbol systems more than a typical English-speaker might. This initial flagging was a judgment call based on how I would expect English-speaking secondary students to articulate their thinking. Another way I identified turns to flag during this initial pass was to consider how much I needed to rely on their use of NEL symbol systems to articulate their meaning. If I needed to rely on their use of NEL symbol systems more than seemed typical for an English speaker, I flagged the turn to examine closer. After I had accumulated a list of these flagged turns, I reexamined them to determine what it was about the way the ELs used NEL symbol systems that was different than what an English-speaker would probably do.

I started creating rough descriptions of the different ways the ELs used NEL symbol systems to support their spoken English in the turns I had flagged. I categorized these descriptions to combine similar practices. I recoded all the data using these categories. As I coded I refined the categories until I had created three categories (*replace*, *augment*, and *learn*) and articulated the different moves that fell in each category. I use the term *move* to refer to an EL's use of an NEL symbol system to support his English language use. I went through all the data a third time and coded turns looking for evidence in each turn of any of these moves. During this third time through the data I felt that my coding scheme (the three categories and their moves) was adequately descriptive of what I observed and did not require further changes that would affect coding. After the framework was complete I coded ELs' turns and kept track of what portions of the turn received move codes. Turns often received more than one of these codes due to the complex nature of communicating mathematics, particularly for ELs with less sophisticated English literacy.

Finding RAL-reliant purposes. To get an idea of the extent to which replacing, augmenting, or learning (RAL) moves facilitated ELs' engagement in the purposes, I compared the total number of times a purpose occurred with the number of times that purpose occurred during replacing, augmenting, or learning. I excluded the Chinese-Chinese session from this portion of the analysis since they interacted almost entirely in Chinese; I felt the high number of purposes with no chance of being coded as replacing, augmenting, or learning (since there was not any spoken English) would skew the results.

Appendix D and E are products of this portion of the analysis. Appendix D reports the overall occurrence of RAL moves for each purpose. A cell in this table means that for the turns with a particular RAL move code, there were *n* number of those turns that also received a particular purpose code. This analysis is meant to give an idea of the breadth of RAL move codes across the purposes. Appendix E reports the impact of the RAL moves for each purpose. The table reports a count of the number of turns that were coded as having a particular purpose. It also reports the number of turns with that purpose that received RAL move codes. This information was used to calculate the percent of turns with a particular purpose code that also received an RAL move code. This analysis showed the impact of the RAL moves for each purpose. There were many purposes where 50% or more of the turns with that purpose received RAL move codes. Those purposes are what I call RAL-reliant purposes.

CHAPTER FOUR: ELS' MATHEMATICAL PURPOSES AND SYMBOL SYSTEM USE

This chapter answers the research question, *What mathematical purposes do ELs accomplish when using literacy with various symbol systems while engaging in mathematical acts during mathematical Discourse?* I first briefly discuss the symbol systems ELs in this study were using to participate in the mathematical Discourse. This list of symbol systems made up the symbol use component of their mathematical acts during this study. It also answers the first part of the above question by showing what literacies ELs used to engage in the purposes that I describe later in the chapter. I then describe and illustrate the purposes ELs engaged in using these systems.

Symbol Systems

Analysis of sign use, the first of the three components of a mathematical act, allowed me to compile a list of the literacies that ELs used as they participated in mathematical Discourse. The symbol systems they used are graphs, symbols in the form of equations, symbols like letters and numbers that are used as labels or in lists, drawings and diagrams, gestures, arrows that are used to organize thinking or emphasize particular portions of a student's work (arrows that are used outside of drawings and diagrams), calculator keystrokes (coded as a symbol system when students have their partner watch them type things into a calculator as part of a description of their thinking), annotations, spoken English, written English, and their first language. No tables were used for the tasks in this study but could easily be added to the list of symbol systems for future research. I categorize these symbol systems into three useful categories: a student's first language (spoken and written), symbols that rely heavily on the use of the English language (spoken and written English), and those that do not (the rest). When I refer to the third category, I will call them the "non-language" symbol systems, meaning they are symbol systems that do not rely heavily on the use of language (English or otherwise). As described previously, NEL symbol systems are all symbol systems excluding spoken and written English.

The students in this study used the various symbol systems (non-language as well as language) in varying degrees to do meaningful mathematics during mathematical Discourse. They usually used multiple symbol systems in their mathematical acts, and as they did so, engaged in many different purposes. The next section describes what purposes students in this study engaged in through the use of these symbol systems.

Mathematical Purposes Accomplished Through Symbol Use

The purposes ELs were engaged in during this study provide evidence that ELs can meaningfully participate in mathematical Discourse. Analysis of purpose, the third component of mathematical acts, revealed a detailed picture of the kinds of authentic activity ELs were involved in during their participation in mathematical Discourse. ELs engaged in these purposes both as they interacted with their partner and during individual work that helped them prepare to contribute to the ongoing mathematical discussion. The purposes ELs engaged in fall into five categories. In this section, I first give an overview of the five purpose categories and their prevalence, and then describe each categories in more detail. I was able to identify 49 purposes in which the ELs in this study engaged, far too many to describe in this chapter. Because of that, I include descriptions of a sample of the purposes in each category to give a sense of the types of purposes that comprise that category. For detailed definitions of each purpose the reader may refer to Appendix C.

Overview of the Purpose Categories

The purposes that ELs in this study accomplished as they used symbol systems to participate in mathematical Discourse fall into the following five categories. The first category is *make sense of problems and persevere in solving them.* The second is *model and reason abstractly and quantitatively.* The third is *construct viable arguments and critique the reasoning of others.* The fourth category is *facilitate communication and cooperative work*. The final category is *look for and generalize patterns and repeated reasoning*. These categories are listed in Table 1. The ELs in this study took a total of 947 turns. These turns received a total of 1834 purpose codes. The table lists the number of these codes from each category and the percent of total codes that came from that category. Percentages are rounded to the nearest whole.

Table 1:
Overview of Purpose Categories

Purpose Category	Number of codes	% of total codes
1. Make sense of problems and persevere in solving them	498	27%
2. Model and reason abstractly and quantitatively	507	28%
3. Construct viable arguments and critique the reasoning of others	442	24%
4. Facilitate communication and cooperative work	340	19%
5. Look for and generalize patterns and repeated reasoning	47	3%

Category 1 is listed first because many of its purposes are ones that ELs engaged in as they began work on the problem; these purposes usually showed up first in ELs' turns as they worked on a task. Categories 2-5 are then listed according to their prevalence. The most prevalent purpose category is category 2, *model and reason abstractly and quantitatively*. A close second, however, is category 1, *make sense of problems and persevere in solving them*. It is important to notice that the frequency of purpose codes is fairly evenly distributed throughout the first four categories. This shows that the ELs in this study were engaged in mathematical Discourse in a variety of meaningful ways that one would expect from any student participating in mathematical Discourse; their activity was not isolated to a small selection of purposes.

Category 5, *Look for and generalize patterns and repeated reasoning,* was by far the least prevalent purpose category. I do not believe that this suggests that ELs are less capable of generalizing than they are of engaging in the other four categories. It is likely that the tasks themselves and the lack of emphasis on generalization in the questions influenced this low number. If ELs were observed doing tasks that were designed to involve more generalization, I suspect ELs would have engaged in purposes in this category far more frequently.

Descriptions and Examples of the Purpose Categories

In this section I describe each purpose category in more detail. For each category, I present a description of the category, a statement of how the category corresponds to the CCSSM practice standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), a table of the purposes in that category and their frequencies, examples of turns with purpose codes from that category, and a discussion concerning how the purposes in that category were accomplished. Tables for each category report the percent of the 947 turns ELs took that were coded for each purpose in the category. The sum of the percentages exceeds the percent of turns that received a code within this category because some turns received more than one code from each category. Statements of how each category relates to the CCSSM practice standards are provided to show how purposes in the category relate to what the field considers to be meaningful and important mathematical activity. A more detailed discussion of how the purpose categories align with the CCSSM practice standards is given in the summary at the end of this chapter. Examples of purposes in each category were chosen to illustrate the breadth of the category, and include purposes of varying degrees of prevalence. Each example consists of a turn taken by an EL while working on one of the four tasks (see Figure 1) in pairs.

Category 1: Make sense of problems and persevere in solving them. The first category is *make sense of problems and persevere in solving them*. This category is based on the CCSSM practice standard by the same name (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). Students who engage in purposes within this category work to understand questions and problem situations, plan solution strategies, analyze known information, make hypotheses, and review their own work. Table 2 is a list of the purposes that ELs in this study engaged in that fall in this category.

Table 2:
Purpose Category 1 Prevalence

1. Make sense of problems and persevere in solving them	% of turns with this code
Far more than typical	
Propose a solution/answer or make a hypothesis	12%
Read and understand the question/problem	10%
Slightly more than typical	
Establish a plan—how should we try to solve this problem?	5%
Try to remember a formula	5%
Self correction	5%
Typical prevalence	
Decide the next step needed to solve the problem	3%
Identify “givens”	3%
Establish a “big picture” goal—what is needed to solve the problem?	3%
Review own work	2%
Classify problem	1%
Obtain additional resources	1%
Decide if a strategy is applicable	1%
Test a representation or hypothesis using a simpler case	<1%

The majority of the purposes ELs engaged in across all purpose categories occurred in 1%-3% of turns. Purposes in this range occurred in what is considered typical prevalence. In all purpose categories, the purposes that occurred in 4%-7% of turns are considered ones that occurred slightly more often than was typical. The few purposes across all the categories that occurred in more than 8% of turns are considered ones that were far more prevalent than was typical. The majority of the purposes in category 1 are in the typical range. The two category 1 purposes that occurred in the highest percent of turns are *read and understand the question/problem* and *propose a solution/answer or make a hypothesis*, and occurred far more than was typical. The least common one was *test a representation or hypothesis using a simpler case,* which only applied to 3 of the 947 turns. Overall, the purposes in this category seemed to occur in a typical percent of turns compared to all of the purposes that occurred during the study.

I now include two examples of ELs making sense of problems and persevering in solving them to illustrate the kinds of purposes that the ELs were engaged in within this category. The examples show how students engaged in purposes associated with making sense of problems and purposes associated with persevering in solving the problems. The turns described in these examples received additional codes that are not a part of this category; for clarity the other codes will not be discussed here. I also include a brief report regarding how students *read and understood the question/problem* for the average speed task.

The first example illustrates a common way that participants engaged in the purpose *read and understand the question/problem.* This particular purpose illustrates the type of purposes associated with the first part of the title for this category, namely making sense of the problem. CK-K was working on the handshake task and had just gotten to the problem where there are 10 people at the party. He read the question mostly silently to himself, and as he read he underlined "10 people," "the party," "handshakes," and "happen." ELs in this study often annotated the problems themselves as they read and attempted to understand them.

The second example is of an EL engaged in the purpose *test a representation or hypothesis using a simpler case.* This purpose illustrates one way that ELs persevered in problem solving—figuring out if a representation they were trying to use was a viable way to solve the problem or if they should abandon it. It occurred shortly after the previous example when CK-K was working on solving how many handshakes there would be if there were 10 people at a party. Throughout his work thus far on the handshake task he had struggled deciding which of two representations he thought accurately represented this problem. (In this case, the two solution methods he was deciding between were made of representations that were part of two different symbol systems.) The general description of his first representation is $\# \ of \ handshakes = \frac{n!}{(n-1)!}$, though he had only used specific values for n and had not represented the general case in writing yet. The second representation he tried to use was made by writing the letters “A,” “B,” “C,” “D,” and “E” in a line, and then drawing lines to connect them in as many combinations as possible. The two representations were not consistently giving him the same answers, which confused him. He had written $\frac{5*4*3*2*1}{4*3*2*1}$ to represent the situation with 5 people, and after some work on the situation with 10 people he went back and crossed out the denominator $(4*3*2*1)$. To try to resolve this dissonance, he went to the side of his paper, wrote “A” and “B” next to each other, wrote “$= \frac{2*1}{1} = 2$,” and then he drew a line connecting A and B in the other representation. As he did this he tested his representations using a simpler case. He ultimately decided to use the representation that took the form $\# \ of \ handshakes = \frac{n!}{(n-1)!}$, and used that throughout the rest of the handshake task. CK-K and CK-C did not interact at all as they worked on his task, which is why this example only involves one student’s individual work and does not involve CK-C at all. While CK-K engaged in this purpose while working individually, it is reasonable to assume that ELs could also engage in this purpose while working on problems together.

I made two observations regarding the nature of the way the purposes in this category were accomplished. First, the ELs in this study were able to *read and understand the question/problem* well enough to engage in the task for the handshake task, the pathway task, and the broken tree task, but many struggled understanding the question posed on the average speed task. Some of the ELs in this study struggled with the average speed task because they did not know what the phrase "average speed" meant. The Spanish-Spanish pairing and MP-P in the Mathematics Teacher-Portuguese pairing asked me to explain what "average speed" means as they tried to understand the problem. I tried to explain it various ways; the most helpful way for both pairs seemed to be to relate the idea of average speed to their understanding of velocities. The Spanish-Spanish pairing never seemed confident in their understanding of what "average speed" meant, but did their best to solve the problem according to their understanding of the question. MP-P seemed to understand it after a brief conversation with MP-M and me. The Chinese-Chinese pairing discussed the meaning of "average speed" with each other before they moved ahead with the problem. The other pairs did not give evidence that they needed to spend time figuring out what the phrase meant. As I tried to explain what "average speed" meant to the various ELs, I found it difficult to just give a definition independent of authentic activity to support the meaning I was trying to explain. For example, building an understanding of "average speed" as it relates to a car's trip through a town may have illuminated the meaning of "average speed" far better than my attempts at definitions. During the interviews many of the ELs explained that they learn mathematical vocabulary best when they used it repeatedly as they worked on related mathematics problems which supported the idea that if these students had worked on a task like I just described prior to getting the average speed task in this study, they may have been better equipped to understand the problem.

Second, NEL symbol systems seemed to play an important role in accomplishing the purposes within this category. During their interviews, many ELs told me that the symbol systems, particularly the ones *provided* in the task, helped them succeed in doing the problems. More specifically, many talked about how the NEL symbol systems that were provided facilitated their entry into the tasks because they helped them understand the problem scenarios and the questions being asked. For example, the diagram offered in the broken tree task helped many students understand the problem being posed and helped them more easily figure out how they might want to solve the problem.

Category 2: Model and reason abstractly and quantitatively. The second category is *model and reason abstractly and quantitatively*. This category is based on the CCSSM practice standards "reason abstractly and quantitatively" and "model with mathematics." Purposes in this category include modeling real world situations with mathematics as well as modeling mathematical situations using different symbol systems. Students then use the representations they create to reason through mathematics they see as problematic. Students who are engaged in activity within this category decontextualize problems, manipulate symbols, and re-contextualize symbols by attending to their meaning. They also reason quantitatively by performing calculations and analyzing relationships between quantities. Table 3 includes a list of the purposes the ELs in this study engaged in that fall in this category.

Table 3:
Purpose Category 2 Prevalence

2. Model and reason abstractly and quantitatively	% of turns with this code
Far more than typical	
Use a representation of the problem to model the situation/problem	20%
Solve an arithmetic equation	12%
Slightly more than typical	
Apply a general formula to a specific situation	7%
Use representation of problem to generate an arithmetic or algebraic representation of a scenario	4%
Typical prevalence	
Create a representation of a problem/situation	3%
Represent a scenario arithmetically based on a recalled formula	3%
Algebraic manipulation	3%
Contextualize solution	1%

Recall that most of all of the purposes ELs engaged in occurred in 1%-3% of turns. Half of the purposes in this category fall into that range. The purposes that occur in 4%-7% of turns, which make up a quarter of the turns in this category, occurred slightly more often than was typical. The few purposes that occurred in more than 8% of turns were far more prevalent than was typical. One quarter of the turns in this category fall into that high range. The two purposes that occurred in the highest percent of turns are *solve an arithmetic equation* and *use a representation of the problem to model the situation/problem.* The least common purpose was *contextualize solution* which only occurred in 13 turns. Overall, half of the purposes in this category occurred more than typical when compared to all of the purposes. This suggests that perhaps the tasks posed lent themselves well to these purposes or that perhaps the ELs were better at pursuing this category of purposes than other categories. The ELs in this study were often engaged in using a representation to model a problem, solving arithmetic equations, applying general formulas to specific situations, and using representations of problems to generate arithmetic or algebraic representations. Many of these purposes may seem like one would need to rely heavily on language to engage in them, but this study gives evidence that ELs at varying levels of English proficiency were still able to engage in these purposes often as they participated in mathematical Discourse.

I now include two examples of times when the ELs were modeling and reasoning abstractly and quantitatively to illustrate the kinds of purposes that the ELs were engaged in within this category. The first example highlights in particular how ELs reasoned abstractly and quantitatively. The second shows how ELs created and used representations to model abstractly about problem scenarios. The turns described in this example received additional codes that are not a part of category 2; for clarity those codes will not be discussed here.

