2008-07-11

Complete Tropical Bezout's Theorem and Intersection Theory in the Tropical Projective Plane

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COMPLETE TROPICAL BÉZOUT'S THEOREM AND INTERSECTION THEORY IN THE TROPICAL PROJECTIVE PLANE

by

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A dissertation submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

Department of Mathematics

Brigham Young University

August 2008

BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a dissertation submitted by

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ABSTRACT

COMPLETE TROPICAL BÉZOUT'S THEOREM AND INTERSECTION THEORY IN THE TROPICAL PROJECTIVE PLANE

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Doctor of Philosophy

In this dissertation we prove a version of the tropical Bézout's theorem which is applicable to all tropical projective plane curves. There is a version of tropical Bézout's theorem presented in other works which applies in special cases, but we provide a proof of the theorem for all tropical projective plane curves. We provide several different definitions of intersection multiplicity and show that they all agree. Finally, we will use a tropical resultant to determine the intersection multiplicity of points of intersection at infinite distance. Using these new definitions of intersection multiplicity we prove the complete tropical Bézout's theorem.

ACKNOWLEDGMENTS

First, I would like to thank my advisor, Dr. Tyler Jarvis, for his help and insight as I worked on my research. I would also like to thank the other members of my committee, Dr. William Lang, Dr. Darrin Doud, Dr. Jasbir Chahal, and Dr. Xian-Jin Li, for their help, support, and feedback on my research. I also never would have understood tropical mathematics as well as I do without the help of the undergraduate and graduate students and other clever mathematicians that researched with me, so I would like to thank Michael Anderson, Christopher Cornwell, Ryan Coatney, Amanda Ellis, Nathan Grigg, Dr. Tracy Hall, Aaron Hill, Mark Kempton, Nathan Manwaring, Daniel Philips, Brian Rushton, Julian Tay, Sean Tibbits, Darryl Wade, and Natalie Wilde. I would like to especially thank Nathan Grigg for his tropical Maple program [7], with which I drew most of my graphs and computed my resultants. Next, I would like to thank Lonette Stoddard and Ivan Andrus for their help and patience with me as I worked with LaTeχ, with special thanks to Ivan for making my proof box. My dissertation looks better because of them. I would also like to thank all of the professor and graduate students in the math department at BYU, and anyone else, who kindly talked tropical math with me. Finally, and most importantly, I would like to thank my family for their continued love and support as I completed my degree. Without their love, faith, and prayers I never would have made it.

Table of Contents

List of Tables

List of Figures

1 Introduction

In this dissertation we prove a complete version of the tropical Bézout's theorem, as stated below.

Theorem 51. *Let $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ be two tropical projective plane curves of degree d and e respectively. Then $\mathcal{Z}(f)$ stably intersects $\mathcal{Z}(g)$ in $d \cdot e$ points, counting multiplicity.*

In [6], [16], and [19] a tropical version of Bézout's theorem is presented, but only for polynomials that satisfy certain restrictive conditions. The Bézout's theorem presented in those articles is applicable to curves of *full support* which are in *general position* with respect to each other. Those restrictions result in the fact that all of intersections happen in the tropical affine plane. This is a strong restriction and does not explain how curves might intersect at infinite distance. We extend the theorem to all tropical projective plane curves, using a different method of proof than that found in [6], [16], and [19]. We also provide solid foundations and tools for working with intersections of tropical plane curves and intersection multiplicity in both the tropical affine plane and the tropical projective plane.

We will first give some basic background information in the area of tropical mathematics. There are many articles (for example, [2],[4],[6],[8],[11],[17],[12],[13],[16],[18], etc.) that discuss background information in tropical mathematics, and may be of interest for either a different perspective or more information than is given here on certain topics. Although these articles provide good background information, we will need to discuss more fully some of the topics that are presented in them such as functional equivalence of tropical polynomials, tropical linear algebra, a tropical version of Cramer's rule, and the behavior of tropical plane curves in the projective plane. We will then discuss some new ideas, such as a tropical plane curve deformations, a tropical resultant, and tropical intersection theory at infinite distance.

Although many tropical theorems appear to be quite similar to classical theorems, the proofs are in general quite different and do not translate from the classical setting to the tropical setting easily. Although we are often interested in determining which of the classical theorems have tropical analogues, the proofs of the classical theorems rarely give us any insight into how to proceed in proving the tropical versions. For this reason, we provide the tropical background and proofs of several theorems and lemmas throughout the dissertation. These proofs are necessary to build up to the proof of the complete tropical Bézout's theorem.

We will first discuss tropical semi-rings and some of their properties. We will be most interested in the tropical semi-field $\mathcal{Q}$ and the tropical polynomial semi-rings $\mathcal{Q}[x_1, \ldots, x_n]$. Using homogeneous tropical polynomials in $\mathcal{Q}[x, y, z]$, we will define tropical projective plane curves. We will then consider the intersections of tropical projective plane curves. In order to understand all of the points of intersection, not just those in the tropical affine plane, we will use a tropical resultant. In order to define the resultant we will need to build up a foundation of tropical linear algebra. Once we have the tropical linear algebra in place, we will be able to define our resultant, and use it to determine points of intersection of two tropical plane curves. We will also use the resultant to define intersection multiplicity in a way which applies to all tropical projective plane curves, regardless of where they intersect in the tropical projective plane. In [6], [16], and [19], two definitions were given for intersection multiplicity. These definitions apply in the special cases when the plane curves are in general position to each other and when all of the points of intersection are in the affine plane. We will show that the definition of intersection multiplicity that we give agrees with these other two definitions, in the cases where they apply. With this definition of intersection multiplicity, we will then be able to prove a complete version of tropical Bézout's theorem, which applies to all tropical projective plane curves.

Finally, we will investigate some other related tropical results. Classically there are some interesting consequences of Bézout's theorem. We will investigate two of those in the tropical setting, namely Pascal's Hexagon and the group law on elliptic curves. Unfortunately, Pascal's Tropical Hexagon is not a straightforward result of tropical Bézout's theorem, although we can still prove it is still true. In the tropical case it is, instead, a consequence of the tropical version of Cramer's Rule. Associativity of the group law on tropical elliptic curves is also not a straightforward result of tropical Bézout's theorem. However, Bézout's theorem and the tropical resultant can be used to show that the integer points on a tropical elliptic curve are closed under addition.

2 Tropical Mathematics

We will begin with a brief overview of tropical mathematics. Tropical algebraic geometry is a fairly recent field of study which can be thought of in several different ways. In this section we will build up one of those ways of looking at tropical algebraic geometry by starting with what is called a *tropical semi-ring*. In this chapter we will explore certain properties of the tropical semi-ring that will help us better understand tropical linear algebra and tropical plane curves in the later chapters.

2.1 Tropical Semi-Rings

Definition 1. 1. A *semi-ring* $\mathcal{R}$ is a set, together with two binary operations $\oplus$ and $\odot$ satisfying the following axioms:

   (a) $\oplus$ is associative, and commutative,

   (b) $\odot$ is associative,

   (c) there exists an element $e_\oplus$ in $\mathcal{R}$ such that $e_\oplus \oplus a = a \oplus e_\oplus = a$ for every a in $\mathcal{R}$,

   (d) the distributive laws hold: $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$ and $c \odot (a \oplus b) = (c \odot a) \oplus (c \odot b)$

2. The semi-ring $\mathcal{R}$ is *commutative* if $\odot$ is commutative.

3. The semi-ring $\mathcal{R}$ is said to have an *identity* (or *contains* $e_\odot$) if there is an element $e_\odot$ in $\mathcal{R}$ such that $e_\odot \odot a = a \odot e_\odot = a$ for every a in $\mathcal{R}$

We note that in a semi-ring the addition is not necessarily invertible, meaning that unlike a ring, where the set $\mathcal{R}$ needs to be a group under the operation $\oplus$, for a semi-ring, $\mathcal{R}$ simply needs to be an additive monoid.

Definition 2. A semi-ring $\mathcal{R}$ with identity $e_\odot$, where $e_\odot \neq e_\oplus$, is called a *semi-division ring* if every element $a \neq e_\oplus$ in $\mathcal{R}$ has a multiplicative inverse, i.e. there exists $b \in \mathcal{R}$ such that $a \odot b = b \odot a = e_\odot$. A commutative semi-division ring is called a *semi-field*. In the semi-field $\mathcal{R}$ we will use $c \oslash a$ to denote division, that is $c \oslash a = c \odot b$ where b is the multiplicative inverse of a.

Although there are many examples of semi-rings, we will be interested only in a special set of semi-rings, called the *tropical semi-rings*. The tropical semi-rings we will consider are those semi-rings which are subsemi-rings of $(\mathcal{R}, \oplus \odot)$ and polynomial and matrix semi-rings defined over the subsemi-rings of $(\mathcal{R}, \oplus, \odot)$, where$\mathcal{R} = \mathbb{R} \cup \{\infty\}$, $a \oplus b = \min\{a, b\}$, with $\infty = e_\oplus$ so $a \oplus \infty = a$ for all a, and $a \odot b = a + b$, with $a \odot \infty = \infty$. It is possible to define a tropical semi-ring that does not have $e_\oplus = \infty$, as shown in Section 2.2.

We note here that we will often suppress the multiplication sign $\odot$ for simplicity. But, since we will still on occasion need to use the classical multiplication as well, we will always use a dot $\cdot$ to represent classical multiplication. So, if we have ab, then we will assume this to mean $a \odot b$, and if we want the classical product, we will write $a \cdot b$.

We give now a few examples of tropical semi-rings.

Example 1. Consider $(\mathcal{N}, \oplus, \odot)$, where $\mathcal{N} = \mathbb{N} \cup \{0\} \cup \{\infty\}$ and for any $a, b \in \mathcal{N}$, $a \oplus b = \min\{a, b\}$ and $a \odot b = a + b$, and $a \odot \infty = \infty$. We will show that $(\mathcal{N}, \oplus, \odot)$ is a semi-ring.

To show that $\mathcal{N}$ is a semi-ring, we need to show that it satisfies Definition 1.

1. (a) $\oplus$ is associative, and commutative:

   Let $a, b, c \in \mathcal{N}$. Then

$$
\begin{aligned}
(a \oplus b) \oplus c &= \min\{\min\{a, b\}, c\} \\
&= \min\{a, b, c\} \\
&= \min\{a, \min\{b, c\}\} \\
&= a \oplus (b \oplus c)
\end{aligned}
$$

   and $a \oplus b = \min\{a, b\} = \min\{b, a\} = b \oplus a$.

   (b) $\odot$ is associative:

   Let $a, b, c \in \mathcal{N}$. Then $(a \odot b) \odot c = (a+b)+c = a+(b+c) = a \odot (b \odot c)$.

   (c) There exists an element $e_{\oplus}$ in $\mathcal{R}$ such that $e_{\oplus} \oplus a = a \oplus e_{\oplus} = a$ for every a in $\mathcal{R}$:

   Let $a \in \mathcal{N}$. Then $a \oplus \infty = a$ and $\infty \oplus a = a$, so $\infty = e_{\oplus}$ in $\mathcal{N}$.

   (d) The distributive laws hold: $(a \oplus b) \odot c = (a \odot c) \oplus (b \odot c)$ and $c \odot (a \oplus b) = (c \odot a) \oplus (c \odot b)$:

   Let $a, b, c \in \mathcal{N}$. Then

$$
\begin{aligned}
(a \oplus b) \odot c &= (\min\{a, b\}) + c \\
&= \min\{a + c, b + c\} \\
&= (a \odot c) \oplus (b \odot c)
\end{aligned}
$$

   and

$$
\begin{aligned}
c \odot (a \oplus b) &= c + \min\{a, b\} \\
&= \min\{c + a, c + b\} \\
&= (c \odot a) \oplus (c \odot b).
\end{aligned}
$$

2. We note that $\mathcal{N}$ is in fact a commutative semi-ring, since

$$\begin{aligned} a \odot b &= a + b \\ &= b + a \\ &= b \odot a. \end{aligned}$$

3. The multiplicative identity in $\mathcal{N}$ is 0 since $a \odot 0 = a + 0 = a$ for every $a \in \mathcal{N}$.

4. Although not part of the definition, we prove now as well the the additive identity and the multiplicative identity are unique in any tropical semi-ring.

   Suppose the $\mathcal{T}$ has two elements $e_{\oplus}$ and $e'_{\oplus}$ which are both additive identities. Then $e_{\oplus} = e_{\oplus} \oplus e'_{\oplus} = e'_{\oplus}$, so these two elements are the same, and the additive identity is unique.

   Now suppose that there are two elements in $\mathcal{T}$ $e_{\odot}$ and $e'_{\odot}$ which are both multiplicative identities. Then $e_{\odot} = e_{\odot} e'_{\odot} = e'_{\odot}$, so these two elements are the same, and the multiplicative identity, if it exists, is unique.

$\mathcal{N}$ is in fact a semi-ring, and not a semi-field since $\mathbb{N}$ does not contain any of the classical additive inverses, therefore $\mathcal{N}$ does not have tropical multiplicative inverses. In Section 2.2 we will discuss in more detail the *tropical arithmetic* of this semi-ring, as it is the tropical semi-ring which is analogous to the ring of integers $\mathbb{Z}$.

Example 2. The sets of positive integers, positive rationals, or positive reals together with ∞ with the given operations are tropical semi-rings. Similarly, the negative integers, negative rationals, and negative reals together with ∞ are tropical semi-rings. These tropical semi-rings do not have multiplicative identities. However, the sets of non-positive or non-negative integers, rationals or reals together with ∞ can be made into tropical semi-rings with multiplicative identities.

Consider the set $T = \{x \in \mathbb{Z} | x \geq a, \text{ for some number } a \in \mathbb{Z}\}$. If $a = 0$, then $T \cup \{\infty\}$ is the tropical semi-ring defined on the set of non-negative integers. If $a = 1$ then $T \cup \{\infty\}$ the tropical semi-ring defined on the positive integers. If $a > 1$, then $T \cup \{\infty\}$ a tropical semi-ring defined on those integers which are greater than or equal to a, because this set is closed under minimum and under addition. But, if $a < 0$, then we do not have a semi-ring. Take, for example, $a = -1$. Then $a \odot a$ must be present to have a semi-ring. But $-1 \odot -1 = -1 + -1 = -2$, but $-2 \notin \{x \in \mathbb{Z} | x \geq -1\}$. Thus, for $a < 0$, $T \cup \{\infty\}$ is not a tropical semi-ring. For the case of the integers (the rationals, the reals) $T_+ = \{x \in \mathbb{Z}\ (\mathbb{Q}, \mathbb{R}) | x \geq a, \text{ for some number } a \in \mathbb{Z}\ (\mathbb{Q}, \mathbb{R})\} \cup \{\infty\}$ with the given operations is a tropical semi-ring for any $a \geq 0$. Similarly, the set $T_- = \{x \in \mathbb{Z}\ (\mathbb{Q}, \mathbb{R}) | x \leq a, \text{ for some number } a \in \mathbb{Z}\ (\mathbb{Q}, \mathbb{R})\} \cup \{\infty\}$ with the given operations is a tropical semi-ring for any $a \leq 0$.

Example 3. $(\mathcal{Z}, \oplus, \odot)$, $(\mathcal{Q}, \oplus, \odot)$, and $(\mathcal{R}, \oplus, \odot)$ are all tropical semi-fields where $\mathcal{Z} = \mathbb{Z} \cup \{\infty\}$, $\mathcal{Q} = \mathbb{Q} \cup \{\infty\}$, and $\mathcal{R} = \mathbb{R} \cup \{\infty\}$. These are semi-fields and not just semi-rings because they all contain the classical additive inverses, so they contain the tropical multiplicative inverses.

In all of the above cases, the arguments used in Example 1 verify that we have a semi-ring.

Definition 3. Let $\mathcal{R}$ be a semi-ring.

1. An element $a \neq e_\oplus$ of $\mathcal{R}$ is called a *zero divisor* if there is an element $b \neq e_\oplus$ in $\mathcal{R}$ such that either $a \odot b = e_\oplus$ or $b \odot a = e_\oplus$.

2. Assume $\mathcal{R}$ has an identity $e_\odot \neq e_\oplus$. An element u of $\mathcal{R}$ is called a *unit* in $\mathcal{R}$ if there is some v in $\mathcal{R}$ such that $u \odot v = v \odot u = e_\odot$. The set of units in $\mathcal{R}$ is denoted $\mathcal{R}^\times$.

Definition 4. A commutative semi-ring with identity $e_\odot \neq e_\oplus$ is called a *semi-integral domain* if it has no zero divisors.

In the tropical semi-rings which have ∞ as their additive identity, there are never any tropical zero divisors, since the only way to have $a \odot b = \infty$ is to have either a or b equal to infinity. Thus all of those tropical semi-rings are in fact tropical semi-integral domains.

Proposition 1. *Assume a, b and c are elements of a tropical semi-ring with a not a zero divisor. If $a \odot b = a \odot c$ then either $a = e_\oplus$ or $b = c$. In particular, if a, b, c are any elements in a tropical semi-integral domain and $a \odot b = a \odot c$ then either $a = e_\oplus$ or $b = c$.*

Proof. In a tropical semi-ring, if $a \odot b = a \odot c$ then $a + b = a + c$. If $a \neq \infty$, then since we have the condition in terms of classical addition, we can subtract a from both sides, so we get $b = c$.

Definition 5. A *subsemi-ring* of the semi-ring $\mathcal{R}$ is a submonoid of $\mathcal{R}$ that is closed under multiplication.

Just as in the case of rings, a subsemi-ring $\mathcal{S}$ of $\mathcal{R}$ is a subset of $\mathcal{R}$ which has the structure of a semi-ring when the operations of $\mathcal{R}$ are restricted to $\mathcal{S}$.

Example 4. Let $\mathcal{R}$ be the real tropical semi-ring. Then $\mathcal{Q}$, the rational tropical semi-ring, is a subsemi-ring or $\mathcal{R}$, and $\mathcal{Z}$, the integer tropical semi-ring is a subsemi-ring of $\mathcal{Q}$ and $\mathcal{R}$.

Example 5. Let $a \in \mathbb{R}$. The set $T_a = \{a \cdot n | n \in \mathbb{Z}\}$ can be used to make a tropical semi-ring $a \cdot \mathcal{Z} = T_a \cup \{\infty\}$. Let us take for an example $a = 3$. Then we have for our set T all classical integer multiples of 3. The tropical addition is closed, since the minimum of two classical multiples of three is a classical multiple of three. Also, the tropical multiplication is closed, since the classical sum of two classical multiples of 3 is a classical multiple of three. And all of the other properties needed for a semi-ring hold by the same arguments as given in Example 1. But a does not need to be an integer. We could have $a = \frac{1}{2}$. Then we would have all rational numbers which can be written as $\frac{b}{2}$ for $b \in \mathbb{Z}$. Again, for the same reasons, this is a semi-ring. Or we could have $a = \pi$, and then we would have all integer multiples of π in our set T. If we have $a = 0$, then we have the semi-field with only 0 and ∞. And we note that if $a \in \mathbb{Z}$, then $\mathcal{T}_a$ is a subsemi-ring of $\mathcal{Z}$, and by extension of $\mathcal{Q}$ and $\mathcal{R}$, if $a \in \mathbb{Q}$, then $\mathcal{T}_a$ is a subsemi-ring of $\mathcal{Q}$ and $\mathcal{R}$, and if $a \in \mathbb{R}$, then $\mathcal{T}_a$ is a subsemi-ring of $\mathcal{R}$.

Example 6. We may also make tropical polynomial rings in one or more variables, by taking a tropical semi-ring or tropical semi-field and adjoining indeterminates and proceeding in the normal way, only using the tropical operations for everything, where when we have x^n we mean repeated multiplication, so $x^n = x \odot x \odot \cdots \odot x = n \cdot x$. Tropical polynomial semi-rings will be discussed in detail in Section 2.4

Example 7. Just as in the classical setting, we can create tropical matrix semi-rings. Fix an arbitrary tropical semi-ring $\mathcal{T}$ with or tropical polynomial semi-ring and let n be any positive integer. Let $\mathcal{M}_n(\mathcal{T})$ be the set of all $n \times n$ matrices with entries from $\mathcal{T}$. As usual, we will denote a matrix in $\mathcal{M}_n(T)$ by (a_{ij}) where the entry in row i and column j is a_{ij}. We define the addition and multiplication just as they are classically, only using the tropical operations in place of the classical operations. More specifically, the addition is componentwise, so the i, j entry of $(a_{ij}) \oplus (b_{ij})$ is $a_{ij} \oplus b_{ij}$, and the multiplication is given by the i, j entry of $(a_{ij})\,(b_{ij})$ is $\bigoplus_{k_1}^{n} a_{ik}b_{kj}$. Using these two operations, it is straightforward to show that $\mathcal{M}_n(\mathcal{T})$ is a tropical semi-ring, however, just as in the classical setting, it is not commutative. The additive identity in $\mathcal{M}_n(\mathcal{T})$ is the matrix all of whose entries are the additive identity of $\mathcal{T}$, and the multiplicative identity is the matrix with diagonal entries $e_\odot$ and all off diagonal entries $e_\oplus$. We note that $\mathcal{M}_n(\mathcal{T})$ will only have a multiplicative identity if $\mathcal{T}$ has one.

Not only can we have tropical matrix semi-rings, but we can also start with a tropical semi-field, and build a tropical vector space. This allows us to do a lot of *tropical linear algebra*, which will be discussed in more detail in Chapter 4

2.2 Tropical Arithmetic in $\mathcal{N}$

We give some properties of the tropical natural numbers. Recall that we define $(\mathcal{N}, \oplus, \odot)$ to be the tropical semi-ring where $\mathcal{N} = \mathbb{N} \cup \{0\} \cup \{\infty\}$. This tropical semi-ring is the natural analogue to $\mathbb{Z}$, since it is in fact a semi-ring, instead of a semi-field. We will provide some basic definitions and properties which are analogous to some for the classical integers, along with some more specific results for the tropical setting.

1. Since the tropical natural numbers are a subset of the integers, the Well Ordering property of $\mathbb{Z}$ applies to them. Classically, this property says that any nonempty subset of $\mathbb{Z}^+$ has a minimal element. Tropically this property says that for any nonempty subset A of $\mathcal{N}$, there is some element $m \in A$ such

that $m \leq a$ for every $a \in A$. In the tropical setting, this means that for that m, $m \oplus a = m$ for every $a \in A$.

2. If $a, b \in \mathcal{N}$, we say a *divides* b if there is an element $c \in \mathcal{N}$ such that $b = a \odot c$. We will write $a \mid b$ if a divides b, and $a \nmid b$ otherwise.

Lemma 2. *Let $a, b \in \mathcal{N}$. Then $a \mid b$ if and only if $a \oplus b = a$.*

Proof. Suppose $a \mid b$. Then there is some $c \in \mathcal{N}$ such that $b = a \odot c$, but $a \odot c = a + c$, and $c \geq 0$, so $a + c \geq a$, so $a = \min\{a, a+c\} = \min\{a, b\} = a \oplus b$. Suppose $a \oplus b = a$. Then $b \geq a$, so $c = b - a \geq 0$, so $c \in \mathcal{N}$, and $b = a \odot c$, so by definition $a \mid b$.

3. For $a, b \in \mathcal{N}$ there is a unique tropical natural number d, called the *greatest common divisor of a and b* (or g.c.d. of a and b, denoted $(a, b)_{\mathbb{T}}$) that satisfies:

   (a) $d \mid a$ and $d \mid b$, and

   (b) if $e \mid a$ and $e \mid b$, then $e \mid d$.

Lemma 3. *Let $a, b \in \mathcal{N}$. Then $(a, b)_{\mathbb{T}} = a \oplus b$.*

Proof. Clearly, if $a \mid b$, then $(a, b)_{\mathbb{T}} = a = a \oplus b$, because a is a divisor of both a and b, but nothing larger than a divides a, as we see from Lemma 2. Now suppose that $a \nmid b$. Then by Lemma 2, we see that $a \oplus b = b$ and $b \mid a$, so $(a, b)_{\mathbb{T}} = b = a \oplus b$.

If $(a, b)_{\mathbb{T}} = 0$ we say a and b are *tropically relatively prime.* We note that if $(a, b)_{\mathbb{T}} = 0$ then $a = 0$ or $b = 0$, so no two numbers $a \neq 0$, $b \neq 0$ are tropically relatively prime.

4. For $a, b \in \mathcal{N}$ there is a unique tropical natural number l, called the *least common multiple of* a *and* b (or l.c.m. of a and b), that satisfies:

   (a) $a \mid l$ and $b \mid l$, and

   (b) if $a \mid m$ and $b \mid m$, then $l \mid m$.

   Lemma 4. *Let $a, b \in \mathcal{N}$. If $a \oplus b = a$, then the l.c.m. of a and b is equal to b.*

   Proof. If $a \oplus b = a$, then by Lemma 2 $a \mid b$, and we know that $b \mid b$. Suppose there is some other number m such that $a \mid m$ and $b \mid m$. Then $m \geq b$ by Lemma 2. Thus b must be the l.c.m. of a and b.

   In the classical setting the connection between d, the g.c.d. of a and b, and l, the l.c.m. of a and b, is given by $dl = ab$. This also holds tropically, but much more trivially, since $a = d$ and $b = l$ (if $a \oplus b = a$).

5. The *Division Algorithm* and the *Euclidean Algorithm* are not nearly as interesting in the tropical setting as they are in the classical setting, but they do exist. For $a, b \in \mathcal{N}$, then there exist $q, r \in \mathcal{N}$ such that

$$a = q \odot b \oplus r.$$

   The difference is that there is no uniqueness, and there is no bound on what values r can take on.

   Let's look at an example of this. Consider the numbers $a = 7$ and $b = 3$. Then, it is clear that $7 = 4 \odot 3 \oplus \infty$. But it is also true that $7 = 4 \odot 3 \oplus 8$, and that $7 = 5 \odot 3 \oplus 7$. Now, of course, the *best* solution would probably be considered $7 = 4 \odot 3 \oplus \infty$, as it mirrors that classical version slightly better, since 7 is in fact a multiple of 3. Now set $a = 3$ and $b = 7$. We note that 3 is not a multiple of 7, so we should have some finite remainder in this case. For example, $3 = \infty \odot 7 \oplus 3$, but we also have $3 = 0 \odot 7 \oplus 3$. Again, we might pick $3 = \infty \odot 7 \oplus 3$ as the best possible answer, as it mirrors the classical case, but that is not forced upon us. As for the Euclidean Algorithm, if you use the best possible version of the division algorithm, and you make sure to start with $a \geq b$, then one step produces the g.c.d. However, there is no real point to the algorithm, since the g.c.d. is just the tropical sum of the two numbers.

6. An element $p \in \mathcal{N}$ is said to be *tropically prime* if $p > 0, p \neq \infty$ and the only divisors of p are 0 and p itself. All other numbers greater then 0 that are not prime are called *tropically composite.*

Lemma 5. *The only tropical prime number is 1.*

Proof. For any $a > 0$, $a \oplus 1 = 1$, so $1 \mid a$. Thus, the only number greater than zero that has only 0 and itself for divisors is 1.

Because of this, it is easy to show that the *Fundamental Theorem of Tropical Arithmetic* is true. Since any number $n \in \mathcal{N}, n > 0, n \neq \infty$ can be written as the classical sum of n 1's, it has a unique prime factorization, even if it is not a very interesting factorization.

Example 8. Let us consider, instead of the tropical semi-ring $\mathcal{N}$, the tropical semi-ring $\mathcal{T}$ we get from the set $T = \{n \in \mathcal{N} | n = 0 \text{ or } n \geq a \text{ for some } a \in \mathcal{N}, a > 0\}$. If $a = 1$, then we just have $\mathcal{N}$ again, but if, for example, $a = 5$, then we have a different tropical semi-ring. In this semi-ring, the numbers that have for divisors only 0 and themselves are 5,6,7,8, and 9. Even though $5 \mid 6$ in $\mathcal{N}$, $5 \nmid 6$ in $\mathcal{T}$. From this we see that although 1 is the only tropical prime, we do have the notion of irreducibility in other tropical semi-rings.

The *Tropical Euler $\varphi_{\mathbb{T}}$-function* is also not nearly as interesting, because for any $n \in \mathcal{N}$ the only number $a \leq n$ for which $(n,a)_{\mathbb{T}} = 0$ is in fact 0. So, for any $n \in \mathcal{N}$, $\varphi_{\mathbb{T}}(n) = 1$.

With these properties in place, we would now like to discuss the tropical analogue of $\mathbb{Z}/n\mathbb{Z}$.

Let n be a fixed element of $\mathcal{N}$. Define a relation on $\mathcal{N}$ by

$$a \sim b \text{ if and only if } a \oplus n = b \oplus n.$$

Clearly $a \sim a$, and $a \sim b$ implies $b \sim a$ for any tropical natural numbers a and b, so the relation is reflexive and symmetric. If $a \sim b$ and $b \sim c$, then $a \oplus n = b \oplus n$ and $b \oplus n = c \oplus n$, so $a \oplus n = c \oplus n$, and $a \sim c$, so the relation is transitive. Thus we have an equivalence relation. This relation is the tropical version of the classical relation of *congruence modulo* n, and so we will refer to it as *tropical congruence modulo* n and will write $a \equiv b \pmod{n}$ if $a \sim b$.

Interestingly enough, instead of there being n equivalence classes as there are classically, there are $n+1$ equivalence classes tmod n, namely the classes

$$\bar{0}, \bar{1}, \bar{2}, \ldots, \overline{n-1}, \bar{n}.$$

We note that if we use the best choice of q and r for the division algorithm, then these classes correspond with the possible remainders of dividing a number by n, with the exception of $\bar{n}$. But the equivalence class of $\bar{n}$ would be the same as that of $\overline{\infty}$, and so we do have the correspondence. We will denote the set of equivalence classes under this relation by $\mathcal{N}/n\mathcal{N}$.

For the elements of $\mathcal{N}/n\mathcal{N}$, we will define *tropical modular arithmetic* as follows: for $\bar{a}, \bar{b} \in \mathcal{N}/n\mathcal{N}$, define the tropical sum and tropical product by

$$\bar{a} \oplus \bar{b} = \overline{a \oplus b} \quad \text{and} \quad \bar{a} \odot \bar{b} = \overline{a \odot b}.$$

Theorem 6. *The operations defined above on $\mathcal{N}/n\mathcal{N}$ are well defined.*

Proof. Suppose that $a_1 \equiv b_1 \pmod{n}$ (tmod) —

that $a_1 \oplus a_2 \equiv b_1 \oplus b_2$ (tmod n) and that $a_1 \odot a_2 \equiv b_1 \odot b_2$ (tmod n). To show that $a_1 \oplus a_2 \equiv b_1 \oplus b_2$ (tmod n), we need to show that $(a_1 \oplus a_2) \oplus n = (b_1 \oplus b_2) \oplus n$. Well,

$$\begin{aligned}(a_1 \oplus a_2) \oplus n &= a_1 \oplus n \oplus a_2 \oplus n \\ &= b_1 \oplus n \oplus b_2 \oplus n \\ &= (b_1 \oplus b_n) \oplus n,\end{aligned}$$

so $a_1 \oplus a_2 \equiv b_1 \oplus b_2$ (tmod n).

Now, to show that $a_1 \odot a_2 \equiv b_1 \odot b_2$ (tmod n), we need to show that $(a_1 \odot a_2) \oplus n = (b_1 \odot b_2) \oplus n$. We will consider three cases to show this.

Case 1: Assume that $a_1 \oplus n = a_1 \neq n$ and $a_2 \oplus n = a_2 \neq n$. This means that $a_1 < n$ and $a_2 < n$. Since $a_i \equiv b_i$ (tmod n), $a_i = a_i \oplus n = b_i \oplus n$, which means that $b_i = a_i$. Thus, $a_1 \odot a_2 = b_1 \odot b_2$ and it is clear that $(a_1 \odot a_2) \oplus n = (b_1 \odot b_2) \oplus n$.

Case 2: Assume that $a_1 \oplus n = n$ and $a_2 \oplus n = n$. This means that $a_1 \geq n$ and $a_2 \geq n$. Since $a_i \equiv b_i$ (tmod n), $n = a_i \oplus n = b_i \oplus n$, which means that $b_i \geq n$. as well. Now, since $a_i, b_i \geq n$, then $a_1 \odot a_2 = a_1 + a_2 > n$ and $b_1 \odot b_2 = b_1 + b_2 > n$. Thus $(a_1 \odot a_2) \oplus n = (b_1 \odot b_2) \oplus n$.

Case 3: Without loss of generality, assume that $a_1 \oplus n = a_1 \neq n$ and $a_2 \oplus n = n$. Then, from the arguments given above, $a_1 = b_1$, $a_2 \geq n$, and $b_2 \geq n$. Since $a_1, b_1 \in \mathcal{N}$, then $a_1, b_1 \geq 0$. Now, since $a_2 \geq n$ and $a_1 \geq 0$, $a_1 + a_2 \geq n$. Similarly, $b_1 + b_2 \geq n$. Thus, $(a_1 \odot a_2) \oplus n = n = (b_1 \odot b_2) \oplus n$.

Thus, in every possible case $(a_1 \odot a_2) \oplus n = (b_1 \odot b_2) \oplus n$, so $a_1 \odot a_2 \equiv b_1 \odot b_2 \pmod{n}$ (tmod n). Thus the operations are well defined, and we do in fact have a tropical semi-ring $\mathcal{N}/n\mathcal{N}$.

We do note that in the tropical semi-ring $\mathcal{N}/n\mathcal{N}$, we do not have the element ∞. But $\bar{n}$ is the additive identity, and $\bar{a} \odot \bar{n} = a \mathbin{\bar{\odot}} n = a \mathbin{\bar{+}} n = \bar{n}$, so $\bar{n}$ does satisfy the required properties and we do have a tropical semi-ring.

2.3 Tropical Exponents

Due to the difference in the operations, we do need to discuss the properties of tropical exponents. We will use the classical properties of exponents as a guide to find tropical exponential properties.

Let's look at a^n where $n \in \mathbb{Z}^+$. As in the classical case, we think of a^n, where n is a positive integer, as repeated multiplication, so $a^n = a \odot a \odot \cdots \odot a$, where there are n copies of a that are being multiplied. So,

$$\begin{aligned} a^n &= a \odot a \odot \cdots \odot a \\ &= a + a + \cdots + a \\ &= n \cdot a. \end{aligned}$$

Now let's look at $a^{1/n}$, where $n \in \mathbb{Z}^+$. Just as in the classical setting, we want this to represent the nth root of a, which to say, we want $a^{1/n} = b$ where $b^n = a$.

Well, $b^n = n \cdot b$, so if $b^n = a$, then we have $n \cdot b = a$, but this is a real equation, which means we can solve for b, and we get $b = \frac{1}{n} \cdot a$, and so we see that $a^{1/n} = \frac{1}{n} \cdot a$.

Using the two properties above, we see that $a^{p/q} = \frac{p}{q} \cdot a$ for any positive rational $\frac{p}{q}$. Now we can use sequences of rational numbers that converge to any irrational numbers, as in the classical setting, to see that for any positive real number $a^r = r \cdot a$.

Using this information, we prove some properties of exponents, assuming the exponents are positive real numbers. These properties will then help us extend to non-positive real exponents.

Proposition 7. *If $a, b \in \mathcal{Q}$ and let $s, t \in \mathbb{R}^+$, then*

1. $a^s a^t = a^{s+t}$.
2. $(a^s)^t = a^{s \cdot t}$.
3. $(ab)^s = a^s b^s$.
4. $0^s = 0$.
5. $a^s \oslash a^t = a^{s-t}$, *for* $a \neq \infty$.

Proof.