The following example is of a time when an EL applied a general formula to a specific situation and then did some algebraic manipulation to ultimately solve an equation. This example shows a wide range of the purposes within this category, particularly ones that involve reasoning abstractly and quantitatively. The EL in this example was MP-P, the Portuguese speaker who was paired with the English-speaking math teacher. This example occurred as they worked on finding the bird's average speed from 3.5 to 4 seconds in the average speed task. They had been successful with their strategy that can be summarized as $avg\ speed = \frac{change\ in\ distance}{change\ in\ time}$. They were working individually to apply that strategy to this problem. When MP-P plugged $t = 3.5$ into $f(t)$, she realized she needed to calculate 3.5^2. She decided to use the formula $(a+b)^2 = a^2 + 2ab + b^2$ so she wrote it on her paper and then applied it to this situation. She wrote out "$\left(3+\frac{1}{2}\right)^2 = 3^2 + 2 * 3 * \frac{1}{2} + \left(\frac{1}{2}\right)^2$." She then did some arithmetic as well as some algebraic manipulation to simplify and solve this problem. She wrote, "$\left(3+\frac{1}{2}\right)^2 = 3^2 + 2*3*\frac{1}{2} + \left(\frac{1}{2}\right)^2$," below it wrote "$9+3+\frac{1}{4}$," below that wrote "$12+\frac{1}{4}$," and then wrote "$= 12{,}25$ (sic)." When she applied $(a+b)^2 = a^2 + 2ab + b^2$ to the situation where $a = 3\ and\ b = \frac{1}{2}$, she was *applying a general formula to a specific situation*. When she manipulated the numbers *simplifying* the expression, her purpose was coded as *algebraic manipulation.* Since she ultimately solved the equation she originally set up and since she did arithmetic along the way, this turn was also coded as *solve an arithmetic equation*. It was often the case that multiple purposes within this category appeared on the same turns. For example, algebraic manipulation and/or solving an arithmetic equation often showed up on the same turns that had the purpose *apply a general formula to a specific situation.*

This final example is one where an EL created a representation of a problem and then used that representation to model a specific situation. This example occurred during the session with a Spanish-speaker and an English-speaking undergraduate student. They were working on the handshake task when there were 5 people at the party. SE-S explained that he thought they should multiply 5 by 4 to get the number of handshakes. SE-E wondered aloud if that would be counting repeats—in other words, if two people shook hands, would $5*4$ count 2 handshakes, or just one? SE-S replied,

> This person comes, right? [draws a dot]. Then 1, 2, 3, 4 more people [writes "1," "2," "3," and "4" below the dot]. So is 4. 4 shakes ["air traces" lines between the dot and each number] for every person. You can verify four five [writes "$4*5=$"].

During this turn he created a representation for how many handshakes one person "shakes" at a party with 5 people. He used that representation to model how many handshakes occurred for one person, and then incorrectly assumed that each person would have 4 distinct handshakes because each person's handshakes could be represented with his model. SE-S used his representation to model quantitatively about the situation where there were 5 people at a party.

Regarding the nature of how purposes in this category were accomplished, I focus specifically on situations like the one in the last example where ELs engage in purposes within this category but come to incorrect mathematical conclusions. NEL symbol systems, particularly the representation SE-S used to model the problem, seemed to play an important role in his engagement in the purposes in this category. He was able to engage in the Discourse using this representation even though he did not fully understand the mathematics in the task. Some careful questioning and support from a teacher could have helped SE-S recognize what was problematic in his reasoning and could have helped him understand the mathematics correctly by helping him build on his representation to correctly model the problem. A teacher's support could have helped him resolve his mathematical issues because he was already able to engage in the purposes in this category through the use of his representation.

Category 3: Construct viable arguments and critique the reasoning of others. The third category is *construct viable arguments and critique the reasoning of others*, which is based on the CCSSM Practice Standard by the same name. Students who engage in activity within this category construct arguments based on givens, known definitions, representations they are using, and others' input. They give justifications, critique others' reasoning, and ask clarification and probing questions. Table 4 includes a list of the purposes the ELs in this study engaged in that fall in this category.

Table 4:
Purpose Category 3 Prevalence

3. Construct viable arguments and critique the reasoning of others	% of turns with this code
Far more than typical	
Confirm a hypothesis or claim	13%
Slightly more than typical	
Ask for clarification or help	6%
Explain or demonstrate a strategy	5%
Typical prevalence	
Explain why a strategy is applicable	3%
Add to or refine someone else's idea	3%
Use current representations to justify a claim	3%
Explain the elements of a formula	2%
Ask a question to challenge thinking	2%
Explain why a claim/explanation is not true or why a plan is not wise	2%
Respond to a request for clarification	2%
Offer an alternative claim	2%
Use an example to justify a claim	1%
Explain how to use the representation to model the problem	1%
Request elaboration	1%
Incorporate another's input into a strategy	1%

Most of all of the purposes ELs engaged in occurred in 1%-3% of turns. Most of the purposes in this category fall into that "typical" range. The purposes that occur in 4%-7% of turns occurred slightly more often than was typical. In this category, that applies to two of the purposes. The few purposes that occurred in more than 8% of turns were far more prevalent than was typical. One purpose in this category, *confirm a hypothesis or claim*, falls into that range. This suggests that perhaps ELs are likely to engage and good at engaging in the practice of listening to another student's strategies and confirming if they agree with the reasoning. It is also interesting that the purpose *explain or demonstrate a strategy*, which occurs when students are explaining their *own* thinking, *also* occurs more than purposes in the "typical" range. This gives evidence that even students who are less proficient in the English language still find ways to engage in the practice of explaining their own strategies and reasoning. The least common purpose was *use an example to justify a claim*, which occurred in only 10 turns. This may provide evidence that the ELs in this study are competent enough at participating in mathematical Discourse that they might realize that using examples as justifications for claims is not typically adequate. It might also suggest that only a few of the ELs in the study engaged in this purpose, which could be the result of other ELs knowing this type of proof is not typically adequate.

I now include two examples of times when the ELs constructed viable arguments and critiqued the reasoning of others to illustrate the kinds of purposes that the ELs were engaged in within this category. One example shows how sometimes many of these purposes occurred during one turn as an EL explained her own thinking and critiqued the work of another student during the same turn. The other example illustrates how a student requested elaboration for another student so he could critique the other student's claim. The turns described in this example received additional codes that are not a part of category 3; for clarity those codes will not be discussed here.

The following example shows how an EL in this study engaged in the four purposes, *explain why a strategy is applicable*, *explain or demonstrate a strategy*, *explain why a claim/explanation is not true or why a plan is not wise*, and *offer an alternative claim*. This example occurred during the session with two Spanish-speaking ELs as they worked on the pathway task. SS-1's strategy was to find the length of each stretch of pathway and add them all up. SS-2's strategy was to count the number of 2 m × 2 m "boxes" along the pathway, since each one had 2 meters worth of the traveled path in it, excluding the last one which has 1 meter of pathway traveled (see Figure 2). They did some work individually using their own strategies, and then they came together to talk about their work. SS-1 explained that the figure was 14 meters wide, but to find the length of the first stretch of path, one "cannot count" the portion of the figure where the path turns (see b in Figure 2). He further explained that the stretch of path in that section would be accounted for by counting the vertical part of the path in the length of the next stretch of path. SS-2 misinterpreted his statement, thinking that he meant that one "cannot count" the 2 m × 2 m box in the top right corner the same way one counts the others. SS-2 responded by saying,

> Yeah, you can. For each of these squares [draws a 2 m × 2 m square on the first two meters of the path (see *a*)] you can count 2 meters. In this square [traces the 2 m × 2 m square at the top right corner (see *b*)] you also have two meters. But, you have one meters heres (sic) [traces the first, horizontal meter in that square (see *b*)] and one meters heres [traces the second, vertical meter of pathway (see *b*)]. So the only thing you have to do is count the numbers of squares [touches each of the first few squares along the pathway (see *a*)]. And the-- because the corners also have two meters [points out the 2 meters of path in the 2 m × 2 m square on the second corner (see *c*)]. One meter here [traces the first segment of path in that square (see *c*)], one meter here [traces the second segment of path in that square (see *c*)].

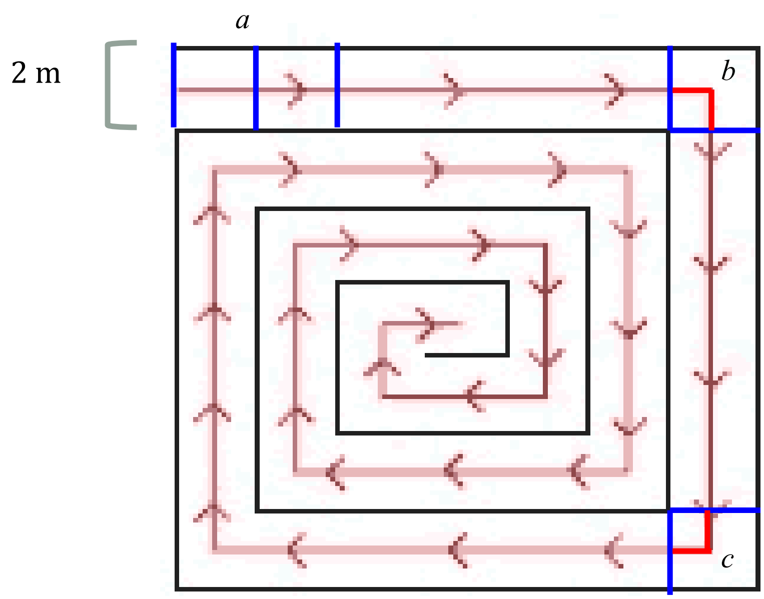


Figure 2. Illustration of SS-2's turn

Although SS-2 has misunderstood SS-1's claim, SS-2's turn includes many of the purposes in this category. He refuted the perceived claim that one cannot count the corner box like the rest of the boxes and explained (using the diagram) why one actually can. Because of that, this turn was coded with the purposes *explain why a claim…is not true…* and *explain why a strategy is applicable.* He also briefly explained his strategy (hence the code *explain or demonstrate a strategy*) and offered the entire explanation as an alternative claim to what he thought the first student believed (hence the code *offer an alternative claim*).

The final example is of *request elaboration*. This example occurred in the same session with the two Spanish speakers. They were working on the average speed task and were having trouble coming up with a way to calculate the average speed from 1 second to 4 seconds. SS-2 decided that if he could not devise a way to calculate the average speed, he could try to devise a way to estimate what it would be close to. He hypothesized that if he could find the exact speed at the time exactly halfway between 1 second and 4 seconds, that value would approximate the average speed on the interval. As he talked about this idea, SS-1 interjected and asked, "but why do you want to find the middle?" This is an example of a turn where a student requests elaboration; he wanted the student to explain in more detail the reasoning behind this proposed strategy.

Regarding the nature of how purposes in this category were accomplished, it was often the case for multiple purposes from this category to happen during a single turn. This provides evidence that constructing arguments and critiquing the arguments of others are complex practices that ELs engaged in. It also suggests that perhaps turns that were coded with purposes in this category were longer and more complex than turns that perhaps only had one or two purpose codes, like *show they are listening and engaged in the conversation*, and that these turns occurred often.

Category 4: Facilitate communication and cooperative work. The fourth category is *facilitate communication and cooperative work*. The purposes in this category are ways of recording and communicating to others information that is not core to the mathematical ideas at hand, but is influential to the way the ELs will interact while they do mathematics. The purposes in this category facilitate smooth communication and support cooperative work on the mathematics. They often occurred concurrently with purposes related to problem solving, argumentation, and critiquing, but focused instead on facilitating communication rather than communicating mathematics specifically. Turns in which ELs gave anecdotal or non-mathematical opinions of the problems they were working on were also included in this category, since the purpose of such opinions often seemed to be to build familiarity and consensus. Table 5 includes a list of the purposes the ELs in this study engaged in that fall in this category.

Table 5:
Purpose Category 4 Prevalence

4. Facilitate communication and cooperative work	% of turns with this code
Far more than typical	
Show that they are listening or engaged in the conversation	10%
Slightly more than typical	
Comment on own knowledge or progress	7%
Request for validation	5%
Record an established answer	5%
Typical prevalence	
Give an opinion of the problem	2%
Create a system to organize thinking and data	2%
Compare work or ideas	2%
Restate an explanation to demonstrate understanding	1%
Establish a common starting point	1%

Most of all of the purposes ELs engaged in occurred in 1%-3% of turns. Just over half of the purposes in this category fall into that typical range. The purposes that occur in 4%-7% of turns occurred slightly more often than was typical. Just under half of the purposes in this category fall into that range. The few purposes that occurred in more than 8% of turns were far more prevalent than was typical. In this category, the purpose *show that they are listening and engaged in the conversation* is the only purpose in that high range. ELs were engaged in many of the purposes in this category more than typical. This shows that ELs in this study were often engaged in activity to support and reinforce the conversation and interaction that was taking place as they worked on tasks with their partner. It also shows that they often requested validation from their partner when they felt unsure either with their mathematics or their language use.

I now include two examples of times when the ELs were engaged in purposes that facilitated communication and cooperative work. The first example shows how an EL engaged in purposes in this category to encourage communication in general. The second example shows how an EL engaged in purposes in this category to strengthen her mathematical collaboration with her partner. Some of the turns described in these examples received other codes that are not from category 4; for clarity those codes will not be discussed here.

The first example is of a time when a student *gave an opinion of the problem* she was working on. In this example, the two Chinese speakers were working on the average speed task (task 2). When they first read over the task, CC-2 said, "this looks difficult." Here she gave her opinion of the problem in a way that might build her relationship with the other EL she was to work with during their session. The two students eventually decided that to calculate the average speed over an interval they would use derivatives to calculate the exact speed at each endpoint of the interval, and then average those two values. The turn described above that CC-2 took served to encourage communication between her and CC-1 as they worked on the task. The "small talk" they had that was not core to the mathematics they were working on, but seemed to keep the lines of communication open so they didn't just work on the tasks on their own and forget to work together.

This final example illustrates an ELs' use of the purposes *compare work or ideas* and *comment on own knowledge or progress* to encourage mathematical collaboration. This example was also used previously to illustrate purposes in category 2. The EL was MP-P, the Portuguese-speaker from Brazil who was paired with the English-speaking math teacher. This example shows how MP-M and MP-P often reported their progress to each other and compared their work to the other person's so that they could each continue to contribute to their collaborative efforts to solve the problems. As they worked on finding the bird's average speed from 3.5 to 4 seconds in the average speed task, they both calculated 3.5^2. MP-P had calculated $\left(3+\frac{1}{2}\right)^2 = 3^2 + 2 * 3 * \frac{1}{2} + \left(\frac{1}{2}\right)^2$. MP-M had calculated $\left(\frac{7}{2}\right)^2$. They each continued working; MP-M finished calculating the average speed, but MP-P had accidentally stopped early and had a final answer that didn't represent the average speed. When they came back together to compare their work, MP-P looked over MP-M's work and said, "yeah we did the same thing but I did it … with plus [traces the "+" in $\left(3+\frac{1}{2}\right)^2$]. That (referring to her own work) is more complicated actually. So I kept doing it-- I kept doing it-- but it must be wrong, I believe." During this turn the EL was comparing her work with her partner's, and also commenting on her own work (coded in the category *comment on own knowledge or progress*) to facilitate the productive communication and cooperative work they had been engaged in during their session. She reviewed her own work and found where she thought she made a mistake publically so that MP-M could understand MP-P's thinking. Doing this let the work on the mathematics continue to be a joint endeavor where they continually checked to see if they were on the same page before they progressed.

Category 5: Look for and generalize patterns and repeated reasoning. The fifth category is *look for and generalize patterns and repeated reasoning.* The questions on the tasks students worked on asked few questions that would require students to generalize patterns, the exception to this being the handshake task. The last part of the average speed task could have been solved using some ideas related to generalizing patterns, but most students did not solve it this way. Because of these and perhaps other reasons, purposes in this category occurred least often; this is the last category and will be discussed least. It is related to the CCSSM practice standards *look for and make use of structure* and *look for and express regularity in repeated* *reasoning.* When students engage in activity that falls within this category they represent patterns, general methods, and repeated reasoning symbolically in some way. Purposes in this category occurred during mathematical acts that were part of the process of noticing these patterns and repeated reasoning and creating a representation of them. Table 6 includes a list of the purposes the ELs in this study engaged in that fall in this category. Most of all of the purposes ELs engaged in occurred in 1%-3% of turns. All the purposes in this category fall into that "typical" range.

Table 6:
Purpose Category 5 Prevalence

5. Look for and generalize patterns and repeated reasoning	% of turns with this code
Typical prevalence	
Use the established cases to come up with a general pattern/formula	2%
Use the representation to find a general pattern/formula	1%
Represent a generalized pattern or formula using symbolic notation	1%
Establish base cases	1%

I now include an example of a time when the ELs were generalizing to illustrate the kinds of purposes that the ELs were engaged in within this category. This turn was chosen because the EL in the example engaged in three of the four generalizing purposes; seeing each of the purposes and distinguishing their differences can be done while they are juxtaposed in the same example. The turn described in this example likely received additional codes that are not a part of category 5, but for clarity they will not be discussed here.

This example illustrates how a student used a representation to find a general pattern/formula, represented a generalized pattern or formula using symbolic notation, and used the established cases to come up with a general pattern/formula. It occurred in the session with two Spanish-speakers. Throughout their work on the handshake task, the students were representing the different scenarios with a diagram they had created. The diagram had as many dots as there were people at the party, and they would connect the dots in all possible combinations to represent the number of handshakes. The students had been working on the last part of the handshake task where they were asked to represent how many handshakes would happen if there were n people at a party. The two students in this pair had previously established their base cases, that when there are no people or there is only one person at the party, there are no handshakes; when there are two people at the party there is one handshake; when there are three people there are three handshakes; and so on for four, five, and 10 people.