1. $a^s a^t = (s \cdot a) + (t \cdot a) = (s + t) \cdot a = a^{s+t}$.
2. $(a^s)^t = (s \cdot a)^t = t \cdot (s \cdot a) = (t \cdot s) \cdot a = a^{t \cdot s} = a^{s \cdot t}$.
3. $(ab)^s = s \cdot (a + b) = s \cdot a + s \cdot b = a^s \oplus b^s$.
4. $0^s = s \cdot 0 = 0$.
5. $a^s \oslash a^t = s \cdot a - t \cdot a = (s - t) \cdot a = a^{s-t}$.

We use property 5 to extend to the non-positive real exponents, giving us the following properties.

Proposition 8. *6. $a^0 = 0$ for $a \neq \infty$.*

7. $a^{-s} = 0/a^s$ for $a \neq \infty$.

Proof. 6. $a^0 = a^{s-s} = a^s \oslash a^s = s \cdot a - s \cdot a = 0$.

7. Let $u, v \in \mathbb{R}^+$ such that $u - v = -s$ (for example, $u = 1, v = 1 + s$). Then

$$a^{-s} = a^{u-v} = a^u \oslash a^v = u \cdot a - v \cdot a = (u - v) \cdot a = -s \cdot a = 0 - s \cdot a = 0/a^s.$$

Thus we note that for any real number r and any tropical number a, $a^r = r \cdot a$.

Now let's consider the expansion of a binomial $(a \oplus b)^n$. We can look at this expansion in two different ways. First, we can think of $(a \oplus b)$ itself as a number, and then raise it to a power, or we can think of it as a binomial and use the distributive law and properties of exponents to expand the expression. Let us look at both versions for different powers of n.

For $n = 1$, we have $(a \oplus b)^1 = 1 \cdot \min\{a, b\} = \min\{a, b\} = a \oplus b$, as desired both expressions are the same.

For $n = 2$, if we think of $(a \oplus b)$ as one number and use our property of exponents on it we get $(a \oplus b)^2 = 2 \cdot \min\{a, b\} = \min\{2 \cdot a, 2 \cdot b\} = a^2 \oplus b^2$. But now if we expand $(a \oplus b)^2$ as $(a \oplus b)(a \oplus b)$ then we get $(a \oplus b)^2 = (a \oplus b)(a \oplus b) = a^2 \oplus ab \oplus ba \oplus b^2 = a^2 \oplus ab \oplus b^2$. This, however, does not seem to be the same as the answer we got using the other property. It is in fact equal, so let us consider why. If $a^2 \oplus ab \oplus b^2 = a^2$, then $a^2 \leq b^2$, so $a^2 \oplus b^2 = a^2$, so they are equal when $a^2 \oplus ab \oplus b^2 = a^2$. Similarly, if $a^2 \oplus ab \oplus b^2 = b^2$, the two expressions are again equal.

Now suppose that $a^2 \oplus ab \oplus b^2 = ab$, where $ab \neq a^2$ and $ab \neq b^2$. Thus, $a + b < 2a$, which means that $b < a$. But this implies that $2b < a+b$, which means $ab \oplus b^2 = b^2$, but this is a contradiction, as $a^2 \oplus ab \oplus b^2 = ab \neq b^2$. Thus, $a^2 \oplus ab \oplus b^2$ can only equal ab if $a^2 = ab = b^2$. And so we see that the values $a^2 \oplus b^2$ and $a^2 \oplus ab \oplus b^2$ are in fact equal.

For $n = n$, a similar argument shows that

$$a^n \oplus b^n = (a \oplus b)^n = a^n \oplus a^{n-1}b \oplus \cdots \oplus ab^{n-1} \oplus b^n.$$

We will see in Section 2.4 that the full expansion gives what we call the *least coefficient form*, so there will be times when we will want that expansion. Of course, it is easier to evaluate the form $a^n \oplus b^n$, so there will also be times when we prefer that representation.

2.4 Tropical Polynomial Semi-Rings

Fix a commutative tropical semi-ring $\mathcal{T}$ with identity. Let x be an indeterminate. The formal sum

$$a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus \ldots \oplus a_1 \odot x \oplus a_0$$

with $n \in \mathbb{Z}$, $n \geq 0$ and each $a_i \in \mathcal{T}$ is called a *tropical polynomial* in x with coefficients a_i in $\mathcal{T}$. If $a_n \neq \infty$, then the polynomial is said to be of *degree* n and $a_n \odot x^n$ is called the *leading term*. The polynomial is monic if $a_n = 0$. The set of all such polynomials is called the semi-ring of *tropical polynomials in the variable x with coefficients in $\mathcal{T}$* and will be denoted $\mathcal{T}[x]$.

The operations of addition and multiplication which make $\mathcal{T}[x]$ into a tropical semi-ring are the same operations as those of any tropical semi-ring, where the addition is componentwise, as shown:

$$(a_n \odot x^n \oplus a_{n-1} \odot x^{n-1} \oplus ... \oplus a_1 \odot x \oplus a_0) \oplus (b_n \odot x^n \oplus b_{n-1} \odot x^{n-1} \oplus \ldots \oplus b_1 \odot x \oplus b_0)$$

$$= (a_n \oplus b_n) \odot x^n \oplus (a_{n-1} \oplus b_{n-1}) \odot x^{n-1} \oplus \ldots \oplus (a_1 \oplus b_1) \odot x \oplus (a_0 \oplus b_0)$$

(here a_n or b_n may be ∞ in order for addition of polynomials of different degrees to be defined). Multiplication is performed by first defining $(a \odot x^i) \odot (b \odot x^j) = a \odot b \odot x^{i+j}$ for *monomials* (polynomials with only one non-infinite term), where $i + j$ is the addition from the ring $\mathbb{Z}$, and then extending to all polynomials by distributive laws (usually referred to as "expanding out and collecting like terms"):

$$(a_0 \oplus a_1 \odot x \oplus a_2 \odot x^2 \oplus ...) \odot (b_0 \oplus b_1 \odot x \oplus b_2 \odot x^2 \oplus ...)$$

$$= (a_0 \odot b_0) \oplus (a_0 \odot b_1 \oplus a_1 \odot b_0) \odot x \oplus (a_0 \odot b_2 \oplus a_1 \odot b_1 \oplus a_2 \odot b_0) \odot x^2 \oplus ...$$

(in general, the coefficients of x^k in the product will be $\bigoplus_{i=0}^{k} a_i \odot b_{k-i}$ where $\bigoplus_{i=0}^{k}$ is tropical summation notation). These operations make sense since $\mathcal{T}$ is a tropical semi-ring, so sums and products of the coefficients are defined. A straightforward argument shows that with these definitions of addition and multiplication, $\mathcal{T}[x]$ is a tropical semi-ring.

The semi-ring $\mathcal{T}$ is a subsemi-ring of $\mathcal{T}[x]$, and appears as the *constant polynomials*. Note that by definition of multiplication, $\mathcal{T}[x]$ is a commutative semi-ring with identity (the identity from $\mathcal{T}$).

Proposition 9. *Let $\mathcal{T}$ be a tropical semi-integral domain and let $p(x),\ q(x) \in \mathcal{T}[x]$. Then*

1. $\deg(p(x) \odot q(x)) = \deg p(x) + \deg q(x)$, *where this addition is that in $\mathbb{Z}$,*

2. *the units of $\mathcal{T}[x]$ are just the units of $\mathcal{T}$,*

3. $\mathcal{T}[x]$ *is a semi-integral domain.*

Proof. 1. If $p(x)$ and $q(x)$ are polynomials with leading terms $a_n \odot x^n$ and $b_m \odot x^m$, respectively, then the leading term of $p(x) \odot q(x)$ is $a_n \odot b_m \odot x^{n+m}$. But, since $a_n \neq \infty$ and $b_m \neq \infty$, and $\mathcal{T}$ is a semi-integral domain, $a_n \odot b_m \neq \infty$, so $\deg(p(x) \odot q(x)) = n + m = \deg p(x) + \deg q(x)$.

2. Since $\mathcal{T} \subset \mathcal{T}[x]$, it is clear that $\mathcal{T}^\times \subset \mathcal{T}[x]^\times$. Suppose $u(x)$ is a unit in $\mathcal{T}[x]$. Then there is a $v(x) \in \mathcal{T}[x]$ such that $u(x) \odot v(x) = 0$, the multiplicative identity in a tropical semi-ring. But from part (1) we see that $\deg(u(x) \odot v(x)) = \deg u(x) + \deg v(x)$, so $0 = \deg 0 = \deg u(x) + \deg v(x)$. But since $\deg u(x) \geq 0$ for all $u(x) \in \mathcal{T}[x]$, it follows that $\deg u(x) = \deg v(x) = 0$, so $u(x),\ v(x) \in \mathcal{T}$ and thus in $\mathcal{T}^\times$, so $\mathcal{T}^\times = \mathcal{T}[x]^\times$.

3. Suppose $\mathcal{T}[x]$ is not a semi-integral domain. Then there are $p(x),\ q(x) \in \mathcal{T}[x]$ with $p(x) \neq \infty$ and $q(x) \neq \infty$, such that $p(x) \odot q(x) = \infty$. Let $p(x) = a_n \odot x^n \oplus ... \oplus a_0$ and $q(x) = b_m \odot x^m \oplus ... \oplus b_0$, where $a_n \neq \infty$ and $b_m \neq \infty$. Then $\infty = p(x) \odot q(x) = a_n \odot b_m \odot x^{n+m} \oplus ... \oplus a_0 \odot b_0$. But in order for this to be true, each $a_i \odot b_j = \infty$. But this is a contradiction since $\mathcal{T}$ is a semi-integral domain. Thus there are no such $p(x)$ and $q(x)$, so $\mathcal{T}[x]$ is also a semi-integral domain.

We can also define tropical polynomial semi-rings in more than one variable.

Definition 6. The *tropical polynomial semi-ring in the variables* $x_1, ..., x_n$ *with coefficients in* $\mathcal{T}$, denoted $\mathcal{T}[x_1, ..., x_n]$ is defined inductively by $\mathcal{T}[x_1, ..., x_n] = \mathcal{T}[x_1, ..., x_{n-1}][x_n]$.

Explicitly, a polynomial in $x_1, ..., x_n$ with coefficients in $\mathcal{T}$ is a finite sum of *monomial* terms for the form $ax_1^{d_1}x_2^{d_2}\cdots x_n^{d_n}$ where $a \in \mathcal{T}$ and the d_i are non-negative integers. The exponent d_i is called the *degree in* x_i of the monomial and the sum $d = d_1 + \cdots + d_n$ is called the *degree* of the monomial. The ordered n-tuple $(d_1, d_2, ..., d_n)$ is the *multi-degree* of the monomial. The *degree* of a polynomial is the largest degree of any of its monomial terms. A polynomial is called *homogeneous* or a *form* if all its monomials have the same degree. If f is a polynomial in n variables, the sum of all the monomials of f of degree k is called the *homogeneous component of f of degree k*. If f has degree d then f maybe written uniquely as the sum $f_0 \oplus f_1 \oplus \cdots \oplus f_d$ where f_k is the homogeneous component of f of degree k, for $0 \leq k \leq d$ (where some f_k may be infinite).

We will use $\mathcal{Q}$ most often for our base semi-field, so the remainder of our discussions about tropical polynomials and concepts related to tropical polynomials will be about the tropical polynomial semi-rings $\mathcal{Q}[x_1, \ldots, x_n]$.

2.5 Formal Polynomials and Functionally Equivalent Polynomials

We can think of tropical polynomials in two different ways, just as we do classically; the formal sums, and the associated functions. Classically there is a one-to-one correspondence between the formal sums and the functions, but this is not the case tropically. As a formal sum, for $f(x_1, ..., x_n) \in \mathcal{Q}[x_1, ..., x_n]$ we have

$$f(x_1, ..., x_n) = \bigoplus_i a_i x_1^{i_1} \cdots x_n^{i_n}.$$

x	$f(x)$	$g(x)$
$x < 1$	$\min\{2 \cdot x, 1 + x, 2\} = 2 \cdot x$	$\min\{2 \cdot x, 2 + x, 2\} = 2 \cdot x$
$x = 1$	$\min\{2 \cdot 1, 1 + 1, 2\} = 2$	$\min\{2 \cdot 1, 2 + 1, 2\} = 2$
$x > 1$	$\min\{2 \cdot x, 1 + x, 2\} = 2$	$\min\{2 \cdot x, 2 + x, 2\} = 2$

Table 1: Functional Values for $f(x) = x^2 + 1x + 2$ and $g(x) = x^2 + 2x + 2$

But if we think of $f(x_1, ..., x_n)$ as a function we have

$$\begin{aligned} f(x_1, ..., x_n) &= \bigoplus_i a_i x_1^{i_1} \cdots x_n^{i_n} \\ &= \min_i \{a_i + i_1 x_1 + \cdots i_n x_n\}. \end{aligned}$$

It is possible for two different formal sums to give the same function. For example, let $f(x), g(x) \in \mathcal{Q}[x]$ be given by $f(x) = x^2 + 1x + 2$ and $g(x) = x^2 + 2x + 2$. Table 1 shows the functional values of the two function for all possible x values. From this we see that although the two formal polynomials are different, the two functions are the same. We use this to define an equivalence, which is easy to show is an equivalence relation.

Definition 7. Let $f(x_1, ...x_n), g(x_1, ..., x_n) \in \mathcal{Q}[x_1, ..., x_n]$ be tropical polynomials in n variables. We say that f is *functionally equivalent* to g, denoted $f \sim g$, if $f(P) = g(P)$ for every $P \in \mathcal{Q}^n$.

Theorem 10. *Functional equivalence $\sim$, as defined in Definition 7, is an equivalence relation.*

Proof. To show that $\sim$ is an equivalence relation we must show that $\sim$ is reflexive, symmetric and transitive. Let $f(x_1, ..., x_n), g(x_1, ..., x_n)$ and $h(x_1, ..., x_n)$ be polynomials in $\mathcal{Q}[x_1, ..., x_n]$. Since $f(P) = f(P)$ for every $P \in \mathcal{Q}^n$, $f \sim f$ and $\sim$ is reflexive. If $f \sim g$ then by definition, $f(P) = g(P)$ for every $P \in \mathcal{Q}^n$. But, then $g(P) = f(P)$ for every $P \in Q^n$, so $g \sim f$, and $\sim$ is symmetric. Now suppose that $f \sim g$ and $g \sim h$. We want to show that $f \sim h$. Well, we know that $f(P) = g(P)$ for every $P \in \mathcal{Q}^n$, and that $g(P) = h(P)$ for every $P \in Q^n$. Thus $f(P) = g(P) = h(P)$ for every $P \in \mathcal{Q}^n$, which means $f \sim h$, and $\sim$ is transitive. Thus $\sim$ is an equivalence relation.

Theorem 11. *Let $f, g, a \in \mathcal{Q}[x_1, ..., x_n]$. If $f \sim g$ then $f \oplus a \sim g \oplus a$ and $fa \sim ga$.*

Proof. If $f \sim g$, then $f(P) = g(P)$ for every $P \in \mathcal{Q}^n$. Consider $f \oplus a$ and $g \oplus a$. Now, $(f \oplus a)(P) = f(P) \oplus a(P) = \min\{f(P), a(P)\}$. Similarly, $(g \oplus a)(P) = \min\{g(P), a(P)\}$. If for a given P $(f \oplus a)(P) = a(P)$, then $(g \oplus a)(P)$ must also equal $a(P)$, since $g(P) = f(P)$. If $(f \oplus a)(P) = f(P)$, then again $(g \oplus a)(P) = g(P) = f(P)$. So, in both cases $(f \oplus a)(P) = (g \oplus a)(P)$, which are the only two possible cases, so it is true for every $P \in \mathcal{Q}^n$ Thus, $f \oplus a \sim g \oplus a$.

Now consider fa and ga. $(fa)(P) = f(P) + a(P)$ and $(ga)(P) = g(P) + a(P)$. But $f(P) = g(P)$ for every $P \in \mathcal{Q}^n$, so $f(P) + a(P) = g(P) + a(P)$ for every $P \in \mathcal{Q}^n$, and $fa \sim ga$.

Definition 8. Let $\mathcal{P}(\mathcal{Q}[x_1, ..., x_n])$ be the power set of $\mathcal{Q}[x_1, ..., x_n]$ where we consider $\mathcal{Q}[x_1, ..., x_n]$ as a set, and $\psi : \mathcal{Q}[x_1, ..., x_n] \to \mathcal{P}(\mathcal{Q}[x_1, ..., x_n])$ be defined by $f \mapsto \{f' \in \mathcal{Q}[x_1, ..., x_n] | f' \sim f\} = [f]$. We define $\mathcal{Q}\langle x_1, ..., x_n\rangle$ to be the image of ψ, that is $\mathcal{Q}\langle x_1, ..., x_n\rangle = im(\psi)$. We note that $\psi(f) = \psi(g)$ if and only if $f \sim g$.

Theorem 12. *$(\mathcal{Q}\langle x_1, ..., x_n\rangle, \oplus, \odot)$ with operations $\oplus$ and $\odot$ defined as $[f] \oplus [g] = [f \oplus g]$ and $[f] \odot [g] = [f \odot g]$ is a tropical semi-ring.*

Proof. To show that $\mathcal{Q}\langle x_1, ..., x_n\rangle$ is in fact a tropical semi-ring, we simply need to show that the addition and multiplication are well defined, the rest follows directly from the fact that $\mathcal{Q}[x_1, ..., x_n]$ is a tropical semi-ring.

Let $f, f', g, g' \in \mathcal{Q}[x_1, ..., x_n]$ such that $f \sim f'$ and $g \sim g'$. We need to show that $f \oplus g \sim f' \oplus g'$ in order to show that the addition is well defined. From Theorem 11 we know that since $f \sim f'$, $f \oplus g \sim f' \oplus g$, and since $g \sim g'$, $f' \oplus g \sim f' \oplus g'$. By transitivity of the equivalence relation we have $f \oplus g \sim f' \oplus g'$. So the addition is well defined.

Now consider fg and $f'g'$. Again by Theorem 11 we have that since $f \sim f'$, $fg \sim f'g$, and since $g \sim g'$, $f'g \sim f'g'$. Again by transitivity we have $fg \sim f'g'$, so the multiplication is well defined. ∎

Since we are interested mostly in the functional behavior of the polynomials, we will deal with the polynomial semi-ring $\mathcal{Q}\langle x_1, x_2, \ldots, x_n\rangle$ instead of $\mathcal{Q}[x_1, x_2, \ldots, x_n]$. It is convenient to have an individual element of each equivalence class to be the representative. As explained in [8] one good candidate for the representative is what is we call the *least-coefficient polynomial.*

Definition 9. Let $f(x) = a_nx^n \oplus a_{n-1}x^{n-1} \oplus \cdots \oplus a_ix^i \oplus \cdots \oplus a_1x \oplus a_0$. A coefficient a_i of $f(x)$ is a *least coefficient* if for any $b \in \mathcal{Q}$ with $b < a_i$, the polynomial $g(x) = a_nx^n \oplus a_{n-1}x^{n-1} \oplus \cdots \oplus bx^i \oplus \cdots \oplus a_1x \oplus a_0$, formed by replacing a_i with b, is not functionally equivalent to $f(x)$. A polynomial is a *least-coefficient polynomial* if all of its coefficients are least coefficients. A term whose coefficient can not be changed without effecting the functional value of the polynomial is called a *contributing term.* A term whose coefficient can be raised without effecting the functional value of the polynomial is called a *non-contributing term.*

In [8] it is shown that there is a unique least-coefficient polynomial for each equivalence class, as well as an algorithm for finding the least coefficient polynomial. It is also shown that for polynomials in $\mathcal{T}[x]$ where $\mathcal{T} \subseteq \mathcal{Q}$, the least coefficient polynomial is in $\mathcal{Q}\langle x \rangle$. Finally, the following theorem is proved in [8] about least-coefficient polynomials in $\mathcal{Q}\langle x \rangle$.

Theorem 13. *(The Fundamental Theorem of Tropical Algebra) Let $f(x) = a_n x^n \oplus a_{n-1}x^{n-1} \oplus \cdots \oplus a_{r+1}x^{r+1} \oplus a_r x^r$ be a least-coefficients polynomial in $\mathcal{Q}[x]$. Then $f(x)$ can be written uniquely as the product of linear factors $a_n x^r (x \oplus d_n)(x \oplus d_{n-1}) \cdots (x \oplus d_{r+1})$ where $d_i = a_{i-1} \oslash a_i = a_{i-1} - a_i$.*

We now include a lemma about the multiplicity of the factors of a polynomial in one variable, which will be helpful to us later on.

Lemma 14. *If $f(x) = a_n x^n \oplus a_{n-1}x^{n-1} \oplus \cdots \oplus a_1 x \oplus a_0$ has a non-contributing term, then $f(x)$ has a factor of multiplicity greater than one.*

Proof. First suppose that $f(x)$ has only one non-contributing term $a_i x^i$, for some i. Then, following the algorithm given in [8], we can rewrite $f(x)$ in a functionally equivalent form where we replace a_i with $\frac{1}{2} \cdot (a_{i+1} + a_{i-1})$, so $f(x) \sim a_n x^n \oplus a_{n-1}x^{n-1} \oplus \cdots \oplus a_{i+1}x^{i+1} \oplus \left(\frac{1}{2} \cdot (a_{i+1} + a_{i-1})\right) x^i \oplus a_{i-1}x^{i-1} \oplus \cdots \oplus a_1 x \oplus a_0$. Now we have a least coefficient representation for $f(x)$ and by Theorem 13, we can factor $f(x)$ uniquely into linear factors of the form $x \oplus d_j$, where $d_j = a_{j-1} - a_j$. Thus, in our case, $d_{i+1} = \frac{1}{2} \cdot (a_{i+1} + a_{i-1}) - a_{i+1} = \frac{1}{2}(a_{i-1} - a_{i+1})$ and $d_i = a_{i-1} - \frac{1}{2}(a_{i+1} + a_{i-1}) = \frac{1}{2}(a_{i-1} - a_{i+1})$. Thus, $d_{i+1} = d_i$, and we have a repeated root.

The algorithm given in [8] together with an inductive argument show that if there are k non-contributing consecutive terms in $f(x)$, then $f(x)$ has a factor of multiplicity $k+1$. ✪

2.6 Tropical Roots and Factors

In the classical setting the factors of the polynomial are in correspondence with the roots the polynomial. As we try to develop a similar correspondence tropically, we notice that there need to be some modifications made.

In Theorem 13 any polynomial in $\mathcal{Q}\langle x\rangle$ can be factored uniquely into linear factors. So now we will consider the roots of the polynomial to determine how to create a correspondence between the factors and the roots.

Classically, the roots of a polynomial $f(x) \in k[x]$ are the set of points $p \in k$ for which $f(p) = 0$. So, if we consider the analogous definition, we would define the roots of a tropical polynomial $f(x) \in \mathcal{Q}[x]$ to be the set of points $p \in \mathcal{Q}$ for which $f(p) = \infty$, since ∞ is our additive identity. Recall that when we think of $f(x)$ as a function, we have $f(x) = \bigoplus_{i=0}^{n} a_i x^i = \min\{a_0, a_1 + x, a_2 + 2x, \ldots, a_n + nx\}$. So, to find the roots of $f(x)$ then we would need to find the points $p \in \mathcal{Q}$ so that $\min\{a_0, a_1 + p, a_2 + 2p, \ldots, a_n + np\} = \infty$. First, if $a_0 \neq \infty$, then the minimum can never be ∞, so there are no $p \in \mathcal{Q}$ for which $f(p) = \infty$, and the set of roots is empty. This means that for all polynomials with a finite constant, the set of roots is the empty set. Now suppose that $a_0 = \infty$, which is the same as saying that there is no constant term. Now, we want to know when $\min\{a_1 + p, a_2 + 2p, \ldots, a_n + np\} = \infty$. If any of the $a_i \neq \infty$, then this only happens when $p = \infty$. So, for all polynomials with no finite constant, but some finite coefficients, the set of roots is just the single point $\{\infty\}$. The only remaining set of points that can be the roots in this setting is the entire set $\mathcal{Q}$, as any element of $\mathcal{Q}$ is a root of the ∞-polynomial (the analogous polynomial to the zero-polynomial) which has all coefficients infinite. So we see that the direct analogue of the roots of a polynomial are not interesting to us in the

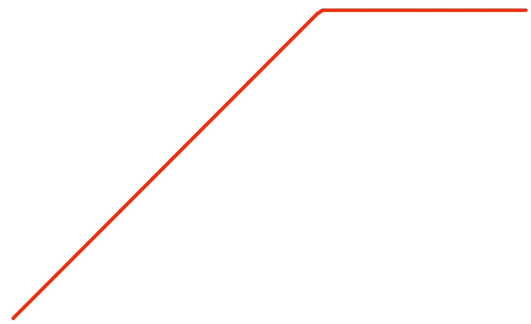


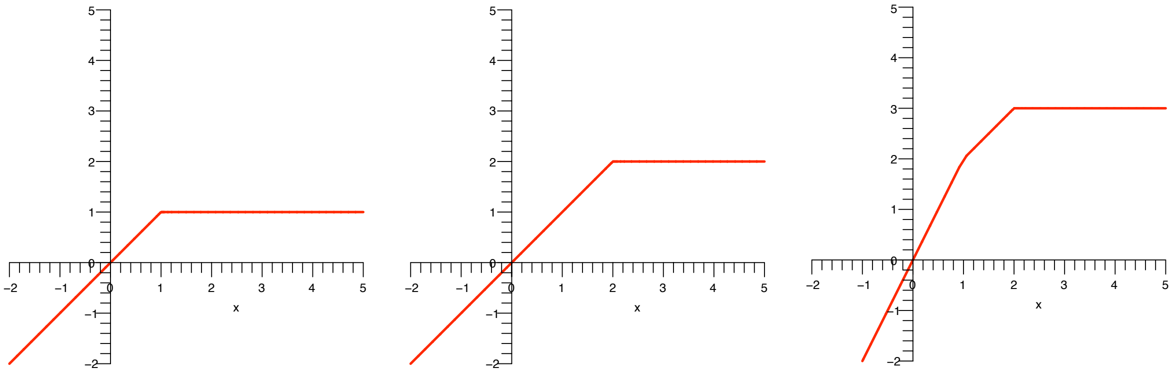
Figure 1: $f(x) = x \oplus a$

same way as they are classically.

Instead, let's look at the roots from the direction of the factors. Classically, c is a root of the polynomial if and only if $x - c$ is a factor of that polynomial. It turns out that tropically, if $x \oplus c$ is a factor of a polynomial, then the points c is the analogous object to the root of the polynomial.

Let $f(x) = x \oplus a$. As far as the function is concerned, when p is less than a, then $f(p) = p$, when p is greater than a, then $f(p) = a$, and when $p = a$ then $p = f(p) = a$. So, for all $p < a$ the minimum is attained by the monomial x, for all $p > a$, the minimum is attained by the monomial a, and when $p = a$, the minimum is attained by both of the monomials. We notice, when we look at the graph of this function, that this point is also where our function is not classically locally linear, as shown in Figure 1.

Now let's consider what happens when we have the product of two linear binomials. We will use specific examples to simplify the graphs of the functional values



(a) $g(x) = x \oplus 1$ (b) $h(x) = x \oplus 2$ (c) $g(x)h(x) = x^2 \oplus 1x \oplus 3$

Figure 2: Graphs of $g(x)$, $h(x)$, and $gh(x)$

when we look at them, but these ideas are easily generalized.

Let $g(x) = x \oplus 1$ and $h(x) = x \oplus 2$. Then

$$\begin{aligned} g(x)h(x) &= (x \oplus 1)(x \oplus 2) \\ &= x^2 \oplus (1 \oplus 2)x \oplus 3 \\ &= x^2 \oplus 1x \oplus 3 \end{aligned}$$

In Figure 2, we see the graphs of the functional values of $g(x), h(x)$, and $gh(x)$.

For our function $g(x) = x \oplus 1$, we note that the functional value is attained by the x term for $x < 1$, but the 1 term for $x > 1$ and by both of the terms for $x = 1$. Similarly with $h(x)$ and the point 2. Now, when we consider the functional values of $g(x)h(x)$, we note that for $x < 1$ the minimum, or functional value, is attained by the x^2 term alone, for $1 < x < 2$ it is attained by the $1x$ term alone, and for $x > 2$ it is attained by the constant term 3 alone. However, for $x = 1$ the value is attained by both the term x^2 and the term $1x$, and for $x = 2$ the value is attained by both the $1x$ term and the constant term 3. We notice that the two places where two terms attained a minimum in our linear polynomials were preserved in the product, or, in other words, the corners of the product are at the same places as the corners of our two original linear functions. Thus, if the roots of the polynomials really are the corners, as we are tempted to believe, it did follow in this example that the roots of the product $g(x)h(x)$ are the roots of $g(x)$ together with the roots of $h(x)$.

These corner points of our graph do indeed fill the roll of the roots, as explained by Theorem 17

Definition 10. Let $\mathcal{T}$ be a tropical semi-ring or a tropical polynomial semi-ring. A *tropical expression* is a sum of tropical monomials $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ where $a_i \in \mathcal{T}$. A tropical expression $t = a_1 \oplus a_2 \oplus \cdots \oplus a_n$ is said to *vanish tropically* if the value of t is attained by at least two of its monomials, which is to say there are some i, j such that $i \neq j$, $a_i = a_j$ and $a_i \leq a_k$ for all k, or if the value of the expression is ∞.

This idea is one of the more subtle problems with tropical mathematics. We notice from this definition that $1 \oplus 1$ vanishes tropically, while 1 does not. But $1 \oplus 1 = 1$, so there seems to be some ambiguity. It is true that this can cause of small amount of confusion, however it is generally clear when we want to simplify $1 \oplus 1 = 1$ so that it does not vanish tropically, and when we want to leave it expanded so that we can see when it does. In the case of polynomials in one variable, when we want to graph the functional values of the polynomial, then we simplify the expression. However, if we wish to determine whether or not a point is one of the corners, then we do not simplify the expression, but we keep all of the terms to see if the minimum shows up twice. As we discuss more ideas we will often mention how the question of simplifying the expression or not affects the outcome. In general, if we are dealing simply with an expression but are not really concerned with a corner locus or determining what points make something vanish tropically, we will simplify the expression so that $1 \oplus 1 = 1$, for example. But there will be times when we are looking for a corner locus or similar object when we need to keep the expression expanded. In [11], the tropically vanishing is always noted, so that $1 \oplus 1 \neq 1$, but instead, regardless of the situation, $1 \oplus 1$ vanishes tropically. As a result, in that tropical semi-ring there is an additional copy of $\mathbb{R}$ for the terms $a \oplus a$. In such a tropical semi-ring, the polynomial $1x \oplus 1x \oplus 4$ vanishes tropically at $x = 3$, but also for any $x \leq 3$, and so we choose not to adopt this convention. We will make it clear though out the remainder of the text when we need to consider $a \oplus a$ as a, and when we need to consider $a \oplus a$ as vanishing tropically.

We also have the following two lemmas about the behavior of the expressions that vanish tropically.

Lemma 15. *If h is a tropical expression that vanishes tropically, then the tropical expression hg vanishes tropically for any tropical expression g.*

Proof. If $h = \infty$, then the result follows trivially from the fact that $a \odot \infty = \infty$. If $h \neq \infty$ and $h = \bigoplus_{i=1}^{m} h_i$ vanishes tropically, then two of its monomials are minimum together, h_k and h_l. Since $g = \bigoplus_{j=1}^{n} g_j$ is a tropical expression it has at least one term that attains the minimum, g_p. Then in the product $hg = \bigoplus_{i=1}^{m} \bigoplus_{j=1}^{n} h_i g_j$ the terms $h_k g_p$ and $h_l g_p$ are equal and minimum, so hg vanishes tropically. ✧

Lemma 16. *Let g and h be tropical expressions in a tropical semi-field. If gh vanishes tropically, then either g vanishes tropically, or h does.*

Proof. If $gh = \infty$, then either $g = \infty$ or $h = \infty$, since we are in a semi-field. So, suppose that $gh \neq \infty$. Let $g = g_1 \oplus g_2 \oplus \cdots \oplus g_n$ and $h = h_1 \oplus h_2 \oplus \cdots \oplus h_m$. Then

$gh = \bigoplus_{i=1}^{n} \bigoplus_{j=1}^{m} g_i h_j$. We know that gh vanishes tropically, so we will consider two cases.

First suppose that $gh = g_i h_j = g_i h_k$, which is to say the two terms that attain the minimum together in gh are $g_i h_j$ and $g_i h_k$. Since $g_i h_j = g_i h_k \leq g_r h_s$ for all r, s, we know that $g_i h_j = g_i h_k \leq g_i h_s$ for all s, which means that $h_j = h_k \leq h_s$ for all s, since we can cancel the g_i from every term. Thus, h vanishes tropically.

Now, suppose that $g_i h_j = g_k h_l$ for $i \neq k$ and $j \neq l$, and suppose that neither of g and h vanishes tropically. Then since $g_i h_j$ participates in attaining the minimum, $g_i h_j \leq g_i h_s$ for every s. But, this equality must be strict since h does not vanish tropically, so $g_i h_j < g_i h_s$ for every $s \neq j$, which means that $h_j < h_s$ for every $s \neq j$. Similarly, we have $g_i < g_r$ for every $r \neq i$. Thus $g_i h_j < g_i h_l < g_k h_l$, so we don't have equality, which is a contradiction. Thus, either h or g must vanish tropically. ✿

We see from these lemmas that tropical vanishing has that same annihilating property as classical zeros. We note that ∞ has these two properties, but as noted is not nearly as interesting in the polynomial setting. Thus, instead of defining a zero locus of a polynomial, the set of all points where the polynomial attains the value of the additive identity, we will define the *corner locus* of a tropical polynomial, which is in fact the analogous object.

Finally, we prove the correspondence between the tropical vanishing of the polynomial and its factors.

Theorem 17. *Let* $f(x) = a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_1 x \oplus a_0 \in \mathcal{Q}\langle x \rangle$. $f(x)$ *vanishes tropically at* $p \in \mathcal{Q}$ *if and only if* $x \oplus p$ *is a factor of* $f(x)$.

Proof. Suppose that $x \oplus p$ is a factor of $f(x)$, so $f(x) = (x \oplus p)h(x)$ for some polynomial $h(x)$. Clearly, $x \oplus p$ vanishes tropically for $x = p$, so by Lemma 15 $f(x)$

also vanishes tropically at p.

Now suppose $f(x)$ vanishes tropically at p. We know that there is a least-coefficients polynomial in $\mathcal{Q}\langle x\rangle$ which is equivalent to $f(x)$ and which by the Fundamental Theorem of Tropical Algebra, factors uniquely as $a_n(x \oplus d_n)(x \oplus d_{n-1}) \cdots (c \oplus d_1)$. Since $f(x)$ vanishes tropically at p, then $a_n(p \oplus d_n)(p \oplus d_{n-1}) \cdots (p \oplus d_1)$ vanishes tropically, so by Lemma 16, one of the factors must vanish tropically as well. Suppose that the factor that vanishes tropically is $p \oplus d_j$ for some j. Then, since $p \oplus d_j$ vanishes tropically, $p = d_j$. Thus $x \oplus p$ is a factor of $f(x)$

So we see that the corners are in fact the analogous object to the roots of the polynomial in relation to the factors.

All of these observations motivate us to give the following definition.

Definition 11. Let $f(x_1, ..., x_n) \in \mathcal{Q}[x_1, ..., x_n]$ be a tropical polynomial. The *corner locus* $\mathcal{Z}(f)$ is defined to be the set of all points $p \in \mathcal{Q}^n$ for which the expression $f(p)$ vanishes tropically. This is

$$\mathcal{Z}(f) = \{p \in \mathcal{Q}^n | f(p) \text{ vanishes tropically}\},$$

or, in other words

$$\mathcal{Z}(f) = \{p \in \mathcal{Q}^n | \text{ two monomials of } f(p) \text{ attain the minimum together}\}.$$

We will use this definition, in the case of $\mathcal{Q}[x, y]$, to create tropical plane curves, which we will discuss in more detail in Chapter 3.

An equivalent definition for a tropical corner locus, or *tropical variety* is given in [19], which is stated below. It is a straightforward argument, based on the fact that the tropical polynomials are simply minimums of linear polynomials, to show the two definitions are equivalent. The definition is stated in [19] for polynomials in $\mathcal{R}[x_1, \ldots, x_n]$, but is clearly suitable for polynomials in $\mathcal{Q}[x_1, \ldots, x_n]$.

Definition 12. Let f be a tropical polynomial in n indeterminates. The tropical variety $\mathcal{Z}(f)$ defined by f is the set of points in $\mathcal{Q}^n$ where the associated function $f : \mathcal{Q}^n \to \mathcal{Q}$ is not linear. If f is homogeneous, we can regard $\mathcal{Z}(f)$ as a tropical projective variety in $\mathbb{TP}^{n-1}$.

Lemma 18. *Let $f, g \in \mathcal{Q}[x_1, x_2, \ldots, x_n]$ such that $f \sim g$. Then $\mathcal{Z}(f) = \mathcal{Z}(g)$.*

Proof. Suppose that $\mathcal{Z}(f) \neq \mathcal{Z}(g)$. Then there is some point $P \in \mathcal{Q}^n$ such that $f(P)$ vanishes tropically, but $g(P)$ does not. By Definition 12 the graph of g is locally linear at P, but the graph of f is not. But this is a contradiction because $f \sim g$, so they always take on the same functional values. Thus they must both be locally linear at a point, or both not. Thus $\mathcal{Z}(f) = \mathcal{Z}(g)$. ∎

Since two curves that are functionally equivalent have the same corner locus, we can simply consider our polynomials as being in $\mathcal{Q}\langle x_1, \ldots, x_n\rangle$, and Definition 11 still makes sense.

3 Tropical Plane Curves

We begin with a few comments on *tropical affine spaces* and *tropical projective spaces*. We will then discuss drawing the plane curves, their dual graphs, and some of their general position properties.

3.1 Tropical Affine and Projective Spaces

We define *tropical-affine n-space* over $\mathcal{Q}$, denoted $\mathbb{TA}^n_{\mathcal{Q}}$ or simply $\mathbb{TA}^n$, to be the set of all n-tuples of elements of $\mathcal{Q}$. An element $P \in \mathbb{TA}^n$ will be called a *point*, and if $P = (a_1, ..., a_n)$ with $a_i \in \mathcal{Q}$, then the a_i will be called the *coordinates* of P. Let $A = \mathcal{Q}\langle x_1, ..., x_n\rangle$ be the tropical polynomial semi-ring in n variables over $\mathcal{Q}$. We can interpret the elements of A as functions from the tropical affine n-space to $\mathcal{Q}$, by defining $f(P) = f(a_1, ..., a_n)$, where $f \in A$ and $P \in \mathcal{Q}^n$. We treat the polynomials as discussed in Section 2.4, but we now also can think of them, as noted above, as functions from $\mathbb{TA}^n$ to $\mathcal{Q}$. We will most often think of $\mathbb{TA}^n$ simply as $\mathcal{Q}^n$.

We define *tropical projective space* over $\mathcal{Q}$, denoted $\mathbb{TP}^n_{\mathcal{Q}}$ or simply $\mathbb{TP}^n$, to be the set of equivalence classes of all $(n+1)$-tuples of elements in $\mathcal{Q}$ under the equivalence relation defined by $(a_0, a_1, \ldots, a_n) \sim (b_0, b_1, \ldots, b_n)$ if there is some $\lambda \in \mathcal{Q}$ such that $\lambda \neq \infty$ and $a_i = \lambda b_i$ for every i. If P is a point in $\mathbb{TP}^n$, then any $(n+1)$-tuple $(a_0, a_1, \ldots, a_n)$ in the equivalence class P is called a *set of homogeneous coordinates*. Borrowing the convention, one may want to denote the equivalence class of $(a_0, a_1, \ldots, a_n)$ by $(a_0 : a_1 : \cdots : a_n)$, however we will usually use (x, y, z) for the equivalence class in $\mathbb{TP}^2$. We note that the point $(\infty, \infty, \ldots, \infty)$ is not in $\mathbb{TP}^n$, analogous to $(0, 0, \ldots, 0)$ not being in the classical projective space $\mathbb{P}^n$. Let $S = \mathcal{Q}\langle x_o, x_1, \ldots, x_n\rangle$ be a tropical polynomial semi-ring. Of course we cannot think of polynomials in S as functions from $\mathbb{TP}^n$ to Q, because of the non-uniqueness of the homogeneous coordinates. However, if we consider only homogeneous polynomials of a given degree d, which is to say each monomial in the polynomial is of degree d, then we have that $f(\lambda a_0, \lambda a_1, \ldots, \lambda a_n) = \lambda^d f(a_0, a_1, \ldots, a_n)$. By Lemma 15, we see that if f vanishes tropically, then so does $\lambda^d f$. Thus we see that tropical vanishing depends only on the equivalence class of $(a_0, a_1, \ldots, a_n)$, and we can talk about the corner locus of a tropical polynomial.

Definition 13. Let $f(x_0 : x_1 : \cdots : x_n)$ be a homogeneous polynomial. We define the corner locus $\mathcal{Z}(f)$ by

$$\mathcal{Z}(f) = \{P \in \mathbb{TP}^n | f(P) \text{ vanishes tropiclly}\}.$$

Depending on the situation, we will use affine coordinates or projective coordinates. To go from a polynomial in $\mathcal{Q}\langle x_1, \ldots, x_n\rangle$ to a homogeneous polynomial in $\mathcal{Q}\langle x_0, x_1, \ldots, x_n\rangle$, we will follow the usual procedure. Since we are interested in plane curves, we will consider the polynomial ring $\mathcal{Q}\langle x, y\rangle$ and the homogeneous polynomial ring $\mathcal{Q}\langle x, y, z\rangle$. We will start with a polynomial $f(x, y) \in \mathcal{Q}\langle x, y\rangle$ where $f(x, y) = \bigoplus a_i x^{d_i} y^{e_i}$ of degree n, and produce the homogeneous polynomial by simplifying $z^n \odot f\left(\frac{x}{z}, \frac{y}{z}\right)$, which will give us $\tilde{f}(x, y, z) = \bigoplus a_i x^{d_i} y^{e_i} z^{n-d_i-e_i}$, so every term has degree n. In essence, we just multiply each term by the appropriate power of z to make its degree n. Since we will usually need to consider our plane curves in the tropical projective plane $\mathbb{TP}^2$, which we explain below, we will most often use the homogeneous version of the polynomial. When we graph the corner locus of a homogeneous polynomial in $\mathcal{Q}\langle x, y, z\rangle$ we will graph it with points of the form $(x, y, 0)$. To dehomogenize a polynomial with respect to one if its variables, we will simply assume the corresponding coordinate is not taking on an infinite value,

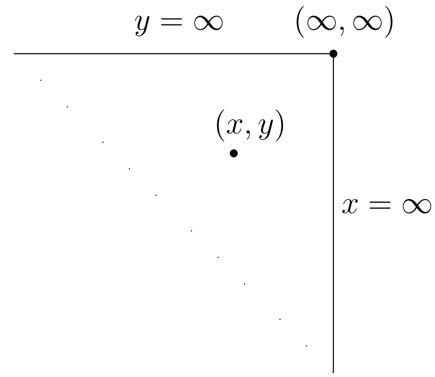


Figure 3: Triangle Model of $\mathbb{TA}^2$.

and therefore points can be scaled so that that coordinate is 0.

When we think about $\mathbb{TA}^n$, we have the plane $\mathbb{Q}^2$ together with the boundaries of the form (a, ∞) and (∞, b). So we can think of it as the half closed region shown in Figure 3.

When we move into the projective plane, we think of this as adding the *line at infinity*, which in our case means we add on a line where the third coordinate z is actually equal to ∞. This adds to our model another edge and two more vertices at infinity, as shown in Figure 4.

3.2 Tropical Plane Curves and Dual Graphs

In this section we will discuss how to determine a plane curve and its *dual graph*. We will do this in the affine case, and then later switch to the more general projective case. We will then discuss the behavior of the plane curves relative to the tropical projective model.

We recall from Section 2.4 that the corner locus of a tropical polynomial is the

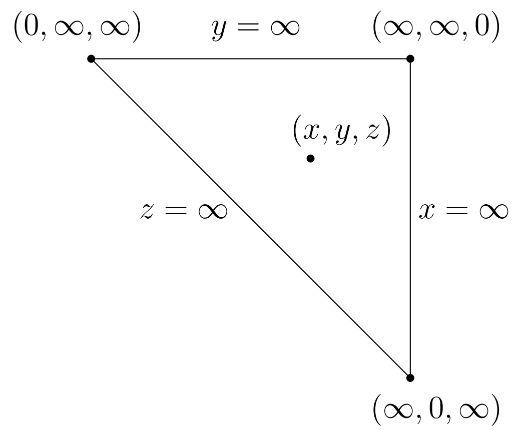
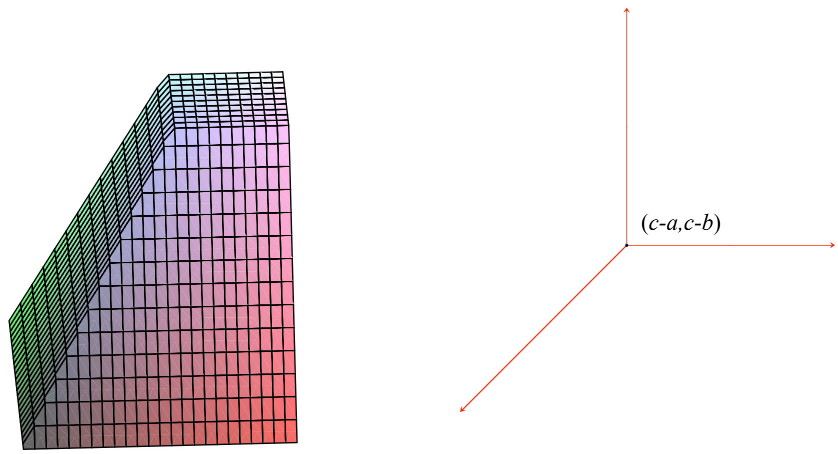


Figure 4: Triangle Model of $\mathbb{TP}^2$.

object which is analogous to the zero locus of a polynomial. The zero locus of a polynomial in two variables (or three homogeneous variables) gives us a plane curve, and the corner locus of a polynomial in two variables gives us the a *tropical plane curve*. We recall below the definition of corner locus, here stated in terms of polynomial in two variables, or three homogeneous variables.

Definition 14. Let $f(x,y) \in \mathcal{Q}\langle x,y\rangle$ be a tropical polynomial. The *corner locus* $\mathcal{Z}(f)$ is defined to be the set of all points $p \in \mathcal{Q}^2$ for which the expression $f(p)$ vanishes tropically. The corner locus of a tropical polynomial in two variables is called a *tropical plane curve*. A tropical plane curve is said to be a *curve of degree d* if $f(x,y)$ is a polynomial of degree d.

Example 9. Let us first consider the example of the linear polynomial $f(x,y) = ax \oplus by \oplus c$, for $a,b,c \neq \infty$. We could, as we did in the one variable case, graph the three planes $z = a+x, z = b+y$ and $z = c$, and then find where they intersect, and then project this down to the xy-plane, as shown in Figure 5.



(a) Graph of $z = ax \oplus by \oplus c$ (b) $\mathcal{Z}(ax \oplus by \oplus c)$

Figure 5: Example of a Tropical Line

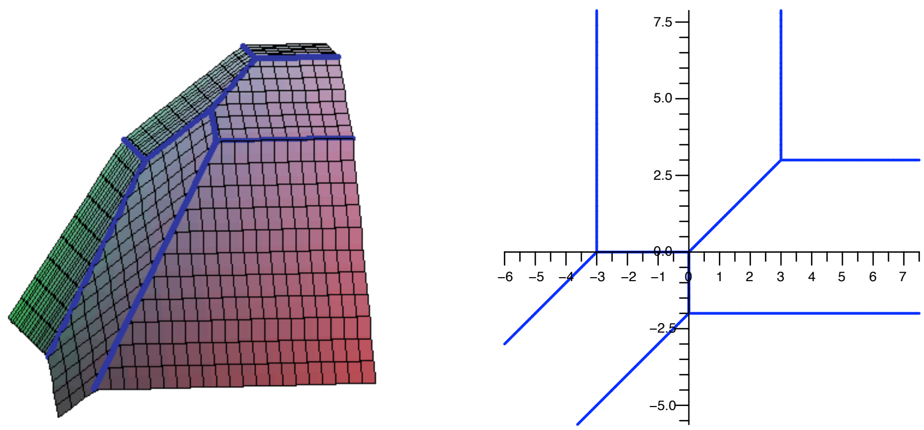
Or we can use Definition 14 directly. In this case we see that the corner locus is the set of all points in $\mathcal{Q}^2$ such that

$$\begin{array}{rcccl} a+x & = & b+y & \leq & c \\ a+x & = & c & \leq & b+y \\ b+y & = & c & \leq & a+x \end{array}$$

So we have the union of the line $y = x + a - b$ for $x \leq c - a$ (or $y \leq c - b$), the line $x = c - a$ for $y \geq c - b$, and the line $y = c - b$ for $x \geq c - a$, as shown in Figure 5.

A similar process is used to determine the plane curve associated with any polynomial, where there are more planes to intersect or more inequalities to consider for higher degree polynomials.

Example 10. Consider the polynomial $h(x, y) = 3x^2 \oplus xy \oplus 2y^2 \oplus x \oplus y \oplus 3$. We could again graph the planes $z = 3+2\cdot x$, $z = x+y$, $z = 2+2\cdot y$, $z = x$, $z = y$, and

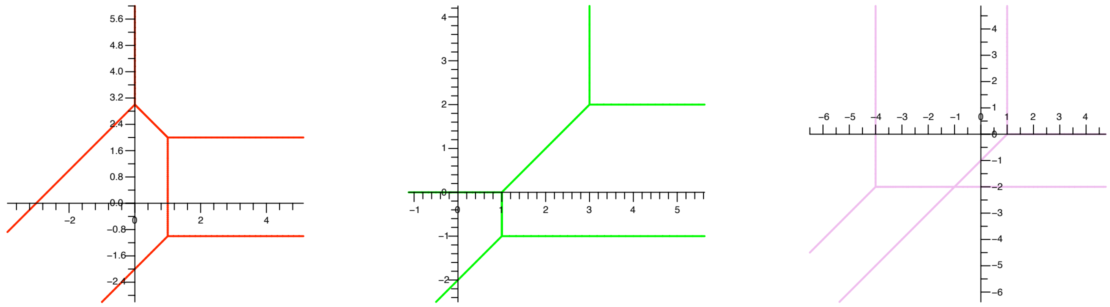


(a) Planes from $h(x, y)$ (b) $\mathcal{Z}(h)$

Figure 6: An Example of a Tropical Conic

$z = 3$, and consider the corners, as shown in Figure 6, from [7]. Or we could consider the inequalities from setting two monomials equal to each other and smaller than or equal to the remaining monomials. (A generalized example of this is shown in [4]). In which case we again have the conic shown in Figure 6.

We notice in Example 20, the line has one trivalent vertex, and three unbounded rays, one in each of the positive x, positive y, and negative $x = y$ directions. In Example 10, the conic has four trivalent vertices and six unbounded rays, two in each of the positive x, positive y, and negative $x = y$ directions. It is not always the case however that all vertices will be trivalent and that there will be n unbounded rays in each three directions listed above. Consider the line given by $ax \oplus by$. For this the minimum is attained by two monomials anytime the two monomials are equal to each other, since they do not need to be compared to any other monomial. So, we have for our plane curve the points (x, y) such that $a+x = b+y$, which is just



(a) $\mathcal{Z}(g)$ (b) $\mathcal{Z}(k)$ (c) $\mathcal{Z}(l)$

Figure 7: Some Tropical Conics

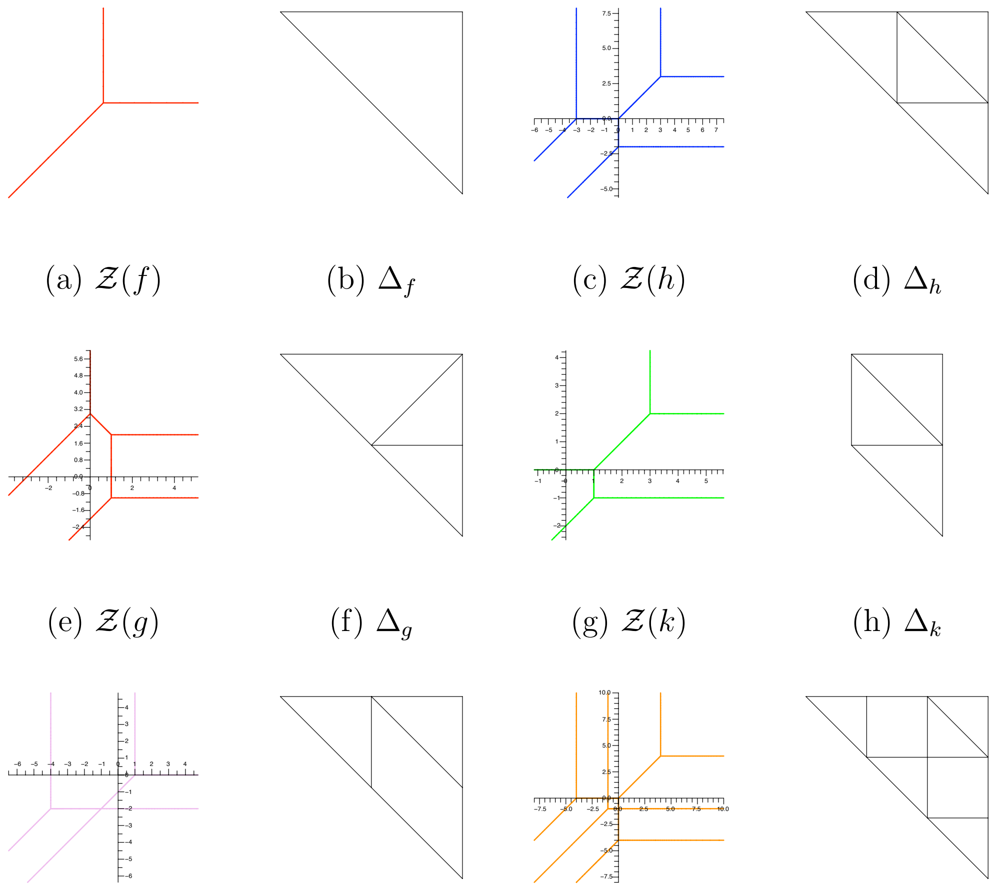
the line $y = x+a-b$. This seems to be a degenerate case of sorts, since the constant term is infinite, so it can only participate in the minimum when both x and y are infinite. But degenerate or not, it does come from a polynomial of degree one and should therefore be considered a tropical line. Thus there are four possible *shapes* for lines, the one as in Figure 5, classical lines of slope one, classical lines of slope zero, and classical lines of slope infinity. Of course, these last three are isomorphic to each other by a change of variables, so we really only have two possible lines. The conics are classified in [4], and there are many more, with more interesting results of degeneracy. For example, the polynomials $g(x,y) = 3x^2 \oplus xy \oplus 2y^2 \oplus 3x \oplus 1y \oplus 3$, $k(x,y) = xy \oplus 2y^2 \oplus x \oplus 1y \oplus 3$, and $l(x,y) = 4x^2 \oplus 2xy \oplus 3y^2 \oplus x \oplus 1y \oplus 1$ give the conics shown in Figure 7.

Unlike $\mathcal{Z}(h)$, $\mathcal{Z}(g)$ has only five unbounded rays, because it does not have two unbounded rays in the positive y direction, and it has only three trivalent vertices, instead of four. And $\mathcal{Z}(k)$ again has only five unbounded rays, and instead of simply missing one in one of the three expected directions, one ray extends in the negative

y direction, and it has only three trivalent vertices. $\mathcal{Z}(l)$ has six unbounded rays, extending as expected, but it only has three vertices, and one of them in four-valent. As we will see below, the one ray of $\mathcal{Z}(g)$ that extends in the positive y direction is actually a double ray, so if we count with multiplicity then we still have two unbounded rays in that direction. In $\mathcal{Z}(k)$ we have the only ray extending in an unexpected direction because of the lack of the x^2 term in the polynomial (this is similar to the lines that come from the polynomials that only have two of the terms present). And, it is clear that $Z(l)$ is the union of two tropical lines, and the one four-valent vertex is the point of intersection of the two individual lines. These are just some examples of the kinds of things that happen for different polynomials of degree 2. As mentioned earlier, in [4] there is a complete classification of tropical conics. There are 21 different combinatorial shapes that the conics can assume. For higher degree polynomials, there are clearly even more possible shapes with more variations in the kinds of interesting things that happen. We will discuss some of these things, and what their implications are in later sections. Now, we will discuss another important object that can be associated to a plane curve.

We can think of our plane curves as planar graphs if we like. We first note that at the *end* of the unbounded rays in the positive x and y directions, there actually is a well defined point of the tropical plane, since $\infty \in \mathcal{Q}$. For example, at the end of those two rays of a line are the points $(\infty, c-b)$ and $(c-a, \infty)$. In the tropical projective plane these points become $(\infty, c-b, 0)$ and $(c-a, \infty, 0)$ and there is a point at the end of the unbounded ray that extends in the negative $y = x$ direction, the point $(c-a, c-b, \infty)$. Every unbounded ray will have some *infinite point* at its end in the tropical projective plane, which is on the boundary of $\mathbb{TP}^2$. With this in mind, then we can let each vertex in the plane curve, and each of the end points of unbounded rays, be vertices in a graph, and each segment or ray in the plane curve be an edge in a graph, which graph we will call $\mathfrak{G}(f)$. We do require, however, that the edges in $\mathfrak{G}(f)$ maintain the same slope and length as their counterparts in $\mathcal{Z}(f)$. We then create a *dual graph* Δ_f for our curve, as explained in [16]. We will form the dual graph in the standard way, with one slight modification. First, we will have a vertex in our dual graph for each connected component of $\mathbb{TP}^2 \setminus \mathfrak{G}(f) = \mathbb{TP}^2 \setminus \mathcal{Z}(f)$ and an edge in Δ_f for each edge $\mathfrak{G}(f)$, where there is an edge in Δ_f between two vertices if there is an edge in $\mathfrak{G}(f)$ between the corresponding components of $\mathbb{TP}^2 \setminus \mathfrak{G}(f)$. The modification that we use is that we require the edges of Δ_f to be perpendicular to the edges of $\mathfrak{G}(f)$. In this way we will still be able to preserve information regarding the the slope of each of the rays and segments as well as which vertices they are connected to. We note that the dual graph may not always fit directly over the curve, because of the added stipulation that the edges are perpendicular to those of the curve, but it does give us the combinatorial shape of the curve. For each curve, there is a unique dual graph, but for each dual graph there are an infinite number of curves, since the dual graph tells you nothing about either position in the plane, or length of finite segments. Figure 8 shows the curves we have looked at so far along with their dual graphs, along with that of the cubic $c(x, y) = 5x^3 \oplus 1x^2y \oplus 1xy^2 \oplus 5y^3 \oplus 1x^2 + \oplus xy \oplus 1y^2 \oplus x \oplus y \oplus 4$.

Definition 15. We say a tropical plane curve of degree n is a *full support tropical plane curve* or *has full support* if in the polynomial of which it is the corner locus, all possible terms of degree n are present and have finite coefficients. In particular, all three pure terms, namely x^n, y^n, and z^n, must have finite coefficients. Similarly, a dual graph is called a *full support dual graph* if the curve to which it is associated



(a) $\mathcal{Z}(f)$ (b) Δ_f (c) $\mathcal{Z}(h)$ (d) Δ_h

(e) $\mathcal{Z}(g)$ (f) Δ_g (g) $\mathcal{Z}(k)$ (h) Δ_k

(i) $\mathcal{Z}(l)$ (j) Δ_l (k) $\mathcal{Z}(c)$ (l) Δ_c

Figure 8: Plane Curves and Dual Graphs

is a full support curve. The boundary ∂^n of a full support dual graph associated with a polynomial of degree n is called the *limiting triangle of degree* n.

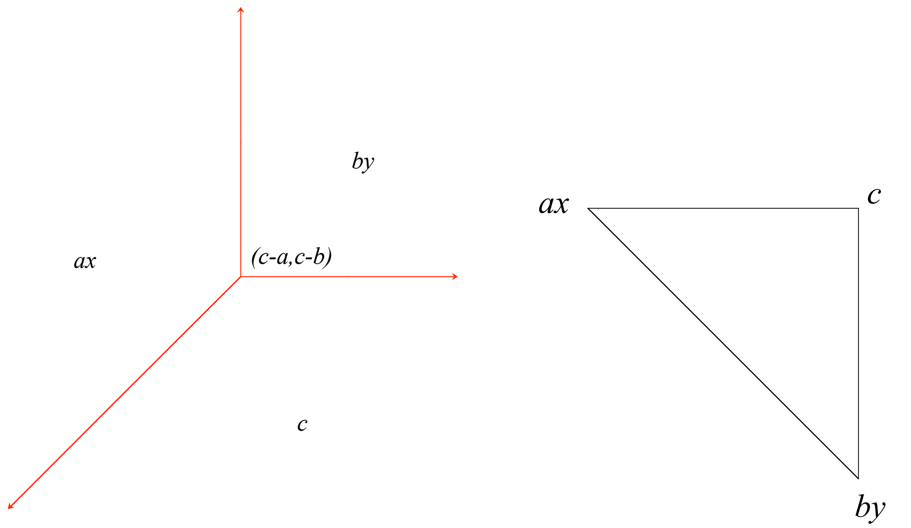
We note that if a polynomial does not have all of the possible terms of degree n, but it does have all three pure terms x^n, y^n, and z^n, then it is functionally equivalent to a polynomial which has all possible term of degree n present with finite coefficients.

We will now note a few properties of tropical plane curves, and then we will discuss the behavior of the curves with regard to the model of the tropical projective plane given earlier.

Lemma 19. *(Due to Aaron Hill) Let $f(x, y)$ and $g(x, y)$ be two tropical polynomial in two variables. Then $\mathcal{Z}(f) \cup \mathcal{Z}(g) = \mathcal{Z}(fg)$.*

Proof. We need to show that the two sets contain each other. Suppose there is a point $p \in \mathcal{Z}(f) \cup \mathcal{Z}(g)$. Then either f or g vanishes tropically at p. Thus, by Lemma 15, fg vanishes tropically at p. Now suppose there is a point p in $\mathcal{Z}(fg)$. Then fg vanishes tropically at p. But then either f or g must vanish tropically by Lemma 16, so p must be in $\mathcal{Z}(f)$ or $\mathcal{Z}(g)$.

It follows from this, as explained in [18], that Δ_{fg} is simply a Minkowski sum of Δ_f and Δ_g. This means that for every polygon of Δ_f and Δ_g are present in Δ_{fg}, but they been combined in such a way as to represent the product of the two. Each vertex of $\mathcal{Z}(fg)$ is either a vertex of $\mathcal{Z}(f)$, a vertex of $\mathcal{Z}(g)$, or a point of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$. We recall the $\mathcal{Z}(l)$ in Figure 8 was the product of two lines, so its dual graph is a Minkowski sum of the dual graph of two lines. We notice that there is parallelogram in Δ_l, which represents the point of intersection of the two lines.



(a) Line with Monomials

(b) Dual Graph of Line with Monomials

Figure 9: Monomials which correspond to Components of $\mathbb{TP}^2 \setminus \mathcal{Z}(f)$

We also note that each of the components of $\mathbb{TP}^2 \setminus \mathcal{Z}(f)$ is where one monomial of $f(x, y, z)$ attains the minimum alone. For example, in Figure 9, we see which of the monomials attains the minimum in each of those components for $f(x, y, z) = ax \oplus by \oplus cz$. For each of these components there is a vertex in the dual graph, so we can see the correlation between the monomials of the polynomial and the vertices of its dual graph, as displayed for the line in Figure 9.

Definition 16. Let $I + \{f_1, f_2, \ldots, f_s\}$ be a set of polynomials in $\mathcal{Q}\langle x, y\rangle$. Then

$$\mathcal{Z}(I) = \{P \in \mathcal{Q}^2 | f(P) \text{ vanishes tropically for every } f(x, y) \in I\}.$$

We will also denote $\mathcal{Z}(I)$ by $\mathcal{Z}(f_1, \ldots, f_s)$.

Lemma 20. *Let $f(x,y)$ and $g(x,y)$ be tropical polynomials in $\mathcal{Q}\langle x,y\rangle$. Then $\mathcal{Z}(f)\cap \mathcal{Z}(g) = \mathcal{Z}(f,g)$.*

Proof. Let $P \in \mathcal{Z}(f)\cap\mathcal{Z}(g)$, then $f(P)$ vanishes tropically, and $g(P)$ also vanishes tropically, so $P \in \mathcal{Z}(f,g)$. If $P \in \mathcal{Z}(f,g)$ then $f(P)$ and $g(P)$ both vanish tropically, so $P \in \mathcal{Z}(f)\cap\mathcal{Z}(g)$.

We will discuss in more detail what we mean by the intersection of two tropical plane curves, as it is not as straightforward as the intersection of two classical curves.

The behavior of the unbounded rays of a tropical plane curve as one of the variables approaches infinity is an important concept to understand, especially for a complete tropical Bézout's theorem. Suppose r is an unbounded ray on a tropical curve (starting at an affine point) containing a point $p = (p_0, p_1, p_2) \in \partial\mathbb{TP}^2$. Also, assume that $p_2 \neq \infty$ (that is, p is not on the $z = \infty$ part of the projective simplex). We can normalize so that $p_2 = 0$ and for every $(x,y,z) \in r$, $z = 0$. There are two (distinct) monomials $a_{ij}x^iy^j$ and $a_{mn}x^my^n$ that attain the minimum together at every point on r. Thus, for all $(x,y,0) \in r$,

$$a_{ij} + ix + jy = a_{mn} + mx + ny,$$

which implies

$$(i-m)x = a_{mn} - a_{ij} + (n-j)y. \tag{1}$$

So, if there is a rational $q \in \mathbb{Q}$ such that $(\infty, q, 0) \in r$ then since the limit of the right hand side of (1) as y approaches q is finite and x approaches ∞, it must be that $i - m = 0$. Examining the dual graph, this implies that the angle θ between r and the x-axis is 0.

Similar reasoning shows that if there is $q \in \mathbb{Q}$ such that $(q, \infty, 0) \in r$ then $n - j = 0$, implying that $\theta = \dfrac{\pi}{2}$. These two facts together show that if r has angle

Direction θ	Intersection with $\partial\mathbb{TP}^2$
$\theta = 0$	(∞, y, z)
$\theta = \pi/2$	(x, ∞, z)
$\theta = -3\pi/4$	(x, y, ∞)
$-3\pi/4 < \theta < 0$	(∞, y, ∞)
$0 < \theta < \pi/2$	(∞, ∞, z)
$\pi/2 < \theta < 5\pi/4$	(x, ∞, ∞)

Table 2: End Behavior of Unbounded Rays

of inclination $0 < \theta < \dfrac{\pi}{2}$, then r hits the boundary of $\mathbb{TP}^2$ at $(\infty, \infty, 0)$. (It is clear that if $\dfrac{\pi}{2} < \theta < 2\pi$ then r cannot hit $(\infty, \infty, 0)$, and we proved that it doesn't hit any points of the type $(q, \infty, 0)$ or $(\infty, q, 0)$. So these types of rays do not avoid $z = \infty$).

The cases where r avoids $x = \infty$ and $y = \infty$ are similar and we have Table 2 detailing the end behavior of tropical curves.

As shown in [4], there are two kinds of invertible linear transformations of the tropical projective plane; namely rotations and reflections due to change of variables and translations due to tropically scaling one or more of the variables. There are six possible changes of variable for this function, which are the six elements of S_3, and an infinite number of translations. We give some examples to see the effects of these transformations on tropical plane curves.

Example 11. Let $f(x, y, z) = ax \oplus by \oplus cz$. Clearly any of the changes of variables leaves us with a line, where the coefficients have been permuted. Thus, even though we still have a line, it's possible that the vertex has moved. For example, if we do the change of variables, $x \mapsto y, y \mapsto z, z \mapsto x$, then we have the equation $cx \oplus ay \oplus bz$, which is a line, but now has vertex at $(b-c, b-a, 0)$ instead of $(c-a, c-b, 0)$. So, unless $a = b = c$, the vertex has moved. In order to translate our curve, we simply multiply at least one of the variables by some scalar. For example, we could have $a\alpha x \oplus by \oplus cz$. Of course we will have a line, but now the vertex is at $(c-a-\alpha, c-b, 0)$, so the entire line has been shifted to the left α units. If we have $ax \oplus b\beta y \oplus cz$, then the vertex has moved to $(c-a, c-b-\beta, 0)$, and the whole thing has shifted down by β. Now if we have $a\alpha x \oplus b\beta y \oplus cz$, then we move the vertex to $c-a-\alpha, c-b-\beta, 0)$, so we have shifted down and to the right. If $\alpha = \beta$, then the vertex is $(c-a-\alpha, c-b-\alpha, 0) = (c-\alpha-a, c-\alpha-b, 0)$, which is the vertex for the line $ax \oplus by \oplus c(-\alpha)z$, which is simply a translation in the z direction. Clearly, if we translate the line in all three directions, this is the same as translating it in just two of the directions, by a different amount. If we have $a\alpha x \oplus b\beta y \oplus c\gamma z$, then we can factor out one of the translations, and see have $\gamma(a(\alpha \oslash \gamma))x \oplus b(\beta \oslash \gamma)y \oplus cz)$, which has the same corner locus as $a(\alpha \oslash \gamma))x \oplus b(\beta \oslash \gamma)y \oplus cz$. Figure 10, we see the original line $\mathcal{Z}(f)$ and the translated line $\mathcal{Z}(a\alpha x \oplus by \oplus cz)$.

Example 12. Let $f(x, y, z) = 3x^2 \oplus xy \oplus 2y^2 \oplus 1xz \oplus 1yz \oplus 1z^2$. In Figure 11, we see the six different conics we get from the six changes of variables possible, which we label according the cycle notation.

We note that these conics are all of the same *combinatorial type*, because they all have the same dual graph up to a change of variables. And not only are they the same combinatorial type, but these curves would clearly be isomorphic, since we know the invertible linear map that takes one to another one.