In this particular turn, SS-2 said, "Okay, there is a pattern here." He went on to show how their diagram proved some of their base cases. He then began generalizing and said, "The first person can chains $n - 1$ [writes "$1 \rightarrow n - 1$"]. The second person can chain hands $n - 2$ [writes "$2 \rightarrow n - 2$"]. And so forth until [the last person]." He then elaborated more on how this pattern he was seeing applied to the case where there are 10 people at the party, most likely as a way to help the other student understand. As this student began to explain the general pattern he saw, he used both the previously established base cases as well as the representation they had created to explain what he saw. He than began to represent the pattern symbolically when he wrote "$1 \rightarrow n - 1$" and "$2 \rightarrow n - 2$" to mean that for the first person there are $n - 1$ handshakes, and for the second person there are $n - 2$ handshakes.

ELs did not engage in purposes in this category enough to notice significant patterns regarding the nature of how the purposes were accomplished. As mentioned previously, generalization was not heavily emphasized through all the tasks given to the ELs. The last part of the handshake task asked students to generalize for n people. When students assumed they had memorized the correct formula to calculate the number of handshakes at a party as was the case with CC-1 and CC-2 and CK-C, they were not motivated to re-generalize based on their previous responses. Other students engaged in purposes within this category for this problem, but the percent of turns that make up their work on this problem is small. Given tasks with more emphasis on generalization and describing patterns, I assume ELs would have engaged more in the purposes in this category and more conclusions may have been possible.

Summary

In this chapter I described what purposes ELs were engaged in as they participated in mathematical Discourse during this study. They were able to do meaningful mathematics by making sense of problems and persevering in solving them, reasoning abstractly and quantitatively, constructing viable arguments and critiquing the reasoning of others, facilitating communication and cooperative work, and looking for and generalizing patterns. Engaging in this authentic mathematical activity allowed the ELs to successfully participate in the mathematical Discourse necessary to work toward solving the problems they were posed and having productive mathematical discussion.

Discussion

Based on the results in this chapter I claim that the purposes ELs engaged in are consistent with meaningful participation in mathematical Discourse because they align well with the CCSSM practice standards. I also claim that ELs' participation in mathematical Discourse is essential for their English language development.

How the results address the research question and problem

The question this chapter answers is *What mathematical purposes do ELs accomplish when using literacy with various symbol systems while engaging in mathematical acts during* *mathematical Discourse?* The ELs in this study were able to use a variety of symbol systems to engage in purposes that are a meaningful part of the mathematical Discourse in which they need to be participants.

ELs' literacy with a wide variety of symbol systems facilitated the mathematical acts they used to participate in mathematical Discourse. ELs' symbol use often varied according to the task and the person, but they all exhibited literacy with a wide variety of symbol systems. The use of symbol systems was how the ELs communicated the information in each mathematical act; their participation in the Discourse would not have been possible otherwise. Each turn was analyzed according to the three components of a mathematical act: the symbol use, the meaning, and the purposes. By definition, every turn an EL took used at least one symbol system to communicate the meaning and to engage in the intended purpose. The examples in this chapter illustrate the kinds of symbol systems ELs used for their turns; turns commonly involved more than one symbol system. Without the use of symbol systems (including spoken and written English, their first language, and the non-language symbol systems), ELs would have had no way to communicate their meanings, no way to engage in the purposes, and as a result would not have been able to participate in mathematical Discourse.

The results in this chapter suggest that ELs with basic literacy in conversational English and NEL symbol systems used to do mathematics can participate meaningfully in mathematical Discourse. The ELs in this study all had at least basic literacy in conversational English. Analysis of the variety of purposes each of the ELs engaged in during their mathematical acts produced the purposes and purpose categories discussed in this chapter. Furthermore, most of the ELs in this study participated in mathematical Discourse in English, which is obviously a valuable skill when in an English mathematics class. Only the Chinese-Chinese pairing chose to speak their native language while solving the tasks together. Participants in the other pairings all had varying levels of English literacy and experience with mathematics, but they were all able to participate meaningfully in mathematical Discourse by engaging in the purpose categories. The purpose categories presented in this chapter show what meaningful activities the ELs engaged in as they participated in mathematical Discourse.

The ELs in this study engaged in many purposes that are a meaningful part of the activity in mathematical Discourse. In fact, the purposes in the five categories presented in this chapter are consistent with practices that are valued in the CCSSM, which describes the meaningful mathematical activity that is valued by leaders in the field of mathematics education as crucial to productive mathematical Discourse. The purpose categories align well with practice standards MP1, MP2, MP3, MP4, MP7, and MP8. The purposes in category 1 and 3, *make sense of problems and persevere in solving them,* and *construct viable arguments and critique the reasoning of others,* encompass the same types of activity described in the practice standard MP1 and MP3, respectively. They aligned well enough that I named the categories after the respective practice standards. Purpose category 2, *model and reason abstractly and quantitatively,* encompasses the kinds of activity that fall within MP2 with a particular emphasis on modeling mathematical situations. Category 2 also includes the types of activity that fall within MP4 that describe modeling real life situations. For the purposes of this study, I was interested in how ELs modeled with and used a variety of symbol systems whether or not the situation they were modeling was a real-world scenario or a mathematical scenario, so I combined these two CCSSM practice standards instead of categorizing them separately. Purpose category 5, *look for and generalize patterns and repeated reasoning,* includes the observable portions of MP7 and MP8. Significant portions of MP7, *look for and make use of structure*, and MP8, *look for and* *express regularity in repeated reasoning*, are not easily observable. The *looking* components of these standards are often difficult to observe and document without a researcher probing for information as the students work on tasks. Since the ELs in this study did not always make their thinking public, purpose category 5 includes purposes that are consistent with the observable parts of this kind of activity. In contrast, purpose category 4, *facilitate communication and cooperative work,* is not reflected in or aligned with any of the practice standards. This category points to a practice that has yet to receive attention by the field, namely to create and foster an environment in which activity in the other four purpose categories can thrive. Figure 3 summarizes this information and visually organizes how these two constructs align.

Standards for Mathematical Practice	Purpose Categories	Explanation
MP1 Make sense of problems and persevere in solving them	1. Make sense of problems and persevere in solving them	Purpose category 1 encompasses the same types of activity as MP1.
MP2 Reason abstractly and quantitatively	2. Model and reason abstractly and quantitatively	Purpose category 2 encompasses all kinds of activity involved in MP2 with an emphasis on modeling mathematical situations.
MP3 Construct viable argument and critique the reasoning of others	3. Construct viable arguments and critique the reasoning of others	Purpose category 3 encompasses the same types of activity as MP3.
MP4 Model with mathematics	2. Model and reason abstractly and quantitatively	Purpose category 2 also incorporates modeling real-life situations.
MP5 Use appropriate tools strategically		*I address some of the "appropriate tools" ELs use in this study in the discussion of the various symbol systems they use to do mathematics; I consider the various symbol systems tools.*
MP6 Attend to precision		*ELs attended to precision as they used symbol systems to support and clarify their mathematical communication.*
MP7 Look for and make use of structure	5. Look for and generalize patterns and repeated reasoning	A good portion of MP7 is not easily observable without a researcher probing for information as the participants worked on the tasks, since participants do not often comment on what aspects of the mathematics they are attending to or make their thought processes public without prompting. The observable portions of ELs' looking for and making use of structure are encompassed in category 5.
MP8 Look for and express regularity in repeated reasoning	5. Look for and generalize patterns and repeated reasoning	A good portion of MP8 is not easily observable (see above cell.) Activity having to do with how participants express regularity in repeated reasoning is more easily observable, however, and is included in category 5.
	4. Facilitate communication and cooperative work	This category did not align well with any of the CCSSM practice standards; purposes in this category create and foster a productive environment for the other purposes to occur.

Figure 3. How the CCSSM practice standards align with the 5 purpose categories

Although ELs in this study used a variety of symbol systems to participate meaningfully in purposes that are a crucial part of mathematical Discourse, this study does not claim to make connections between particular symbol systems and purposes. Trying to find patterns between the uses of 11 symbol systems across 49 purposes became unwieldy for such a small data set, especially since each turn often included multiple purposes and symbol systems. Therefore, I cannot make any claims that particular purposes were typically accomplished with a certain symbol system.

Results compared to other studies and theories

In this section I discuss how the results in this chapter compare to other relevant studies and theories. In particular, I make claims related to three prevalent ways of thinking that were found in the framework and literature review in chapter two. First, I claim that if ELs are encouraged to participate in mathematical Discourse with appropriate tasks and encouraged to work with a partner, they are more likely to engage in the language domains that facilitate English development. Second, I claim that a focus on mathematical Discourse gives ELs the opportunity to engage in meaningful mathematics and develop English in such a way that cannot be replicated outside of mathematical Discourse. Finally I discuss what this chapter contributes to understanding ELs' use of many symbol systems, and motivate the next chapter by describing the need to understand *how* ELs use them to participate in mathematical Discourse.

Emphasis on the four language domains. As ELs in this study were encouraged to participate in mathematical Discourse, given tasks designed to focus on important mathematics, encouraged to work with a partner, and supported as they did so, they engaged in three of the four language domains that facilitate English development. The purposes ELs engaged in as they participated in mathematical Discourse involved reading, listening, and speaking. There is evidence in the very first purpose category that ELs read; there were many instances of ELs engaged in the purpose *read and understand the question/problem.* There were not many other times during their work on the tasks that they needed to read English (primarily due to a decreased emphasis on writing, and therefore decreased opportunity to read the writing of others), but they read each time they began a problem or worked to understand a problem. There are many purposes in the third purpose category where ELs engaged in activity that required them to listen; ELs engaged in a lot of listening as they critiqued the reasoning of others. Purposes such as *confirm a hypothesis or claim* and *add to or refine someone else's idea* often involved listening as they served to progress the mathematical conversation.

All of the ELs in this study spoke (and therefore listened to) English to some degree during the course of their work on the tasks. The Chinese-Chinese pairing spoke the least English, and then only when they talked to me. The Chinese-Korean pairing spoke only slightly more; they spoke to me in English and then they spoke small amounts of English when they interacted during the second half of the broken tree task. Based on the interview responses from CC-1, CC-2, CK-C, and CK-K, there was little evidence to suggest they had lower English literacy than the other ELs, preventing any assumptions that lower English literacy contributed to their less frequent use of spoken English. All the other pairings spoke a great deal of English as they participated in the mathematical Discourse. The Spanish-Spanish pairing spoke in English for the entire session whereas the other same-language pairing, the Chinese-Chinese pairing, chose not to use much spoken English. SS-1 assumed that since the study was about how ELs did mathematics, was being overseen by a native English speaker (me), and involved questions written in English, he needed to speak in English to do the problems. I assume that the Chinese-Chinese pairing did not have this same assumption, so their primary use of spoken English was when they spoke to me. I do not have very much insight as to why the Chinese-Korean pair did not interact much and therefore did not use spoken English very much. I suspect it was mostly because they each felt like they could solve the problems more efficiently on their own, but other contributing factors could be their different first languages, their differing genders, or their level of comfort speaking about mathematics on camera. Regardless of why they did not interact very much using spoken English, when they *did* interact as CK-C struggled with the broken tree task, they used spoken English (requiring someone to also be listening to English) along with a variety of symbol systems to engage in the purposes. Each of the ELs in the MathEd-Portuguese and the Spanish-English pairings spoke and listened to a great deal of English in large part because they were paired with native-English speaking partners. As demonstrated by the descriptions in this paragraph and the examples previously described in this chapter, it is clear that as the ELs in this study engaged in the purposes described in this chapter, they engaged in the learning domains of speaking and listening.

It is important to note that ELs' activity within these language domains did not occur in a situation that was solely focused on engaging them in the four language domains; it stemmed from an emphasis on doing mathematics within mathematical Discourse in a situation that was conducive to English interaction. I did not design the sessions specifically with the goal to get ELs to engage in the language domains. Instead, I focused on facilitating participation in mathematical Discourse in a way that incorporated both language and mathematical goals for ELs. ELs' language use happened naturally when given good tasks that were carefully designed to provide opportunities for them to think about and work on important mathematics, and when they were encouraged to work with a partner.

Although the ELs did not engage in very many purposes through the use of written English, the fact that the tasks did not prompt them to do so prevents any conclusion that they were unable to write, or that they would not have written if prompted to do so. If tasks were designed so that ELs engaged in purposes within the five purpose categories with an additional writing component, I suspect the ELs would have been involved with a lot more writing; I see no evidence that suggests otherwise. Those who may be particularly focused on having ELs use the four language domains should note that the use of these domains occurs frequently as ELs participate in mathematical Discourse.

The Necessity of a Discourse perspective. A focus on getting ELs to engage in mathematical Discourse is productive because it supports students' participation in meaningful mathematical activity and their English development in worthwhile ways that cannot be replicated outside of mathematical Discourse. The purposes ELs engaged in align with the goals of the mathematics education community, which I discussed above. ELs use of three language domains during mathematical Discourse is consistent with many goals of EL educators who focus on ELs' participation in mathematical activity within these domains. Furthermore, any language instruction outside of mathematical Discourse would not allow students to engage in the types of language use that facilitate the purposes in mathematical Discourse. For example, the only way for ELs to learn to use language in a way that helps *explain why a strategy is applicable* is for them to work on a mathematical task, have a strategy that they understand well enough to know why it applies to a particular problem, and then have occasion to articulate why the strategy is applicable well enough for someone else to understand. Or consider, for example, the kind of language use required to *use a representation of the problem to model the situation/problem.* The way language is used in this purposes cannot be taught outside of a situation in which students are actually using a representation to model a problem they are working on. Participating in mathematical Discourse is the only way that all the necessary components would be in place for ELs to learn and use language in a way consistent with how they need to during mathematical Discourse.

Emphasis on the use of multiple representations. While this research does show what symbol systems ELs use to participate in mathematical Discourse, the unique finding in this study is that ELs used a wide variety of NEL symbol systems and did not seem to be at all limited in their use. Inspection of the wide variety of symbol systems that were used to engage in the purposes described in this chapter gives an idea of *what* ELs used the symbol systems to accomplish. The symbol systems they used allowed them to communicate their mathematical acts, during which they engaged in meaningful purposes. While knowing *what* ELs can accomplish during mathematical Discourse is important, it only describes what did and can happen; it does not do much to further our understanding of how to support ELs' participation in mathematical Discourse. To more thoroughly address the issue of how teachers should support ELs, it is important to understand *how* ELs use NEL symbol systems to participate in mathematical Discourse. Chapter five addresses this issue by answering questions about how ELs' use of NEL symbol systems supports their use of spoken English.

CHAPTER FIVE: RAL FRAMEWORK

Throughout this study I have claimed that ELs should be able to participate in mathematical Discourse because of the many symbol systems used in mathematics that are not heavily language-laden. In the last results chapter I compiled a list of the various symbol systems the ELs in this study used as they worked on mathematical tasks with their partners. I also showed that they were engaged meaningfully in mathematical Discourse by describing the kinds of activity they engaged in. A logical question, then, is to wonder *how* the ELs engaged in the Discourse and how the non-English language (NEL) symbol systems helped them to do so. In this chapter, I describe how the ELs in this study used fluency with NEL symbol systems to make up for a lack of fluency with spoken English. This chapter answers the question, *How do ELs use literacy with NEL symbol systems to support their spoken English?*

Replace-Augment-Learn (RAL) Framework

As I examined ELs' turns to understand how they used NEL symbol systems to compensate for a lack of fluency in spoken English, I was able to identify three different kinds of moves. As explained before, I use the term *move* to refer to an EL's use of an NEL symbol system to support his English language use. The three categories of moves are *Replace*, *Augment*, and *Learn:* ELs used NEL symbol systems to *replace* spoken English, to *augment* their spoken English, and to help them *learn* and develop their spoken English. I refer to these categories and the specific types of moves in each category as the RAL Framework. Figure 4 provides an overview of each category and their subcategories. Note that because this study did not address ELs' use of written English, the moves that comprise the RAL Framework consist entirely of ways ELs used NEL symbol systems to support *spoken* English

REPLACE	AUGMENT	LEARN
Predominantly use symbols	Use symbol systems as referents	Use own symbol use to improve English
Symbol replaces operation	Correctly represent something being said incorrectly	Interpret another's symbol use to improve English
Predominantly use first language	Legitimize non-standard uses of English	
Silence; non-interaction	Illustrate words or phrases	

Figure 4. RAL Framework

I begin this chapter by describing each of these categories and the moves that comprise them. I then tell *The "Squared" Story* to illustrate how these categories interact during Discourse and how the use of NEL symbol systems supports language development. Later on in the chapter I talk about what can be learned from closer analysis of how ELs used the categories in this framework. Before reading further, the reader should review the four tasks the students worked on during this study. Figure 1 (found in the methodology chapter) can be used as a reference to help the reader understand the context of the examples I present throughout this chapter.

Replace

The first way that ELs in this study used NEL symbol systems to support their spoken English was that they used them to *replace* spoken English. Replacing exhibited the least sophisticated English use of the three RAL categories. When ELs replaced spoken English, they switched back and forth between speaking English and only using NEL symbol systems—the use was not concurrent. This non-concurrency is small scale and does not mean that only NEL symbol systems *or* spoken English were used during an entire turn. I considered not the turn but smaller phrases and meaningful chunks within the turn to determine if ELs were using NEL symbol systems to replace spoken English. The four ways that ELs replaced spoken English by using NEL symbol systems are as follows. First, ELs used NEL symbol systems to replace a major part of an explanation. Second, ELs used NEL symbol systems to replace saying the name of an operation. Third, ELs interacted solely in their first language. Fourth, ELs worked silently with no interaction. Refer to Figure 4 or Table 7 for a summary of the four types of replacing. An EL's turn can include more than one type of replacement. Occasionally replacement moves were not successful, which is discussed following the descriptions of the types of replacement.