When we translate this conic, the same thing happens as in the case of the line,

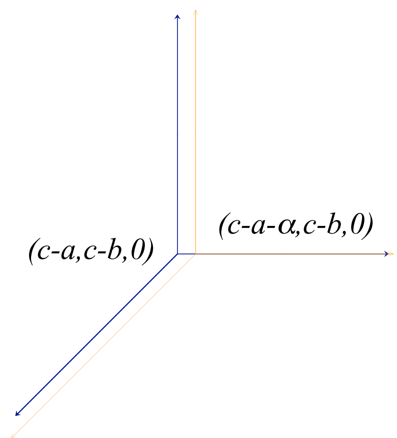


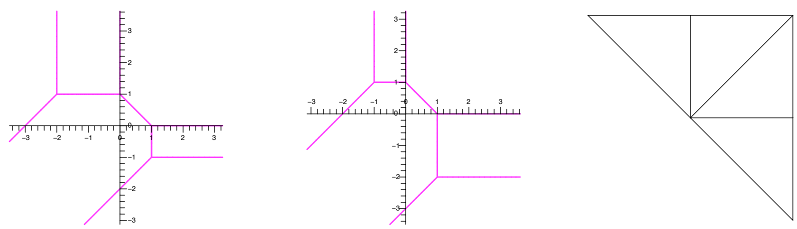
Figure 10: Translation of $ax \oplus by \oplus cz$ to $a\alpha x \oplus by \oplus cz$

it simply moves the conic up or down, or left or right, or up or down a line of slope one. We do this in the same way, be scaling one of the variables. So, for example, if we wanted to shift left or right, we would simply scale the x by some scalar α. In such a case our polynomial would become $3\alpha^2x^2 \oplus \alpha xy \oplus 2y^2 \oplus 1\alpha xz \oplus 1yz \oplus 1z^2$, but clearly all this does is slide all of the vertices α to the left or right. We could also translate in both the x and y directions, as above, by two different amounts α and β. For our polynomial we would then get $3\alpha^2x^2 \oplus \alpha\beta xy \oplus 2\beta^2y^2 \oplus 1\alpha xz \oplus 1\beta yz \oplus 1z^2$. Both of these translations are shown in Figure 12.

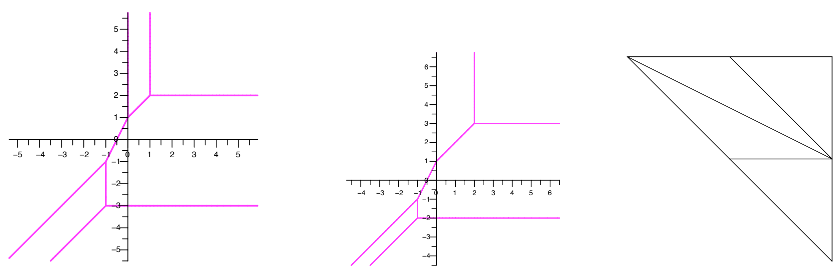
We note that since translations are continuous in the classical setting, since we are using the Euclidean topology, then the translations are still continuous.

3.3 Intersections of Curves in General Positions

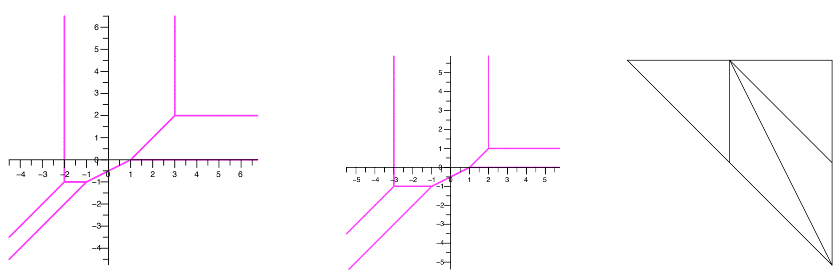
In order to understand and prove the complete tropical Bézout's Theorem we need to understand what *stable intersections* are, and how to count the *intersection*



(a) Identity (b) (xy) (c)

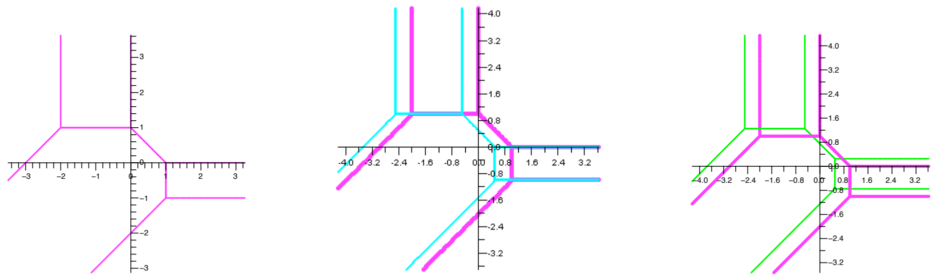


(d) (xyz) (e) (xz) (f)



(g) (xzy) (h) (yz) (i)

Figure 11: Conic Transformations under Change of Variables



(a) Original Conic (b) $x \mapsto \alpha x$ (c) $x \mapsto \alpha x, y \mapsto \beta y$

Figure 12: Conic Transformations under Translations

multiplicity of a given intersection.

As in the classical setting we will first consider two curves which are in general position to each other. Classically, this simply means that the two curves do not share a common component. Tropically, however, this is slightly more involved. Take for example the two lines $\mathcal{Z}(ax \oplus by \oplus cz)$ and $\mathcal{Z}(a\alpha x \oplus by \oplus cz)$ shown in Figure 13. These two lines do not have a component in common, since they are not the same line. But, they intersect in an infinite number of points. We do not want to consider such a pair of curves as being in general position. One way of saying this is to say that the curves are in general position to each other if all of the points of intersection of the two curves are *transverse* intersections, which we define below. In Figure 14(a), we see that these two conics are not in general position to each other, but those in Figure 14(b) are.

Definition 17. Let $\mathcal{F}$ and $\mathcal{G}$ be tropical plane curves, and let $P \in \mathcal{F} \cap \mathcal{G}$. P is said to be a *transverse intersection* if there is some open neighborhood U_P of P, in the Euclidean topology, such that $U_p \cap \mathcal{F} \cap \mathcal{G} = P$ and P lies in the relative interior of an edge of $\mathcal{F}$ and in the relative interior of an edge of $\mathcal{G}$. In other words,

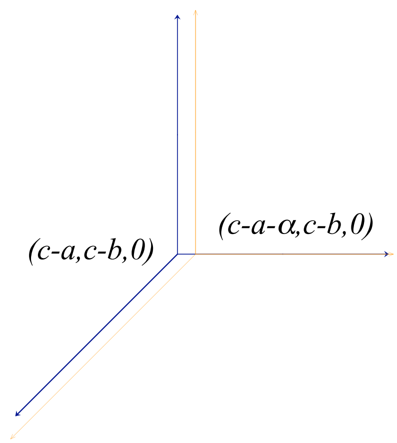
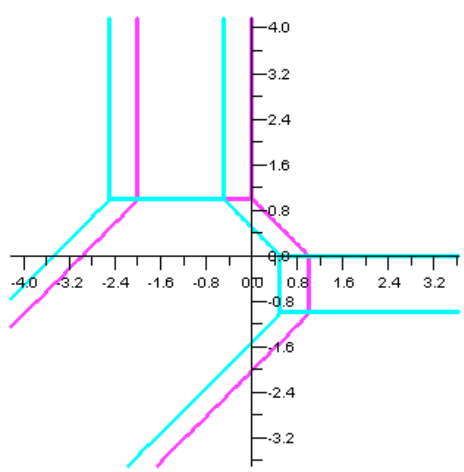
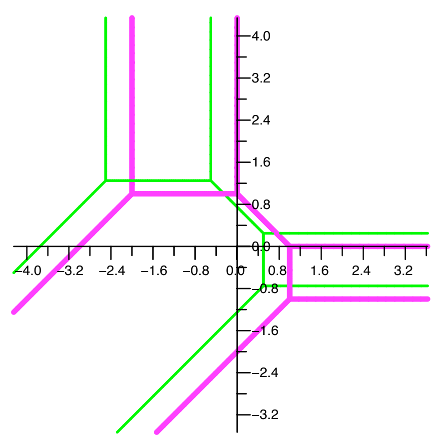


Figure 13: Translation of $ax \oplus by \oplus cz$ to $a\alpha x \oplus by \oplus cz$



(a) Conics with non-transverse intersection



(b) Conics with transverse intersection

Figure 14: Conic Transformations under Translations

an intersection is transverse if locally it is the intersection of two classical lines of unequal slopes. Two curves are *in general position with respect to each other* if all of their intersections are transverse intersections.

If two curves are in general position, then it is clear by looking at the graphs where their points of intersection are.

Lemma 21. *Any two curves C_1 and C_2 can be put into general position with respect to each other by using a finite number of translations.*

Proof. It's easy to see that two curves with no vertices can be put in general position. So suppose at least one of them has a vertex. If P is an intersection point that does not satisfy the condition, then either P is a vertex or there is a subset of $C_1 \cap C_2$ that contains P and is homeomorphic to an interval. If there is a subset homeomorphic to an interval, since the curves are unions of rectilinear lines, either C_1 or C_2 has a vertex in $C_1 \cap C_2$. In either case, there is a vertex in $C_1 \cap C_2$. So if we can shift the curves a bit so that no vertex is in the intersection, then we're done.

Let v be the closest such vertex to p. Without loss of generality, we may assume that that $v \in C_1$. The rays of C_1 emanating from v have positive span the plane. Choose two rays (vectors) at v that are linearly independent from the vector vp (note that at least two such vectors exist) and translate C_1 a small amount in the direction of the sum of the two vectors. That is, you can find α and β so that the map $(x, y, z) \mapsto (\alpha x, \beta y, z)$, when applied to C_1, translates C_1 in the desired direction and a desired Euclidean distance away from the original curve. It is possible to choose α and β in such a way that this translation of C_1 makes v no longer in the intersection. Since there are only a finite number of vertices, one can also make the magnitude of the translation small enough so that every vertex not previously in the intersection is still not in the intersection. Repeat the process the requisite number of times and the two curves are in general position.

As with classical curves, it is not only how the curves intersect, but *where* that can be a problem. If two curves intersect at infinite distance, then it is useful to move that point of intersection to a finite point. This is done with an easy transformation of the plane classically. Tropically the points of intersection at infinite distance all happen at a corner of the model shown in Figure 4. So, to move an intersection at infinite distance to a point at finite distance we would need to move a corner into the interior of the complex somehow. There are only two kinds of linear transformations of $\mathbb{TP}^2$, as shown in [4], namely translations and rotations achieved by change of variables. Unfortunately, both of these transformations map the corners to corners, so the points of intersection at infinite distance stay at infinite distance. Thus, in order to consider these intersection points at finite distance we deform the curves so that they do not intersect the boundary at a corner. This will force the intersections to finite points, as the curves will no longer extend to places where infinite intersections are possible. In Chapter 5 we will deform our curves in the manner explained Lemma 22 and Corollary 23 to verify that we are counting the infinite intersection multiplicity correctly.

Lemma 22. *Every dual graph is the (not necessarily proper) subset of at least one full support dual graph.*

Proof. Suppose $f(x, y, z)$ is a polynomial of full support. Then the dual graph Δ_f of $f(x, y, z)$ is a full support dual graph, and we're done.

Now, suppose that $f(x, y, z)$ is not a polynomial of full support, but in only missing one pure term. Without loss of generality suppose that $f(x, y, z)$ is only missing the y^n of the pure terms, but has both the x^n and z^n terms. This means that Δ_f has both of the upper corners, but is missing the lower corner. To find the full support dual graph that has this dual graph as a subset, we will start with Δ_f and expand it. We will do this by staring with the vertex associated to the term of the form $x^i y^{n-i}$ which has the smallest i. We will number this vertex v_0, and then number the vertices along the boundary of the dual graph in a counterclockwise direction form there, with v_s being the vertex associated to the term of the form $y^j z^{n-j}$ which has the largest j. From v_1 we will drop a vertical segment to the boundary of the limiting triangle. Then, from each of the vertices $v_2, \ldots v_s$ we will drop vertical segments of the same length. At the end of each segment will be a new vertex v'_k. where $v'_0 = v_0$. Now, connect v'_k to v'_{k+1}, for $k = 0, ..., s-1$. This gives a new dual graph that has the original dual graph of $f(x, y, z)$ as a subset. Now relabel the vertices and start again, only this time you will have vertices $v_1, \ldots v_{s-1}$. Repeat this process until the entire limiting triangle is full. Now we have a full support dual graph of which the dual graph of $f(x, y, z)$ as a subset.

Now suppose that two of the pure terms are missing, x^n and y^n. We proceed in the same way as described above, filling in the y^n corner of the dual graph. We then do a change of variables so the y^n is again the missing term and fill the corner in as explained. If all three pure terms are missing, we simply perform one more change of variables, and fill in the last corner as well. Now you have a full support dual graph of which our original dual is a subset.

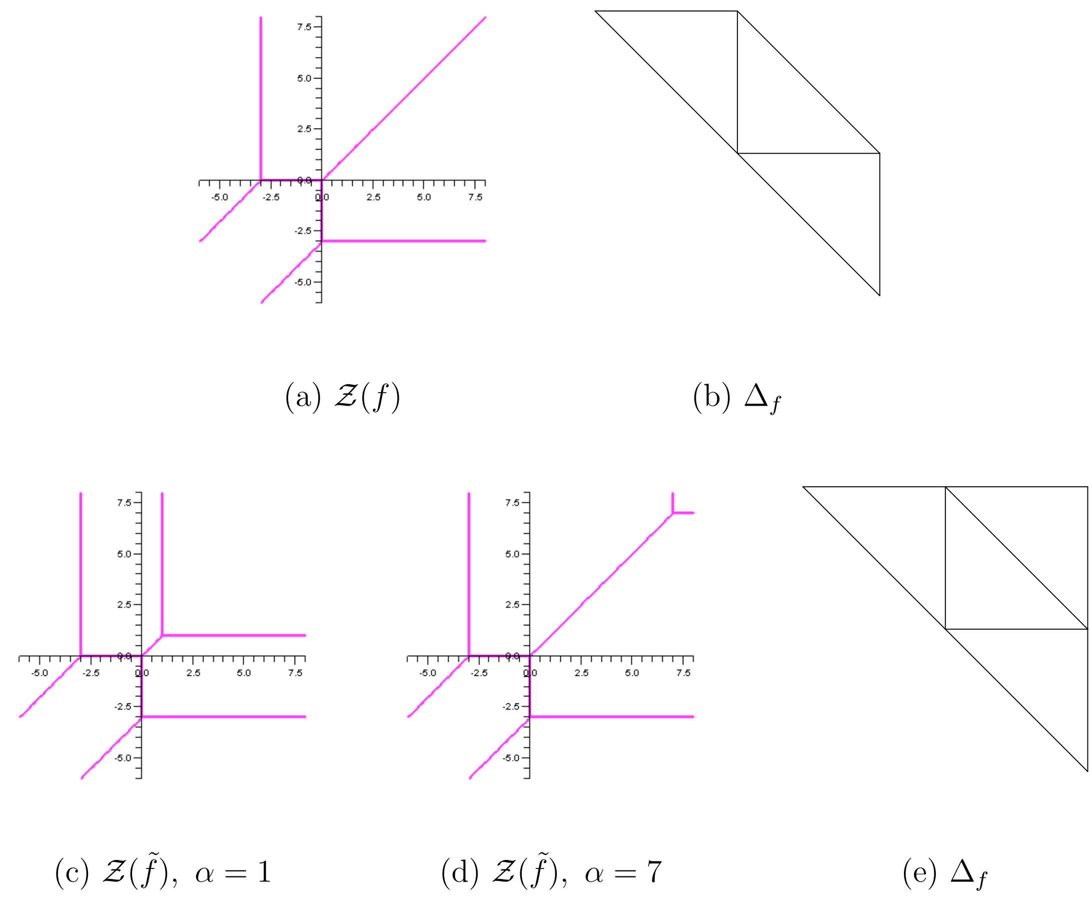
It is true that this is not the only way to create a full support dual graph that has the dual graph of $f(x, y, z)$ as a subset. We could have just as easily dropped a segment from each of $v_1, \ldots, v_s$ to the limiting triangle, and then connected the vertices at their endpoints. However, the method described will be useful to us when we look at the infinite intersections moved to finite points. This method allows us to look at the infinite intersections one at a time.

Corollary 23. *Every tropical plane curve associated to a tropical polynomial $f(x, y, z)$ is equal to a tropical plane curve of full support associated to a full support tropical polynomial $\tilde{f}(x, y, z)$ in some bounded region of the $\mathbb{TP}^2$.*

Proof. Since every dual graph is the subset of a dual graph of full support, and since the dual graph gives you the basic shape of the curve, we can fix the original bounded portion of our curve, and then add onto it to fill in the full support curve. All tropical plane curves are made up of vertices, segments and rays, where the segments have finite length and do not intersect the boundary of $\mathbb{TP}^2$ and the rays extend infinitely in a given direction and do intersect the boundary of $\mathbb{TP}^2$. Thus, there is some ball that contains all of the segments, and parts of the rays. It is this portion of the graph that we fix, but then we are able to split the rays of our original curve by using the full support dual graph we found to create a full support curve which is equal to our original curve inside the ball.

In essence, we deform a curve that is not of full support by splitting the rays that extend in non-desirable directions. This produces for us a curve of full support, which does not intersect $\partial\mathbb{TP}^2$ in any of the corners.

Example 13. Let $f(x, y, z) = 3x^2 \oplus xy \oplus 3y^2 \oplus xz \oplus yz$. This curve, as shown in Figure 15, is not a curve of full support as the polynomial is missing the z^2 term. However, we can deform this polynomial by adding αz^2 for any $\alpha > 0$, so we have the polynomial $\tilde{f}(x, y, z) = 3x^2 \oplus xy \oplus 3y^2 \oplus xz \oplus yz \oplus \alpha z^2$, which does produce a



(a) $\mathcal{Z}(f)$ (b) Δ_f

(c) $\mathcal{Z}(\tilde{f}),\ \alpha = 1$ (d) $\mathcal{Z}(\tilde{f}),\ \alpha = 7$ (e) Δ_f

Figure 15: Deforming Curves of Non-full Support

curve of full support. The larger α the farther then new vertex is from the original vertices.

3.4 Affine Intersections of Tropical Plane Curves

As discussed our plane curves maybe in general position to each other, or they may not be. We will first discuss *intersection multiplicity* for two curves that are in general position, and then we will discuss curves that are not in general position. This discussion will follow that given in [16].

For transverse intersections, we need to know some information about the slopes of the segments or rays that are intersecting in order to find the intersection multiplicity of the points of intersection. As explained in [6],[13],[14], and[16] we can also think of our tropical plane curves as weighted balanced graphs. In this way there is a weight and a integral direction vector associated with each segment. Suppose the two segments l_1 and l_2 and suppose that l_i has integral direction vector $(u_{i,1}, u_{i,2}, u_{i,3})$ and weight m_i. The following definition of *intersection multiplicity* is given in [16].

Definition 18. Let $f(x, y, z)$ and $g(x, y, z)$ be two homogenous polynomials in general position with respect to each other, and let $P \in \mathbb{TA}^2$ be a point of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$. The *intersection multiplicity* i_P for the point P is defined to be the absolute value of the classical determinant of the matrix

$$\begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ 1 & 1 & 1 \end{pmatrix}$$

times $m_1 \cdot m_2$. That is

$$i_P = \begin{vmatrix} u_{1,1} & u_{1,2} & u_{1,3} \\ u_{2,1} & u_{2,2} & u_{2,3} \\ 1 & 1 & 1 \end{vmatrix} \cdot m_1 \cdot m_2$$

We recall that our dual graph is made on a lattice with points that correspond to possible monomials in our polynomial, so there is a point in the lattice for each term of the form $x^i y^j z^k$. If we define the distance between points of the form $x^{i+1} yj z^k$ and $x^i y^j z^k$ to be one, and the distance between terms of the form $x^i y^{j+1} z^k$ and

$x^i y^j z^k$ to also be one, then using the Euclidean metric from there, we have the distance from $x^i y^j z^{k+1}$ to $x^i y^j z^k$ being $\sqrt{2}$.

Definition 19. Let $f(x, y, z)$ and $g(x, y, z)$ be two homogenous polynomials in general position with respect to each other, and let $P \in \mathbb{TA}^2$ be a point of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$. The *dual graph intersection multiplicity* of the point P is defined the be the area of the parallelogram of Δ_{fg} that corresponds to P.

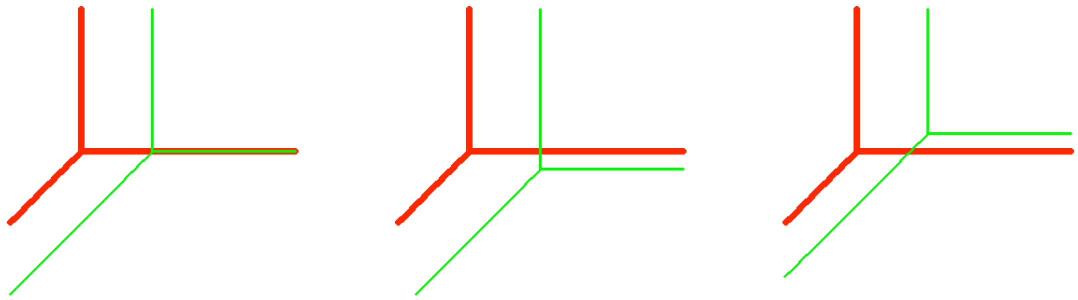
Lemma 24. *The intersection multiplicity defined in Definition 18 agrees with the dual graph intersection multiplicity defined in Definition 19 in the cases where both of them apply.*

Proof. It is straightforward to show that the intersection multiplicity given in Definition 18 the same as the area of the associated parallelogram in the dual graph of the product of the original polynomials. This is because the segments in the dual graph are perpendicular to the segments of the curve. The determinant above just gives the area of the parallelogram determined by the integral direction vectors, so using perpendicular vectors will provide the same area. Also, as explained in [16] the lattice length of the segment in the dual associated with the segment in the curve tells us the weights m_i of the segments in the curve. Thus, the area of the parallelogram is precisely the classical determinant of the above matrix times the weights $m_1 \cdots m_2$.

We note that the area of such a parallelogram will only be greater than one if there are more than four points of the lattice contained in the interior or the boundary of the parallelogram. For example, the sides have lattice length greater than one, in which case there would be additional lattice points on the edges of the parallelogram. An other possibility is that all the sides have lattice length one, but that there is a point of the lattice interior to the parallelogram. In all of these cases, there is a term of the product that is non-contributing, which is why there are addition points of the lattice involved in the parallelogram.

When we have two curves that are not in general position to each other, meaning they have an infinite intersection without having a common component, or they have a vertex in the intersection, it is not as clear what the intersection is or what the intersection multiplicity is. Since there is a vertex of one of the curves in the intersection, there is not a well defined segment of that curve which is participating in the intersection. Thus, there is no way to determine the integral direction vectors, so Definition 18 does not apply. Similarly, since the curves are not in general position, the intersections are not transverse, which means that the polygon in the dual graph which corresponds to the point of intersection is not a parallelogram, and corresponds not only to the point of intersection, but to the original vertex as well, so finding its area does not give the intersection multiplicity either. As shown Lemma 21, we can translate one of the two curves so that the two curves are in general position to each other. When the curves are in general position to each other it becomes clear what the intersections are. So, we determine the points of intersection, and then consider the limit of the intersections of the two curves as we translate the given back from its translated position to its original positions, as explained in [16]. Since the translations are continuous, the limits are well defined, so we have well defined points of intersection, called the *stable intersections.*

Definition 20. Let $f(x, y, z)$ and $g(x, y, z)$ be homogenous polynomials. The *stable intersection* of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ is defined to be the set of points which are the limits of the points of intersection of $\mathcal{Z}(f_{\alpha,\beta})$ and $\mathcal{Z}(g)$ where $f_{\alpha,\beta}(x, y, z) = f(\alpha x, \beta y, z)$,



(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$ (b) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$ (c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$

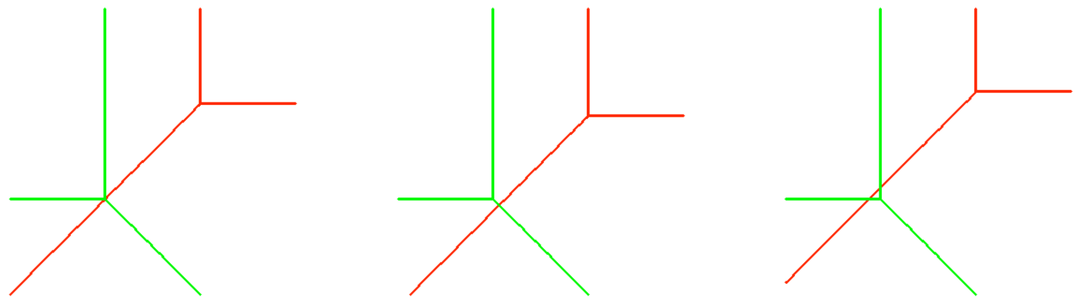
Figure 16: Stable Intersections

for some $\alpha, \beta \in \mathbb{Q}$ and we are taking the limits as $\alpha, \beta \to 0$.

Proposition 25. *Let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous polynomials of degree d and e respecitively. The stable intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ given in Definition 20 is well defined.*

Proof. Since the translations of tropical plane curves are continuous functions in the Euclidean topology, the points of intersection vary continuously as α and β vary. Thus any sequence of intersections points must converge to the same points, so the points are well defined. Because the translation of one curve puts our curves into general position with respect to each other, we can use Definition 18 to see that we have $d \cdot e$ points of intersection for $\alpha, \beta \neq 0$. But, since the points of intersection vary continuously, when $\alpha, \beta \to 0$, we have $d \cdot e$ such points.

Example 14. Let us consider the two lines as shown in Figure 16, $f(x, y, z) = ax \oplus by \oplus cz$ and $g(x, y, z) = ax \oplus sy \oplus cz$. If we shift g under the translation $y \mapsto \beta y$, then the green line either moves up or down, and are curves are in general position to each other, as shown in Figure 16.



(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$ (b) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$ (c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$

Figure 17: Stable Vertex Intersections

We see that under these translations there is one clear point of intersection. As we allow the green line to return to its original position, the point of intersections converges to the vertex of the green line.

Recall that when we are translating one of the two curves, we want to do so by a very small α and β, so that our curves will stay in a relatively similar position. We just want to move them enough so that they are now in general position. If we do this then we can define the intersection multiplicity of a stable intersection to be the the sum of the intersection multiplicities of all the points of transverse intersection that limit to our stable intersection when we translate in a given direction.

Example 15. Let us consider the two lines as shown in Figure 17, $f(x, y, z) = x \oplus y \oplus 4z$ and $g(x, y, z) = xy \oplus 2xz \oplus 4z^2$. If we shift f under the translation $y \mapsto \beta y$, then the red line either moves up or down, and are curves are in general position to each other, as shown in Figure 17.

Here we see that when the red line shifts up, then there are two points of intersection, but when it shifts down there is only one. However, this one point of interstice does indeed have intersection multiplicity two. This is because as we translated the curve that was shifted up back down to its original position, the two points of intersection converge to the same point, thus giving a point of intersection multiplicity two.

The version of Bézout's theorem that is given in [16] uses only curves that have full support, which we recall means that all three pure terms x^n, y^n, and z^n have finite coefficients, and which are in general position to each other. Suppose we have two polynomials, $f(x, y, z)$ and $g(x, y, z)$, both of full support, of degree d and e respectively. Then, using the distance convention given above, Δ_f has area $d^2/2$, since it is a isosceles right triangle of leg length d. Similarly, Δ_g has area $e^2/2$, and Δ_{fg} has area $(d + e)^2/2 = d^2/2 + e^2/2 + d \cdot e$. Note that is does not matter the relative position of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$, since fg is a full support polynomial of degree $d+e$. So, in such a case, the area of Δ_{fg} is the sum of the area of Δ_f, the area of Δ_g and $\deg f \cdot \deg g$. But the area in Δ_{fg} that is not from Δ_f and Δ_g is associated with the vertices of the product which are the points of intersection of the two curves. And that area, as explained above, is the intersection multiplicity of the points of intersection of the two curves. Thus, for the special case where the two curves are both curves of full support, tropical Bézout's theorem is true. But, we note that for polynomials that are full support, all of the points of intersection are in the tropical affine plane, but only the finite points in the affine plane. Although this is an important result, in Chapter 5 we extend this result to all polynomials, not just those of full support.

4 Tropical Linear Algebra

In this section we want to consider the tropical analogue of a vector space. A tropical vector is simply an n-tuple with tropical entries, the analogous object to a vector in $\mathbb{R}^n$, for example. Through out the section $\mathcal{T}$ will be any tropical semi-ring which has ∞ as its additive identity.

Definition 21. Let $\mathbf{u}$ and $\mathbf{v}$ be tropical vectors in $\mathcal{T}^n$ and let α be a scalar in $\mathcal{T}$. We define $\mathbf{u} \oplus \mathbf{v}$ to be the componentwise sum of $\mathbf{u}$ and $\mathbf{v}$, and we define $c \odot \mathbf{u}$ to be the vector found by multiplying each entry of $\mathbf{u}$ by the scalar c.

Using Definition 21, it is straightforward to show that all of the properties of a semi-vector space are satisfied by our collection of vectors, where a semi-vector space has all of the conditions of a vector space, with the exception of additive inverses. We note we will write our vectors interchangably as row or column vectors when it is not ambiguous to save space.

Let us consider $\mathcal{Q}^2$ for an example. As far as the geometric interpretation of a vector is concerned, it is the same as in the classical setting. So, the vector $(1, 20)$ is the same in both $\mathbb{R}^2$ and $\mathcal{Q}^2$. However, the geometric interpretation of the vector space operations is not the same. For example, the tropical sum of two vectors can not be realized in the same geometric manner as in the classical case. Consider the vectors $\mathbf{u} = (3, 2)$ and $\mathbf{v} = (1, 4)$. Then $\mathbf{u} \oplus \mathbf{v} = (3 \oplus 1, 2 \oplus 4) = (1, 2)$. Thus the sum simply lowers each coordinate to the least value of all the values for that coordinate in the vectors we add. Although we will not have geometric interpretations to match the classical setting, the formal operations on the vectors are well defined, and many of the operations with both vectors and matrices will still be meaningful.

4.1 Linear Independence and Singularity

We now give some basic definitions and properties of vectors and matrices with entries in $\mathcal{T}$. The following definitions and proofs basically follow definitions and proofs of similar theorems in [11], only with a more universal notation of tropical vanishing. We do deviate from his proof for the proof of Lemma 32

We recall that a tropical expression *vanishes tropically* if two of the terms in the expression are minimal together. We also recall that if g is a tropical expression that vanishes, then gh also vanishes for any tropical expression h.

Definition 22. A tropical vector expression is said to *vanish tropically* if each entry of the resulting tropical vector sum vanishes tropically. Explicitly, let $\mathbf{v}_i = (v_{i1}, v_{i2}, ..., v_{in}) \in \mathcal{T}^n$ where $v_{ij} \in \mathcal{T}$. Then

$$\mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \cdots \oplus \mathbf{v}_m = \begin{pmatrix} v_{11} \\ v_{12} \\ \vdots \\ v_{1n} \end{pmatrix} \oplus \begin{pmatrix} v_{21} \\ v_{22} \\ \vdots \\ v_{2n} \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} v_{m1} \\ v_{m2} \\ \vdots \\ v_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} v_{11} \oplus v_{21} \oplus \cdots \oplus v_{m1} \\ v_{12} \oplus v_{22} \oplus \cdots \oplus v_{m2} \\ \vdots \\ v_{1n} \oplus v_{2n} \oplus \cdots \oplus v_{mn} \end{pmatrix},$$

and $\mathbf{v}_1 \oplus \mathbf{v}_2 \oplus \cdots \oplus \mathbf{v}_m$ vanishes tropically if $v_{1i} \oplus v_{2i} \oplus \cdots \oplus v_{mi}$ vanishes tropically for every i.

Definition 23. Let $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ be a collection of m vectors in $\mathcal{T}^n$. These vectors are said to be *tropically linearly dependent* if for some $\lambda_1, ..., \lambda_m \in \mathcal{Q}$, not all infinite,

$\lambda_1\mathbf{v}_1 \oplus \lambda_2\mathbf{v}_2 \oplus \cdots \oplus \lambda_m\mathbf{v}_m$, vanishes tropically. If no such constants λ_i exist so that the sum vanishes tropically, then the vectors are said to be *tropically linearly independent*.

Definition 24. The *rank* $\mathbb{T}Rank(A)$ of a tropical matrix is defined to be the maximum number of its independent columns.

There are three different definitions for the rank of a tropical matrix, as given in [2]. We will show later that the above definition of the rank of a tropical matrix is equivalent to what is called the *tropical rank* of a matrix in that paper, as given below.

Definition 25. Let A be a tropical $n \times n$ matrix with entries $a_{i,j}$ and let S_n be the collection of permutations of $\{1, ..., n\}$. The *tropical determinant* of A is defined to be

$$|A|_{\mathbb{T}} = \bigoplus_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}.$$

A is said to be *tropically singular* if the expression $\bigoplus\limits_{\sigma \in S_n} a_{1,\sigma(1)} \cdots a_{n,\sigma(n)}$ vanishes tropically.

We note that tropical determinants can be determined using the method of co-factors, with all addition and subtraction being replaced by tropical addition.

Definition 26. Let A be a tropical matrix with entries in $\mathcal{T}$. The *tropical rank* of A is defined to be the largest integer r such that A has a a non-singular $r \times r$ minor.

To show that the two definitions of rank are equivalent, we first need the following definitions and lemmas that relate the linear independence of the columns of a matrix and the singularity of a square matrix.

Definition 27. The *directed graph* G_A *associated to an* $n \times n$ *matrix* A is the directed graph with n vertices, $v_1, \ldots, v_n$, indexed by the rows/columns of the matrix, and edges satisfying the condition that there is an edge from v_i to v_j if a_{ij} has a given property P.

Unless stated otherwise, the property P that we will be dealing with is minimum among the row entries. So, if $a_{ij} \leq a_{ik}$ for all $k = 1, ..., n$, then there will be an edge from v_i to v_j.

Lemma 26. *Suppose* A *is an* $n \times n$ *matrix with entries in* $\mathcal{Q}$*. Then* G_A*, under this property* P*, contains at least one cycle.*

Proof. Suppose that G_A does not a cycle. Since the entries of A are in $\mathcal{T}$, then each row has an entry that is minimum. Thus, in row b_1, a_{b_1,b_2} is minimum, so we have an edge from v_{b_1} to v_{b_2}. But row b_2 also has a minimum, say a_{b_2,b_3}, so there is an edge from v_{b_2} to v_{b_3}, and so on. But, if there is no cycle in G_A, then $b_i \neq b_j$ for any $i \neq j$. But row b_n also has a minimum in it, $a_{b_n,b_{n+1}}$. But there are only n vertices, so $b_{n+1} = b_j$ for some j. Thus there is at least one cycle in G_A.

We note that it is possible for the cycle in G_A to be a *self-loop*, from v_i to itself.

Lemma 27. *Suppose that* A *is an* $n \times n$ *matrix with entries in* $\mathcal{Q}$*, and suppose that the cycle resulting from Lemma 26 has length* r*. Then there is some permutation* $\sigma \in S_n$ *such that* $\sigma(\alpha_i) = \alpha_{i+1}$ *for indices in the cycle, and* $\sigma(j) = j$ *for indices not in the cycle, where* $\alpha_{r+1} = \alpha_1$*.*

Proof. This is clear.

Lemma 28. *If* A *is an* $n \times n$ *matrix with entries in* $\mathcal{T}$ *and if* $|A|_{\mathbb{T}} = \bigodot_{i=1}^{n} a_{i,\mu(i)}$ *for some permutation* μ*, then at least one of* $a_{i,\mu(i)}$ *is minimum among its row.*

Proof. Without loss of generality, suppose μ is the identity permutation. Our matrix still satisfies the conditions for Lemma 27, so there is a permutation σ corresponding to some cycle of G_A, as described. But for those indices in the cycle, it is true that $a_{i,\sigma(i)} \leq a_{i,\mu(i)}$, and for those indices not in the cycle, $a_{i,\sigma(i)} = a_{i,\mu(i)}$. Thus $\bigodot_{i=1}^{n} a_{i,\sigma(i)} \leq \bigodot_{i=1}^{n} a_{i,\mu(i)}$. If equality holds, then the permutation σ also produces the determinant, and we know that $a_{i,\sigma(i)}$ is minimum among its row for indices in the cycle. If equality does not hold, then we have a contradiction, as $\bigodot_{i=1}^{n} a_{i,\mu(i)}$ is minimal over all the terms in the determinant. Thus, it follows that $a_{i,\mu(i)}$ is minimum among its row for at least one i. ✰

Lemma 29. *Let A be an $n \times n$ matrix with entries in $\mathcal{Q}$ and suppose each row has two distinct entries that attain the minimum of that row; that is there are indices $i_1 \neq i_2$ such that $a_{i,i_1} = a_{i,i_2} \leq a_{i,j}$ for every $j = 1, ..., n$. Then G_A has two cycles with no edges in common.*

Proof. With the minima a_{i,i_1}, by Lemma 26 we have one cycle c_1. But, using the minima a_{i,i_2}, Lemma 26 gives us another cycle c_2. Suppose that c_1 and c_2 have an edge in common, say $\overrightarrow{v_j v_k}$, choosing j minimally among all common edges of c_1 and c_1. Then a_{jk} is minimal in row j. By our assumption, there is some $k' \neq k$ such that $a_{jk'}$ is also minimal in row j. We can thus replace the edge $\overrightarrow{v_j v_k}$ in c_1 with the edge $\overrightarrow{v_j v_{k'}}$ and continue building c_1 in the same manner as described in Lemma 26, where we choose the edge not contained in c_2 at each vertex. This gives two cycles in G_A with no edges in common. ✰

Lemma 30. *Let A be an $n \times n$ matrix with entries in $\mathcal{T}$ and suppose each row has two distinct entries that attain the minimum in that row, as in Lemma 29. Then A is tropically singular.*

Proof. From the proof of Lemma 29 we see that for every cycle c of G_A, there is another cycle c' of G_A such that c and c' do not share an edge. By Lemma 26 we have a cycle c_1 associated with one set of minima a_{ii_1} from the rows. By Lemma 27 there is a permutation σ_1 associated with c_1 as described. By the proof of Lemma 28, we see the determinant must be attained by a permutation (called σ_1) that contains such a cycle of G_A. Assume σ_1 is the identity permutation. Now we have another cycle c_2 with no edges in common with c_1. Thus by Lemma 27 we have another permutation σ_2 associated to c_2. As in the proof of Lemma 28, we can use this permutation to to get $\bigodot_{i=1}^{n} a_{i,\sigma_2(i)} \leq \bigodot_{i=1}^{n} a_{i,\sigma_1(i)}$. But $|A|_{\mathbb{T}} = \bigodot_{i=1}^{n} a_{i,\sigma_1(i)}$, so we can not have a strict inequality. Thus $\bigodot_{i=1}^{n} a_{i,\sigma_2(i)} = \bigodot_{i=1}^{n} a_{i,\sigma_1(i)}$, and $|A|_{\mathbb{T}}$ vanishes tropically, and A is tropically singular.

Lemma 31. *Suppose A is an $n \times n$ matrix whose entries are tropical expressions where the monomials in each expression are in $\mathcal{T}$, and suppose that each row of A either has two distinct entries that attain the minimum of that row, or one entry that is an expression which vanishes tropically and is minimum in the row. Then A is singular.*

Proof. If each row has at least two distinct entries which attain the minimum we're done by Lemma 30. So, suppose there is some row where one of the entries vanishes tropically and is smaller than the other entries in the row, say a_{jk}. If $|A|_{\mathbb{T}} = \odot_{i=1}^{n} a_{i,\sigma(i)}$ where $\sigma(j) = k$, which is to say the entry that vanishes tropically is in the determinant, then we are done by Lemma 15. If there is no such entry in that permutation, then there is an entry in each row that is not on the diagonal that attains the minimum, so by Lemma 30 we can find another permutation that attains the minimum and we are done. Thus A is singular.

Lemma 32. *Let A be a tropical $n \times n$ matrix such that $|A|_{\mathbb{T}}$ vanishes tropically. Then the columns $C_1, ..., C_n$ of A (or rows of A) are tropically linearly dependent.*

Proof. Consider a $n \times n$ tropical matrix A which is singular. We note that the determinant can be found by cofactor expansion around any row or column. Now, either every possible $(n-1)$ minor of A is infinite, or there is at least one $(n-1)$ which has a finite value.

First suppose that every $(n-1)$ minor is infinite. Then A is singular because $|A|_{\mathbb{T}} = \infty$, and every term is infinite. This means that every term in $\bigoplus_{\sigma \in S_n} \bigodot_{i=1}^{n} a_{i,\sigma(i)}$ must have an infinite factor. This implies that either an entire row or column of A is infinite, or that there are two rows or columns each with one finite entry in the same position. If one entire row or column is infinite, then clearly the rows or columns are linearly dependent. Similarly, if two rows or columns each only have one finite entry in the same position, then one is a multiple of the other, and again the rows or columns are linearly dependent.

Now suppose that at least one of the $(n-1)$ minors from this expansion is not infinite, say the minor where the first column and kth row are deleted. We then define a mapping which takes in $n-1$ vectors and returns a tropical expression in $\mathcal{T}$ to be the determinant of the matrix attained by deleting the k entry from each vector and letting those be the columns. Now consider the following tropical sum:

$$\bigoplus_{i=1}^{n} f(C_1, ..., \hat{C}_i, ..., C_n)C_i.$$

This sum is a tropical vector with n entries. We want to show that this sum vanishes tropically. If this is true, then we have found suitable $\lambda_i = f(C_1, ..., \hat{C}_i, ..., C_n)$ to provide the linear dependence of the columns of A. Consider the jth entry in the resulting sum, where $j \neq k$. We need to show that this entry vanishes tropically.

We will show that for every term of the sum, there is another term with the same value, thus those two terms together vanish tropically. Thus, the entire sum for the entry is in fact a sum of terms that vanish tropically, and thus the sum must vanish tropically. A term in the sum for the jth entry looks like $c_{m,j} \bigodot c_{i,\sigma(i)}$ where $c_{m,j}$ is the jth entry of the mth column, and the product never has $i = m$ or $\sigma(i) = k$ (since the kth row has been deleted). But each product must have a term from each row, which means that it must have a term from the the jth row, namely $c_{\sigma^{-1}(j),j}$. But then $f(C_1, ..., \hat{C_{\sigma^{-1}(j)}}, ..., C_n)C_{\sigma^{-1}(j)}$ has the exact same term with the difference that $c_{m,j}$ is the term in the product and $c_{\sigma^{-1}(j),j}$ is the term from the vector. Thus each term has another term that matches it, and the entries all vanish tropically for $j \neq k$. Now, the kth entry is simply the cofactor expansion $|A|_{\mathbb{T}}$ around the kth row. But, by hypothesis, this vanishes tropically. We note that at least $\lambda_1 \neq \infty$, since $f(C_2, ..., C_n)$ is the minor that was not infinite. Thus, every entry vanishes tropically, and we have found suitable λ_i for our columns to be tropically linearly dependent, our desired result.

Theorem 33. *Let A be an $n \times n$ square matrix whose entries are tropical expressions. $\mathbb{T}Rank(A) < n$ if and only if A is tropically singular.*

Proof. Suppose $\mathbb{T}Rank(A) < n$. This means that there are $\lambda_1, ..., \lambda_n \in \mathcal{Q}$, not all infinite, such that $\lambda_1 c_1 \oplus \lambda_2 c_2 \oplus \cdots \oplus \lambda_n c_n$, vanishes tropically, where c_j is the jth column of A. Since $|\lambda_j a_{ij}|_{\mathbb{T}} = (\bigodot_{j=1}^{n} \lambda_j) \odot |A|_{\mathbb{T}}$, if all of the λ_i are finite, $|\lambda_j a_{ij}|$ vanishes tropically if and only if $|A|$ vanishes tropically, so we may consider $[\lambda_j a_{ij}]$ as A and disregard the λ_j for simplicity. Since A is tropically singular, each entry of $c_1 \oplus c_2 \oplus ... \oplus c_n$ vanishes tropically, which means that each row either has two distinct entries that attain the minimum together, or has an entry with vanishes tropically and is minimum in the row. But we can now apply Lemma 31, and we see that A is singular.

We claim that if $\lambda_i = \infty$ for any i, then $|A|_{\mathbb{T}}$ vanishes tropically. Suppose without loss of generality that $\lambda_1 = \infty$, but that $\lambda_i \neq \infty$ for $i \geq 2$. Then all of the $(n-1)$-minors associated with $c_2, ..., c_n$ are in the situation described above, so they all vanish tropically. Thus, if we do the cofactor expansion around the first column, then each summand of the determinant will be the product of an entry from c_1 and an expression that vanishes tropically. Thus, each summand vanishes tropically, and the entire determinant vanishes tropically. A similar argument shows that if k of the λ_i's are infinite, then since certain $(n-k)$ minors still vanish tropically, there is a cofactor expansion which vanishes tropically, and thus the determinant does as well.

Conversely, if A is singular, then by Lemma 32, the columns of A are linearly dependent, so $\mathbb{T}Rank(A) < n$.

We note here that since $|A|_{\mathbb{T}} = |A^T|_{\mathbb{T}}$, we can use either the rows or the columns of an $n \times n$ matrix to determine the rank.

Lemma 34. *The definition of rank of a tropical matrix as given in Definition 24 is equivalent to that given in Definition 26. More specifically, the maximum number of independent columns of A is r if and only if r is the largest integer such that A has a non-singular $r \times r$ minor.*

Proof. Suppose that the maximum number of independent columns of A is r, and suppose that there is a non-singular minor of size $s \times s$ such that $s > r$. Then by Theorem 33, the columns of this $s \times s$ matrix are linearly independent. Which means that the s columns from A associated with the columns of the $s \times s$ minor are also linearly independent (since it doesn't matter what we add on in the other rows, the rows represented by the $s \times s$ minor are linearly independent). But this is a contradiction, since the maximum number of independent columns of A is $r < s$.

Now suppose that r is the largest integer such that A has a non-singular $r \times r$ minor, and suppose that A has s linearly independent columns where $s > r$. Thus at least one of the rows does not have two entries which are equal and minimum. Consider any $s \times s$ minor which is made up of a set of s rows of these columns, where the row that does not have two minimums is contained in the set. These truncated columns are still linearly, so by Theorem 33, this minor is non-singular. But this is a contradiction, since the largest non-singular minor was $r \times r$ where $r < s$.

Thus the two definitions are equivalent.

4.2 Tropical Cramer's Rule

In the classical setting we solve systems of linear equations using matrices. This is often done by row reducing the matrices to being as close to the identity matrix as possible. Unfortunately for the tropical setting, this generally involves subtraction, which we do not have. However, classically is it also possible to solve the system of equations using Cramer's Rule, for which there is a tropical analogue. We will look at a few examples before we give the theorem and it's proof.

Example 16. Let us first consider the question of trying to determine whether or not three points are collinear. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be three points in the plane. These three points are tropically collinear if there are scalars a, b, c so that $ax_i \oplus by_i \oplus c$ vanishes tropically for each i. But this is the same as saying the three vectors

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

are linearly dependent with scalars a, b, c. But these three vectors are linearly dependent if and only if the matrix

$$\begin{pmatrix} x_1 & y_1 & 0 \\ x_2 & y_2 & 0 \\ x_3 & y_3 & 0 \end{pmatrix}$$

is singular. We can of course generalize this to a projective setting, where instead of looking at points (x_i, y_i) in the tropical affine plane, we can look at points (x_i, y_i, z_i) in the tropical projective plane. These three points are then tropically collinear if the matrix

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

is tropically singular.

We can use this same method to find the line that contains two points. Let (x_1, y_1, z_1) and (x_2, y_2, z_2) be two points in the tropical projective plane. Then the set of all points (x, y, z) that are collinear with these two points is the same as the set of all points (x, y, z) for which

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x & y & z \end{pmatrix}$$

is tropically singular. We recall that this matrix is tropically singular when the determinant vanishes tropically. If we calculate the determinant of this matrix using cofactor expansion around the last row, then we get

$$(y_1z_2 \oplus y_2z_1)x \oplus (x_1z_2 \oplus x_2z_1)y \oplus (x_1y_2 \oplus x_2y_1)z.$$

The points (x, y, z) at which this vanishes tropically are the same as the corner loci of the linear equation given by the determinant. This gives us a method to determine the linear equation for the line between two points.

As in the classical setting, the points in $\mathbb{TP}^2$ are in one-to-one correspondence with the lines in $\mathbb{TP}^2$. Thus this method can also be used to determine the point of intersection of two lines $ax \oplus by \oplus cz$ and $dx \oplus ey \oplus fz$ in $\mathbb{TP}^2$. To find the point of intersection, we first find consider the points (a, b, c) and (d, e, f) in $\mathbb{TP}^2$ that correspond to the lines, and then find the line that passes through these points. The point that corresponds to this line is the point of intersection of the two original lines. Thus, when we take the determinant of the matrix

$$\begin{pmatrix} a & b & c \\ d & e & f \\ x & y & z \end{pmatrix},$$

instead of the 2×2 minors giving the coefficients for the linear equation, they give the coordinates of the point where the lines intersect.

As we mentioned above, the lines in $\mathbb{TP}^2$ are in one-to-one correspondence with the points in $\mathbb{TP}^2$. This extends naturally to the points in $\mathbb{TP}^n$ being in one-to-one correspondence with the hyperplanes in $\mathbb{TP}^n$. We can use the same methods as we did in Example 16 to determine where n hyperplanes in $\mathbb{TP}^n$ intersect, or, equivalently, whether or not n points in $\mathbb{TP}^n$ are co-hyperplanar. Thus we have the following theorem, which is the tropical analogue of Cramer's Rule.

Theorem 35. *Let $p_i = (x_{1,i}, x_{2,i}, \ldots, x_{n+1,i})$, $i = 1 \ldots n+1$ be $n+1$ points in $\mathbb{TP}^n$. These $n+1$ points lie on the same tropical hyperplane, if and only if the matrix*

$$\begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{n+1,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{n+1,2} \\ \vdots & \vdots & & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n+1,n} \\ x_{1,n+1} & x_{2,n+1} & \cdots & x_{n+1,n+1} \end{pmatrix}$$

is tropically singular.

Proof. Suppose that the $n+1$ points lie on a tropical hyperplane. Then they all satisfy the same equation of the form $a_1x_1 \oplus a_2x_2 \oplus \cdots a_nx_n \oplus a_{n+1}x_{n+1}$, which means that the scalars $a_1, a_2, \ldots, a_n, a_{n+1}$ are the appropriate scalars to satisfy the linear dependence relation for the columns of the above matrix. Thus, the matrix is singular.

Now suppose the matrix is singular. Then the columns are linearly dependent, so there exist scalars $a_1, a_2, \ldots, a_n, a_{n+1}$ which satisfy the linear dependence relation of the columns of the above matrix. But then these scalars are the appropriate coefficients for the linear equation $a_1x_1 \oplus a_2x_2 \oplus \cdots a_nx_n \oplus a_{n+1}x_{n+1}$ for a hyperplane, which each point must then satisfy. Thus the $n+1$ points lie on the same hyperplane.

Corollary 36. *The hyperplane in $\mathbb{TP}^n$ which contains the points*

$p_i = (x_{1,i}, x_{2,i}, \ldots, x_{n+1,i})$, $i = 1 \ldots n$, is the corner locus of the $|A|_{\mathbb{T}}$ where

$$A = \begin{pmatrix} x_{1,1} & x_{2,1} & \cdots & x_{n+1,1} \\ x_{1,2} & x_{2,2} & \cdots & x_{n+1,2} \\ \vdots & \vdots & & \vdots \\ x_{1,n} & x_{2,n} & \cdots & x_{n+1,n} \\ x_1 & x_2 & \cdots & x_{n+1} \end{pmatrix},$$

where the last row is simply made up of the $n+1$ indeterminants of $\mathcal{Q}\langle x_1, x_2, \ldots, x_{n+1}\rangle$.

Proof. The set of all points $(x_1, x_2, \ldots, x_{n+1})$ that lie on a hyperplane with the given n points are those for which the matrix above is singular. Which is to say the set of points $(x_1, x_2, \ldots, x_{n+1})$ for which $|A|_{\mathbb{T}}$ vanishes tropically. But $|A|_{\mathbb{T}}$ is simply a linear polynomial in $n+1$ indeterminates, so the points that make it vanish are the same as the points in its corner locus.

In the classical setting, these arguments are easily extended to help us find curves of certain degrees passing through a given number of points in the plane. As we have seen, the line that passes through two points in the plane can be determined in this manner. Similarly, we know that if we are given five points in the plane, then we can find the conic that passes through those points. This is because classically the coefficients of a polynomial determine the polynomial and associated curve. There are a total of six terms of degree two in three homogenous polynomials, namely, $x^2, xy, y^2, xz, yz, z^2$. Thus, each homogeneous quadratic polynomial in three variables has associated to it a point in $\mathbb{P}^5$. But more than that, these polynomials are actually in one-to-one correspondence with the points of $\mathbb{P}^5$ since each polynomial and its associated curve is uniquely determined by its coefficients. Thus for each point in $\mathbb{P}^5$ we have a unique conic plane curve. Unfortunately, this is not the case in the tropical setting. Although it is true that each homogeneous quadratic tropical polynomial in three variables gives a point in $\mathbb{TP}^5$, and each point in $\mathbb{TP}^5$ gives a homogenous quadratic tropical polynomial, since a tropical conic curve in the plane does not necessarily have a unique polynomial associated with it because of functional equivalence, there is not a one-to-one correspondence of points in $\mathbb{TP}^5$ and tropical conic plane curves. But, even though the condition is not as strong, we can still us the fact that to every point in $\mathbb{TP}^5$ there is an associated conic, and for every quadratic polynomial there is a point in $\mathbb{TP}^5$ to find the conic that passes through five given points. This, of course, can be extended to higher degree polynomials and the number of points required to determine such curves. The only thing that we are missing due to functional equivalence is uniqueness. But, just as in the classical setting, since there are $\dfrac{n(n+3)}{2}+1$ terms of degree n in three homogeneous variables, the polynomial of degree n can be associated with a point of $\mathbb{TP}^N$ where $N=\dfrac{n(n+3)}{2}$. Thus, if we find the hyperplane in $\mathbb{TP}^N$ that passes through the appropriate N points, we will have a polynomial in three homogeneous variables whose corner locus contains the N original points. We thus have two additional corollaries to Theorem 35

Corollary 37. *$N+1=\dfrac{n(n+3)}{2}+1$ points $(x_i,y_i,z_i), i=1,\dots,n$ in $\mathbb{TP}^2$ lie on a tropical plane curve of degree n if and only if the matrix*

$$\begin{pmatrix} x_1^n & x_1^{n-1}y_1 & \cdots & x_1^{n-1}z_1 & x_1^{n-2}y_1z_1 & \cdots & y_1z_1^{n-1} & z_1^n \\ x_2^n & x_2^{n-1}y_2 & \cdots & x_2^{n-1}z_2 & x_2^{n-2}y_2z_2 & \cdots & y_2z_2^{n-1} & z_2^n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots \\ x_N^n & x_N^{n-1}y_N & \cdots & x_N^{n-1}z_N & x_N^{n-2}y_Nz_N & \cdots & y_Nz_N^{n-1} & z_{N+1}^n \\ x_{N+1}^n & x_{N+1}^{n-1}y_{N+1} & \cdots & x_{N+1}^{n-1}z_{N+1} & x_{N+1}^{n-2}y_{N+1}z_{N+1} & \cdots & y_{N+1}z_{N+1}^{n-1} & z_{N+1}^n \end{pmatrix}$$

is troically singular.

We note that the entries of a the jth row of the matrix are simply the different possible monic monomials of degree n in three variables evaluated at the point (x_j, y_j, z_j).

Corollary 38. *For any* $N = \dfrac{n(n+3)}{2}$ *points* $(x_i, y_i, z_i) \in \mathbb{TP}^2$*, let*

$$A = \begin{pmatrix} x_1^n & x_1^{n-1}y_1 & \cdots & x_1^{n-1}z_1 & x_1^{n-2}y_1z_1 & \cdots & x_1z_1^{n-1} & y_1z_1^{n-1} & z_1^n \\ x_2^n & x_2^{n-1}y_2 & \cdots & x_2^{n-1}z_2 & x_2^{n-2}y_2z_2 & \cdots & x_2z_2^{n-1} & y_2z_2^{n-1} & z_2^n \\ \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots \\ x_N^n & x_N^{n-1}y_N & \cdots & x_N^{n-1}z_N & x_N^{n-2}y_Nz_N & \cdots & x_Nz_N^{n-1} & y_Nz_N^{n-1} & z_{N+1}^n \\ x^n & x^{n-1}y & \cdots & x^{n-1}z & x^{n-2}yz & \cdots & xz^{n-1} & yz^{n-1} & z^n \end{pmatrix}.$$

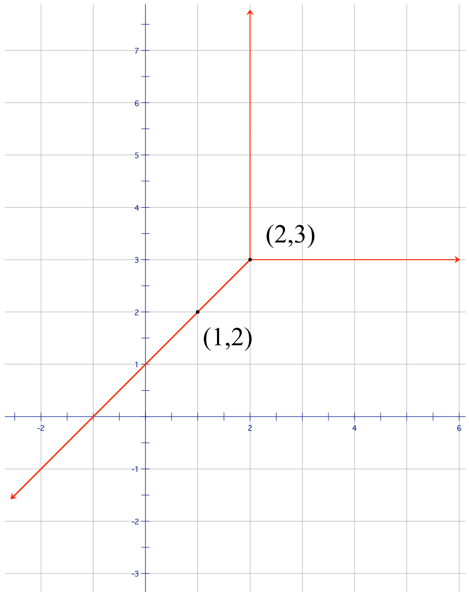
Then $|A|_{\mathbb{T}}$ *is a polynomial of degree* n *whose corner locus passes through our* N *points.*

Of course, as we mentioned before, this polynomial may not be the only polynomial whose corner locus passes through the given points. Often it will happen that the different polynomials whose corner loci pass through this point will all have the same corner locus. However, it also happens that sometimes there are different corner loci that pass through given points. Let us consider an example

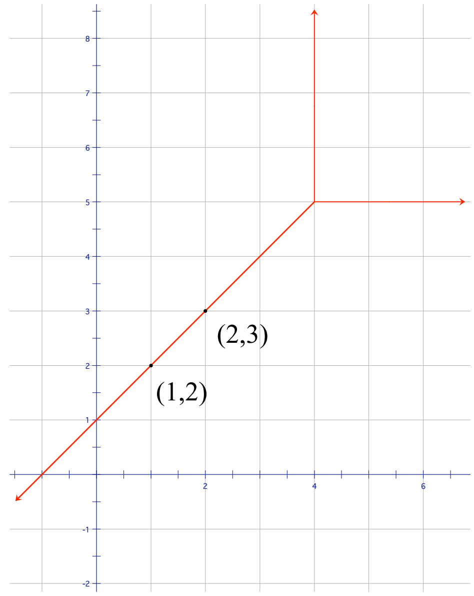
Example 17. Let $p_1 = (1, 2, 0)$ and $p_2 = (2, 3, 0)$. Then from Corollary 38 we get the line that passes through these two points by finding the determinate of the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ x & y & z \end{pmatrix}.$$

We get for this determinant the equation $2x \oplus 1y \oplus 4x$, and tne corner locus of this polynomial does in fact pass through the points as desired, as shown in Figure 18.



(a) $\mathcal{Z}(2x \oplus 1y \oplus 4z)$



(b) $\mathcal{Z}(2x \oplus 1y \oplus 6z)$

Figure 18: Lines Through $(1, 2)$ and $(2, 3)$

The polynomial $2x \oplus 1y \oplus 6z$ is also satisfied by the two points. In fact, any polynomial of the form $2x \oplus 1y \oplus \alpha z$ where $\alpha \geq 4$ has a corner locus which passes through these two points.

This happens because the two points are not in *general position* to each other. For lines this means that the two points lie on a classical line of slope $0, 1$ or ∞. When this happens, then there is not a unique line that passes through the two points, as we see in our example. But we can also see this from our matrix, if we don't want to determine the classical line on which our two points lie. If one of the 2×2 minors used to calculate the polynomial is singular, then our points are not in general position to each other. In our example, the determinant in its non-simplified form is $(2 \oplus 3)x \oplus (1 \oplus 2)y \oplus (4 \oplus 4)z$, which means that the minor

$$\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

is singular. If we think about when our matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 2 & 3 & 0 \\ x & y & z \end{pmatrix}$$

is singular, we then have either that two of the monomials are equal and minimal, which gave us the line $\mathcal{Z}(2x \oplus 1y \oplus 4z)$. But, it could also have happened the there was some point so that the $(4 \oplus 4)z$ term was smaller than the x and y terms, which would have also made the entire determinant vanish tropically. Thus, when one of the 2×2 minors is singular, the line that passes through the two points is not unique.

Example 18. Let us consider the five points $p_1 = (3, 5, 0)$, $p_2 = (7, 3, 0)$, $p_3 = (0, -2, 0)$, $p_4 = (-2, -5, 0)$, and $p_5 = (-3, -2, 0)$. We will use the matrix

$$A = \begin{pmatrix} 6 & 8 & 10 & 3 & 5 & 0 \\ 14 & 10 & 6 & 7 & 3 & 0 \\ 0 & -2 & -4 & 0 & -2 & 0 \\ -4 & -7 & -10 & -2 & -5 & 0 \\ -6 & -5 & -4 & -3 & -2 & 0 \\ x^2 & xy & y^2 & xz & yz & z^2 \end{pmatrix}$$

to calculate the quadratic polynomial that passes through these five points. We get as a result

$$|A|_{\mathbb{T}} = 3x^2 \oplus xy \oplus 3y^2 \oplus xz \oplus yz \oplus 3z^2.$$

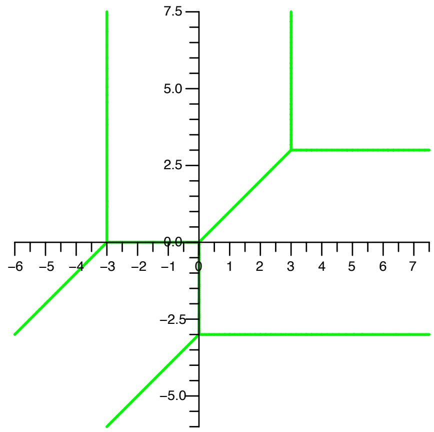


Figure 19: Conic Through Five Points

None of the minors are singular, and we do in fact have a unique tropical conic that passes through these five points, as shown in Figure 19.

Example 19. Let us consider one more example, in the case of a cubic. Let us start with the nine points $p_1 = (-2, 0, 0)$, $p_2 = (0, -3, 0)$, $p_3 = (3, 3, 0)$, $p_4 = (-3, 0, 0)$, $p_5 = (0, -2, 0)$, $p_6 = (1, 1, 0)$, $p_7 = (-2, -2, 0)$, $p_8 = (2, -1, 0)$, and

$p_9 = (-1, 2, 0)$. We now use the matrix

$$A = \begin{pmatrix} -6 & -4 & -2 & 0 & -4 & -2 & 0 & -2 & 0 & 0 \\ 0 & -3 & -6 & -9 & 0 & -3 & -6 & 0 & -3 & 0 \\ 9 & 9 & 9 & 9 & 6 & 6 & 6 & 3 & 3 & 0 \\ -9 & -6 & -3 & 0 & -6 & -3 & 0 & -3 & 0 & 0 \\ 0 & -2 & -4 & -6 & 0 & -2 & -4 & 0 & -2 & 0 \\ 3 & 3 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 0 \\ -6 & -6 & -6 & -6 & -4 & -4 & -4 & -2 & -2 & 0 \\ 6 & 3 & 0 & -3 & 4 & 1 & -2 & 2 & -1 & 0 \\ -3 & 0 & 3 & 6 & -2 & 1 & 4 & -1 & 2 & 0 \\ x^3 & x^2y & xy^2 & y^3 & x^2z & xyz & y^2z & xz^2 & yz^2 & z^3 \end{pmatrix}$$

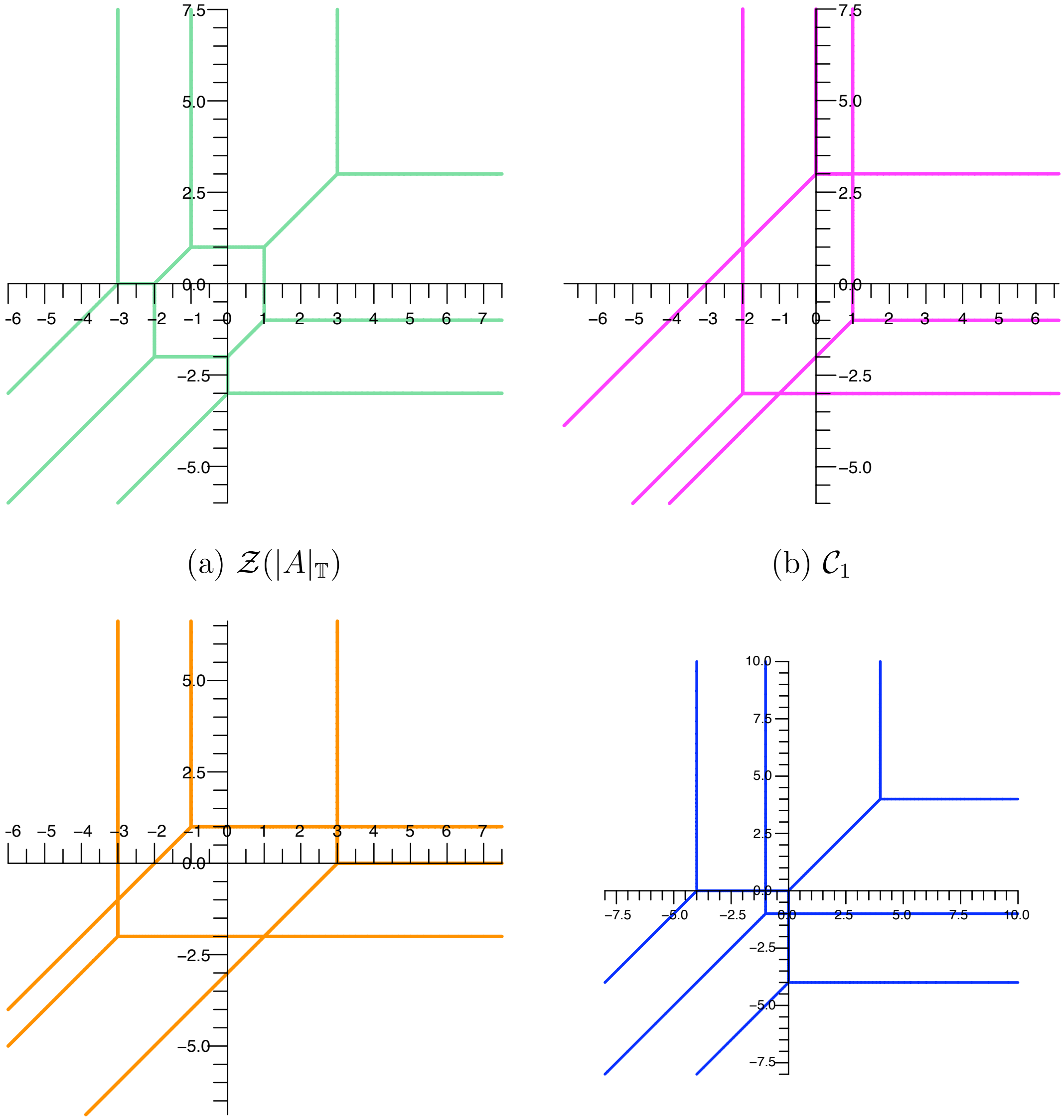
to find the cubic polynomial that passes through the given nine points. From this we get

$$|A|_{\mathbb{T}} = (x, y, z) = 5x^3 \oplus 2x^2y \oplus 2xy^2 \oplus 5y^3 \oplus 2x^2z \oplus xyz \oplus 2y^2z \oplus 1xz^2 \oplus 1yz^2 \oplus 4z^3.$$

In Figure 20 we see this cubic plane curve, passing through the nine points, as well as three other cubic curves that also pass through the nine points.

In turns out in this example that each of the 9×9 minors which give the coefficients for $|A|_{\mathbb{T}}$ are in fact singular. This implies that we can evaluate the final row of A at any point in the plane, and still have a singular matrix. Thus for each point in the plane, there is a cubic that passes through that point, and the nine given points.

These examples leads us to the following lemma about uniqueness of the polynomial and curve passing through a given number of points.



(a) $\mathcal{Z}(|A|_{\mathbb{T}})$

(b) $\mathcal{C}_1$

(c) $\mathcal{C}_2$

(d) $\mathcal{C}_3$

Figure 20: Cubics Through Nine Points

Lemma 39. *Let A be the matrix used in Corollary 38 to determine the polynomial of degree n through $N = \dfrac{n(n+3)}{2}$ points. If all of the minors calculated when expanding around the bottom row are non-singular, then the polynomial and associated curve through the N points is unique, call it $\mathcal{P}_{\mathcal{C}}$.*

Proof. Suppose that all of the minors are non-singular. Then no coefficient of $\mathcal{P}_{\mathcal{C}}$ vanishes tropically on its own. So, each point of the curve through our points must be attained by two distinct monomials. Suppose that there is another polynomial $\mathcal{B}$ whose corner locus contains the given points, but whose corner locus is not equal to that of $\mathcal{P}_{\mathcal{C}}$. Then, there is a point $p \in \mathcal{B}$ such that $p \notin \mathcal{P}_{\mathcal{C}}$. But, then the point p also makes satisfies the condition that if p is substituted into the last row of M, then the determinant of M vanishes tropically. But, if that is the case, then we know that $p \in \mathcal{P}_{\mathcal{C}}$, since $\mathcal{P}_{\mathcal{C}}$ is the set of all points that make that determinant vanish tropically. But, this is a contradiction, so there is no other $\mathcal{B}$, and the polynomial is unique. ∎

5 Complete Tropical Bézout's Theorem

Our goal in this chapter is to prove a complete version of a tropical Bézout's theorem, as stated below.

Theorem 51. *Let $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ be two tropical projective plane curves of degree d and e respectively. Then $\mathcal{Z}(f)$ stably intersects $\mathcal{Z}(g)$ in $d \cdot e$ points, counting multiplicity.*

We will show that the number of points of intersection of two curves in general position to each other is $d \cdot e$, and then we will generalize this to curves in any relative position to each other. We will also prove that the number of intersection points is correct for two curves where at least one of them is a curve of full support, and then extend that to general curves, using the above to help us count the multiplicity of the points of intersection at infinite distance.

To prove the tropical version of Bézout's Theorem, we will use a *tropical resultant*, which we will define and discuss it in the next section. We will then discuss how we define intersection multiplicity using the resultant for several cases. In Section 5.6 we will prove the complete Tropical Bézout's Theorem. Finally, we will discuss some other results relating to the resultant and how they relate to tropical Bézout's theorem.