Table 7:
Prevalence of Replacing Codes

Replace			
Replace Code	# of turns with this code	% of "Replace" turns	% of total turns
Analyzed			
Predominantly use symbols	34	87%	6%
Symbol replaces operation	7	18%	1%
Not analyzed			
Predominantly use first language[1]	--	--	--
Silence; non-interaction[2]	--	--	--

1. This occurred during the session with the Chinese-Chinese pairing and is discussed below.
2. This occurred for the majority of the session with the Chinese-Korean pairing and is discussed below.

In general when I refer to "replacement" in this chapter, I am referring to the first two types of replacement. While the latter two *are* ways that ELs replace spoken English with the use of NEL symbol systems, they are *not* ways that support English language use. When ELs used the first two types of replacement in a turn, they seldom replaced all of the spoken English in that turn by using NEL symbol systems. Thus, these replacement moves supported the spoken English in the turn rather than alleviating entirely the need for spoken English. If the goal for ELs in English mathematics classes is to engage in authentic mathematical activity as well as develop their English fluency, then for the purposes of this study we are interested in the first two types of replacement Table 7 above. When statistics are reported later to show prevalence, I am only reporting the prevalence of the two “analyzed” replace moves.

The prevalence of replace codes varied significantly across pairings. Table 8 reports the prevalence of replacing in each session. For sessions 2 and 3, the percent of total turns is calculated using the total number of turns excluding turns when the ELs were interacting in their own language in the Chinese-Chinese session, and during the long stretch in the Chinese-Korean session when they were not interacting. Since ELs did not attempt to use English during these turns, NEL symbol systems did not have the potential to support their use of spoken English and were therefore not used to analyze the frequency of RAL codes. One exception is the turns when an EL spoke to me in English during these times; those turns are counted in the total. Percentages are rounded to the nearest whole.

Table 8:
Prevalence of Replacing Turns Across Sessions

Replacement		
Session	Replace turns	% of total turns in session
Session 1 (SS)	10	4%
Session 2 (CK)	7	23%
Session 3 (CC)	2	20%
Session 4 (MEP)	14	6%
Session 5 (SE)	6	6%
Total	39	6%

Predominantly use symbols. The first way that ELs in this study used NEL symbol systems to replace spoken English was when the symbol systems alone made up a major part of a phrase, a sentence, an explanation, or a question. This can range from a turn that consists solely of NEL symbol systems, which was uncommon, to turns with limited English use like the use of conjunctions (e.g., and, because, but), articles (e.g., the, a, an) and filler English (e.g., um, well, uh) that have little meaning without the corresponding use of NEL symbol systems. There were 34 turns with the *predominantly use symbols* code, making up 87% of turns with *replace* codes and 6% of the total turns ELs took (see Table 7). Below are three examples when the ELs in this study replaced spoken English in this way. The first two examples describe different ways that the ELs in this study used NEL symbol systems in lieu of other forms of explanation. The last example describes how an EL used NEL symbol systems to effectively replace the majority of an important justification.

One common way this type of replacement occurred was when ELs used NEL symbol systems in lieu of another form of explanation or justification for an answer. In other words, they reasoned and recorded their thinking with NEL symbol systems, wrote the answer, and moved on with no further explanation. An example of this occurred in the Chinese-Chinese pairing as they worked on the handshake task. When she read the first problem, CC-2 said that she knew the answer but did not know how to write an explanation. They thought for a moment and hypothesized that if there were *n* people at a party then there were $5+4+3+2+1=15$ handshakes. Then CC-2 said, "Actually, I can pen a tree here." She drew five dots in a row and began connecting each pair of dots with lines. She crossed out "$5+4+3+2+1=15$," finished drawing lines connecting the pairs of dots, wrote "$4+3+2+1=10$," and moved on to the next problem. She used these various NEL symbol systems (the drawing/diagram and the equations) in lieu of any other form of explanation or justification for the answer "10." After this point in her work, she spoke very little English for the rest of the time and interacted primarily in Chinese.

Similar turns occurred quite often in the Chinese-Korean pairing during the broken tree task. This was the first time these students interacted at all during their session, and one of the first turns during this interaction involved the use of NEL symbol systems to replace spoken English. There came a point where CK-C was stuck and asked for help. CK-K offered help, but the majority of the help he gave was writing and manipulating equations (using the Pythagorean theorem) on CK-C's paper. He simply wrote his explanation in response for her request for help. He used NEL symbol systems to replace a verbal explanation that would have required spoken English.

An example of how ELs used NEL symbol systems to replace the majority of a justification happened during the MP session. They were working on task 4. MP-P wrote "$60^2 + \left(\frac{x}{4}\right)^2 = \left(\frac{3x}{4}\right)^2$." As she finished writing this she said, "This I know how to do. Because, [draws a right angle mark at the base of the tree]." Where an English-speaker may have said something like, "I can do this because this is a right triangle," this EL effectively replaced the justification with the use of NEL symbol systems.

Symbol replaces operation. The second way that ELs in this study used NEL symbol systems to replace spoken English was when they used a symbol system in place of saying the name of an operation as they spoke. This typically happened when a student wrote, pointed at, or gestured to a symbol for an operation instead of saying the name of it as he was speaking. There were 7 turns with this code, making up 18% of turns with *replace* codes and 1% of the total turns ELs took (see Table 7).

The following example shows how ELs in this study used a symbol to replace saying the name of an operation. This example occurred during the Spanish-English session. They were working on the first problem in the handshake task. SE-S had just created a drawing/diagram to represent the situation where there were 5 people at a party. He explained to SE-E how that diagram represented the number of handshakes. He then said, “You can verify 4, 5 [writes "$4 \times 5 =$ "]. How much is it?” It was precisely during the pause (represented by the comma) between saying “4” and “5” that SE-S drew the " $\times$." The way he used this symbol replaced saying, “times” in “four times five.” Another quite similar way that ELs used a symbol to replace saying the name of an operation is when they pointed at a symbol that had already been written in place of saying one (instead of writing it themselves).

Predominantly use first language. The third way that ELs in this study used NEL symbol systems to replace spoken English was when they interacted predominantly in their first language. This could occur for as little as a turn or for as long as an entire session. The *predominantly use first language* code is different than *code-switching,* when an EL occasionally slipped or intentionally said a few words in their first language as they spoke in English. The reason for this distinction is that occasional code-switching does not necessarily detract from English language development. The replacement practice of predominantly using the EL’s first language, however, completely avoids using English, which does not contribute to ELs’ English use and development in the way we are interested in for this study. Code-switching was not studied in depth in this study; I captured instances of code-switching by identifying when ELs used the *first language* symbol system. In this study, *predominant use of the first language* only occurred during the Chinese-Chinese session. CC-1 and CC-2 used verbal English when they spoke to me. Otherwise, they worked on the task in Chinese with the only English use happening when they had to read and understand the questions (which were written in English).

Silence; non-interaction. The fourth way that ELs in this study used non-language symbol systems to replace spoken English was when they worked silently with very little interaction. This category is intended to describe times when major portions of ELs' time spent together passes with no interaction. This is not meant to describe times when students are working on their own with the intention to come together afterwards to discuss their ideas. In this study, this form of replacement occurred during the Chinese-Korean session. The two students did not even attempt to work together during the first three tasks (despite my reminders that they could and should do so). In the middle of the fourth task, CK-K had successfully solved the problem, but CK-C was stuck. She finally reached out and asked for help, and they interacted for the rest of that task.

The majority of turns involving replacement were successful in that the turn contributed to (and did not hinder) the ongoing mathematical conversation, which suggests replacing is a viable way to participate in mathematical Discourse. Some turns involving replacement, however, were unsuccessful in that the EL's meaning was not successfully communicated. An example of when this may have occurred was during the Spanish-English session as they were working on the pathway task. They had started figuring out how to find the length of each stretch of pathway; SE-E had only found the length of the first few stretches and SE-S was trying to understand her strategy. To clarify which lengths she was talking about, SE-E said, "It's 14 by 14 [points to the length and width of the figure]. So yeah [we're finding] this segment, from here to here [points at each end of the first stretch of path]." SE-S responded by saying, "Oh, because [points at the center of the figure]--." He replaced the majority of an explanation using an NEL symbol systems when he stopped after "because" and pointed at the center of the figure. His meaning was not clear and could not be articulated during coding. SE-E did not respond to his comment, presumably because she did not understand what he meant either. Perhaps in that moment she could have asked for clarification to figure out his meaning, but she did not, and the replacing move in that turn may not have been successful; we don't have evidence as to whether or not SE-E understood SE-S's statement. Since replacing moves are potentially less developed and less mature than other turns as far as their use of spoken English, there is a danger of being unsuccessful. There are many important reasons that language use in mathematics is emphasized and its value should not be forgotten. Fortunately, the overwhelming majority of turns that were coded in this category were successful in that the EL's meaning was successfully communicated.

Augment

The second way that ELs in this study used NEL symbol systems to support their spoken English was that they used them to *augment* their spoken English. When ELs augmented their spoken English with NEL symbol systems they used spoken English and the symbol systems concurrently as they said phrases and sentences within a turn. This contrasts with the replacement moves where the symbol systems and verbal English are not used concurrently. Turns with augment moves always involved the use of the *spoken English* symbol system. This English use was typically more sophisticated than the English use in turns with just replacement. The four ways that ELs use NEL symbol systems to augment their spoken English are as follows. First, they use NEL symbol systems as referents. Second, they correctly represent a word or phrase (with NEL symbol systems) that they are saying incorrectly. Third, they use NEL symbol systems to legitimize non-standard uses of English. Fourth, they use NEL symbol systems to illustrate words or phrases. Refer to Table 9 for a summary of the four types of augmenting. An EL's turn can include more than one type of augmenting.

Table 9:
Prevalence of Augmenting Codes

Augment			
Augment code	# of turns with this code	% of "Augment" turns	% of total turns
Use symbol systems as referents	49	49%	8%
Correctly represent something being said incorrectly	10	10%	2%
Legitimize non-standard uses of English	22	22%	4%
Illustrate words or phrases	30	30%	5%

Augmenting was more prevalent than replacing overall, though the prevalence of augment codes varied across sessions. Table 10 reports the prevalence of augmenting in each session. For sessions 2 and 3, the percent of total turns is calculated using the total number of turns excluding turns when the ELs were interacting in their own language in the Chinese-Chinese session, and during the long stretch in the Chinese-Korean session when they were not interacting. Percentages are rounded to the nearest whole.

Table 10:
Prevalence of Augmenting Turns Across Sessions

Augmenting		
Session	Augment turns	% of total turns in session
Session 1 (SS)	46	19%
Session 2 (CK)	9	29%
Session 3 (CC)	1	10%
Session 4 (MEP)	32	15%
Session 5 (SE)	13	12%
Total	101	17%

Use symbol systems as referents. The first way ELs in this study used NEL symbol systems to augment spoken English is when they used NEL symbol systems as referents to pronouns. This category also includes when the EL used the symbol systems as a referent when they were saying a word (not using a pronoun for it), like when they pointed to or circled numbers as they said them. It also includes when the ELs used the symbol systems as a referent when a pronoun was not said but implied. For example, if an EL traced a stretch of pathway and said "14," they did not use a pronoun but a pronoun like "this length" was implied. Doing this type of augmenting often involves a great deal of pointing and tracing as they speak.

This type of augmenting is slightly different than the way that English speakers typically use pronouns. Consider how English speakers might use pronouns and symbol systems as referents when they are having a hard time articulating something verbally; that is what this type of augmenting is. When ELs augment this way they might have a turn that is heavily symbol-laden that would be a lot of work to turn into a solely verbal description. Other times the pronouns refer to more than a simple object or symbol—for example, they might refer to an entire strategy or process. There were 49 turns with this code, making up 49% of turns with *augment* codes and 8% of the total turns ELs took (see Table 9). Using NEL symbol systems as referents is the most common way that ELs in this study supported their spoken English with symbol systems.

I now describe two examples of how ELs in this study augmented their spoken English by using NEL symbol systems as referents. The first example shows how an EL used this type of augmenting in such a way that every noun in his description was a pronoun with NEL symbol system referents. The second example shows how the NEL symbol systems were used to refer to more than just an object—the EL used the NEL symbol systems to refer to an entire strategy and process.

This first example is an example of a turn an EL took that relied almost entirely on using NEL symbol systems as referents for pronouns. During the Spanish-Spanish session, SS-1 and SS-2 were working on the average speed task. They were brainstorming about how velocity (v) and acceleration (a) might help them solve this problem. They had established that $v = \frac{d}{t}$ and $a = \frac{v}{t}$ where d is distance and t is time. SS-1 said, "Yeah, so, having this [points to $a = \frac{v}{t}$], we have that this [writes "a"] is equal [writes " $= \frac{d}{t^2}$"], to this, right?" Clearly SS-1 was using NEL symbol systems (symbols in the form of equations and gestures) to augment his spoken English since every noun in that sentence was represented with a pronoun and a NEL symbol referent.

The final example is of a time when an EL augmented using NEL symbol systems as referents where the thing to which she was referring was more than just an object or a symbol—it was a strategy and a process. This instance occurred during the Mathematics teacher-Portuguese session as they began to work on the pathway task. MP-P spent the first minute or so figuring out what the problem was asking and understanding the diagram. As soon as she did that she said, "Okay, we can do like this [draws lines on left portion of diagram indicating 2-meter widths using the interior edges of the path as guidelines]. Two, two, two, two, two, two [writes "2," labeling each 2 meter width]." She augmented her spoken English by saying "we can do *this*" and then demonstrating the "this" (the strategy) she was referring to with NEL symbol systems and spoken English. The representations she created using NEL symbol systems were the diagram of the path, her annotations on the path, and the symbols (numbers) she used to label the widths.

Correctly represent something being said incorrectly. The second way ELs in this study used NEL symbol systems to augment spoken English is when they correctly represented something with NEL symbol systems that they were saying incorrectly. They may have been saying something incorrectly knowingly or unknowingly. There were 10 turns with this code, making up 10% of turns with *augment* codes and 2% of the total turns ELs took (see Table 9).

I now include two examples of times ELs in this study used NEL symbol systems to represent correctly something they were saying incorrectly. In the first example, an operation was being said incorrectly; in the second example a quantity was being said incorrectly. The first example is of an EL correctly representing the operation "multiplication" she was saying incorrectly. It came from the Mathematics teacher- Portuguese session during their work on the handshake task. They had already figured out that to solve the problem when there are 5 people at a party they could calculate $\frac{5\times 4}{2}$. MP-P looked at the problem asking about if there were 10 people at a party and said, "Oh we do the same thing so 10 **plus** 9 divided by 2 [writes "$\frac{10\times 9}{2} = 5 \times 9 = 45$"] yeah, 45 [nods]." She said "plus" instead of "times," which would be incorrect. But, using the NEL symbol systems she correctly represented her solution. This mistake of saying "plus" instead of "times" also happened frequently for the students SS-1 and SS-2 in the Spanish-Spanish pairing, but they too used NEL symbol systems to correctly represent what they said incorrectly.

The second example came from the Chinese-Korean pairing and shows how an EL used NEL symbol systems to correctly represent the quantity "21.2" he was saying incorrectly. When CK-K was helping CK-C with the broken tree task, he helped her use the Pythagorean theorem to set up an equation she could use to solve for x. She made an error during the simplification and so he helped her do it correctly. They figured out that $x = \sqrt{450}$. He pulled out the calculator, calculated $\sqrt{450}$ for her, and said, "So I think the answer is twenty-one twenty [writes $x = 21.2$ and underlines it]." If an English-speaker said "twenty-one twenty," one could reasonably assume that they meant 2,120 or even possibly $21.20 if the context was something to do with money. But, neither of those answers is correct in this case, and so writing $x = 21.2$ correctly represented something that he was not saying correctly.

Legitimize non-standard uses of English. The third way ELs in this study used NEL symbol systems to augment spoken English was by using the symbol systems to legitimize non-standard uses of English. This type of augmenting is different than the second category where the "wrong" thing they are saying uses standard English, but makes the mathematical claim wrong or vague; in this category they are legitimizing non-standard words. One way legitimizing non-standard uses of English occurred was when ELs used words that described components of an NEL symbol system in a statement instead of saying the "right" words that an English speaker would typically use to describe the same thing. Another way this happened was when ELs used NEL symbol systems to assign meaning to a non-standard word or phrase. The symbol use answers the question, "What do they mean by (*insert word or phrase*)?" for a word without a standard, English, mathematical meaning. There were 22 turns with this code, making up 22% of turns with *augment* codes and 4% of the total turns ELs took (see Table 9).

Regarding the first way that ELs legitimized their non-standard use of English, ELs used words that described the components of two different NEL symbol systems instead of saying words or phrases the way an English speaker might. They coordinated the use of NEL symbol systems and spoken English to develop meaning that an English speaker may have communicated with just spoken English. The first came from the Spanish-Spanish session. The two students had been using a diagram they created to reason about the handshake task. The diagram had as many dots as there were people at a party, and they used line segments to connect all possible pairs of dots. Those segments each represented a handshake. When they got to the third task they were trying to generalize the pattern they saw. SS-2 drew 5 dots and began to recreate the representation from the first part of the task. He said,

> The first person [emphasizes one dot] can **make chains** with one [connects dots 1 and 2], two [connects dots 1 and 3], three [connects dots 1 and 4], four [connects dots 1 and 5]. The first person can **chains** $n - 1$ [writes "$1 \rightarrow\ n - 1$"]. The second person can **chain** hands $n - 2$ [writes "$2 \rightarrow\ n - 2$"].

In this example, instead of saying "the first person shakes hands with…," he says "make chains with" and "chains." He is using language that describes the symbol system he's using (the diagram); he links, or "chains" the dots together to represent the handshakes. This use of the drawing/diagram he had created legitimized the word *chains* as a way to talk about people's handshakes.

The second way ELs used words that described components of an NEL symbol system instead of typical words came from the Mathematics teacher- Portuguese session during their work on the average speed task. They were working to approximate the exact speed at 4 seconds, and they decided to calculate some more average speeds using values of t that got closer and closer to $t = 4$. At one point MP-P said, "Okay. So let's do it with three dot eight [writes "3.8" on x-axis]." For an English-speaker, the standard way to say "3.8" is "three point eight." This EL used language that described the symbol system she was using, with the decimal being "dot," to describe the number "3.8." In many Central and South American countries, students use a comma instead of a decimal to represent a value like 3.8. She switched back and forth between the two as she worked (as if she was attempting to do it the American way but was in the habit of using the comma). But, when it came time to say some values aloud, she did not know the standard way of saying "3.8" and legitimized her way of saying it ("three dot eight") by writing "3.8" as she said it.

The other way ELs legitimized non-standard words and phrases they used was to use the NEL symbol systems to define or assign meaning to a word or phrase without a clear, English, mathematical meaning. The words and phrases to which ELs in this study assigned meaning using NEL symbol systems are *factorize* (to mean *factor*), *summatory* (to mean *sum*, or *summation*), *in a normal way* (to talk about a graph with a *constant velocity*), *as a point of two* (to mean 0.2), *physicist* (to refer to *physics equations*), *divide(d) for* (to mean *divide(d) by*), and *multiplied for* (to mean *multiplied by*). In each case they wrote (or demonstrated in writing), pointed at, or traced symbol systems that made it clear what they meant by their use of that word or phrase. For example, whenever SS-2 said *summatory*, he wrote, traced, or pointed at "$\sum$." What is interesting to note is that rarely did the EL "define" the word/phrase only once; most often they "defined" it nearly every time they used the word or phrase in question.

Illustrate words or phrases. The last way ELs in this study used NEL symbol systems to augment spoken English is by using NEL symbol systems to illustrate words or phrases. This is when an EL used NEL symbol systems to clarify, emphasize, highlight, or embody something they were saying. Often the word or phrase being illustrated would be vague or its use imprecise without the concurrent use of the NEL symbol systems. While the definitions may seem similar, illustrating is not the same as defining. For example, students could repeatedly trace the graph of a parabolic function to illustrate what they mean when they claim an object is moving faster. But, tracing the graph does not define what they mean by "faster," nor does "faster" need defining. There were 30 turns with this code, making up 30% of turns with *augment* codes and 5% of the total turns ELs took (see Table 9). Using NEL symbol systems to illustrate is the second most common way ELs in this study used symbol systems to support spoken English; it is the most common way that does not rely heavily on the use of pronouns.

I now include two examples of how ELs used NEL symbol systems to illustrate. The first example shows a time when an EL used NEL symbol systems to illustrate and therefore clarify what she was saying. This example occurred in the Mathematics teacher - Portuguese session as they worked on finding the bird's average speed from 3.5 to 4 seconds. MP-P had attempted to calculate 3.5^2 using the formula $(a+b)^2 = a^2 + 2ab + b^2$ by calculating $\left(3+\frac{1}{2}\right)^2 = 3^2 + 2 * 3 * \frac{1}{2} + \left(\frac{1}{2}\right)^2$. MP-M had just calculated 3.5^2 by calculating $\left(\frac{7}{2}\right)^2$. When they were comparing their work a bit later, MP-P was going through MP-M's work to see how it compared to hers. When she got to the step where they each calculated 3.5^2, she said (of their strategies as a whole), "Yeah we did the same thing but I did with multiply [points at MP-M's work] like with plus [traces the "+" in $\left(3+\frac{1}{2}\right)^2$]. That is more complicated actually." Tracing the "+" made it clear what she meant when she said "I did it with plus" because it emphasized what strategy she used and was referring to.

The final example of how ELs used NEL symbol systems to illustrate their spoken English occurred in the Spanish-English pairing during their work on the average speed task. They had decided that the average speed from $t = 1$ to $t = 4$ was 1 m/s, from $t = 3$ to $t = 4$ was 1.4 m/s, and from $t = 3.5$ to $t = 4$ was 1.5 m/s. When they moved on to the final question, neither one of them could decide on a strategy they were sure of to approximate the exact speed. After they talked about and attempted a few different strategies, they decided to calculate the average of the three average speeds they had found and use that as the average speed. SE-E calculated the average (1.3 m/s) on her calculator. She reasoned using the graph a bit and said that she was confused because if the average speed had been around 1.3 m/s, then she suspected at 4 seconds the bird would have traveled farther than it had. So, she concluded, the bird must be slowing down over time. After she said this, SE-S said, "Is, is, is it getting, a-slow? Slow? [Traces along the graph of the function.] Should be faster?" He was questioning a claim she made, and tracing the graph with an ever-increasing slope as he asked "should be faster?" helped her understand the justification behind his question and claim that the bird should be getting faster, not slower.

Learn

The third and final way that ELs in this study used NEL symbol systems to support their spoken English was that they used them specifically to *learn* spoken English. Where *replacing* and *augmenting* were typically focused mainly on the mathematics and participating in the mathematical activity, *learning* seemed to involve taking a step back and focusing a bit more on the English language itself. Turns with learn moves always involved the use of the *spoken English* symbol system. The first way ELs intentionally used NEL symbol systems to learn English was when they used their own use of NEL symbol systems with the intention of improving their spoken English. The second way was when ELs interpreted another person's use of NEL symbol systems with the intention of improving their own spoken English. An EL's turn can include one or both types of learning moves. Refer to Table 11 for a summary of these two types of learning.

Table 11:
Prevalence of Learning Codes

Learn			
Learn code	# of turns with this code	% of "Learn" turns	% of total turns
Use own symbol use to improve English	24	73%	4%
Interpret another's symbol use to improve English	11	33%	2%

Learning was the least prevalent of the three RAL categories, and varied greatly across sessions. Table 12 reports the prevalence of learning in each session. For sessions 2 and 3, the percent of total turns was calculated using the total number of turns excluding turns when the ELs were interacting in their own language in the Chinese-Chinese session and during the long stretch in the Chinese-Korean session when they were not interacting. Percentages are rounded to the nearest whole.

Table 12:
Prevalence of Learning Turns Across Sessions

Learning		
Session	Learn turns	% of total turns in session
Session 1 (SS)	15	6%
Session 2 (CK)	1	3%
Session 3 (CC)	2	20%
Session 4 (MEP)	13	6%
Session 5 (SE)	2	2%
Total	33	5%

Use own symbol use to improve English. The first way ELs in this study used NEL symbol systems to help them learn spoken English is when they used their own use of NEL symbol systems to learn and improve their English. Turns that received this code were the turns that included evidence that the EL was intentionally using their use of symbol systems this way. Turns with this type of learning include when ELs wrote a symbol or a number before they said it each time as if it helped them say it right. These turns also include when ELs used a symbol system to demonstrate or describe something they were trying to say. Sometimes they used this description to help themselves think of how to say it, and sometimes they used the description so that somebody else can help them learn to say it in English. There were 24 turns with this code, making up 73% of turns with *learn* codes and 4% of the total turns ELs took (see Table 11). Note that this number is likely an underestimate of how often ELs improved their English through their use of NEL symbol systems, because I did not count turns where there was insufficient evidence that they used the NEL symbol system to learn English. For example, a turn in which an EL pointed to and traced an inscription with her finger as she spoke would not be counted as a learn move even though she may have been using the inscription to help her decide how to express herself in English.

Since this code can be complex, I include three examples that should give the reader an idea of the scope of this code. The first shows how an EL used NEL symbol systems to explain to another EL a word he was trying to figure out how to say. The second shows how an EL used NEL symbol systems to ensure he said something correctly. The final example shows how an EL used NEL symbol systems to create a resource for her to refer to when she spoke about operations.

The first example happened in the Spanish-Spanish session as they worked on the average speed task. This example shows how an EL used NEL symbol systems to explain to someone the word he was trying to learn to say right. Toward the beginning of the task when they were figuring out how they wanted to work on the problem, SS-2 was trying to remember what the average speed was. He said, “So the-- I think, I try to remember-- I think that the, the average speed would be-- it must be the average of the distance divided by total-- total time or average time-- I try to remember. [Writes " $\Rightarrow \bar{r} = \frac{\bar{d}}{total\ time}$ "].” SS-1 then said, “Uh, isn't that uh [points toward the denominator] the-- eh, how do you-- how do you-- eh, I don't know the name of this-- eh, this one? [Writes "$\Delta t.$"] Increase?.” When SS-1 said this, he wrote Δt to try to describe to SS-2 what phrase he was trying to say. After he wrote it he guessed what a word might be for it (increase). He was using the NEL symbols to help develop his English by trying to help them (SS-2 and himself) think of the right phrase for “change in time.” Although it is true that the EL was probably also using the NEL symbols to communicate mathematical information there is evidence that he was simultaneously using them to try to figure out how to communicate the information correctly in English, and thus was developing his spoken English.

The second example also happened in the Spanish-Spanish session, this time as they worked on the handshake task. This example shows how an EL used NEL symbol systems to help himself say something correctly during an explanation. SS-1 and SS-2 were working to solve the problem with n people at a party. SS-2 had the sum $9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 + 0$ written on his paper to represent the number of handshakes that would happen when there were 10 people, and he referred back to it during this turn. During this conversation, SS-2 said the following,

> I, I were thinking the following. You plus 9 [draws a bar connecting 9 and 0 in $9 + \ldots + 0$] mi--plus, [writes “$9 + 0$”] nine plus zero (pauses) is nine [writes “$= 9$”]. Eight plus one [draws a bar connecting 8 and 1 in $9 + \ldots + 0$] is nine [writes “$8 + 1 = 9$”]. Seven plus two [draws a bar connecting 7 and 2 in $9 + \ldots + 0$] [writes “$7 + 2 = 9$”] is 9.

He goes on to describe that given a series of numbers that are spaced evenly, for the formula for the average value one can add the first and the last number together and divide the sum by 2.

The *learn* move happened toward the beginning of SS-2's turn. As he began he had a difficult time figuring out how to talk about the pairs of sums he was describing. He drew a bar connecting the 9 and the 0 in the long sum, but stumbled over his words for a moment ("You plus 9…mi--plus,…"). Then, he wrote "$9 + 0$" and used the structure of the symbol system to structure his verbal English. He said "nine plus 0." He wrote the sums twice more as he said them (i.e., 8+1 and 7+2). But then for the rest of his description he did not need to write the sums (in the form $number + number$) to be able to say them the same way as he did "nine plus zero." Using the symbol systems ("$9 + 0$") likely helped him figure out how to communicate his strategy with spoken English.

This final example comes from the Mathematics teacher-Portuguese session as they worked on the handshake task. This example shows how an EL used NEL symbols she created earlier as a resource to help her say operations correctly. MP-P was trying to figure out a symbolic expression to represent the number of handshakes with 5 people at the party. She wrote "5×4" on her paper earlier. She then said,

> Oh, okay, it's 5, for [points at 5×4], like, it's just [points at 5×4] 5 plus? not plus, it's this sign [points at "$\times$ " in 5×4]. Sorry, I don't know English very well. There are some words, including in math, that I don't know. It's plus [writes "$+$"], menus [writes "$-$"], [writes "$\times$" and "$\div$"], and these two I don't know in English.

As she tried to figure out how to say "times" for "$5 \times 4,$" she went off to the side of her paper and wrote symbols for those four operations, identified what ones she did not know how to say, and looked to MP-M to help her learn the right words based on her symbolic explanations. Later on in the same session when she wanted to say something like 5×4, she would look over at the list she created earlier as if she was using it to remember the symbol names they had discussed. MP-P was essentially creating her own word wall to use as she talked about her strategy for this task. It is likely that this "word wall" was particularly useful to her because she had created it herself while engaged in a mathematical act.

Interpret another's symbol use to improve English. The second way ELs in this study used NEL symbol systems to help them learn spoken English is when they interpreted others' use of the symbol systems for the purpose of learning and improving their own English. When ELs do this they interpret someone else's use of NEL symbol systems in such a way that it is clear they are using them to figure out how to say something, to figure out what a particular word or phrase means, or to think of the best way to describe something. There were likely other times when an EL improved their English by interpreting others' use of symbol systems, but nothing was made public to give an observer evidence of it. This code attempts to capture the turns in which it is clear the EL is doing this. There were 11 turns with this code, making up 33% of turns with *learn* codes and 2% of the total turns ELs took (see Table 11).

An example of how ELs in this study interpreted others' NEL symbol systems to develop their English occurred in the Spanish-Spanish pairing as they began working on the third task. SS-1 looked up at me and asked, "What does it mean if 'wide'?" I used verbal English, gestures, and the diagram to show him what the problem meant when it said that the path was 2 meters wide. As I did, he watched my fingers and the diagram and interpreted my use of those symbol systems to figure out what the problem meant when it said "the path is 2 meters wide." This example shows how an EL might interpret someone's use of NEL symbol systems (e.g,. gestures to the diagram of the path) in response to a request for clarification or help regarding how to say something or what a word means in English.

Another interesting example occurred during this same session as they worked on the handshake task. Recall from the examples in the *legitimize* category of *augmenting*, SS-2 used the word *summatory* for the sum he had constructed to represent the number of handshakes. He defined *summatory* by drawing, pointing at, or tracing the $\sum$ each time he said it. SS-1 interpreted SS-2's use of those symbol systems and adopted the word *summatory* as well. Note that *summatory* is not a real word in English, but because of their use of NEL symbol systems, the ELs were able to create a shared understanding of this word; it was functional to the ELs as they worked on the problem. These examples show how ELs might interpret others students' use of NEL symbol systems to improve their spoken English, perhaps without the other student knowing about it. It seems that given the right situation in which ELs are motivated to use spoken English and are working closely with a knowledgeable other, ELs would be able to effectively learn English vocabulary in this same way as they interpreted others' use of NEL symbol systems to develop their own spoken English.

The "Squared" Story

ELs in this study used NEL symbol systems to support their spoken English by *replacing* spoken English, *augmenting* spoken English, and using them to *learn* spoken English. *The "Squared" Story* provides evidence that replacing and augmenting creates a space for words and phrases to be inserted and learned by ELs during mathematical Discourse. This story shows how one EL used replacing, augmenting, and learning with NEL symbol systems to participate in mathematical Discourse and learn how to correctly use a specific term as he did so. As I tell this story, I describe what happened at each step by describing what led up to the quote being analyzed, and include a figure for each quote that shows the quote itself, how an English speaker might have said the same thing, and a description of the RAL framework analysis for that quote. The figure should enable the reader to easily understand the student's train of thought by referring to how an English speaker may have said the same thing, and compare that to how the EL actually said the quote. The bottom portion of each figure will then clearly depict what portions of the quote received each RAL code.

This story occurred during the Spanish-Spanish session as they worked on the average speed task. SS-1 and SS-2 were trying to figure out how accelerations and velocities could help them solve this problem. It is important to note that even though acceleration is not actually needed to solve this problem, that does not change the fact that SS-1 and SS-2 meaningfully engaged in mathematical Discourse and used non-language symbol systems to support their spoken English as they did so. In the conversation that preceded Quote 1, SS-1 and SS-2 were talking about what the units look like for various characteristics of a graph. They discussed how velocity has "normal units," meaning that they were not dealing with square units. Immediately following his comment about the units for velocity, SS-1 says Quote 1 (see Figure 5) regarding the units of acceleration.

Quote 1	**How an English-speaker might say it**
"...acceleration, remember they give you the unit-- [writes "$a = u^2$"] in this [points to " ^{2}"]. Remember that the units are like, basic, yeah you know? The acceleration, is always [circles " ^{2}"] in this way."	"... For acceleration, remember they give you the units squared. The units are always squared."

Replace-Augment-Learn Coding
"...acceleration, remember they give you the unit-- [writes "$a = u^2$"and pauses] in this [points to " ^{2}"]. Remember that the units are like, basic, yeah you know? The acceleration, is always [circles " ^{2}"] in this way." REPLACE: Predominantly use symbols AUGMENT: Use symbol systems as referents

Figure 5. Squared story, quote 1

It became especially apparent in this moment that SS-1 did not know how to say "squared" when talking about an exponent of 2. But he is able to replace and augment his spoken English to effectively communicate his claim that acceleration is given in units squared.

After this turn, they continued working on the problem. They established that $v = \frac{d}{t}$ and $a = \frac{v}{t}$ where v is velocity, a is acceleration, d is distance, and t is time. Their work progressed until they got to the problem of finding the average speed from $t = 3$ to $t = 4$. At that point, SS-1 said Quote 2 (Figure 6) about how they might apply their formulas to find the acceleration for $t = 3$.

<table>
<tr><th>Quote 2</th><th>How an English-speaker might say it</th></tr>
<tr><td>"So in this we have one d-- [writes "$a = \frac{d}{3^2}$"]. Like this one, right? Yeah, so just we need to [circles d] find this. And this formula [circles $f(t) = \frac{1}{5}t^2$] is un function of t [points at t]. So we just take t as 3 [writes "3" below the t in $f(t)$], and we say that one over five, three [writes " $= \frac{1}{5}(3^2)$"]. Right? And we're gonna get the meters [circles "m" on y-axis]."</td><td>"So for the acceleration formula we now have $a = \frac{d}{3^2}$, right? We just need to find the distance. $f(t)$ is a function of t, so we take $t = 3$ and plug it in. We get $f(3) = \frac{1}{5}(3^2)$ for the distance, right?"</td></tr>
<tr><th colspan="2">Replace-Augment-Learn Coding</th></tr>
<tr><td colspan="2">"So in this we have one d-- [writes "$a = \frac{d}{3^2}$"]. Like this one, right? Yeah, so just we need to [circles d] find this. And this formula [circles $f(t) = \frac{1}{5}t^2$] is un function of t [points at t]. So we just take t as 3 [writes "3" below the t in $f(t)$], and we say that one over five, three [writes " $= \frac{1}{5}(3^2)$"]. Right? And we're gonna get the meters [circles "m" on y-axis]."