5.1 Tropical Resultants

Definition 28. Let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous tropical polynomials of degree d and e respectively. To form the *tropical Sylvester matrix* of f and g with respect to z, we write f and g as polynomials in z with coefficients in $\mathcal{Q}\langle x, y\rangle$. Thus they can be written as $f(x, y, z) = f_0(x, y)z^d \oplus f_1(x, y)z^{d-1} \oplus \cdots \oplus f_{d-1}(x, y)z \oplus$

$f_d(x,y)$ and $g(x,y,z) = g_0(x,y)z^e \oplus g_1(x,y)z^{e-1} \oplus \cdots \oplus g_{e-1}(x,y)z \oplus g_e(x,y)$, where $f_i, g_j \in \mathcal{Q}\langle x \rangle$. It is possible that either of f_0 or g_0 is infinite. Let $f_{d'}(x,y)$ be the coefficient of the highest degree term of f which is not infinite, while thinking of f as a polynomial in z. Similarly let $g_{e'}(x,y)$ be the coefficient of the highest degree term of g, while thought of as a polynomial in z, that is not infinite. That is to say, $f(x,y,z) = f_{d'}z^{d-d'} \oplus f_{d'+1}z^{d-(d'+1)} \oplus \cdots \oplus f_{d-1}z \oplus f_d$. Let $m = d - d'$ and $n = e - e'$.

The *tropical Sylvester matrix* $\mathcal{M}_{f,g,z}$ of f and g with respect to z is an $(m+n) \times (n+m)$ matrix of the form

$$\mathcal{M}_{f,g,z} = \left(\begin{array}{ccccccccc}
f_{d'} & f_{d'+1} & \cdots & f_d & \infty & \infty & \infty & \cdots & \infty \\
\infty & f_{d'} & f_{d'+1} & \cdots & f_d & \infty & \infty & \cdots & \infty \\
\infty & \infty & f_{d'} & f_{d'+1} & \cdots & f_d & \infty & \cdots & \infty \\
 & \ddots & & \ddots & & & \ddots & & \\
 & & \ddots & & \ddots & & & \ddots & \\
\infty & \infty & \infty & \cdots & \infty & f_{d'} & f_{d'+1} & \cdots & f_d \\
\hline
g_{e'} & g_{e'+1} & \cdots & g_e & \infty & \infty & \infty & \cdots & \infty \\
\infty & g_{e'} & g_{e'+1} & \cdots & g_e & \infty & \infty & \cdots & \infty \\
\infty & \infty & g_{e'} & g_{e'+1} & \cdots & g_e & \infty & \cdots & \infty \\
 & \ddots & & \ddots & & & \ddots & & \\
 & & \ddots & & \ddots & & & \ddots & \\
\infty & \infty & \infty & \cdots & \infty & g_{e'} & g_{e'+1} & \cdots & g_e
\end{array}\right)$$

where the upper block of the matrix (B_1) has n rows and the lower block (B_2) has m rows.

Definition 29. Let $f(x,y,z), g(x,y,z)$ be homogeneous polynomials in $\mathcal{Q}\langle x,y,z \rangle$. The *tropical resultant* $\mathcal{R}_{f,g,z}$ *of f and g with respect to z* is defined to be the tropical determinant of the of $\mathcal{M}_{f,g,z}$.

We note that $\mathcal{R}_{f,g,x}$ and $\mathcal{R}_{f,g,y}$, the tropical resultants of f and g with respect to x and y, can be defined in a similar manner, where we for the tropical Sylvester matrix by thinking of f and g as polynomials in x and y respectively.

Lemma 40. *Let $f(x, y, z), g(x, y, z)$ be homogeneous polynomials of degree d and e respectively such that $(\infty, \infty, 0) \notin \mathcal{Z}(f) \cap \mathcal{Z}(g)$. Then $R_{f,g,z}$ is a homogeneous polynomial in x and y of degree $d \cdot e$*

Proof. We form the Sylvester matrix as described in Definition 28, noting that since we have assumed that the point $(\infty, \infty, 0)$ is not in the intersection of the polynomials, so f_0 and g_0 cannot both be infinite. Without loss of generality, let $f_0 \neq \infty$ and let $g_{e'}(x, y)$ be the coefficient of the highest degree term of g, while thought of as a polynomial in z, which is not infinite. Let $d = m$ and $e - e' = n$.

Thus, $\mathcal{M}_{f,g,z}$ is and $(m + n) \times (m + n)$ matrix of the form

$$\mathcal{M}_{f,g,z} = \left(\begin{array}{ccccccccc}
f_0 & f_1 & \cdots & f_d & \infty & \infty & \infty & \cdots & \infty \\
\infty & f_0 & f_1 & \cdots & f_d & \infty & \infty & \cdots & \infty \\
\infty & \infty & f_0 & f_1 & \cdots & f_d & \infty & \cdots & \infty \\
 & \ddots & & \ddots & & & \ddots & & \\
\infty & \infty & \infty & \cdots & \infty & f_0 & f_1 & \cdots & f_d \\
\hline
g_{e'} & g_{e'+1} & \cdots & g_e & \infty & \infty & \infty & \cdots & \infty \\
\infty & g_{e'} & g_{e'+1} & \cdots & g_e & \infty & \infty & \cdots & \infty \\
\infty & \infty & g_{e'} & g_{e'+1} & \cdots & g_e & \infty & \cdots & \infty \\
 & \ddots & & \ddots & & & \ddots & & \\
\infty & \infty & \infty & \cdots & \infty & g_{e'} & g_{e'+1} & \cdots & g_e
\end{array}\right)$$

where the upper block of the matrix (B_1) has n rows and the lower block (B_2) has m rows. In general, let $r_{i,j}$ represent the i,j^{th} entry. Since the entries from g in $\mathcal{M}_{f,g,z}$ start on the $(n+1)^{th}$ row, the terms g_e are on the diagonal of $\mathcal{M}_{f,g,z}$. Thus,

$$\bigodot_{i=1}^{m+n} r_{i,i} = f_0^n g_e^m$$

is the term in the resultant that corresponds to the identity permutation in S_{m+n}. Since $\deg f_0 = 0$ and $\deg g_e = e$, this term has degree $me = de$.

We now show that if $\bigodot_{i=1}^{m+n} r_{i,\sigma(i)} \neq \infty$, then multiplying a permutation σ by a transposition (ij) leaves the degree of the corresponding term in the resultant unchanged, if that term doesn't become ∞. Let $r_{\sigma^{-1}(i),i}$ be the entry in the i^{th} column in this term of $\mathcal{R}_{f,g,z}$ and let $r_{\sigma^{-1}(j),j}$ be that in the j^{th} column, $i < j$. If $\tau = (ij)\sigma$, then $r_{\tau^{-1}(j),j} = r_{\sigma^{-1}(i),j}$, so $\deg\left(r_{\tau^{-1}(j),j}\right) - \deg\left(r_{\sigma^{-1}(i),i}\right) = j - i$. Likewise, $\deg\left(r_{\tau^{-1}(i),i}\right) - \deg\left(r_{\sigma^{-1}(j),j}\right) = i - j$. Thus, the degree of $\bigodot_{i=1}^{m+n} r_{i,\tau(i)} = de + j - i + i - j = de$.

Thus, $\mathcal{R}_{f,g,z}$ is a homogenous polynomial in x and y of degree $d \cdot e$.

It has been shown independently by [8] and [11] that any polynomial in $\mathcal{Q}\langle x\rangle$ has a unique factorization into linear factors. Since $\mathcal{R}_{f,g,z}$ is the homogenization of such a polynomial, it has a unique factorization into linear factors of the form $ax \oplus by$.

We note that if $(0, \infty, \infty) \notin \mathcal{Z}(f) \cap \mathcal{Z}(g)$, then $\mathcal{R}_{f,g,x}$ is a homogeneous polynomial of degree $d \cdot e$ in y and z, and thus has a unique factorization into linear factors of the form $by \oplus cz$. Similarly, if $(\infty, 0, \infty) \notin \mathcal{Z}(f) \cap \mathcal{Z}(g)$ then $\mathcal{R}_{f,g,y}$ is a homogeneous polynomial of degree $d \cdot e$ in x and z, and it has a unique factorization in to linear factors of the form $ax \oplus cz$.

Lemma 41. *let $f(x,y,z)$ and $g(x,y,z)$ be homogeneous tropical polynomials. Then $\mathcal{R}_{f,g,z}(p_{i,0},p_{i,1})$ vanishes tropically for every $p_i=(p_{i,0},p_{i,1},p_{i,2})\in\mathcal{Z}(f)\cap\mathcal{Z}(g)$.*

Proof. Let $p_i=(p_{i,0},p_{i,1},p_{i,2})\in\mathcal{Z}(f)\cap\mathcal{Z}(g)$. If $\mathcal{M}_{f,g,z}$ is evaluated at $(x,y)=(p_{i,0},p_{i,1})$ then every row attains its minimum twice. In other words, if $\mathbf{c}_1,\mathbf{c}_2,\ldots,\mathbf{c}_{n+m}$ are the column vectors of $\mathcal{M}_{f,g,z}(p_i)$, then $\mathbf{c}_1\oplus\mathbf{c}_2\oplus\cdots\oplus\mathbf{c}_{n+m}$ vanishes tropically. By Theorem 33 we get that $\mathcal{R}_{f,g,z}(p_i)$ vanishes tropically as well. So the point $(p_{i,0},p_{i,1},p_{i,2})$ is contained in $Z(a_ix\oplus b_iy)$ for some factor $a_ix\oplus b_iy$ of $\mathcal{R}_{f,g,z}$. This shows there exists some constant $\alpha\in\mathbb{Q}$ such that

$$a_ix\oplus b_iy=\alpha(p_{i,1}x\oplus p_{i,0}y). \tag{2}$$

Lemma 42. *Let f and g be homogeneous polynomials in $\mathcal{Q}\langle x,y\rangle$. Then f and g have a common root if and only if $\mathcal{R}_{f,g,y}(p)$ vanishes tropically for all $p\in\mathbb{Q}$.*

Proof. Suppose f and g have a common root. Using the Fundamental Theorem of Tropical Algebra, [8], f and g have $\deg f=m$ and $\deg g=n$ roots, respectively. Let $a_1,a_2,\ldots,a_r$ be the roots of f that are not roots of g and let $b_1,b_2,\ldots,b_s$ be the roots of g that aren't roots of f. Then $r<m$ and $s<n$ since f and g have a root in common. Also, let f_m and g_n be the leading coefficients of f and g respectively.

Define $\psi=g_n(x\oplus b_1)(x\oplus b_2)\cdots(x\oplus b_s)$ and $\varphi=f_m(x\oplus a_1)(x\oplus a_2)\cdots(x\oplus a_r)$. Then $\psi f=\varphi g$, where we note that their least coefficient forms are formally equal as well. Write ψ and φ in least coefficient forms as

$$\mu_0\oplus\mu_1x\oplus\cdots\oplus\mu_sx^s\quad\text{and}\quad\lambda_0\oplus\lambda_1x\oplus\cdots\oplus\lambda_rx^r.$$

Also, define $\mu_j=\infty$ for $j>s$ and $\lambda_j=\infty$ for $j>r$. One may form the Sylvester matrix of f and g in the same way as above. Let $\mathbf{r_i}$ be the i^{th} row of $\mathcal{M}_{f,g,y}$. Then

$$\mu_0 x^0 \mathbf{r}_1 \oplus \mu_1 x^1 \mathbf{r}_2 \oplus \cdots \oplus \mu_{n-1} x^{n-1} \mathbf{r}_n = \lambda_0 x^0 \mathbf{r}_{n+1} \oplus \lambda_1 x^1 \mathbf{r}_{n+2} \oplus \cdots \oplus \lambda_{m-1} x^{m-1} \mathbf{r}_{n+m},$$

because each entry s_i in the sum is the term of degree $i-1$ in ψf on the left hand side and φh on the right hand side. So for any $x \in \mathbb{Q}$,

$$\mu_0 x^0 \mathbf{r}_1 \oplus \mu_1 x^1 \mathbf{r}_2 \oplus \cdots \oplus \mu_{n-1} x^{n-1} \mathbf{r}_n \oplus \lambda_0 x^0 \mathbf{r}_{n+1} \oplus \lambda_1 x^1 \mathbf{r}_{n+2} \oplus \cdots \oplus \lambda_{m-1} x^{m-1} \mathbf{r}_{n+m} \quad (3)$$

vanishes tropically. Thus, by Theorem 33, $\mathcal{R}_{f,g,y}$ always vanishes tropically.

Conversely, by Theorem 33, if $\mathcal{R}_{f,g,z}(p)$ vanishes tropically for all $p \in \mathbb{Q}$, then there exist $\mu_0, \ldots, \mu_{n-1}$ and $\lambda_0, \ldots, \lambda_{m-1} \in \mathcal{Q}\langle x \rangle$ such that (3) is satisfied. In order for this to happen, we see that the ratio $\dfrac{\mu_0}{\lambda_0}$ must be in $\mathcal{Q}$, since $f_0, g_0 \in \mathcal{Q}$. Thus, we might as well have $\mu_0, \lambda_0 \in \mathcal{Q}$. In fact, in order for each entry of the sum to have the minimum attained twice for every x, each term in the sum must have the same degree of x. And, as the degree increases by one in each entry of the first row, the degree of the μ_i and λ_j must increase in a corresponding way. We thus note that the highest degree a μ_i may have is $n-1$ and the highest degree a λ_j may have is $m-1$ Defining ψ and φ by $\psi = \mu_0 \oplus \cdots \oplus \mu_{n-1}$ and $\varphi = \lambda_0 \oplus \cdots \oplus \lambda_{m-1}$, this says that $\psi f \oplus \varphi g$ vanishes tropically for all x, because their corresponding coefficients match up. But since this is true, they are functionally equivalent and must have the same roots. But $\deg \psi < n = \deg g$ and $\deg \varphi < m = \deg f$. So it must be that f and g share at least one root. ✩

Lemma 43. *If $(x_0, y_0) \in \mathcal{Z}(\mathcal{R}_{f,g,z})$, and neither $x_0 \neq \infty$ nor $y_0 \neq \infty$, then there is some $q \in \mathbb{Q}$ such that $(x, y, q) \in \mathcal{Z}(F) \cap \mathcal{Z}(G)$.*

Proof. Note that, since x and y are not ∞, we have $g_{e'}(x, y) \neq \infty$. Of course, we have assumed already that f_0 is a constant in $\mathbb{Q}$. So

$$f' := f_0(x_0, y_0)z^m \oplus f_1(x_0, y_0)z^{m-1}w \oplus \cdots \oplus f_d(x_0, y_0)w^{m-d} \in \mathcal{Q}\langle z, w\rangle$$

has degree m and

$$g' := g_{e'}(x_0, y_0)z^n \oplus g_{e'+1}(x_0, y_0)z^{n-1}w \oplus \cdots \oplus g_e(x_0, y_0)w^{n-e} \in \mathcal{Q}\langle z, w\rangle$$

has degree n. By hypothesis, $(x_0, y_0) \in \mathcal{Z}(\mathcal{R}_{f,g,z})$. This means that $\mathcal{R}_{f,g,z}(x_0, y_0)$ vanishes tropically. By a similar argument as given above, $\mathcal{R}_{f',g',z}$ is a homogeneous polynomial, but it is only in one variable. Thus, $\mathcal{R}_{f',g',z}$ is only one term, and $\mathcal{R}_{f,g,z}(x_0, y_0)$ is its coefficient. Thus $\mathcal{R}_{f',g',z}$ vanishes tropically for all z. By Lemma 42, f' and g' have a common root $q \in \mathcal{Q}$.

We see from Lemma 43 that the resultants tells us lines in $\mathbb{TP}^2$ that contain points of intersection. For example, since $\mathcal{R}_{f,g,z}$ is a homogenous polynomial in x and y only, then its corner locus, $\mathcal{Z}(\mathcal{R}_{f,g,z})$ is a collection of classical lines of slope one in $\mathbb{TP}^2$, which all intersection in the $(\infty, \infty, 0)$ corner of the projective plane model. For each factor of $\mathcal{R}_{f,g,z}$ there is only line in $\mathcal{Z}(\mathcal{R}_{f,g,z})$, which contains one point of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$. If a factor is repeated (has multiplicity greater than one) then the corresponding line has weight greater than one in $\mathcal{Z}(\mathcal{R}_{f,g,z})$. This means that there are multiple points of intersection on this line. These points of intersection could all be the same point, yielding a point of intersection multiplicity greater than one, or there could be several different points on the line, some of intersection multiplicity one and others with greater intersection multiplicity. But, counting multiplicity, there are the same number of points of intersection on that line as the multiplicity of the factor of $\mathcal{R}_{f,g,z}$. Similarly, $\mathcal{R}_{f,g,x}$ tells us the classical lines of infinite slope that the points of intersection lie on, and $\mathcal{R}_{f,g,y}$ tell us the classical lines of slope zero that the intersections lie on.

5.2 Transverse Points of Intersection

We will now give an algorithm for finding the transverse points of intersection using all $\mathcal{R}_{f,g,x}$, $\mathcal{R}_{f,g,y}$, and $\mathcal{R}_{f,g,z}$.

Algorithm 1. Let $f(x, y, z)$ and $g(x, y, z)$ be to homogenous polynomials of degree d and e respectively, such that $\mathcal{Z}(f)$ is in general position with respect to $\mathcal{Z}(g)$, so that all of their intersection points are transverse.

First we will find all three resultants, $\mathcal{R}_{f,g,x}$, $\mathcal{R}_{f,g,y}$, and $\mathcal{R}_{f,g,z}$, and factor them into linear factors. These factors tell us which lines the points on intersection lie on. If P is a point of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ then it will satisfy all three resultants, and will therefore be a point where all three resultants intersect. We find all points of intersection of all three resultants on a given line. This is easily done using classical techniques, since they are all just classical lines.

Starting with any factor of multiplicity one of any of the resultants, we look at the points of intersection of all three resultants that lie on its corresponding line. Since this line has weight one in the resultant, there is only one point of intersection of all three curves on this line that is also a point of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$, so all but one of the points will be disregarded. Doing this with all the factors of multiplicity one, we will find all such points of intersection multiplicity one. Once we have found all of these points of intersection multiplicity one, we can reduce our resultants by removing the corresponding factor of multiplicity one from each resultant for each line on which we found a point of intersection multiplicity one.

We now consider the reduced resultants. If any of them now have factors with multiplicity one, then we proceed as above to find the points of intersection multiplicity one that correspond to them. If they do not, then we look at the factors of multiplicity two. In order for a point of intersection to have intersection multiplicity two then it must be a point where all three resultants intersect, each from a factor of multiplicity two or greater. So, we start with a factor of multiplicity two in our reduced resultant and find all the points where the remaining reduced resultants both intersect that line. If there is only one point of intersection that is also on both of the original curves, then this point of intersection has intersection multiplicity 2. If there are two points of intersection on this line that are on both curves, then they must each have intersection multiplicity one. In either case we can again reduce our resultants by removing factors with the appropriate multiplicity for each of the points.

We now repeat this step for any factors of the again reduced resultants which have multiplicity one or two. Once we have found all such points of intersection multiplicity one or two, we reduce our resultants and then move on to the factors of multiplicity three. Again, we find all the points of intersection of each of the resultants, and determine whether or not those points are on the original curves. If there is one such point it has intersection multiplicity three. If there are three such points then they have intersection multiplicity one each. If there are two such points then one has intersection multiplicity one and the other has intersection multiplicity two. Pick one of the two points and look at the two lines in the other directions that contain it. These lines must both have at least one more point of intersection, since both of these lines have weight three. If either of these lines has two more points of intersection, then all the points on that line must have intersection multiplicity one, so the original point in consideration does as well. If both of the lines just have one more point of intersection, then we look at the lines that pass through points and determine if those points are guaranteed to have intersection multiplicity one. If not, then we look at the lines through the additional points. Since there are only a finite number of points of intersection, and each line has a finite weight, this process will eventually terminate, allowing us to determine which point has intersection multiplicity one, and which has multiplicity two.

We then reduce our resultants again, and proceed as above starting with the lowest multiplicity of a factor and reducing as we determine points of intersection. Again, as there are only a finite number of points of intersection, this will terminate and we will have found all the points of intersection with multiplicity.

Lemma 44. *Let $f(x, y, z)$ and $g(x, y, z)$ be homogeneous polynomials of degree d and e respectively. The transverse points of intersection of $\mathcal{Z}(f) \cap \mathcal{Z}(g)$ as described in Section 3.4 are precisely those accounted for by the resultants.*

Proof. By Lemma 41 $P = (p_0, p_1, p_2) \in \mathbb{TP}^2$ is in $\mathcal{Z}(f) \cap \mathcal{Z}(g)$m then all three resultants vanish tropically at P. We can use Lemma 43 and Algorithm 1 to use the resultants to find the points of intersection.

Example 20. Let $f(x, y, z) = ax \oplus by \oplus cz$ and $g(x, y, z) = rx \oplus sy \oplus tz$. The Sylvester matrices for these two polynomials are

$$\mathcal{M}_{f,g,x} = \begin{pmatrix} a & by \oplus cz \\ r & sy \oplus tz \end{pmatrix} \quad \mathcal{M}_{f,g,y} = \begin{pmatrix} b & ax \oplus cz \\ s & rx \oplus tz \end{pmatrix} \quad \mathcal{M}_{f,g,z} = \begin{pmatrix} c & ax \oplus by \\ t & rx \oplus sy \end{pmatrix},$$

from which we get

$$\mathcal{R}_{f,g,x} = (as \oplus br)y \oplus (at \oplus cr)z$$

$$\mathcal{R}_{f,g,y} = (br \oplus as)x \oplus (bt \oplus cs)z$$

$$\mathcal{R}_{f,g,z} = (cr \oplus at)x \oplus (cs \oplus bt)y.$$

Thus the point of intersection is a point that satisfies the three lines from the resultants. From $\mathcal{R}_{f,g,x}$ we know that the point must have the form $(x, at \oplus cr, as \oplus br)$, from $\mathcal{R}_{f,g,y}$ we know if must be of the form $(bt \oplus cs, y, br \oplus as)$, and from $\mathcal{R}_{f,g,z}$ we see it must have the form $(cs \oplus bt, cr \oplus at, z)$. The point $(bt \oplus cs, at \oplus cr, as \oplus br)$ satisfies all of these conditions and is the point of intersection.

Alternatively, if we were to use Cramer's Rule instead of the resultant, we would find the determinant of the matrix

$$A = \begin{pmatrix} a & b & c \\ r & s & t \\ x & y & z \end{pmatrix}$$

is $(bt \oplus cs)x \oplus (at \oplus cr)y \oplus (as \oplus br)z$, and so the point of intersection is $(bt \oplus cs, at \oplus cr, as \oplus br)$. So we see that they agree.

Example 21. Let $f(x, y, z) = x \oplus 2y \oplus 4z$ and $g(x, y, z) = 6x^2 \oplus 2xy \oplus 6y^2 \oplus 1xz \oplus 1yz \oplus 4z^2$. To compute $\mathcal{R}_{f,g,x}$, we will find the determinant of the matrix

$$\mathcal{M}_{f,g,x} = \begin{pmatrix} 0 & 2y \oplus 4z & \infty \\ \infty & 0 & 2y \oplus 4z \\ 6 & 2y \oplus 1z & 6y^2 \oplus 1yz \oplus 4z^2 \end{pmatrix},$$

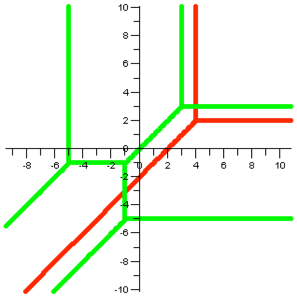
and we see that $\mathcal{R}_{f,g,x} = 4y^2 \oplus 1yz \oplus 4z^2 = 4(y \oplus -3z)(y \oplus 3z)$. Thus, $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ intersect somewhere on the classical line $y = -3$ and somewhere in the line $y = 3$. Since both of these roots have multiplicity one, we know that the points of intersection are also both of intersection multiplicity one. But, we can consider the other resultants as well to verify our result. We get $\mathcal{R}_{f,g,y} = 4x^2 \oplus 3xz \oplus 7z^2 = 4(x \oplus -1z)(x \oplus 4z)$, so we have points of intersection on the classical lines $x = -1$ and $x = 4$. Finally, $\mathcal{R}_{f,g,z} = 4x^2 \oplus 5xy \oplus 7y^2 = 4(x \oplus 1y)(x \oplus 2y)$, so the points of intersection are on the classical lines $y = x-1$ and $y = x-2$. In Figure 21 we see the two curves graphed together so we can see their points of intersection, the dual graph of the curve $\mathcal{Z}(fg)$, the two curves graphed with each of the resultants individually, all three resultants, and the two curves with all three resultants. We notice that the points of intersection are where factors of all three resultants intersect each other.

Normally, when the curves have transverse intersection, it is very straightforward to find and count the intersections with the appropriate multiplicity. When the points of intersection are not transverse, we will use the same algorithm, but it will be slightly more involved.

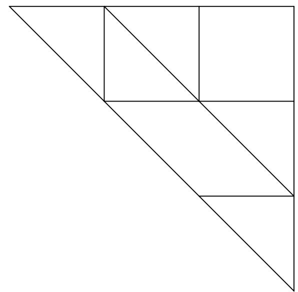
5.3 Stable Points of Intersection

Lemma 45. *Let $f(x, y, z)$ and $g(x, y, z)$ be two homogeneous polynomials of degree d and e respectively. The stable intersection points of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$, as defined in Definition 20, are the same as the points of intersection found by the resultants.*

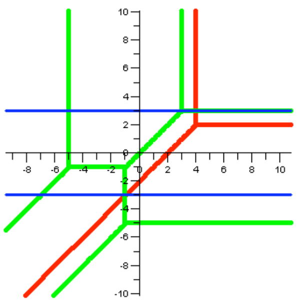
Proof. Assume that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ are not in general position to each other, and suppose that to put them in general position to each other we shift $f(x, y, z)$ to $f(\alpha x, \beta y, z) = \tilde{f}(x, y, z)$. Now by Lemma 40 $\mathcal{R}_{\tilde{f},g,z}$ is a homogeneous polynomial of degree $d \cdot e$ in x and y, but its coefficients now are polynomials in α and β. More specifically, the coefficient of the $x^i y^{d \cdot e - i}$ term in $\mathcal{R}_{f.g.z}$ is some sum of tropical rational numbers, $c_1 \oplus c_2 \oplus \cdots \oplus c_k$, and the coefficient of $x^i y^{d \cdot e - i}$ in $\mathcal{R}_{f,g,z}$ is $\alpha^{r_1} \beta^{s_1} c_1 \oplus \alpha^{r_2} \beta^{s_2} c_2 \oplus \cdots \oplus \alpha^{r_k} \beta^{s_k} c_k$, for r_i, s_j non-negative integers. Now, it is not possible to determine in general what power of α and β will be in each term, as it does depend on the original polynomials, but the c_i will be the same for both resultants. But $\mathcal{R}_{\tilde{f},g,z}$ gives the points of intersection of the translation of $\mathcal{Z}(f)$ and the original



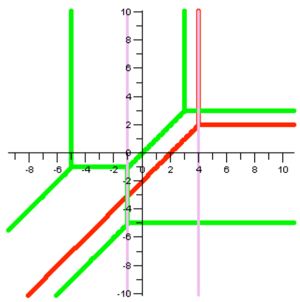
(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g)$



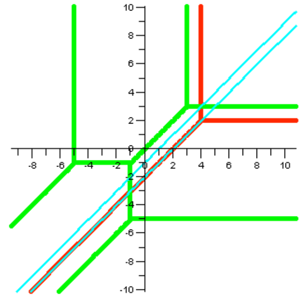
(b) Δ_{fg}



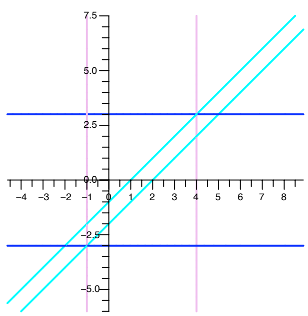
(c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$



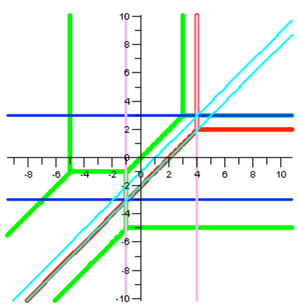
(d) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$



(e) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$



(f) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$



(g) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

Figure 21: Transverse Intersections

curve $\mathcal{Z}(g)$. So, if we now take the limit of this resultant as $\alpha, \beta \to 0$ then we will have the stable intersections, since both the translations and the resultants will vary continuously. But, if we consider $\mathcal{R}_{\tilde{f},g,z}$, and look at the limit of it as $\alpha, \beta \to 0$, then we see that we get $\mathcal{R}_{f,g,z}$ the resultant of the two original curves, because when α and β approach 0, we have our original coefficient. Explicitly, $\lim_{\alpha,\beta \to 0} \alpha^{r_1}\beta^{s_1}c_1 \oplus \alpha^{r_2}\beta^{s_2}c_2 \oplus \cdots \oplus \alpha^{r_k}\beta^{s_k}c_k = c_1 \oplus c_2 \oplus \cdots \oplus c_k$. Thus the resultant gives us the stable intersection as defined in Definition 20.

Example 22. Let $f(x, y, z) = ax \oplus by \oplus cz$ and $g(x, y, z) = rx \oplus by \oplus cz$. This means that the two curves are not in general position to each other because f has its vertex at $(c - a, c - b, 0)$ and g has its vertex at $(c - r, c - b, 0)$, so they overlap on a horizontal ray. To find the stable point of intersection, we first translate one of the curves so that they no longer overlap. Since they share a horizontal ray, we can simply shift one of the two curves, say g, up or down, using the transformation $(x, y, z) \mapsto (x, \beta y, z)$. Now our polynomials become $f(x, y, z) = ax \oplus by \oplus cz$ and $g_\beta(x, y, z) = ra \oplus b\beta y \oplus cz$. For most $\beta \neq 0$ these two curves are now in general position, so we may assume we picked such a β. Now we get

$$\mathcal{M}_{f,g_\beta,x} = \begin{pmatrix} a & by \oplus cz \\ r & b\beta y \oplus cz \end{pmatrix} \quad \mathcal{M}_{f,g_\beta,y} = \begin{pmatrix} b & ax \oplus cz \\ b\beta & rx \oplus cz \end{pmatrix} \quad \mathcal{M}_{f,g_\beta,z} = \begin{pmatrix} c & ax \oplus by \\ c & rx \oplus b\beta y \end{pmatrix},$$

from which we get

$$\begin{aligned}
\mathcal{R}_{f,g_\beta,x} &= (ab\beta \oplus br)y \oplus (ac \oplus cr)z \\
&= b(a\beta \oplus r)y \oplus c(a \oplus r)z \\
\mathcal{R}_{f,g_\beta,y} &= (br \oplus ab\beta)x \oplus (bc \oplus cb\beta)z \\
&= b(r \oplus a\beta)x \oplus bc(0 \oplus \beta)z \\
\mathcal{R}_{f,g_\beta,z} &= (cr \oplus ac)x \oplus (cb\beta \oplus bc)y \\
&= c(r \oplus a)x \oplus bc(\beta \oplus 0).
\end{aligned}$$

We note that here we are in the same situation as we were above, since our curves are in general position, and we get the desired point of intersection, which is at $(bc\beta \oplus bc, ac \oplus cr, ab\beta \oplus br)$. But we really want to know what happens as we allow β to approach 0. Clearly, when $\beta = 0$ we have our original curves, which are not in general position to each other. But, as β approaches 0, we have a continuous translation of g, so the point of intersection moves continuously. Also we can find the limits of the coefficients of the resultants as β approaches 0. When we let β approach 0 in our resultants we get

$$\begin{aligned}
\lim_{\beta \to 0} \mathcal{R}_{f,g_\beta,x} &= (ab + br)y \oplus (ac \oplus cr)z \\
\lim_{\beta \to 0} \mathcal{R}_{f,g_\beta,y} &= (br \oplus ab)x \oplus (bc \oplus cb)z \\
\lim_{\beta \to 0} \mathcal{R}_{f,g_\beta,z} &= (cr \oplus ac)x \oplus (cb \oplus bc)y.
\end{aligned}$$

But these are the same resultants we would find if we computed them without shifting one of the curves.

Now, the limit of our point of intersection as β approaches 0 is $(bc \oplus bc, ac \oplus$

$cr, \oplus ab \oplus br)$, which is also the point of intersection we would get if we used the limited resultants or the resultants where we didn't shift one of the curves.

Now, if we look at what happens when we use Cramer's rule with the shifted polynomials, and then let β approach 0, we have the same result, since we would get $(bc\beta \oplus bc, ac \oplus cr, ab\beta \oplus br)$ as our point of intersection.

But let's examine this point a little more.

$$
\begin{aligned}
(bc \oplus bc, ac \oplus cr, ab \oplus br) &= (bc, c(a \oplus r), b(a \oplus r)) \\
&= (bc \oslash b(a \oplus r), c(a \oplus r) \oslash b(a \oplus r), 0) \\
&= (c \oslash (a \oplus r), c \oslash b, 0) \\
&= (c - \min\{a, r\}, c - b, 0)
\end{aligned}
$$

But $(c - \min\{a, r\}, c - b, 0)$ is the vertex of f or g, depending on which of them has a smaller coefficient of x. So, the point of intersection that the resultant gives us is the same as the point of intersection as the limit of the resultant, and is the vertex where the two curves begin to share an infinite ray, which is the stable intersection of the curves as defined earlier.

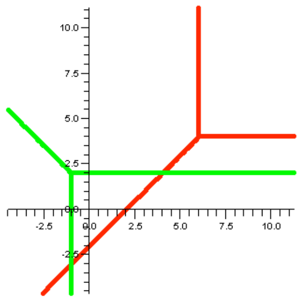
Example 23. Let $f(x, y, z) = x \oplus 2y \oplus 6z$ and $g(x, y, z) = 1xy \oplus yz \oplus 2z^2$. This time we will start with $\mathcal{R}_{f,g,z}$, since $g(x, y, z)$ has the z^2 term, which means that the corner $(\infty, \infty, 0)$ could not possibly have any intersection. We get $\mathcal{R}_{f,g,z} = 2x^2 \oplus 4xy \oplus 6y^2 = 2(x \oplus 2y)^2$. We notice that $\mathcal{R}_{f,g,z}$ has a root of multiplicity two, which means that either there are two distinct points of intersection with multiplicity one on the line $y = x - 2$, or there is one point of intersection on that line with intersection multiplicity two. We compute $\mathcal{R}_{f,g,x}$ and $\mathcal{R}_{f,g,y}$ to determine which case we have. $\mathcal{R}_{f,g,x} = 3y^2 \oplus yz \oplus 2z^2 = 3(y \oplus -3z)(y \oplus 2z)$ and $\mathcal{R}_{f,g,y} =$

$1x^2 \oplus xz \oplus 4z^2 = 1(x \oplus -1z)(x \oplus 4z)$. Neither of these have multiple roots, so we see that there must be two points of intersection on the line $y = x - 2$, which we see in Figure 22.

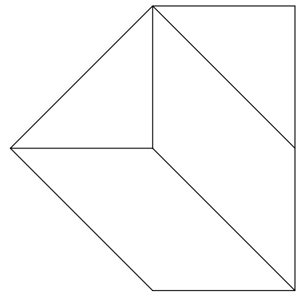
Example 24. Let $f(x, y, z) = xy^2 \oplus y^2z \oplus xyz \oplus yz^2 \oplus 3x^2y \oplus 3x^2z \oplus 3xz^2$ and $g(x, y, z) = 3x^3 \oplus -3x^2y \oplus -6xyz \oplus -3xz^2 \oplus -3y^3 \oplus -6y^2z \oplus z^3$. For our three resultants we get

$$\begin{aligned}
\mathcal{R}_{f,g,x} &= -3y^9 \oplus -6y^8z \oplus -6y^7z^2 \oplus -6y^6z^3 \oplus -6y^5z^4 \\
&\quad \oplus -6y^4z^5 \oplus -6y^3z^6 \oplus -3y^2z^7 \oplus yz^8 \oplus 6z^9 \\
&= -3(y \oplus -3z)(y \oplus z)^5(y \oplus 3z)^2(y \oplus 6z) \\
\mathcal{R}_{f,g,y} &= 3x^9 \oplus -3x^8z \oplus -6x^7z^2 \oplus -9x^6z^3 \oplus -9x^4z^5 \\
&\quad \oplus -9x^3z^6 \oplus -9x^2z^7 \oplus -9xz^8 \oplus -6z^9 \\
&= (x \oplus -6z)(x \oplus -3z)^2(x \oplus z)^5(x \oplus 3z) \\
\mathcal{R}_{f,g,z} &= 9x^9 \oplus 3x^8y \oplus x^7y^2 \oplus -3x^6y^3 \oplus -6x^5y^4 \oplus -9x^4y^5 \\
&\quad \oplus -12x^3y^6 \oplus -12x^2y^7 \oplus -12xy^8 \oplus -9y^9 \\
&= (x \oplus -6y)(x \oplus -3y)^5(x \oplus y)^2(x \oplus 3y).
\end{aligned}$$

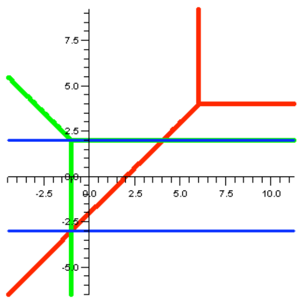
In Figure 23 we see all the graphs as before, but with the added graph of the resultants together where the thickness of the lines is determined by the multiplicity of the root. We notice that each of the resultants has a root of multiplicity 5, which means that it is possible for there to be one point of intersections multiplicity 5. However, this can only happen if all three of the lines associated with these roots intersect in one point, which we see from the graph that they do not. Thus, we know that there is no point of intersection which has intersection multiplicity five. This means that each of the points of intersection can have either multiplicity one



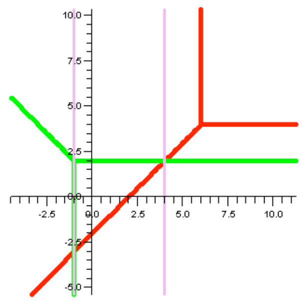
(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g)$



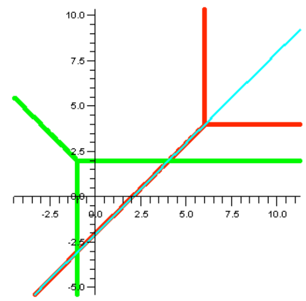
(b) Δ_{fg}



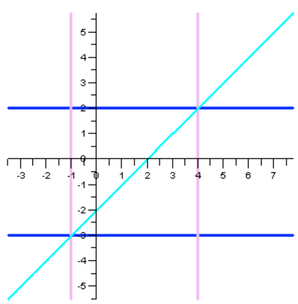
(c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$



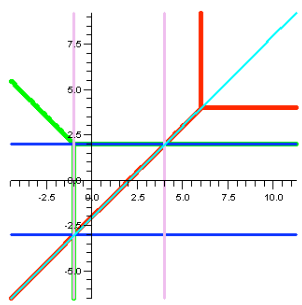
(d) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$



(e) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$



(f) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$



(g) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

Figure 22: Counting Intersection Multiplicity for Transverse Intersections

or two, but nothing else. We look again at the intersections of the three resultants, and see that there are more points of intersection of the three resultants than there should be for the two curves, which means that some of the points of intersection of the resultants are not points of intersection of the two curves. This is normal, since the resultants are giving you entire lines on which the points of intersection lie. So, we pick a point of intersection of all three resultants and determine whether or not it is on both of the original curves. For example, we see in the graph that the point $(-6, -3, 0)$ is on all three resultant lines (we can also see that this point satisfies all three resultants by considering the equations), but it is not on the curve $\mathcal{Z}(g)$ (the green curve). Thus, this is not a point of intersection of the two curves. But there must be a point of intersection multiplicity one on the line $x = -6$, and it must be where all three resultants intersect, which means it must be at the point $(-6, 0, 0)$. But we also know that there is only one point of intersection on the line $y = x + 6$, and this point is, so the other two points on that line are not points of intersection. And there must be one point of intersection on the line $y = 6$, and we have ruled out the point $(0, 6, 0)$, so this point must be $(3, 6, 0)$. But now we know that there is only one point of intersection on the line $x = 3$, so the other two points where all three resultants intersection on that line are not points of intersection. But,there must be one point of intersection on the line $y = x - 3$, so it must be at the point $(0, -3, 0)$. And the line $y = -3$ can only have one, so that point it is, and the other point on the line is also not a point of intersection. So, we now how three points of intersection multiplicity one each, and the resultants all intersect each other in only three more places, and each of those intersections involves lines of multiplicity two or five. But remember that no points have intersection multiplicity five, so they must have intersection multiplicity two for all of the multiplicities to work out as needed. Thus the points $(0,0,0)$, $(0,3,0)$, and $(-3,0,0)$ are all points of intersection multiplicity two. Thus, we have a total of nine points of intersection, when we count with multiplicity.

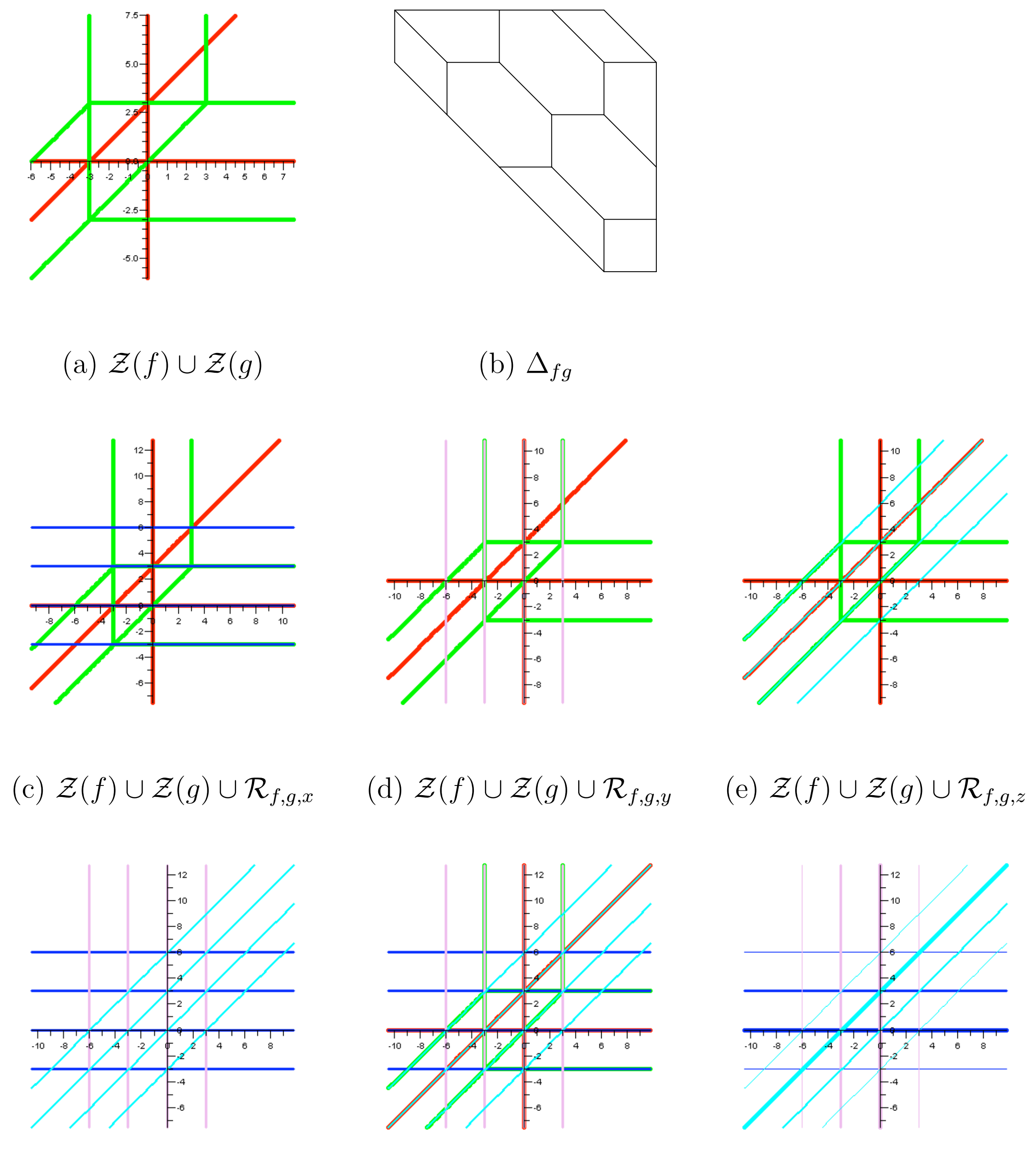
5.4 Resultant and Dual Graph Intersection Multiplicities

Recall that the intersection multiplicity for affine intersections was defined in Section 3.4 using the dual graph as well. We say in that section that the intersection multiplicity for transverse intersection was the same as that given by the area in the dual graph. We will show in this section that the intersection multiplicity given by the resultants is also the same as that given by the area in the dual graph.

Lemma 46. *Let $f(x,y,z)$ and $g(x,y,z)$ be homogenous polynomials of degree d and e respectively such that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ are in general position to each other. If the product $fg(x,y,z)$ has a non-contributing term then all three resultants have similar non-contributing terms.*

Proof. Let $f(x,y,z) = f_0z^d \oplus f_1z^{d-1} \oplus \cdots \oplus f_{d-1}z \oplus f_d$ and let $g(x,y,z) = g_0z^e \oplus g_1z^{e-1} \oplus \cdots \oplus g_{e-1}z \oplus g_e$. Then $fg(x,y,z) = (f_0g_0)z^{d+e} \oplus (f_0g_1 \oplus f_1g_0)z^{d\oplus e-1} \oplus (f_0g_2 \oplus f_1g_1 \oplus f_2g_0)z^{d+e-2} \oplus \cdots \oplus (f_{d_1}g_e \oplus f_dg_{e-1})z \oplus (f_dg_e)$. If $fg(x,y,z)$ has a non-contributing term which does not correspond to a non-contributing term of either f or g, then the corresponding vertex of the lattice is not a vertex in the dual graph Δ_{fg}. This vertex is in the interior of Δ_{fg}, because any non-contributing terms on the boundary of Δ_{fg} correspond to non-contributing terms of either f or g.

The term that is non-contributing is non-contributing among the terms of the same degree, when we think of fg as above. That means that $f_0g_k \oplus f_1g_{k-1} \oplus \cdots \oplus f_{k-1}g_1 \oplus f_kg_0$ has a non-contributing term. But $\mathcal{R}_{f,g,z}$ is a polynomial of the form



(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g)$

(b) Δ_{fg}

(c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$

(d) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$

(e) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$

(f) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

(g) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

(h) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$ with multiplicity

Figure 23: Counting Intersection Multiplicity for Stable Intersections

$\bigoplus f_{\sigma(0)}f_{\sigma(1)}\cdots f_{\sigma(e-1)}g_{\sigma(e)}\cdots g_{\sigma(e+d-1)}$, so it has terms of the form $(f_0g_k \oplus f_1g_{k-1} \oplus \cdots \oplus f_{k-1}g_1 \oplus f_kg_0)\tilde{f}\tilde{g}$ where $\tilde{f}$ is some product of f_i's and $\tilde{g}$ is some product of g_j's. But this means that the same conditions that create a non-contributing term in fg exist in $\mathcal{R}_{f,g,z}$, so it must also have a non-contributing term. Rewriting f and g as polynomial with respect to x and y, the product fg will still have a non-contributing term, so this same argument applies to $\mathcal{R}_{f,g,x}$ and $\mathcal{R}_{f,g,z}$, so all three resultants have non-contributing terms.

If f or g has a non-contributing term, then fg must also have a non-contributing term. This non-contributing term in non-contributing when compared with certain terms in the product. But, as shown above, these terms all show up in the resultants as well, so the term must also be non-contributing all three resultants. ✿

Lemma 47. *The intersection multiplicity found by the resultants is the same the intersection multiplicity as determined by the dual graph as defined in Definition 19.*

Proof. We recall from the discussion in Section 3.4 that if the dual graph of the product has a parallelogram corresponding to a point of intersection which has area greater than one, then there is a non-contributing term of fg. But then by Also, by Lemma 46, the resultants all must have non-contributing terms. But, by Lemma 14 we know that if a homogeneous polynomial in two variables has a non-contributing term, then it has a factor of multiplicity greater than one. So all three resultants have factors of intersection multiplicity greater than one, which must all intersect each other in one place, because their non-contributing terms all correspond to each other. Thus the point of intersection has intersection multiplicity greater than one. We note that for each increase in area of the dual graph, and addition term becomes non-contributing, thus it also does in the resultants, so the multiplicities agree. ✿

In all of the examples that we have given we have included the dual graphs. We see in all of the cases where the intersections were transverse that the area of the parallelogram of the dual graph was in fact equal to the intersection multiplicity found using the resultants.

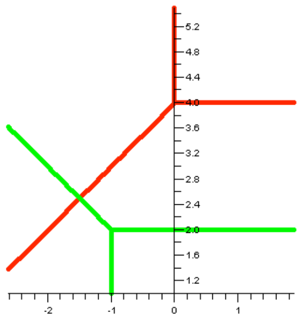
Example 25. For this example, we take the same curves as in Example 23, only we shift the line to the left so that the points of intersection move. So, we now have $f(x, y, z) = 6x \oplus 2y \oplus 6z$, but we sill have $g(x, y, z) = 1xy \oplus yz \oplus 2z^2$. We begin again with the resultant with respect to z and get $\mathcal{R}_{f,g,z} = 14x^2 \oplus 10xy + 6y^2 = 14(x \oplus -4y)^2$. We again look at the other two resultants and get $\mathcal{R}_{f,g,x} = 3y^2 \oplus 6yz \oplus 8z^2 \sim 3y^2 \oplus \frac{11}{2}yz \oplus 8z^2 = 3(y \oplus \frac{5}{2}z)^2$ and $\mathcal{R}_{f,g,y} = 7x^2 \oplus 6xz \oplus 4z^2 \sim 7x^2 \oplus \frac{11}{2}xz \oplus 4z^2 = 7(x \oplus \frac{3}{2}z)^2$. So all three of the resultants have double roots, and we notice in Figure 24 that the three resultants intersect in just one place, as do the curves. Thus, we do have one point of intersection, with multiplicity two.

As noted in Algorithm 1 that if the resultants have factors of multiplicity greater than one, that does not mean that there must be a point of intersection of intersection multiplicity greater than one, it is only a possibility. Example 23 shows such a situation.

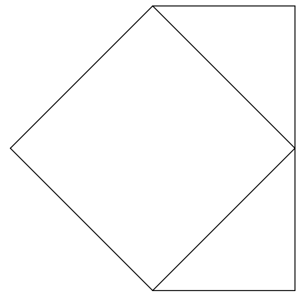
Example 26. Let us look at Example 25 again. We recall that $f(x, y, z) = 6x \oplus 2y \oplus 6z$ and $g(x, y, z) = 1xy \oplus yz \oplus 2z^2$, as shown in Figure 24. The product

$$\begin{aligned} fg(x, y, z) &= 6x^2y \oplus 6xyz \oplus 8xz^2 \oplus 2xy^2 \oplus 2y^2z \oplus 4yz^2 \oplus 7xyz \oplus 6yz^2 \oplus 8z^3 \\ &= 6x^2y \oplus 2xy^2 \oplus 6xyz \oplus 2y^2z \oplus 8xz^2 \oplus 4yz^2 \oplus 8z^3 \\ &\sim 6x^2y \oplus 2xy^2 \oplus 6xyz \oplus 2y^2z \oplus 8xz^2 \oplus 4yz^2 \oplus 8z^3. \end{aligned}$$

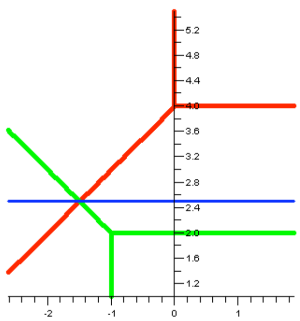
We notice in the last line that we dropped the $6xyz$ term. This is because



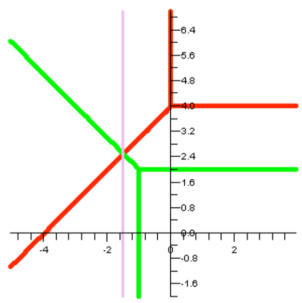
(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g)$



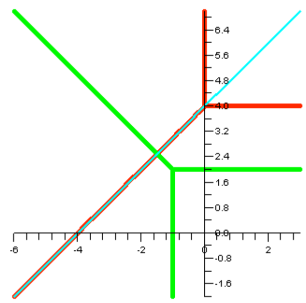
(b) Δ_{fg}



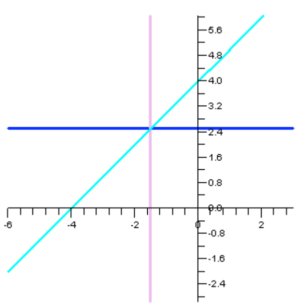
(c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$



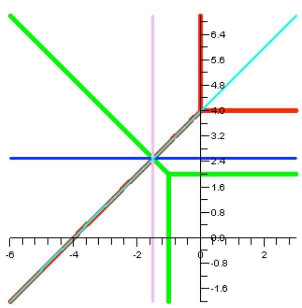
(d) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$



(e) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$



(f) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$



(g) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x} \cup$
$\mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

Figure 24: Transverse Intersection of Intersection Multiplicity Two

$6xzy$ never attains the minimum by itself, so it is not contributing. This is the reason that the corresponding vertex does not show up in the dual graph. Thus, the parallelogram that corresponds to the intersection of the two curves has area two.

Now let's consider the resultants again. We have

$$\begin{aligned}
\mathcal{R}_{f,g,z} &= 14x^2 \oplus 10xy + 6y^2 \\
&\sim 14x^2 \oplus 6y^2 \\
\mathcal{R}_{f,g,x} &= 3y^2 \oplus 6yz \oplus 8z^2 \\
&\sim 3y^2 \oplus 8z^2 \\
\mathcal{R}_{f,g,y} &= 7x^2 \oplus 6xz \oplus 4z^2 \\
&\sim 7x^2 \oplus 4z^2
\end{aligned}$$

since none of the mixed terms are ever contributing. But the fact that they are non-contributing means that the resultants all have double roots. Because when we find the resultant we are finding products of terms of a given degree of the two original polynomials, the behavior of the product of the two polynomials manifests itself in this way. Thus, whenever the resultants all have a root of multiplicity greater than one at the same point, it corresponds to a certain term not contributing, which term has a counterpart in the product of the two functions which also does not contribute.

5.5 Points of Intersection at Infinite Distance

Again recall that $\mathcal{R}_{f,g,z}$ gives us lines of classical slope one in the plane that tell us where our points of intersection are. These lines of slope one in the plane all pass through the vertex $(\infty, \infty, 0)$, as noted in Table 2 and intersecting the edge $z = \infty$ of our model of $\mathbb{TP}^2$. This is one of the reasons that we required that the point $(\infty, \infty, 0)$ not be in the intersection. $\mathcal{R}_{f,g,z}$ does not $(\infty, \infty, 0)$ as a point of intersection because all of lines that have a point of intersection pass through this point. However, the other resultants do recognize $(\infty, \infty, 0)$ as a point of intersection, because only one line of slope zero and only one line of infinite slope pass through $(\infty, \infty, 0)$. Similarly, $\mathcal{R}_{f,g,x}$ gives lines through the point $(0, \infty, \infty)$ and intersecting the edge $x = \infty$ on which there are points of intersection, but fails to recognize the point $(0, \infty, \infty)$ and $\mathcal{R}_{f,g,y}$ gives lines through the point $(\infty, 0, \infty)$ and intersecting the edge $y = \infty$, while ignoring the point $(\infty, 0, \infty)$. Because this the case, we will need to use at least two resultants to appropriate determine the intersection multiplicity at ∞ if the two curves intersect in all three corners.

Lemma 48. *Let $f(x, y, z)$ and $g(x, y, z)$ be two homogeneous polynomials, of degree d and e respectively, such that $\mathcal{Z}(f) \cap \mathcal{Z}(g)$ contains the point $(\infty, \infty, 0)$, but does not $(0, \infty, \infty)$ and $(\infty, 0, \infty)$. Then $\deg(\mathcal{R}_{f,g,z}) < d \cdot e$, and both $R_{f,g,x}$ and $\mathcal{R}_{f,g,y}$ have infinite roots of multiplicity $d \cdot e - \deg(\mathcal{R}_{f,g,z})$.*

Proof. To satisfy the given conditions, f cannot have the term z^d and g can not have the term z^e, but at least one of f and g must have the pure in x term and the pure in y term. But, since f_0 and g_0 are both ∞ there are not e rows in the lower block of $\mathcal{M}_{f,g,z}$, which means that each term of the resultant does not have degree $d \cdot e$. However, since the curves do not both pass through the points $(0, \infty, \infty)$ and $(\infty, 0, \infty)$, then $\mathcal{R}_{f,g,x}$ and $\mathcal{R}_{f,g,y}$ are still both of degree $d \cdot e$, so there are still $d \cdot e$ points of intersection. However, $\mathcal{R}_{f,g,x}$ will not have a term of the form $z^{d \cdot e}$, because there is no z^d term in f and no z^e term in g to contribute to this term in the resultant. It will have, on the other hand the $y^{d \cdot e}$ term, because those terms are present. Thus we will be able to factor y^k out of $R_{f,g,x}$ for some k. But, $y^k = (y \oplus \infty z)^k$, so we have k factors that vanish tropically only when $y = \infty$. Similarly, $\mathcal{R}_{f,g,y}$ will not have the $z^{d \cdot e}$ term, but it will have the $x^{d \cdot e}$ term, so we can factor x^k out and have $(x \oplus \infty z)^k$, giving k factors that vanish tropically only when $x = \infty$. This k must be the same in both cases, since $\mathcal{R}_{f,g,x}$ and $\mathcal{R}_{f,g,y}$ must have the same number of finite points of intersection. But $R_{f,g,z}$ also sees those same finite points of intersection, so $\mathcal{R}_{f,g,z}$ must have degree $d \cdot e - k$.

Definition 30. Let $f(x, y, z)$ and $g(x, y, z)$ be two homogeneous polynomials, of degree d and e respectively, such that $\mathcal{Z}(f) \cap \mathcal{Z}(g)$ contains the point $(\infty, \infty, 0)$, but does not $(0, \infty, \infty)$ and $(\infty, 0, \infty)$. We define the *infinite intersection multiplicity* to be the number $k = d \cdot e - \deg(R_{f,g,z})$ of Lemma 48.

Definition 31. Let $f(x, y, z)$ and $g(x, y, z)$ be two homogeneous polynomials, of degree d and e respectively, such that $\mathcal{Z}(f) \cap \mathcal{Z}(g)$ contains the point $(\infty, \infty, 0)$, but does not $(0, \infty, \infty)$ and $(\infty, 0, \infty)$. Let $\mathcal{Z}(\tilde{f})$ and $\mathcal{Z}(\tilde{g})$ be the curves of full support the agree with f and g respectively in some bounded region, as explained in Corollary 23. The limit of the points of intersection of $\mathcal{Z}(\tilde{f})$ and $\mathcal{Z}(\tilde{g})$ as the coefficients of the additional terms increase without bound are called the *infinite stable intersections.*

Lemma 49. *The points of stable infinite intersection as defined Definition 31 are the same as the infinite points of intersection found by the resultants in Lemma 48 as defined in Definition 30.*

Proof. By Corollary 23, we know we can find $\tilde{f}$ and $\tilde{g}$ so that the bounded region contains all of the finite points of intersection of the $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ and so that the new points of intersection are transverse. Now when we find the resultants, all three of them are of degree $d \cdot e$, and their factors all give points of intersection at finite distance. We note that the original points of intersection at finite distance have not changed, since they were preserved when we found our new curves. Now, $\tilde{f} = f \oplus h_1$ and $\tilde{g} = g \oplus h_2$, where h_1 and h_2 only contain terms that were not present in f and g, but which make $\tilde{f}$ and $\tilde{g}$ into full support polynomials. We note that we have not replaced any of the finite coefficients of f or g in $\tilde{f}$ and $\tilde{g}$. The coefficients of the terms in h_1 and h_2 can be increased without disturbing the original points of intersections of f and g, and without changing the combinatorial type of the curves $\tilde{f}$ and $\tilde{g}$. Let us start with the the z^d term $\tilde{f}$ and the z^e term of $\tilde{g}$. Since the coefficients of these two terms are finite, we have $d \cdot e$ points of intersection at finite distance by Lemma 40. As we allow these two coefficients to increase, we still have those mn intersections at finite distance, but the x and y coordinates of at least one of the points of intersection are getting larger. When we allow these two coefficients to become infinite, then we no longer have $d \cdot e$ intersections at finite distance, because this point now has infinite x and y coordinates. However, the total degree of $\mathcal{R}_{\tilde{f},\tilde{g},z}$ has not changed. But, some of the finite factors have become infinite factors, accounting for the one point of intersection that no longer has finite coordinates. As we allow all of the additional coefficients of $\tilde{f}$ and $\tilde{g}$ to become infinite, we see that all k of the points that we picked up with the full support curves, have in fact moved back to the corner $(\infty, \infty, 0)$, and all of $d \cdot e$ points of intersection are accounted for, and the intersection multiplicity matches that given in Lemma 48.

We note that in the argument given in Lemma 49, we deformed both curves to

Resultant	Factor in Resultant	Corner of Intersection
$\mathcal{R}_{f,g,z}$	x^k	(∞, y, ∞)
	y^k	(x, ∞, ∞)
$\mathcal{R}_{f,g,y}$	x^k	(∞, ∞, z)
	z^k	(x, ∞, ∞)
$\mathcal{R}_{f,g,x}$	y^k	(∞, ∞, z)
	z^k	(∞, y, ∞)

Table 3: Points of Intersection at Infinite Distance

be full support, but it is only necessary to deform one of the two curves to move the points of intersection to finite points.

If the two curves intersect in one of the other corners, then the same arguments as given in Lemma 48 and Lemma 49 can be used to determine the intersection multiplicity at those corners. The results of these arguments are shown in Table 3

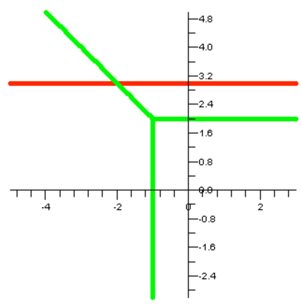
Example 27. Let $f(x, y, z) = 2y \oplus 5z$ and $g(x, y, z) = 1xy \oplus yz \oplus 2z^2$, as shown in Figure 25. We notice first that neither of these equations have a term that is pure in x, which is to say of the form αx^n for some non-negative integer n. So, the three resultants for f and g are

$$\begin{aligned}\mathcal{R}_{f,g,x} &= 2y \oplus 5z\\ &= 2(y \oplus 3z)\end{aligned}$$

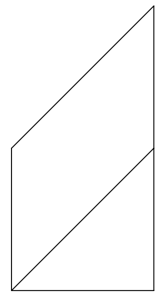
$$
\begin{aligned}
\mathcal{R}_{f,g,y} &= 4z^2 \oplus 6xz \\
&= 4z(z \oplus 2x) \\
&= 4(z \oplus \infty x)(z \oplus 2x) \\
\mathcal{R}_{f,g,z} &= 6y^2 \oplus 11xy \\
&= 6y(y \oplus 5x) \\
&= 6(y \oplus \infty x)(y \oplus 5x).
\end{aligned}
$$

From $\mathcal{R}_{f,g,x}$ we see that f and g only have on intersection at finite distance. However, from $\mathcal{R}_{f,g,y}$ we see that there is a point of intersection when $z = \infty$ and from $\mathcal{R}_{f,g,z}$ we see there is a point of intersection when $y = \infty$. If we look at the $f(x, \infty, z)$ we see that we have $2 \odot \infty \oplus 5z$, and the only way that the two monomials can attain the minimum together is if $z = \infty$ as well. Similarly, $g(x, \infty, z) = 1x \odot \infty \oplus \infty \odot z \oplus 2z^2$, so again for two monomials to attain the minimum together, $z = \infty$. If we consider instead $f(x, y, \infty)2y \oplus 5 \odot \infty$ and $g(x, y, \infty) = 1xy \oplus y \odot \infty \oplus 2 \odot \infty^2$, we see that the common point of the two curves is when $y = \infty$ as well. So, we have a point of intersection at $(0, \infty, \infty)$. We note that the resultant with respect to x does not display the infinite intersection with a factor, but rather with the lack there of.

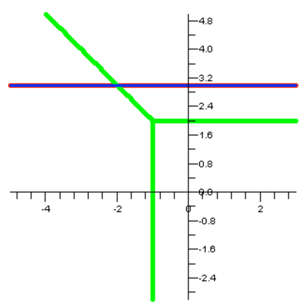
Suppose we deform the curve f, as explained in Lemma 22 and Corollary 23. Then we get $\tilde{f}(x, y, z) = \alpha x \oplus 2y \oplus 5z$, where $\alpha > 7$, as shown in Figure 26. With this choice of α, the finite point of intersection of f and g is not changed, but now



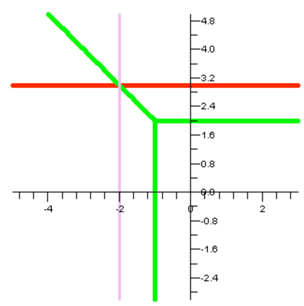
(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g)$



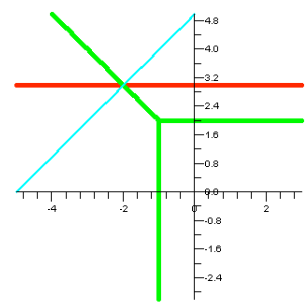
(b) Δ_{fg}



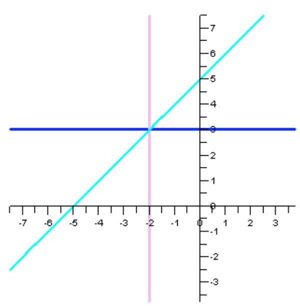
(c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$



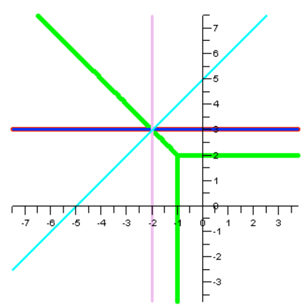
(d) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$



(e) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$



(f) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$



(g) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

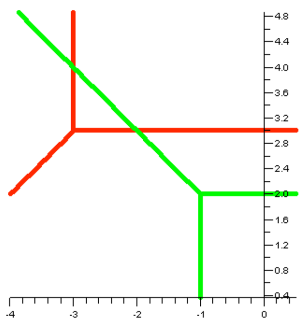
Figure 25: Intersections at Finite and Infinite Distance

$\tilde{f}$ has its pure in x term, so our resultants will change to

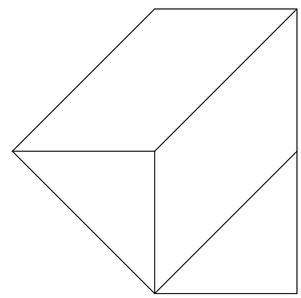
$$\begin{aligned}
\mathcal{R}_{\tilde{f},g,x} &= 3y^2 \oplus (\alpha \oplus 6)yz \oplus 2\alpha z^2 \\
&= 3y^2 \oplus 6yz \oplus 2\alpha z^2 \\
&= 3(y \oplus 3z)(y \oplus (2\alpha \oslash 6)z) \\
\mathcal{R}_{\tilde{f},g,y} &= 4z^2 \oplus (\alpha \oplus 6)xz \oplus 1\alpha x^2 \\
&= 4z^2 \oplus 6xz \oplus 1\alpha x^2 \\
&= 4(z \oplus 2x)(z \oplus (1\alpha \oslash 6)x) \\
\mathcal{R}_{\tilde{f},g,z} &= 6y^2 \oplus (11 \oplus 4\alpha \oplus 5\alpha)xy \oplus 2\alpha^2 x^2 \\
&= 6y^2 \oplus 11xy \oplus 2\alpha^2 x^2 \\
&= 6(y \oplus 5x)(y \oplus (2\alpha^2 \oslash 11)x).
\end{aligned}$$

From $\mathcal{R}_{\tilde{f},g,x}$ we see that our points of intersection must be of the form $(x, 3, 0)$ and $(x, \alpha - 4, 0)$. From $\mathcal{R}_{\tilde{f},g,y}$ we get the points $(-2, y, 0)$ and $(0, y, \alpha - 50$. And from $\mathcal{R}_{\tilde{f},g,z}$ we get $(0, 5, z)$ and $(0, 2\alpha - 9, z)$. From this we see that our points of intersection must be $(-2, 3, 0)$ and $(0, 2\alpha - 9, \alpha - 5)$. Now we will let α approach ∞ so that our curve returns to it's normal self, and we see that all of the resultants lose a pure term, and so they all lose one of the finite roots (the new finite root), and those roots are replaced by infinite when the limit has been completed. And our points of intersection become $(-2, 3, 0)$ and $(0\infty, \infty)$, so we had the correct points of intersection above.

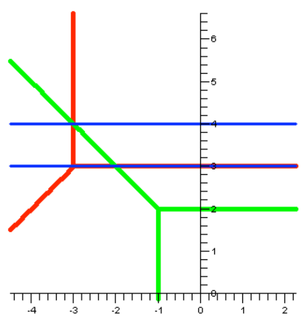
Lemma 50. *Suppose that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ are topical projective plane curves, of degree d and e respectively, that intersect in each of the corners $(\infty, \infty, 0)$, $(\infty, 0, \infty)$, and $(0, \infty, \infty)$. There there are $d \cdot e$ points of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ in $\mathbb{TP}^2$ counting multiplicity.*



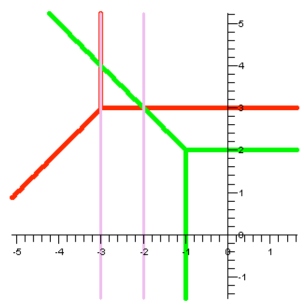
(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g)$



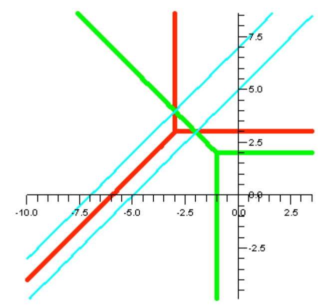
(b) Δ_{fg}



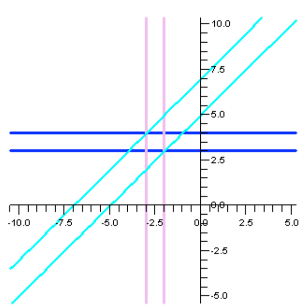
(c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$



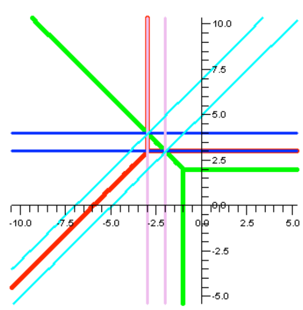
(d) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$



(e) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$



(f) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$



(g) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

Figure 26: Infinite Intersection Deformed to Finite Distance

Proof. Since the curves intersect in all three corners, then none of the resultants will have degree $d \cdot e$. All three resultants will, however, have the same number, r, of finite points of intersection. As shown in Lemma 48, $\mathcal{R}_{f,g,x}$ has an infinite factor of the form $(y \oplus \infty z)^k$ if and only if $\mathcal{R}_{f,g,y}$ has an infinite factor of the form $(x \oplus \infty z)^k$. Similarly, $\mathcal{R}_{f,g,x}$ has an infinite factor of the form $(\infty y \oplus z)^m$ if and only if $\mathcal{R}_{f,g,z}$ has an infinite factor of the form $(x \oplus \infty y)^m$. And, $\mathcal{R}_{f,g,y}$ has an infinite factor of the form $(\infty x \oplus y)^n$ if and only if $\mathcal{R}_{f,g,z}$ has an infinite factor of the form $(\infty x \oplus y)^n$. We can use the same method as explained in Lemma 49, considering one corner at a time. We thus see that there are k points of intersection at $(\infty, \infty, 0)$, m points of intersection at $(\infty, 0, \infty)$ and n points of intersection at $(0, \infty, \infty)$, along with the r points of intersection at finite distance. Thus we have $k + m + n + r$ points of intersection counting multiplicity. But, before we considered the limit, we knew we had $d \cdot e$ points of intersection, and the limiting process did not create any new points of intersection, so $d \cdot e = k + m + n + r$, and we have the desired number of points of intersection.