REPLACE: Symbols replace operation
REPLACE: Predominantly use symbols
AUGMENT: Use symbol systems as referents
AUGMENT: Illustrate words or phrases</td></tr>
</table>

Figure 6. Squared story, quote 2

In Quote 2, SS-1 described how he wanted to apply the acceleration and distance formulas they had for $t = 3$. He was able to effectively communicate this despite his inability to articulate the idea of something being "squared" because he used the non-language symbol systems to support his spoken English as shown in the coding cell in Figure 6 above.

Almost immediately after Quote 2, SS-1 explains his steps aloud as he applies the acceleration formula for $t = 3$. The reader should not dismiss the following dialogue because the students are solving the problem incorrectly; problem solving and attempting to figure out legitimate solution strategies based on previous knowledge are meaningful practices in mathematical Discourse. Neither SS-1 or SS-2 knew how to do this type of problem and were doing their best to figure out a viable solution strategy. Quote 3 (Figure 7) occurred as SS-1 applied this equation.

<table>
<tr><th>Quote 3</th><th>How an English-speaker might say it</th></tr>
<tr><td>"So we just take distance as this [points at $f(t) = \frac{1}{5}t^2$]. It's equal to have one over five uh, multiplied by 3, [writes "$a = \frac{\frac{1}{5}(3^2)}{}$"] this way [points at ^{2}], over [finishes writing "$a = \frac{\frac{1}{5}(3^2)}{3^2}$"]-- right?"</td><td>“So if we just take the distance as $f(t) = \frac{1}{5}t^2$, it’s the same as having $a = \frac{\frac{1}{5}3^2}{3^2}$, isn’t it?”</td></tr>
<tr><th colspan="2">Replace-Augment-Learn Coding</th></tr>
<tr><td colspan="2">"So we just take distance as this [points at $f(t) = \frac{1}{5}t^2$]. It's equal to have one over five uh, multiplied by 3, [writes "$a = \frac{\frac{1}{5}(3^2)}{}$"] this way [points at ^{2}], over [finishes writing "$a = \frac{\frac{1}{5}(3^2)}{3^2}$"]-- right?"

REPLACE: Predominantly use symbols
AUGMENT: Use symbol systems as referents</td></tr>
</table>

Figure 7. Squared story, quote 3

Again, SS-1 uses the non-language symbol systems to make up for the fact he did not know how to talk about a quantity being “squared.”

After Quote 3, SS-1 took some time to rewrite $a = \frac{\frac{1}{5}(3^2)}{3^2}$ by moving the 5 into the denominator to get $a = \frac{3^2}{5(3^2)}$. At this point, SS-2 suggested that if they write the units on the numbers they are coming up with, it will help them know if they have correct and meaningful answers. SS-1 agreed and in Quote 4 (Figure 8) recalled the units for acceleration.

Quote 4	**How an English-speaker might say it**
"The acceleration is meters divided in-- meters divided by seconds-- uh [points near the *s* where the exponent would go] uh-- (SS-2 interrupts and says, "square?") Yeah."	"The acceleration is meters divided by seconds squared."
Replace-Augment-Learn Coding	
"The acceleration is meters divided in-- meters divided by seconds-- uh [points near the *s* where the exponent would go] uh-- (SS-2 interrupts and says, "square?") Yeah." REPLACE: Symbols replace operation	

Figure 8. Squared story, quote 4

As SS-1 recalled that acceleration is in square units, again he uses non-language symbol systems to support his spoken English when his spoken English is not adequate. But the more interesting thing about this moment is that SS-2 suggested the correct way to say "square." This is not coded as "learning" because SS-1's intention was the same as it was all along—just communicate effectively. He did not point at the place where the exponent would go *and* try to get SS-2 to help him think of the right word—he just pointed at it to communicate his intended meaning. SS-2 was also only using spoken English at this point, so it could not be coded as SS-1 interpreting NEL symbol systems as there were not any. Fortunately, the replacing and augmenting that SS-1 had done all along was effective enough that SS-2 knew exactly what SS-1 was talking about. SS-2 decided to interrupt and suggest the right word during this turn.

Almost immediately after this exchange, SS-1 took another turn (Quote 5, Figure 9) to continue talking about how the units show up in the work he was doing.

<table>
<tr><th>Quote 5</th><th>How an English-speaker might say it</th></tr>
<tr><td>"I-- like you can see here [points at "$=\frac{1}{5}3^2$"] it gives you time in-- it is giving you seconds [circles t^2 and draws an arrow out to the side and writes "s^2"]-- square seconds?"</td><td>“You can see here that it’s giving you time in square seconds.”</td></tr>
<tr><th colspan="2">Replace-Augment-Learn Coding</th></tr>
<tr><td colspan="2">"I-- like you can see here [points at "$=\frac{1}{5}3^2$"] it gives you time in-- it is giving you seconds [circles t^2 and draws an arrow out to the side and writes "s^2"]-- square seconds?"

LEARN: Use own symbol
use to improve English</td></tr>
</table>

Figure 9. Squared story, quote 5

In this turn, SS-1 used his own creation of the symbol s^2 to help him say “square seconds,” so this was coded with a Learn code. The way that SS-1 paused, wrote "s^2," and *then* deliberately said, “square seconds” is evidence that his English was developing by learning how to use and say words related to this concept of “squaring,” *as he participated* in mathematical Discourse.

After this, SS-1 only had one more occasion where he would need to say something was “squared.” Quote 6 (Figure 10) shows this and shows that he continued to use the word “square” correctly and did not need to replace or augment like before.

Quote 6	How an English-speaker might say it
"...Here are the seconds-- *square* seconds [as he says that, writes "s^2" in the denominator to get $\frac{\frac{1}{5}3^2}{3^2s^2}$ and $\frac{3^2}{5(3^2)s^2}$], right?"	"These are seconds squared (or square seconds), right? [Writes "s^2" in the denominator to get $\frac{\frac{1}{5}3^2}{3^2s^2}$ and $\frac{3^2}{5(3^2)s^2}$.]"

Replace-Augment-Learn Coding
"...Here are the seconds-- *square* seconds [as he says that, writes "s^2" in the denominator to get $\frac{\frac{1}{5}3^2}{3^2s^2}$ and $\frac{3^2}{5(3^2)s^2}$], right?" LEARN: Use own symbol use to improve English

Figure 10. Squared story, quote 6

The moment that SS-2 suggested the word "square" was the turning point of this story. Before that moment, SS-1 replaced or augmented his spoken English when he was talking about something that was "squared" because he did not know how to talk about it. After that moment, he never again had to replace or augment when he talked about a quantity being "squared"; he used learn moves and spoke about it correctly. SS-1's use of NEL symbol systems to replace and augment his spoken English created a space in which SS-2 was able to introduce correct terminology, and SS-1 was able to learn it and use it effectively.

Interaction of RAL Framework and Purposes

There are many purposes in which ELs' engagement seems to have been facilitated by replacing, augmenting, and learning moves. Since mathematical activity is often so heavily dependent on the use of a variety of symbol systems, we can claim that the purposes associated with any turn that did not only use spoken English were facilitated by the ELs' use of NEL symbol systems. But, it is more enlightening to examine what purposes seemed to be *facilitated*, at least in large part, by the use of NEL symbol systems as they supported spoken English.

In this final section of this chapter I discuss the extent to which the purposes in the five purpose categories seemed to be facilitated by replacing, augmenting, and learning. I first discuss the breadth of RAL framework moves across ELs' participation in mathematical Discourse by looking at the overall prevalence of RAL framework codes across the purposes. I then discuss the purposes for which ELs seemed to rely a lot on NEL symbol systems by discussing the RAL-reliant purposes.

Prevalence of RAL codes across purposes

An overwhelming majority of the mathematical purposes codes were represented on turns with RAL codes. It is quite possible that many of ELs' mathematical acts were made possible, at least in part, through NEL symbol system literacies. The only purpose that never occurred on the same turn with an RAL framework code is *test a representation or hypothesis using a simpler case.* Refer to Appendix D to see the frequency of each RAL framework code for each purpose.

Based on the data, there is not a clear explanation for the times ELs did not replace or augment for a particular purpose (occurrences of "0's" in the replace and augment columns in Appendix E). The lack of replacing or augmenting for a purpose could have occurred because the ELs did not *need* to replace or augment to accomplish the turns. It could also have been that there was just a lower occurrence of turns with those purposes in the first place and therefore fewer chances to replace or augment as they engaged in those purposes. Yet another possible explanation is that maybe some purposes generally require a higher level of English proficiency to achieve. What the data do tell us is what kind of mathematical purposes the ELs in this study were engaged in as they supported their spoken English through the use NEL symbol systems.

Since slightly more purposes occurred during augmenting turns than occurred during replacing turns, and since augmenting moves typically use more English fluency than replacing moves, I suspect that when there were no replace codes for a particular purpose *did* have augment codes, that purpose may have required slightly more sophisticated English use than was present in replacing moves.

RAL-Reliant Purposes

There are many purposes for which ELs seemed to rely particularly heavily on the use of NEL symbol systems. I define an RAL-reliant purpose as one in which at least 50% of the instances of that purpose code were on turns that received one or more RAL framework codes (see Table 13).

Table 13: *RAL Framework Codes that make up more than 50% of total purposes*

RAL-Reliant Purposes	
1. Make sense of problems and persevere in solving them	% of total
Classify problem	67%
Establish a plan- how should we try to solve this problem?	58%
Establish a goal- what is needed to solve the problem?	57%
Identify "givens"	50%
3. Construct viable arguments and critique the reasoning of others	% of total
Explain why a strategy is applicable	83%
Use an example to justify a claim	80%
Use current representations to justify a claim	81%
Explain how to use the representation to model the problem	86%
Explain or demonstrate a strategy	71%
Explain the elements of a formula	86%
Incorporate another's input into a strategy.	50%
Offer an alternative claim.	57%
4. Facilitate communication and cooperative work	% of total
Restate an explanation to demonstrate understanding.	67%
Request for validation	61%
5. Look for and generalize patterns and repeated reasoning	% of total
Represent a generalized pattern or formula using symbolic notation	50%

There were 15 RAL-reliant purposes. This suggests that to a large extent, using NEL symbol systems to support spoken English may have facilitated ELs' engagement in these purposes. Four of the 13 codes in category 1, *Make sense of problems and persevere in solving them,* are RAL-reliant. The same is true for eight of the 15 codes in category 3, *construct viable arguments and critique the reasoning of others*, two of the nine purposes in category 4, *facilitate communication and cooperative work*, and one of the four purposes in category 5, *look for and* *generalize patterns and repeated reasoning*. None of the purposes in category 2, *model and reason abstractly and quantitatively* are RAL-reliant. It is interesting that so many purposes in purpose category 3 fall into this specification. Closer examination shows that for many of the purposes in category 3, more than 80% of the occurrences occur in turns that receive an RAL framework code. This suggests that as ELs' constructed viable arguments and critiqued the reasoning of others, it was in large part facilitated by the use of NEL symbol systems to replace, augment, and learn spoken English. The fact that there were not any purposes in purpose category 2 that fell into this group suggests that ELs seem to be able to model and reason abstractly and quantitatively in such a way that they did not need to replace or augment their spoken English as much. Perhaps the purposes in category 2 did not require as high a level of English literacy as perhaps did purposes in other categories.

In the turns that received RAL-reliant purposes that *were not* coded with an RAL framework code, the ELs in this study were able to use symbol systems (including spoken English) in such a way that their meaning was communicated successfully without needing the NEL symbol systems to make up for a lack of literacy with spoken English. Sometimes this meant that their use of spoken English was good enough in that turn that it did not need to be made up for. Other times it was still apparent that they lacked fluency with spoken English, but they were able to successfully communicate their meaning anyway without replacing or augmenting their spoken English with the NEL symbol systems.

There were many purposes that are not considered RAL-reliant. Most of these purposes still occasionally occurred on turns with RAL framework codes. For a detailed list of the percent of each purpose that occurred on a turn with an RAL framework code, refer to Appendix D.

Discussion

Based on the results in this chapter I claim that ELs are able to engage in mathematical Discourse by using NEL symbol systems to make up for decreased English literacy. I also claim that participation in mathematical Discourse using NEL symbol systems supports ELs' language use, promotes language development, and allows them to participate meaningfully in mathematical Discourse, solving the Discourse participation paradox.

How the results address the research question and problem

The question this chapter addresses is, *How do ELs use literacy with NEL symbol systems to support their spoken English?* I claim that ELs often use NEL symbol systems to make up for deficiencies in their spoken English during mathematical Discourse. The results in this chapter show that ELs' use of NEL symbol systems facilitates their participation in mathematical Discourse; much of ELs' activity within mathematical Discourse seems to be dependent on ELs' use of NEL symbol systems since ELs with only basic English literacy likely will not be able to engage in as many meaningful mathematical acts relying only on English. NEL symbol systems are valuable to any mathematical learner, but their importance is more heavily weighted for ELs who can rely less on their use of English to understand and communicate.

This chapter shows that ELs use NEL symbol systems to support their spoken English during mathematical Discourse. The RAL framework describes the ways ELs use symbol systems to support spoken English. They replace their spoken English with NEL symbol systems, they augment their spoken English with NEL symbol systems, and they capitalize on the use of NEL symbol systems as a means to learn English. The examples in this chapter show that the ELs in this study *replaced*, *augmented*, and *learned* through their use of NEL symbol systems. It is likely that ELs other than the ones in this study would use the RAL framework as they participate in mathematical Discourse. Due to the wide variety of participants' mathematical backgrounds, first languages, and experiences learning English, the results of this study are generalizable to EL populations outside of this study with basic English literacy; it is reasonable to assume that ELs with fluency in basic conversational English would use NEL symbol systems to replace, augment, and learn spoken English as they participate in mathematical Discourse.

A great deal of ELs' participation in mathematical Discourse is dependent on their use of NEL symbol systems to support their spoken English. There are 15 RAL-reliant purposes, which shows that many of the meaningful purposes that ELs engaged in would probably not have happened very often without the use of NEL symbol systems. In particular, eight of the 15 purposes in purpose category 3, *construct viable arguments and critique the reasoning of others*, are RAL-reliant. Not only are these eight purposes RAL-reliant because more than 50% of the purpose codes were on turns with RAL codes, but for five of the 15 codes in category three, more than *80%* of the turns with that purpose code had RAL codes. The ELs relied heavily on NEL symbol systems to engage in purposes related to such an important part of mathematical Discourse. Without the use of NEL symbol systems it is likely that there would have been extremely limited explaining or critiquing of explanations, which severely limits ELs participation in mathematical Discourse.

Results compared to other studies and theories

In this section I discuss how the results in this chapter compare to other relevant studies and theories. In particular, I make claims related to four prevalent ways of thinking that were found in the framework and literature review in chapter two. I first claim that ELs can learn the meaning of words and phrases and how to use them correctly *as* they participate in mathematical Discourse. I then claim that the replacing, augmenting, and learning that ELs engage in using

NEL symbol systems provide significant support to their activity in the *speaking* and *listening* language domains. Third, I claim that ELs' use of RAL framework moves solves the Discourse participation paradox. Finally I emphasize that ELs' use of NEL symbol systems does a great deal to compensate for ELs' decreased English literacy.

Emphasis on vocabulary acquisition. As ELs participate in mathematical Discourse using NEL symbol systems to replace, augment, and learn English, they are able to learn the meaning of and learn to correctly use specific terms they may not have known before. *The "Squared" Story* in this chapter shows how an EL replaced and augmented using NEL symbol systems to make up for not knowing how to say that something was "squared." He was able to participate in the Discourse well using replacing and augmenting. Learning the English language *is* an important goal for ELs; the "Squared" Story showed that as he participated in the Discourse, SS-1 was able to use *learning* moves and the assistance of SS-2 who understood his *replace* and *augment* moves to learn how to correctly use terms associated with a quantity being "squared."

The "Squared" Story suggests that using replacing and augmenting moves creates a space in the ongoing mathematical conversation for a knowledgeable other to introduce correct vocabulary and demonstrate more sophisticated English use. This knowledgeable other could be another EL who has learned particular words already, an English-speaking student, or a teacher. In the case of *The "Squared" Story*, SS-1 was able to communicate effectively through replacing and augmenting moves. But, as time went on he got increasingly impatient with the need to replace or augment. That created a space where SS-2 was able to suggest the correct word. SS-1 picked it up quickly and no longer needed to replace or augment to make up for not knowing how to talk about a quantity being squared.

Emphasis on the four language domains. The replacing, augmenting, and learning that ELs engage in using NEL symbol systems provides significant support to their activity in the *speaking* language domain. The RAL framework identifies and describes the ways that ELs in this study used NEL symbol systems to support their spoken English, which are clearly described and illustrated throughout this chapter. Clearly, then, as ELs used moves on the RAL framework during mathematical Discourse, those NEL symbol systems provided significant support to their activity within the *speaking* language domain.

Perhaps less obvious than the speaking domain, the replacing, augmenting, and learning that ELs engage in using NEL symbol systems also provides significant support to their activity in the *listening* domain. Whenever the ELs in this study were willing to work together and interact in English, they were able to work well together on the tasks, interact successfully, listen to and discuss each others' ideas, and engage in the purposes discussed in chapter four. The RAL framework moves supported the spoken English on many different occasions for the ELs in this study. Since those moves supported their spoken English so they could more successfully communicate and engage in the purposes, the EL listening to a particular turn had a better chance of understanding what was being said. The listener's task would be difficult if the EL who was speaking did not use NEL symbol systems to support her spoken English; the listener would have to try to understand a turn with inadequate and unsupported English. Because of the high prevalence of RAL moves, however, this potential problem for listeners was avoided. Thus, the RAL framework moves supported ELs' activity in the *listening* domain.

The Necessity of a Discourse perspective. Understanding that ELs can use NEL symbol systems to replace, augment, and learn spoken English to participate in mathematical Discourse solves the Discourse participation paradox. Recall that the Discourse participation paradox is that though we claim that ELs should learn English within mathematical Discourse, language is such a valuable component of mathematical acts in Discourse that it seems inequitable to expect ELs to participate in Discourse with decreased English literacy. Fortunately, the RAL framework moves offer the solution to this perceived paradox. ELs are able to participate meaningfully in mathematical Discourse, using these moves when necessary to facilitate their engagement in the purposes. It is not inequitable to expect ELs to participate in mathematical Discourse because the RAL moves facilitate their participation when their English fluency is not enough. In fact, participation in mathematical Discourse is particularly valuable to ELs because of the potential for language development as they engage in *learn* moves and engage in activity within the language domains while they learn and do mathematics.

Emphasis on the use of multiple representations. This study shows that not only is using many symbol systems good for ELs' mathematical understanding like it would be for English speakers, but the symbol systems are perhaps even *more* important for ELs; the symbol systems allow ELs access to activity they may not have had access to otherwise. ELs use of NEL symbol systems partially compensates for their decreased English literacy, allowing them to participate meaningfully in mathematical Discourse. Mathematics education research often focuses on students' use of multiple representations, or symbol systems, as a means to further their mathematical understanding. For the RAL-reliant purposes in particular, the use of NEL symbol systems is crucial for ELs to engage in those purposes. Much of the mathematical development and understanding that comes from, say, *using a representation to justify a claim* might be scarce for ELs without using symbol systems since 81% of the turns with that purpose also had RAL framework codes.

CHAPTER SIX: CONCLUSION

Summary

There are many ELs in mathematics classes today. Teachers need to learn how to support these students in a way that helps the ELs engage meaningfully in mathematical activity and supports appropriate language use and development. Unfortunately, neither EL education literature nor mathematics education literature provides adequate information for how to best support ELs in mathematics classes. Using a Discourse perspective effectively incorporates ELs' mathematical activity as well as language use that will help them participate in the authentic mathematical activity. I studied ELs' participation in mathematical Discourse by analyzing their mathematical acts as they worked on mathematical tasks in pairs. The first major result was the purpose categories that describe what mathematical activity ELs with basic English literacy might be able to engage in during mathematical Discourse. The second major result of this analysis is the RAL framework, which describes how ELs used NEL symbol systems to support their spoken English and facilitate their participation in mathematical Discourse. These results also provide a solution to the Discourse participation paradox by suggesting why it is reasonable and equitable to expect ELs to participate in mathematical Discourse with only basic levels of English literacy.

Contributions

There are five main contributions that this study makes to the fields of mathematics education and EL education regarding how ELs participate in mathematical Discourse. First, this study suggests that ELs can use conversational English and literacy with other symbol systems to begin to participate in mathematical Discourse. Neither EL education literature nor mathematics education literature makes it clear what literacies students needed to have to be able to participate in mathematical Discourse. Both sides claim that teachers should have high expectations for all students. But, that does not make it clear whether absolutely *every* EL regardless of English literacy or literacy with various NEL symbol systems is able to participate meaningfully in mathematical Discourse. Is it the case that students really do need to learn academic discourse and master important vocabulary words in order to participate in mathematics classrooms? Based on the findings of this research, I claim that they do not. If students have literacy with basic conversation English and literacy with some NEL symbol systems, they will be able to participate meaningfully in mathematical Discourse by performing mathematical acts with valuable purposes.

Second, the purposes and the purpose categories described in this study provide an extensive, specific list of the kinds of activity ELs are likely to engage in during mathematical Discourse. Moschkovich (2007a) showed that ELs could describe patterns, justify claims, and clarify descriptions. Chval and Khisty (2009) showed that ELs could write and clarify a written description. But these descriptions of what purposes ELs can engage in were limited. This study provides an extensive list of the specific kinds of activity ELs are likely to engage in during mathematical Discourse. This can broaden and add depth to researchers' and teachers' understanding of what ELs can do during mathematical Discourse, what purposes to try to get students to engage in, and what this kind of activity looks like.

Third, the concept of analyzing ELs' mathematical acts improves the fields' understanding of how to analyze ELs' participation in mathematical Discourse. Current descriptions of ELs' engagement in mathematics classes tend to be unfocused in that researchers describe ELs' general participation, describe perhaps some steps they took to solve a particular problem, and show the final result of written work or an assignment. Analyzing mathematical acts instead of mathematical practices is a way to increase the specificity of discussion regarding the kinds of activity in which ELs should engage, and significantly focuses efforts to study ELs actively engaged in mathematics classrooms. A mathematical act considers their symbol use, the meaning they communicate with the symbol use, and the purpose in which they are engaging through the symbol use. Focusing on the symbol use addresses their language use but does not overemphasize language use to the point that their abilities and knowledge are underestimated. Analysis of the meanings they communicate enables a focus on students' mathematical understanding and contributions to the ongoing mathematical conversation. Analysis of the purposes in which they engage sheds light into what kinds of activity a student might often engage in and describes in detail much of their participation and contribution to the mathematical Discourse. It is far too general and less useful to describe what ELs might do in a math class with descriptions such as "use multiple representations" or "engage in reasoning and proof." It is far more useful to be able to articulate the components of specific, potential mathematical acts and say that ELs should choose from a variety of symbol systems to create a representation that models their problem, use that representation to reason about the problem, explain to someone how to use that representation to model the problem, and then use the representation to justify claims about the problem. The concept of examining ELs' mathematical acts is a significant step forward in productive analysis of their participation in mathematical Discourse.

Fourth, the RAL framework is a significant development in the field's understanding of how ELs use NEL symbol systems to support their spoken English and facilitate their participation in mathematical Discourse. Using a variety of symbol systems is important for any mathematics student since symbol systems are such a valuable part of mathematics. EL educators have known that there were objects like clipart and drawings that could assist ELs' acquisition of vocabulary. Mathematics educators knew that using a variety of symbol systems (representations) was important for mathematics students in general since many symbols systems are an important part of mathematical activity. But, previous research regarding ELs' participation in mathematical Discourse discussed ELs' use of NEL symbol systems anecdotally, usually as just an element of a particular problem students were trying to solve. This study shows that ELs' use of NEL symbol systems is far more than anecdotal; it is crucial to their participation in mathematical Discourse regarding both their mathematical activity and their language use and acquisition.

Finally, the RAL framework and the related analysis of *The "Squared" Story* suggest that authentic mathematical activity might be a productive site for vocabulary acquisition. A great deal of EL education literature is focused on vocabulary acquisition. While I argue that a sole focus on vocabulary acquisition is not a productive way to support ELs' participation in mathematics classes, students need to learn appropriate words and terms to improve their participation in mathematical Discourse. Knowing and being able to use key words will improve students' ability to engage in purposes and may improve their ability to communicate their thinking. This study suggests that authentic mathematical activity may be a productive and alternative site for effective vocabulary acquisition. *The "Squared" Story* shows that as an EL used RAL moves to engage in many purposes, he learned how to talk about quantities being "squared." The replacing and augmenting he engaged in created a space for the introduction and effective learning of correct vocabulary. Not only does this story demonstrate that one EL learned how to use a word during mathematical Discourse, it demonstrates how ELs might effectively learn vocabulary during their time engaging in authentic mathematical activity.

Implications

The two categories of potential implications from this study are implications related to the research of ELs' participation in mathematics classes and implications for the practice of teaching mathematics to ELs. The main implication for research is that the fields of mathematics education and EL education both need more research of ELs' participation in mathematics classes *using a Discourse perspective with a focus on mathematical acts*. All three components of mathematical acts must be taken into consideration if we are to understand how to simultaneously support ELs' English and mathematical development:

- English language use in mathematics is entwined with the use of many other symbol systems. To focus only on an ELs' language use is likely to overlook how students leverage many non-language symbol systems to support language. This study showed how ELs leveraged NEL symbol systems support their English. Focusing only on language instead of ELs' overall symbol use misrepresents their actual activity during mathematical Discourse.
- Attending to the meaning of a students' mathematical acts allows researchers to consider more than just how well ELs use particular symbol systems. They can then consider the understanding that students are expressing and developing. Since the goal of mathematical instruction is to learn mathematics, it is important to attend to more than just how well they use particular symbol systems. Attending to the meaning of a mathematical act balances the attention given to literacy and content learning so that neither is overlooked.
- Addressing the kind of mathematical activity in which ELs engage opens a space for accounting for many of the goals of mathematics educators regarding what meaningful

activity students need to engage in to best learn mathematics. It also allows researchers to attend to the overall activity ELs' turns are contributing to.

A Discourse perspective with a focus on mathematical acts can unify the goals and intentions of both mathematics education researchers and EL education researchers in their quest to better understand how to support ELs in mathematics classes, because it accounts for all three components of the mathematical act as necessary for meaningful participation in mathematical Discourse.

There are three implications of this research for practice. First, ELs' participation in mathematics classes needs to involve the use of a variety of symbol systems; ELs' use of a symbol system should not be isolated from their use of other symbol systems. Some mathematics teachers may focus on getting students to use one or two types of symbol system at a time, and only encourage using many when explicitly making connections across symbol systems (or representations, as many call them) is the goal of a problem. For example, a teacher might want students to explain each answer to a problem about fractions of candy bars three ways: one solution using just words, one using only numbers in equations, and one using only a drawing or diagram. While such a teacher's intentions might be good, isolating symbol systems in this way is not a good way to support ELs' participation in the class. As evidenced by ELs' use of a variety of symbol systems to accomplish the purposes in this study, ELs benefit from using symbol systems to support each other. Of particular importance in an English mathematics class is their use of NEL symbol systems to support spoken English. To prevent ELs from using their literacies with a variety of symbol systems as needed to communicate and accomplish mathematical purposes is inequitable and ineffective. Forcing ELs to isolate the use of one symbol system without the others to support it is likely to hinder their ability to communicate the meaning component of a mathematical act and hinder their ability to engage in meaningful purposes. ELs should be taught and encouraged to use literacies with many symbol systems in ways that allow them to perform mathematical acts for meaningful participation in mathematical Discourse. Trying to get ELs to rely solely on one type of representation to do and learn a particular mathematical concept is artificial and inequitable; it is more important for meanings and ideas to be effectively represented and communicated, perhaps using many symbol systems, than for an idea to be communicated using only one symbol system.

Second, lecture-based mathematics classes are inequitable for ELs. This study shows that ELs' meaningful participation in mathematical Discourse is facilitated in large part by their use of NEL symbol systems to replace, augment, and learn spoken English. The more lecturing that goes on in a classroom, the less opportunity there is for ELs to have any chance to engage in mathematical acts that use RAL moves, simply because they have less opportunity to take turns and use spoken English at all. Also, while reading and listening to English are two language domains that ELs should engage in and may still have opportunity to engage in during lecture-based classes, the nature of a lecture severely limits ELs' opportunity to speak and write English. Based on the results of this study, it is particularly important for ELs to have the chance to use spoken English along with NEL symbol systems so that they can engage in mathematical purposes.

Lastly, teachers need to use multiple symbol systems when presenting problems. When possible, teachers should include NEL symbol systems along with the spoken and written English they might use to present problems. Consider, for instance, the broken tree task. The drawing of the broken tree provided a representation of the problem at hand for students to use to understand the situation and the question. That problem could have been stated entirely in written English and all the information could have been provided. But, ELs would likely have struggled significantly since they used the drawing/diagram of the tree to make sense of what the problem was saying even before they began to solve it. There may be times when the creation of a representation is an important part of the mathematical goals for the students for a particular problem. In cases like this, consider using another NEL symbol system to create a helpful representation. Or, consider giving students a similar task beforehand to help them begin to learn and use the vocabulary that will be important in the new problem statement.

Limitations and Directions for Future Research

As with any study, this study has some limitations. First, the age and experience of many participants in this study does not fully represent the students in secondary mathematics classes. Some of the students in this study were the age of many high school seniors, but many were in their 20's and 30's. The concern might be that because of their age or experience, these students would be able to do many things that secondary ELs would not. It is true that these ELs might have stronger literacies with many NEL symbol systems based on their prior experience. Fortunately, the tasks these students worked on were all secondary-level mathematics; the literacies ELs used to solve them demonstrate the kinds of solutions, strategies, and activity that secondary students have the potential to use on the same problems. Another important characteristic of the participants to note is that there were many who were not mathematically sophisticated. Many had not taken very high levels of mathematics classes during their secondary or college experience. Some ELs were also many years removed from any mathematical tutelage. These characteristics of the students likely made their participation in mathematical Discourse very similar to many secondary ELs' potential participation.

Future research should investigate pairings of secondary-age students as they work on tasks in a similar manner to the participants in this study. Researchers should investigate what purposes secondary-age ELs engage in to see how consistent they are with the purposes ELs engaged in during this study. The purpose categories created in this study are general enough that they can structure this investigation, though it is possible that secondary ELs may engage in some types of activity that would refine the descriptions of these categories. Future research could also investigate how secondary ELs use NEL symbol systems to support their spoken English. I suspect that they will replace, augment, and learn spoken English using NEL symbol systems like the ELs in this study, but this hypothesis could be researched and verified.

The second potential limitation of this study is that the small sample size may be perceived as under representative of the possible situations in a secondary mathematics class. I addressed this issue by carefully selecting ELs with a variety of first languages. I chose pairings to represent many situations that could occur in a secondary class including two pairs of ELs with the same first language, a pair of ELs with different first languages, an EL paired with an English-speaking student, and an EL paired with a mathematics teacher who did not speak the EL's first language. Future research using larger sample sizes could potentially verify the results of this study and provide the ability to do more comparison across pairing types. As discussed with the first limitation, this study provided a great deal of information about what is important to study with regards to ELs' participation in mathematical Discourse, but it did not lead to any conclusions regarding secondary-age students, which is the population we are most interested in.

The third limitation of this study is that this study examined students' work in pairs, which represents only some of the activity within mathematical Discourse. This study examined in depth ELs' work on mathematical tasks as they worked in pairs, which represents only a portion of the social interactions that could take place in productive mathematical Discourse. Future research could study what purposes ELs engage in and how ELs use NEL symbol systems during small-group work, whole-class discussion, and in situations where an EL presents work to a group.

Fourth, this study investigated and described how ELs used NEL symbol systems to support their *spoken* English but did not address how they used NEL symbol systems to support *written* English. This was due in large part to the nature of the tasks that did not emphasize the use of written English, and due in part to the emphasis on interaction. Most of the English used in this study was *spoken*. Writing is an important component in mathematics and for ELs' language development, however, so future research should investigate how ELs use NEL symbol systems to support their *written* English.

Finally, while this study suggests that RAL moves can support language use and promote language development, it is not clear just how a student may need to progress through the different levels in order to learn English. As discussed in chapter five, it seems that replace and augment moves in particular create openings for the introduction of vocabulary and opportunities for the introduction of more sophisticated language use. What is not clear, however, is just how ELs might need to use the RAL moves to learn English during mathematical Discourse; this study just suggests that English learning can take place through RAL moves. It may be that ELs need to progress from replace moves to augment moves, have vocabulary introduced by some knowledgeable other, and then use learn moves to solidify their newly learned language. It may be that language can be effectively introduced and learned at any point during the use of replace and augment moves. Future research should investigate in detail what role each stage of the RAL framework plays in language development; the findings would have major implications for ELs' language development in mathematics classrooms and for how teachers could support that development.

Conclusion

While there is a great deal of research left to fully understand how to support ELs in mathematics classrooms, this study has made significant progress in understanding how ELs can engage in mathematical Discourse. It has shown that using a Discourse perspective is an effective way to incorporate both mathematical content learning and English language literacy. Studying ELs' mathematical acts attends to the many symbol systems that can be used and leveraged to support other symbol systems when doing and communicating mathematics, looking beyond mere proficiency with symbol use, and attending to students' mathematical activity. ELs are able to participate meaningfully in mathematical Discourse using NEL symbol systems to support their spoken English and facilitate their engagement in mathematical purposes. Teachers should support the use of a variety of symbol systems in their classrooms by encouraging the use of many symbol systems to solve problems and by using many symbol systems to present problems to students. This is an important first step in supporting EL's participation in mathematical Discourse.

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APPENDIX A: TASKS

Task 1

Handshakes

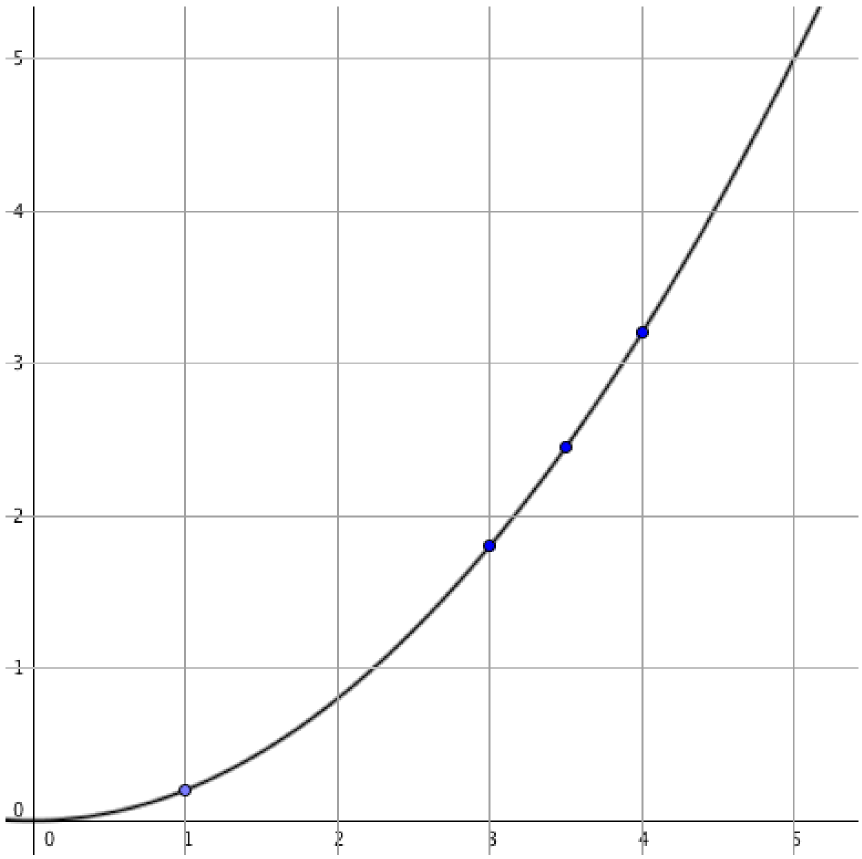
There are 5 people at a party. Everybody shakes everybody else's hand. How many handshakes happen at this party?

What if there are 10 people at the party? How many handshakes happen?

What if there are n people at the party? How many handshakes will happen?

Task 2

This graph shows a bird's distance (in meters) from a tree over time (in seconds). $f(t) = \frac{1}{5}t^2$



What is the bird's average speed from t=1 to t=4?

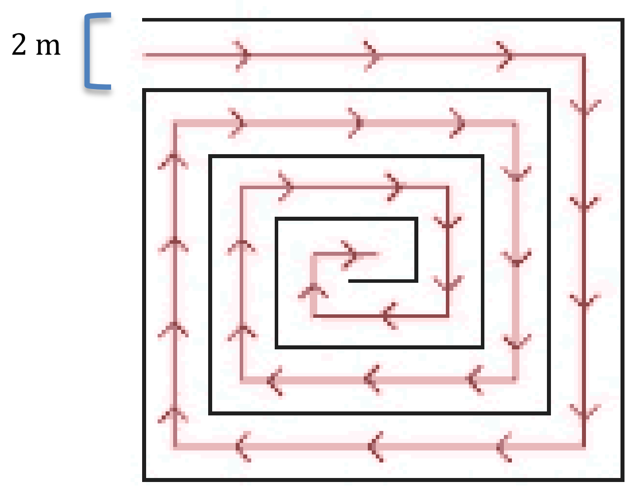
What is the bird's average speed from t=3 to t=4?

What is the bird's average speed from t=3.5 to t=4?

Use average speeds to approximate the bird's speed at exactly 4 seconds.

Task 3

Jon walks along the spiral path on the red line. The line is in the middle of the path. If the path is 2 meters wide, how far does Jon walk?



Task 4

A tree broke ¼ of the distance up the trunk (from the ground). The tree fell so that the top landed 60 feet away from the base. How tall was the tree before it fell?

APPENDIX B: INTERVIEW QUESTIONS

Interview

1. The purpose of this interview is to find out more about your English language experience, your mathematical background, and ask you some questions about the task you did.
2. What is/are your native language(s)?
3. What is your experience learning English? How long have you spoken English? How long have you been in America or another English speaking country?
4. What is your mathematical background? What mathematics classes have you taken in (your native language)? What classes have you taken in English/in America?
5. How comfortable were you with the language you needed to speak during these mathematical tasks? How did that feel compared to the English you use in social settings?
6. (If applicable.) I am going to show you a clip of your video.
    a. In this clip you seemed to struggle expressing yourself. What were you thinking? What were you trying to do? Was the mathematics difficult or did you just struggle explaining your thinking?
    b. In this clip I interpreted that you meant _____. Is that correct?
7. (Alternate if 6 was not necessary) Here are the problems you did. Were there any times that you struggled to explain your thinking? Were you struggling to use English or was it just the math concept you were struggling with?
8. Was there anything you wish you could have done that you couldn't that would have helped you solve these problems?
9. What are some things that help you when you do math in English?
10. If you had to write a mathematical explanation in English, what would you use to help you do that?
11. Is there anything else you would like to say about your experience doing mathematics?

APPENDIX C: PURPOSE CODES AND DEFINITIONS

1. Make sense of problems and persevere in solving them	
Purpose code	Definition
Propose a solution/answer or make a hypothesis	When a student proposes an answer or solution to a problem or makes a hypothesis about the solution to the problem. This includes when a student makes claims about the "answer" to a major portion of the problem, even if it isn't the final solution.
Read and understand the question/problem	Student tries to read, interpret, and make sense of the problem statement and/or what is being asked.
Establish a plan—how should we try to solve this problem?	When a student works to decide on or makes clear what they plan to do to solve the problem. Can include questions such as "Should we do this to solve the problem?"
Try to remember a formula	When a student tries to recall a formula they think is applicable to the current problem. This includes when a student is trying to remember a formula using case-specific numbers (non-generalized).
Self correction	When a student corrects or clarifies (unprompted) their work or explanations.
Decide the next step needed to solve the problem	When students work to decide what the next step needs to be to solve the problem. This can be deciding what to do next to follow their plan.
Identify “givens”	When a student identifies the information that can be assumed in a problem, or what information can now be taken as "given" after some work has been done.
Establish a “big picture” goal—what is needed to solve the problem?	When a student works to decide on or makes clear what is needed to solve the problem.
Review own work	When a student reviews their own work to check for accuracy, find a mistake, or review their strategy.
Classify problem	When a student identifies a problem as a particular type of problem or as similar to a problem type that they are familiar with.
Obtain additional resources	When a student requests or searches for additional resources.
Decide if a strategy is applicable	When a student works to decide if a particular strategy is applicable or viable to help solve the problem.

2. Model and reason abstractly and quantitatively	
Purpose code	Definition
Use a representation of the problem to model the situation/problem	When a student uses the representation (perhaps one they've created) to model some particular event or scenario occurring in the problem situation, or to reason about the given problem.
Solve an arithmetic equation	When a student solves an arithmetic equation/problem.
Apply a general formula to a specific situation	When a student applies a general formula to a specific problem situation. When functions are involved, this includes plugging values into a function (e.g., to find $f(1)$).
Use representation of problem to generate an arithmetic or algebraic representation of a scenario	When students use their work with the representation to create an arithmetic or algebraic representation of a scenario.
Create a representation of a problem/situation	When a student tries to come up with a diagram, figure, or picture that can be used to represent the problem situation.
Represent a scenario arithmetically based on a recalled formula	When a student creates an arithmetic representation of a situation based on a recalled formula (not based a drawing/diagram they've created, for instance). This includes when a student writes something like a limit based on a recalled formula for a limit, even though there are still "variables" in the representation.
Algebraic manipulation	When a student rearranges and re-labels numbers, variables, and other letters. For this study this also includes simplification of radicals, and changing expressions using calculus operations (e.g., taking the derivative of a function).
Contextualize solution	When a student interprets or presents a solution in the context of the problem.

3. Construct viable arguments and critique the reasoning of others	
Purpose code	Definition
Confirm a hypothesis or claim	When a student approves of, supports, or agrees with a hypothesis or claim (that typically comes from another student though it could come from the own student's prior work). This can include "yes" responses to requests for validation, but only if there is sufficient evidence that this is their purpose (otherwise the purpose would be "show they are listening and engaged.")
Ask for clarification or help	When a student asks someone for clarification. (This can also be to clarify expectations, ask for help, request someone to repeat a previous statement, or express general confusion.)
Explain or demonstrate a strategy	When a student explains what their strategy is, demonstrates how to do a particular strategy, or explains how that strategy works.
Explain why a strategy is applicable	When a student explains why a particular strategy is applicable as a way to help solve the problem.
Add to or refine someone else’s idea	When a student adds to or refines another person's idea or claim. This includes when a student completes another's sentence or corrects another student.
Use current representations to justify a claim	When a student uses the structure of the representations being used to justify a claim.
Explain the elements of a formula	When a student explains elements of a formula or describes the formula as a whole.
Ask a question to challenge thinking	When a student asks a question to probe or challenge someone's thinking about a particular piece of mathematics, zeroing in on that particular part. For example, a student may ask, "why did you do this move?" or they may ask, "what kind of function is this?"
Explain why a claim/explanation is not true or why a plan is not wise	When a student explains why another's claim or explanation is not true, or why another's plan is not wise.
Respond to a request for clarification	When a student responds to a request for clarification or repeats something they said previously upon request.
Offer an alternative claim	When a student offers an alternative claim in response to a claim. This includes when a student corrects another student's work.

Use an example to justify a claim	When a student comes up with a new example or uses an existing example to justify a claim.
Explain how to use the representation to model the problem	When a student explains how to use the representation they've created or been given to model the problem, or explains why the representation fits the problem. This explanation is usually, but not always, directed at someone else. This includes when a student explains or clarifies something about how the representation works.
Request elaboration	When a student requests further discussion or extended explanation about a hypothesis, claim, or strategy.
Incorporate another’s input into a strategy	When a student takes up and incorporates another’s input into their own strategy.

4. Productive communication and mathematical disposition	
Purpose Code	Definition
Show that they are listening or engaged in the conversation	When a student uses conversational cues to show they are engaged in the conversation (e.g., says "okay"). This includes general agreement when it is not clear whether or not they are confirming a particular claim. This also includes comments of general understanding, like "I got it" when it's not clear what in particular they are claiming they understand.
Comment on own knowledge or progress	When a student comments on his or her own knowledge of a particular mathematical concept relevant to the problem or when they comment on their progress in solving the problem.
Request for validation	When a student looks to the other student or to the teacher for validation regarding an answer, a claim, a step, or a strategy. This also includes when a student checks to see if the other person is understanding or following along, i.e., "Does that make sense?".
Record an established answer	When a student records an established or agreed-upon answer. This includes when they write "Ans=" for instance based on their own work--the answer is "established" in their own mind.
Give an opinion of the problem	When a student expresses an opinion about the problem or the perceived expectations.
Create a system to organize thinking	When a student creates labels, categories, or some other system to organize their thinking/scratch work.
Compare work or ideas	When a student attempts to compare their work, ideas, or solution with another student's.
Restate an explanation to demonstrate understanding	When a student restates or revoices an explanation to demonstrate understanding.
Establish a common starting point	This is when students pay particular attention to agreeing upon a *common* strategy, goal, or way to start solving the problem.

5. Look for and generalize patterns and repeated reasoning	
Purpose code	Definition
Use the established cases to come up with a general pattern/formula	When a student uses the results of base cases to try to find a general pattern or formula. This includes attempts to generalize based on the arithmetic work they have already done.
Use the representation to find a general pattern/formula	When a student attempts to use the representation (its structure, patterns, etc.) to find a general pattern or formula.
Represent a generalized pattern or formula using symbolic notation	When students attempt to write a general pattern or formula they observe using symbolic notation.
Establish base cases	When a student identifies or creates concrete examples to use as base cases (typically for later generalization).

APPENDIX D: OCCURRENCE OF PURPOSES IN RAL FRAMEWORK CODES

Occurrence of purposes in RAL framework codes			
1. Make sense of problems and persevere in solving them	**Replace**	**Augment**	**Learn**
Read and understand the question/problem	5	9	6
Classify problem	0	1	1
Establish a plan- how should we try to solve this problem?	5	15	2
Establish a goal- what is needed to solve the problem?	7	7	1
Decide what to do next to solve the problem	3	7	2
Try to remember a formula	3	8	8
Identify "givens"	4	6	0
Obtain additional resources	1	2	0
Decide if a strategy is applicable	2	2	0
Test a representation or hypothesis using a simpler case	0	0	0
Self correction	2	6	1
Review own work	2	5	0
Propose a solution/answer or make a hypothesis	5	19	5
2. Model and reason abstractly and quantitatively	**Replace**	**Augment**	**Learn**
Create a representation of a problem/situation.	4	6	0
Solve an arithmetic equation.	8	16	3
Use representation of the problem to model the situation/problem.	13	42	4
Apply a general formula to a specific situation.	8	16	3
Use representation of problem to generate an arithmetic or algebraic representation of that specific scenario.	2	6	1
Represent a scenario arithmetically based on recalled formula.	1	2	1
Algebraic manipulation	3	5	0
Contextualize solution	0	2	1
3. Construct viable arguments and critique the reasoning of others	**Replace**	**Augment**	**Learn**
Explain why a strategy is applicable	5	17	3
Use an example to justify a claim	1	5	2
Use current representations to justify a claim	6	11	3
Respond to a request for clarification	1	3	4
Explain how to use the representation to model the problem	2	6	1
Explain or demonstrate a strategy	8	19	9
Explain the elements of a formula	4	8	4
Incorporate another's input into a strategy.	3	4	1

Explain why a claim/explanation is not true or why a plan is not wise.	0	5	2
Add to or refine someone else's idea.	2	8	3
Offer an alternative claim.	1	6	1
Ask for clarification or help	3	12	3
Question to challenge thinking	1	4	0
Request to elaborate	0	1	0
Confirm a hypothesis or claim.	5	10	2
4. Productive communication and mathematical disposition	**Replace**	**Augment**	**Learn**
Give an opinion of the problem	1	1	0
Establish a common starting point	0	1	0
Create a system to organize thinking	0	1	0
Comment on own knowledge or progress	4	7	1
Restate an explanation to demonstrate understanding.	0	1	3
Compare work or ideas	5	6	1
Record an established answer	0	3	0
Show that they are listening or engaged in the conversation	0	0	2
Request for validation	7	13	3
5. Generalization	**Replace**	**Augment**	**Learn**
Establish base cases	0	2	2
Use the representation to find a general pattern/formula	0	1	1
Represent a generalized pattern or formula using symbolic notation	1	3	1
Use the established cases to come up with a general pattern/formula	1	4	1

The number “5” in the “Read and understand the question/problem” row, in the “Replace” column means that there were 5 turns that were coded as “replace” that had the purpose “read and understand the question/problem.” They 9 “Augment” turns on this same row may have some turns that are also accounted for in the “replace” column if a particular turn had both replacing and augmenting moves.

APPENDIX E: IMPACT OF RAL FRAMEWORK CODES BY PURPOSE

Impact of RAL Framework Codes by Purpose			
1. Make sense of problems and persevere in solving them	**Total**	**RAL Framework**	**% of total**
Read and understand the question/problem	70	15	21%
Classify problem	3	2	67%
Establish a plan- how should we try to solve this problem?	31	18	58%
Establish a goal- what is needed to solve the problem?	23	13	57%
Decide what to do next to solve the problem	19	9	47%
Try to remember a formula	40	15	38%
Identify "givens"	14	7	50%
Obtain additional resources	8	2	25%
Decide if a strategy is applicable	7	3	43%
Test a representation or hypothesis using a simpler case	3	0	0%
Self correction	40	7	18%
Review own work	23	6	26%
Propose a solution/answer or make a hypothesis	88	26	30%
2. Model and reason abstractly and quantitatively	**Total**	**RAL Framework**	**% of total**
Create a representation of a problem/situation.	25	6	24%
Solve an arithmetic equation.	91	22	24%
Use representation of the problem to model the situation/problem.	166	49	30%
Apply a general formula to a specific situation.	54	22	41%
Use representation of problem to generate an arithmetic or algebraic representation of that specific scenario.	32	9	28%
Represent a scenario arithmetically based on recalled formula.	18	3	17%
Algebraic manipulation	20	5	25%
Contextualize solution	12	3	25%
3. Construct viable arguments and critique the reasoning of others	**Total**	**RAL Framework**	**% of total**
Explain why a strategy is applicable	24	20	83%
Use an example to justify a claim	10	8	80%
Use current representations to justify a claim	21	17	81%
Respond to a request for clarification	17	6	35%
Explain how to use the representation to model the problem	7	6	86%
Explain or demonstrate a strategy	38	27	71%
Explain the elements of a formula	14	12	86%
Incorporate another's input into a strategy.	10	5	50%
Explain why a claim/explanation is not true or why a plan	15	6	40%

is not wise.			
Add to or refine someone else's idea.	24	10	42%
Offer an alternative claim.	14	8	57%
Ask for clarification or help	45	15	33%
Question to challenge thinking	10	4	40%
Request to elaborate	9	1	11%
Confirm a hypothesis or claim.	112	13	12%
4. Productive communication and mathematical disposition	**Total**	**RAL Framework**	**% of total**
Give an opinion of the problem	7	2	29%
Establish a common starting point	2	0	0%
Create a system to organize thinking	11	1	9%
Comment on own knowledge or progress	52	10	19%
Restate an explanation to demonstrate understanding.	6	4	67%
Compare work or ideas	16	7	44%
Record an established answer	38	3	8%
Show that they are listening or engaged in the conversation	90	2	2%
Request for validation	31	19	61%
5. Generalization	**Total**	**RAL Framework**	**% of total**
Establish base cases	11	2	18%
Use the representation to find a general pattern/formula	5	2	40%
Represent a generalized pattern or formula using symbolic notation	8	4	50%
Use the established cases to come up with a general pattern/formula	20	6	30%