Example 28. In this example we will look at two curves which have intersections at all three corners. Let $f(x, y, z) = 3xz \oplus 2yz \oplus xy$ and $g(x, y, z) = 1xz^2 \oplus 1yz^2 \oplus$

$xyz \oplus 2y^2z \oplus 3xy^2$, as shown in Figure 27. The resultants are given by

$$\begin{aligned}
\mathcal{R}_{f,g,x} &= 2y^3z \oplus 1y^2z^2 \oplus 3yz^3 \\
&= yz(2y^2 \oplus 1yz \oplus 3z^2) \\
&= 2(y \oplus \infty z)(\infty y \oplus z)(y \oplus -1z)(y \oplus 2z) \\
\mathcal{R}_{f,g,y} &= 1x^3z^2 \oplus 3x^2z^3 \oplus 5xz^4 \\
&= xz^2(1x^2 \oplus 3xz \oplus 5z^2) \\
&= 1(x \oplus \infty z)(\infty x \oplus z)^2(x \oplus 2z)^2 \\
\mathcal{R}_{f,g,z} &= 1x^3y^2 \oplus 1x^2y^3 \oplus 4xy^4 \\
&= xy^2(1x^2 \oplus 1xy \oplus 4y^2) \\
&= 1(x \oplus \infty y)(\infty x \oplus y)^2(x \oplus y)(x \oplus 3y).
\end{aligned}$$

From our resultants we see that we have two finite points of intersection and some infinite points of intersection. Starting with $\mathcal{R}_{f,g,x}$, we see that there is an infinite point of intersection when $y = \infty$ and when $z = \infty$. If we look at $f(x, \infty, z) = 3xz \oplus \infty \oplus \infty$ and $g(x, \infty, z) = 1xz^2 \oplus \infty \oplus \infty \oplus \infty \oplus \infty$ we see that we need either $x = \infty$ or $z = \infty$. But we remember the $\mathcal{R}_{f,g,x}$ gives us the lines of slope zero through the point $(0, \infty, \infty)$, so it doesn't see the intersections at that point. Which means in this case that $x = \infty$, and we have an infinite point of intersection at $(\infty, \infty, 0)$. Now if we consider $f(x, z, \infty) = \infty \oplus \infty \oplus xy$ and $g(x, y, \infty) = \infty \oplus \infty \oplus \infty \oplus \infty \oplus 3xy^2$, we see that $x = \infty$ or $y = \infty$. But again, since $\mathcal{R}_{f,g,x}$ doesn't recognize the point $(0, \infty, \infty)$, we see that $x = \infty$, and we have an infinite intersection at the point $(\infty, 0, \infty)$. The point $(\infty, \infty, 0)$ is also seen in $\mathcal{R}_{f,g,y}$ from the factor $(x \oplus \infty z)$. And the point $(\infty, 0\infty)$ can be seen in $\mathcal{R}_{f,g,z}$ from the factor $(x \oplus \infty y)$. This leaves the factors $(\infty x \oplus y)^2$ from $\mathcal{R}_{f,g,z}$ and $(\infty x \oplus z)^2$ from $\mathcal{R}_{f,g,y}$, which both yield a infinite point of intersection multiplicity two at $(0, \infty, \infty)$.

To verify this let $\tilde{f} = \alpha x^2 \oplus xy \oplus \beta y^2 \oplus 3xz \oplus 2yz \oplus \gamma z^2$ where $\alpha > 4$, $\beta > 5$, and $\gamma > 5$, as shown in Figure 28. Then $\tilde{f}$ is a full support curve, so all of the intersections will be finite. We then find these intersections, and then consider the points as α, β, and γ approach ∞ individually to see that the infinite intersections appear as we have calculated them.

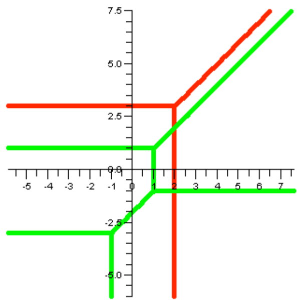
5.6 Complete Tropical Bézout's Theorem

Theorem 51. *Let $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ be two tropical projective plane curves of degree d and e respectively. Then $\mathcal{Z}(f)$ stably intersects $\mathcal{Z}(g)$ in $d \cdot e$ points, counting multiplicity.*

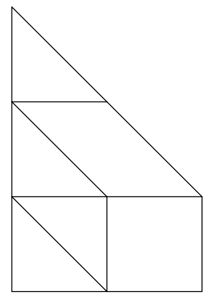
Proof. First suppose that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ do not intersect in any of the corners of $\mathbb{TP}^2$, and that all of the intersections are transverse. Then, by Lemma 41, the resultants vanish tropically at every point of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$. But by Lemma 40, the resultants are all homogeneous polynomials in two variables of degree $d \cdot e$. Thus, they have $d \cdot e$ roots counting multiplicity. Thus, there are $d \cdot e$ points of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$.

Now suppose that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ do not intersect in any of the corners of $\mathbb{TP}^2$, and but that some of the intersections are not transverse. By Lemma 40, the resultants still have degree $d \cdot e$, and by Lemma 45 the resultants give the stable points of intersection, so there are $d \cdot e$ stable points of intersection.

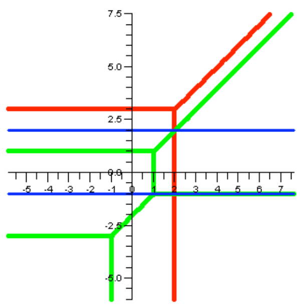
Now suppose that $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ intersect in some corners of $\mathbb{TP}^2$. Then, by Lemma 48 or Lemma 50 (depending on the case), there are $d \cdot e$ points of intersection of $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ in $\mathbb{TP}^2$.



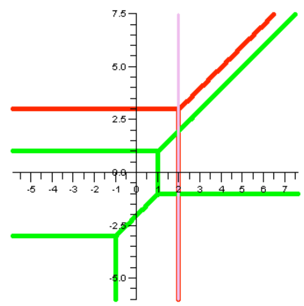
(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g)$



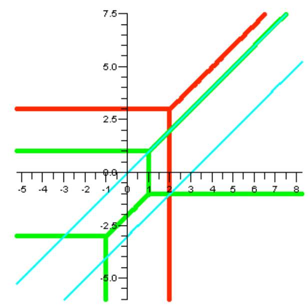
(b) Δ_{fg}



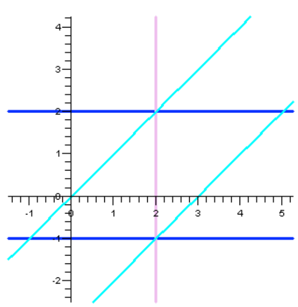
(c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$



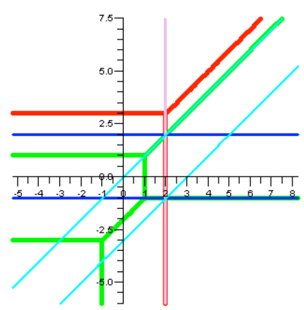
(d) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$



(e) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$

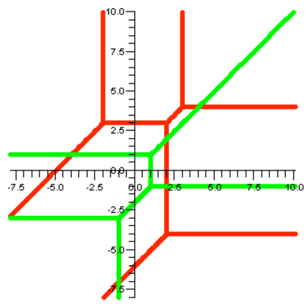


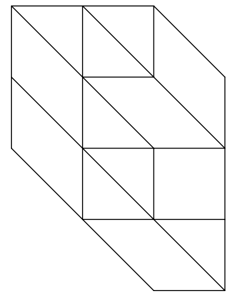
(f) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$



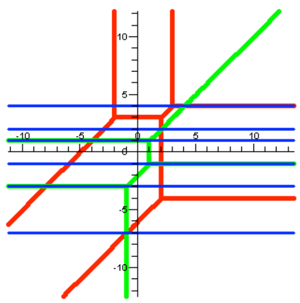
(g) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x} \cup$
$\mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

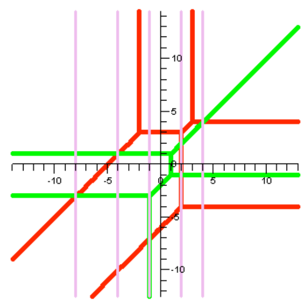
Figure 27: Intersection Multiplicity at Infinite Distance

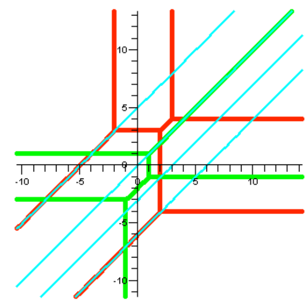




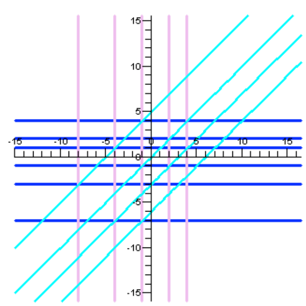
(a) $\mathcal{Z}(f) \cup \mathcal{Z}(g)$ (b) Δ_{fg}

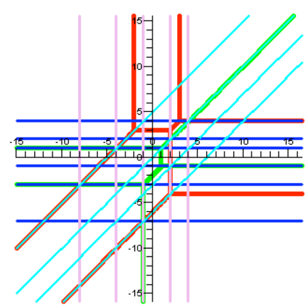






(c) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x}$ (d) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,y}$ (e) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,z}$





(f) $\mathcal{R}_{f,g,x} \cup \mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$ (g) $\mathcal{Z}(f) \cup \mathcal{Z}(g) \cup \mathcal{R}_{f,g,x} \cup$

$\mathcal{R}_{f,g,y} \cup \mathcal{R}_{f,g,z}$

Figure 28: Infinite Intersection Multiplicity Deformed to Finite Distance

Thus, in all cases, $\mathcal{Z}(f)$ stably intersects $\mathcal{Z}(g)$ in $d \cdot e$ points.

5.7 Complete Tropical Vanishing of the Resultant

As shown above, the resultant does in fact vanish whenever the two curves have a point in common. This means that the resultant must also vanish for all of the points that the curves have in common, even when they are not in general position to each other. Although this seems contradictory, since we just showed using the resultant, that there are $d \cdot e$ points of stable intersection, it is not. The resultants can be thought of in two ways, because the determinant of a matrix can be thought of in two ways. If we have an $n \times n$ matrix, then its determinant can either be thought of as an expression, which we simplify, or as a polynomial in n^2 variables; one variable for each entry of our matrix. When we simplify the determinant of the tropical Sylvester matrix, then we get our resultant as a homogeneous polynomial of degree $d \cdot e$, which has $d \cdot e$ roots. However, if we think of the determinant as a polynomial in n^2 variables, then it can vanish tropically in more places than those of the simplified resultant. Thus, if we treat our determinant in this way, we will have the resultant vanishing at all the points that are common to both curves, by Lemma 41

Classically, if two curves have a component in common, then the resultant vanishes completely. This is also the case tropically, but we must think of the resultant in its non-simplified form, as explained above. Lemma 52 explains why this is. This lemma applies to curves that have a common component, since if they do, the first row of the upper block of the Sylvester matrix will be a multiple of the first row of the lower block of the Sylvester matrix.

Lemma 52. *Let A be an $n \times n$ matrix with columns $C_1, \ldots, C_n$ such that $C_i = C_j$ for some i and j. Then $|A|_{\mathbb{T}}$ vanishes tropically. Moreover, for every term $\bigodot_{k=1}^{n} a_{k,\sigma(k)}$ in $|A|_{\mathbb{T}}$ there is another term $\bigodot_{k=1}^{n} a_{k,\tau(k)}$ such that $\bigodot_{k=1}^{n} a_{k,\sigma(k)} = \bigodot_{k=1}^{n} a_{k,\tau(k)}$.*

Proof. Let $\sigma \in S_n$ be any permutation, and let τ be the permutation obtained by first applying the transposition (ij) and then the permutation σ, so $\tau = (\sigma)(ij)$. Then $\sigma(i) = \tau(j), \sigma(j) = \tau(i)$ and $\sigma(k) = \tau(k)$ for all $k \neq i, j$. So,

$$
\begin{aligned}
&a_{1,\sigma(1)} \cdots a_{1,\sigma(i)} \cdots a_{j,\sigma(j)} \cdots a_{n,\sigma(n)} \\
&\qquad = a_{1,\sigma(1)} \cdots a_{i,\tau(j)} \cdots a_{j,\tau(i)} \cdots a_{n,\sigma(n)} \\
&\qquad = a_{1,\tau(1)} \cdots a_{i,\tau(i)} \cdots a_{j,\tau(j)} \cdots a_{n,\tau(n)}.
\end{aligned}
$$

Thus, for every term term $\bigodot_{k=1}^{n} a_{k,\sigma(k)}$ in $|A|_{\mathbb{T}}$ there is another term $\bigodot_{k=1}^{n} a_{k,\tau(k)}$ such that $\bigodot_{k=1}^{n} a_{k,\sigma(k)} = \bigodot_{k=1}^{n} a_{k,\tau(k)}$, and $|A|_{\mathbb{T}}$ is the sum of things that vanish tropically, and so vanishes tropically.

Thus, it follows that if the two curves do have a component in common, the resultant vanishes tropically at every point, but we can only see it if we keep track of all of the terms. If we do not keep track of all of the terms, then the resultant gives us the vertices of the common component as the points of intersection, which are the points of stable intersection.

Example 29. Let us consider the intersection of one line with itself. Let $f(x, y, z) = ax \oplus by \oplus cz$. For the three resultants we get

$$
\begin{aligned}
\mathcal{R}_{f,f,x} &= (ab \oplus ba)y \oplus (ac \oplus ca)z \\
\mathcal{R}_{f,f,y} &= (ba \oplus ab)x \oplus (bc \oplus cb)z \\
\mathcal{R}_{f,f,z} &= (ca \oplus ac)x \oplus (cb \oplus bc)y.
\end{aligned}
$$

We note that both of the coefficients of each of the resultants is are expressions that vanish tropically. This means that each resultant vanishes tropically for every point (x, y, z) in $\mathbb{TP}^2$. That means that on every line of slope one, every line of slope zero, and every line of infinite slope there is a point that is common to both of the polynomials (which is clear, since they are the same line). But, if we simplify the resultants we appear to have a different result. In that case we have

$$\mathcal{R}_{f,f,x} = (ab)y \oplus (ac)z$$
$$\mathcal{R}_{f,f,y} = (ab)x \oplus (bc)z$$
$$\mathcal{R}_{f,f,z} = (ac)x \oplus (bc)y.$$

These resultants give us one point of intersection, which is the point $(bc, ac, ab) = (bc \oslash ab, ac \oslash ab, 0) = (c-a, c-b, 0)$, which is in fact the vertex of the line $f(x, y, z) = ax \oplus by \oplus cx$.

Similarly, if they two curves do not have a component in common, but do intersect in an infinite number of places, the resultant does vanish tropically on all of those points, but only if we keep track of the terms again. The points that the resultant picks out are the points where the segments begin to be common to both curves, generally a vertex of one or both of the curves.

Example 30. Let us look again at the two curves given in the second half of Example 22. For this example we had $f(x, y, z) = ax \oplus by \oplus cz$ and $g(x, y, z) =$

$rx \oplus by \oplus cz$, and our resultants were

$$\mathcal{R}_{f,g,x} = (ab + br)y \oplus (ac \oplus cr)z$$

$$\mathcal{R}_{f,g,y} = (br \oplus ab)x \oplus (bc \oplus cb)z$$

$$\mathcal{R}_{f,g,z} = (cr \oplus ac)x \oplus (cb \oplus bc)y.$$

Here we notice that the coefficients of $\mathcal{R}_{f,g,x}$ do not vanish tropically, so there is only one line of slope zero that has a point of intersection. However, in $\mathcal{R}_{f,g,y}$, the coefficient of z does vanish tropically. So, if we pick a point so that $(br \oplus ab)x \oplus bcz = bcz$, then the minimum will be attained twice. So, for any point of the form $(\alpha, y, 0)$ where $\alpha \geq c - \min\{a, r\}$, $\mathcal{R}_{f,g,y}$ vanishes tropically. So, every line of the form $x = \alpha$, where α is as above has a point of intersection on it. Similarly from $\mathcal{R}_{f,g,z}$ we see that any line of the form $y = x - \gamma$ where $\gamma \leq \min\{a, r\} - b$ has a point of intersection on it.

We recall that when we simplified these resultants in Example 22, we got the point $(c - \min\{a, r\}, c - b, 0)$, as the point of intersection, which as the left most vertex of the two lines.

Finally, it is unfortunately possible for all three resultants to vanish tropically for every point (x, y, z) in $\mathbb{TP}^2$, even if the two curves do not have a component in common. In the case of two lines, if all three resultants vanish for every (x, y, z), then we know that the two lines are in fact the same line. Also, if one of the two curves is a single line, and all three resultants vanish tropically everywhere, then we know that the line is a component of the other curve. However, if the two curves are both of degree greater than one, then it is possible for all three resultants to vanish tropically, even if the two curves do not share a common component.

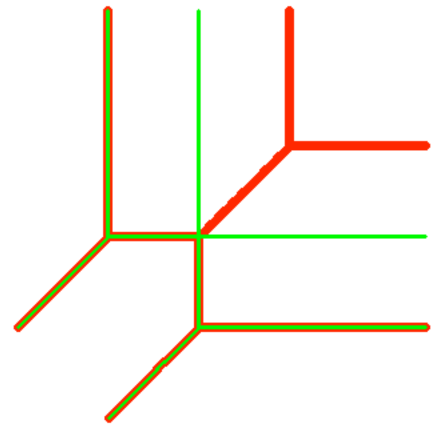


Figure 29: Completely Vanishing Resultants

Example 31. Let $f(x,y,z) = 1x^2 \oplus xy \oplus 1y^2 \oplus xz \oplus yz \oplus 1z^2$ and $g(x,y,z) = 1x^2 \oplus xy \oplus 1y^2 \oplus xz \oplus yz \oplus z^2$, as shown in Figure 29. We first note that f is irreducible, so if g is a component of f, then it must be all of f. But, these are clearly not the same curve. However, for every line of slope one, slope zero, or infinite slope, the two curves do in fact have a point in common. Thus, all three resultants must vanish tropically for every (x,y,z). If we calculate the resultants, keeping track of the repeated minimal terms, we get

$$\begin{aligned}
\mathcal{R}_{f,g,x} =&(2 \oplus 2)y^4 \oplus (1 \oplus 1)y^3z \oplus (1 \oplus 1)y^2z^2 \oplus (1 \oplus 1)yz^3 \oplus (1 \oplus 1)z^4 \\
\mathcal{R}_{f,g,y} =&(2 \oplus 2)x^4 \oplus (1 \oplus 1)x^3z \oplus (1 \oplus 1)x^2z^2 \oplus (1 \oplus 1)xz^3 \oplus (1 \oplus 1)z^4 \\
\mathcal{R}_{f.g,z} =&(1 \oplus 1)x^4 \oplus (0 \oplus 0)x^3y \oplus (0 \oplus 0)x^2y^2 \oplus (0 \oplus 0)xy^3 \oplus (1 \oplus 1)y^4.
\end{aligned}$$

6 Related Tropical Results

Now that we have a complete tropical Bézout's Theorem, the natural thing to do is determine which of its classical consequences have tropical analogues. We will consider two such consequences, but it we will see that they are not consequences of Bézout's theorem in the the tropical setting.

6.1 Tropical Pascal's Hexagon

At the age of 16 Blaise Pascal proved the following theorem, of which we include a proof, since it shows the motivation for some of the things we do later.

Theorem 53. *For a hexagon inscribed in an irreducible conic Q, the three points of intersections of the pairs of opposite sides are collinear.*

Proof. Let Q be an irreducible conic, and let $p_1, p_2, \ldots, p_6$ be the six vertices of the hexagon on the conic. Let l_i be the line between p_i and p_{i+1} for $i = 1, \ldots, 6$, where $p_7 = p_1$, and let q_j be the point of intersection of lines l_j and $l_j + 3$ for $j = 1, \ldots, 3$. We note that p_j does not lie on Q for any j, for it it did, then $l_j \cap Q \supseteq \{p_j, p_{j+1}, q_j\}$. But by Bézout's Theorem, $l_j \cap Q$ should only contain two points, or l_j and Q must share a component. But Q is irreducible, and l_j is a line, so they do not have a component in common. Let $\mathcal{C}_1 = l_1 \cup l_3 \cup l_5$ and $\mathcal{C}_2 = l_2 \cup l_4 \cup l_6$. $\mathcal{C}_1$ and $\mathcal{C}_2$ are two cubic plane curves, which intersect in the nine points $p_1, \ldots, p_6, q_1, q_2, q_3$. There are no other points of intersection, by Bézout's Theorem. Now let us consider the pencil of cubic plane curves given by $\lambda\mathcal{C}_1 + \mu\mathcal{C}_2$. Since this is a pencil of cubics, we know that every curve in the pencil contains the nine points of intersection of the two original cubic curves, and that for any point in the plane, there is a cubic in the pencil that passes through that point. So, let P be some point on Q such that

$P \neq p_i$ for any i. Then there is some cubic $\mathcal{C}$ that is in the given pencil that passes through P. But then $\mathcal{C} \cap Q \supseteq \{p_1, \ldots, p_6, P\}$. But that means there are seven points of intersection of the cubic $\mathcal{C}$ and the conic Q. But by Bézout's Theorem there can only be six points of intersection, unless the two curves have a component in common. But Q is irreducible, so the only way for the two curves to have a component in common is for $Q \subset \mathcal{C}$. Thus, $\mathcal{C} = Q \cup \mathcal{L}$ for some line $\mathcal{L}$. Since none of the q_1, q_2, q_3 lie on Q, but they are points of $\mathcal{C}$, they must all lie on $\mathcal{L}$, so they are collinear.

This theorem is extended to an arbitrary hexagon inscribed in an arbitrary conic, and so Pappus' Theorem is a special case of Pascal's Theorem. There is a natural tropical analogue to this theorem, which we state in its general form here for reference, but will prove later.

Theorem 56. *For an arbitrary tropical hexagon inscribed in an arbitrary tropical conic Q, the three points of intersection of the pairs of opposite sides are tropically collinear.*

In order to follow the same method of proof as the classical proof given, we need to to have a pencil of curves with which to work. Classically, a pencil satisfies the conditions that all of the curves in the pencil pass through any points of intersection of the two original curves. Also, since the curves in the pencil are of the form $\lambda f + \mu g$, there is a curve in the pencil for every point in $\mathbb{P}^1$. We also know that for any point $(a, b, c) \in \mathbb{P}^2$, that there is a curve in the pencil that passes through this point. If, for example, (a, b, c) is on one of the two original curves, say f, then we may have $\lambda = 1$ and $\mu = 0$. If (a, b, c) is not on either of the curves, then $f(a, b, c) \neq 0$ and $g(a, b, c) \neq 0$, so we may let $\lambda = 1$ and solve $\lambda f(a, b, c) + \mu g(a, b, c) = 0$ for μ, and we will have the desired point, $(\lambda, \mu) = \left(1, -\dfrac{f(a,b,c)}{g(a,b,c)}\right)$, so that our curve passes through the point (a, b, c).

In the tropical case we will take two curves of the same degree, and consider the set of all curves of the form $\lambda f \oplus \mu g$, and see if it has these same nice properties.

Lemma 54. *Let $f, g \in \mathcal{Q}\langle x, y, z\rangle$ be two homogeneous polynomials, and let $P \in \mathcal{Z}(f) \cap \mathcal{Z}(g)$. Then $P \in \mathcal{Z}(\lambda f \oplus \mu g)$ for any $\lambda, \mu \in \mathcal{Q}$.*

Proof. We have a few cases to consider. First, if $\lambda = \mu = \infty$ then every point of $\mathcal{Q}$ is in $\mathcal{Z}(\lambda f \oplus \mu g)$, so we're done. Without loss of generality, if $\lambda = \infty$ and $\mu \neq \infty$, then $\lambda f \oplus \mu g = \mu g$, and since $\mathcal{Z}(\mu g) = \mathcal{Z}(g)$ for any $\mu \neq \infty$, and $P \in \mathcal{Z}(g)$, then $P \in \mathcal{Z}(\lambda f \oplus \mu g)$. Finally we have the case where $\lambda \neq \infty$ and $\mu \neq \infty$. But, as we mentioned above, $\mathcal{Z}(\mu g) = \mathcal{Z}(g)$ for any $\mu \neq \infty$, so we can simply consider $f \oplus g$.

Let f_{p_1}, f_{p_2} be two monomials of f that attain the minimum together at p and let g_{p_1}, g_{p_2} be two monomials of g that attain the minimum together at p. Without loss of generality assume that $f(P) \leq g(P)$. Assume that some monomial g_0 of g replaces the monomial f_{p_1} in 5he sum $f \oplus g$. This means that $f_{p_1} = \alpha x^r y^s z^t$ and $g_0 = \beta x^r y^s z^t$ where $\beta < \alpha$, and that $f_{p_1}(x_0, y_0, z_0) > g_0(x_0, y_0, z_0)$ for every $(x_0, y_0, z_0) \in \mathcal{Q}^3$. But we know that $f(P) \geq f_{p_1}(p) > g_0(p) \geq g(p)$. But this is a contradiction, as $f(p) \leq g(p)$. Thus, the monomial f_{p_1} could not have been replaced. Similarly for f_{p_2}. Thus, $f \oplus g$ still has the two monomials that attain the minimum at p and $P \in \mathcal{Z}(f \oplus g)$. ✧

So we see that this version of a pencil does in fact pass through all the points of intersection of the two curves. Unfortunately there are curves f and g for which there are points in the plane so that no curve in the pencil $\lambda f \oplus \mu g$ passes through the given point.

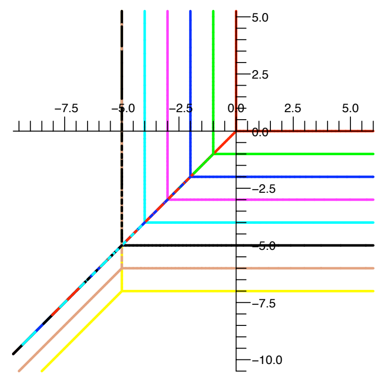


Figure 30: Some line in the pencil $\lambda f \oplus \mu g$

Example 32. Let $f(x) = x \oplus y \oplus z$ and $g(x) = 5x \oplus 7y \oplus z$. Figure 30 shows some of the curves in the pencil $\lambda f \oplus \mu g$.

We will just consider the case where $\mu = 0$, since we would be able to scale one of the two of them out anyway. We note that in this particular case, we will not ever need to have $\mu = \infty$ to get the μg term to drop out, because all the coefficients have finite coefficients, so there is some finite μ which makes μg larger than f. We can write down a piecewise function for our sum, as given below.

$$\lambda f \oplus g = \begin{cases} \lambda x \oplus \lambda y \oplus \lambda z & \lambda \leq 0 \\ \lambda x \oplus \lambda y \oplus z & 0 \leq \lambda \leq 5 \\ 5x \oplus \lambda y \oplus z & 5 \leq \lambda \leq 7 \\ 5x \oplus 7y \oplus z & \lambda \geq 7 \end{cases}$$

We note that $\mathcal{Z}(\lambda f) = \mathcal{Z}(f)$, so we just have lines between $\mathcal{Z}(f)$ and $\mathcal{Z}(g)$ as shown in Figure 30. Naturally, we only have a few values of λ, and chose integers for simplicity.

But maybe we can find a curve of the form $\lambda f \oplus \mu g$ that goes through any point we want, for example, the point $(2, 2, 0)$. For this to happen, we need $\lambda f \oplus g$ to vanish tropically at $(2, 2, 0)$. $\lambda f \oplus g = (\lambda \oplus 5)x \oplus (\lambda \oplus 7)y \oplus (\lambda \oplus 0)z$. We need to find λ so that

$$2 + \min\{\lambda, 5\} = 2 + \min\{\lambda, 7\} \leq \min\{\lambda, 0\} \text{ or}$$

$$2 + \min\{\lambda, 5\} = \min\{\lambda, 0\} \leq 2 + \min\{\lambda, 7\} \text{ or}$$

$$2 + \min\{\lambda, 7\} = \min\{\lambda, 0\} \leq 2 + \min\{\lambda, 5\}.$$

It is straight forward to see that this never happens. So, there is not curve in the pencil that passes through the point $(2, 2, 0)$.

In the case of Pascal's theorem, we can create the two cubics by taking the appropriate product of lines, and make a pencil out of them. The curves in this tropical pencil do pass through the nine desired points, but we cannot guarantee that we will be able to find a curve in the pencil that passes through another good point on the conic. But, suppose that we could. We would then, as desired, have the seven points of intersection with a cubic and the conic. However, this does not guarantee in any way that the conic is a component of the cubic, since it might simply mean that they have a segment or ray in common. And even if we can show that all three resultants vanish tropically everywhere, this also does not guarantee that the conic is a component of the cubic, as we saw in Example 31. So, we can not prove Pascal's theorem in this way.

We might be able to consider a different form of pencil of curves, where instead of taking the curves that are dependent on two curves of the same degree, we take all curves of a given degree through a fixed number of points. If we use the nine points which are the points of intersection of our two cubics, it turns out that for any point in the plane, there is a cubic that passes through that point and the nine original points. However, there is no way, again, to determine whether or not the cubic you find contains the conic as a component, so this does not help either. It seems that there is no way to prove this theorem using similar methods to the classical proof, if the classical proof relies on begin able to determine when a conic is a component of a cubic. Thus, we turn to a different method to prove this theorem.

Suppose that six points $p_i = (x_i, y_i, z_i)$ lie on a conic. We recall that

$$Q = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1z_1 & y_1z_1 & z_1^2 \\ x_2^2 & x_2y_2 & y_2^2 & x_2z_2 & y_2z_2 & z_2^2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3z_3 & y_3z_3 & z_3^2 \\ x_4^2 & x_4y_4 & y_4^2 & x_4z_4 & y_4z_4 & z_4^2 \\ x_5^2 & x_5y_5 & y_5^2 & x_5z_5 & y_5z_5 & z_5^2 \\ x_6^2 & x_6y_6 & y_6^2 & x_6z_6 & y_6z_6 & z_6^2 \end{pmatrix}$$

is a singular matrix, which is to say that it's determinant vanishes tropically.

We also know that three points $q_i = (x_i, y_i, z_i)$ are collinear when

$$\begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

is a singular matrix. We will use these two facts to prove Pascal's Theorem for hexagons.

Lemma 55. *Let $p_i = (x_i, y_i, z_i)$ be six points in $\mathbb{TP}^2$. If*

$$Q = \begin{pmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1z_1 & y_1z_1 & z_1^2 \\ x_2^2 & x_2y_2 & y_2^2 & x_2z_2 & y_2z_2 & z_2^2 \\ x_3^2 & x_3y_3 & y_3^2 & x_3z_3 & y_3z_3 & z_3^2 \\ x_4^2 & x_4y_4 & y_4^2 & x_4z_4 & y_4z_4 & z_4^2 \\ x_5^2 & x_5y_5 & y_5^2 & x_5z_5 & y_5z_5 & z_5^2 \\ x_6^2 & x_6y_6 & y_6^2 & x_6z_6 & y_6z_6 & z_6^2 \end{pmatrix}$$

is tropically singular, then

$$L = \begin{pmatrix}
\begin{vmatrix} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ \begin{vmatrix} x_4 & z_4 \\ x_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ \begin{vmatrix} y_4 & z_4 \\ y_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \\ \begin{vmatrix} y_4 & z_4 \\ y_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & z_4 \\ x_5 & z_5 \end{vmatrix} \end{vmatrix} \\
\begin{vmatrix} \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \\ \begin{vmatrix} x_5 & z_5 \\ x_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & y_5 \\ x_6 & y_6 \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \\ \begin{vmatrix} y_5 & z_5 \\ y_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & y_5 \\ x_6 & y_6 \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} \\ \begin{vmatrix} y_5 & z_5 \\ y_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & z_5 \\ x_6 & z_6 \end{vmatrix} \end{vmatrix} \\
\begin{vmatrix} \begin{vmatrix} x_3 & z_3 \\ x_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} \\ \begin{vmatrix} x_6 & z_6 \\ x_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & y_6 \\ x_1 & y_1 \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} y_3 & z_3 \\ y_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} \\ \begin{vmatrix} y_6 & z_6 \\ y_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & y_6 \\ x_1 & y_1 \end{vmatrix} \end{vmatrix} &
\begin{vmatrix} \begin{vmatrix} y_3 & z_3 \\ y_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & z_3 \\ x_4 & z_4 \end{vmatrix} \\ \begin{vmatrix} y_6 & z_6 \\ y_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & z_6 \\ x_1 & z_1 \end{vmatrix} \end{vmatrix}
\end{pmatrix}$$

is tropically singular.

Proof. A straightforward, although tedious, computation and comparison of the terms of $|Q|_{\mathbb{T}}$ and $|L|_{\mathbb{T}}$ shows that $|L|_{\mathbb{T}} = |Q|_{\mathbb{T}} \oplus H$, where H is a tropical expression which vanishes tropically as every term in repeated at least twice. Thus, if $|Q|_{\mathbb{T}}$ vanishes tropically, then $|L|_{\mathbb{T}}$ vanishes tropically, as it is the sum of two expressions that vanish tropically.

Note that this computation can be using a computational program by computing the classical permanent $L_{\mathcal{P}}$ and $Q_{\mathcal{P}}$ of each matrix (which is simply the determinant but with out the sign changes for the permutations), and finding the difference $L_{\mathcal{P}} - Q_{\mathcal{P}}$. The difference can then be reduced modulo 2, producing a result of 0. This implies that the terms of $L_{\mathcal{P}} - Q_{\mathcal{P}}$ all appear an even number of times, which implies that tropically the expression must vanish.

Theorem 56. *For an arbitrary tropical hexagon inscribed in an arbitrary tropical conic Q, the three points of intersection of the pairs of opposite sides are tropically collinear.*

Proof. Let $p_1 = (x_1, y_1, z_1), \ldots, p_6 = (x_6, y_6, z_6)$ be the six points on the hexagon. The line that contains p_i and p_{i+1} is given by

$$\begin{vmatrix} x_i & y_i & z_i \\ x_{i+1} & y_{i+1} & z_{i+1} \\ x & y & z \end{vmatrix}.$$

So, we have

$$l_1 = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} x \oplus \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} y \oplus \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} z,$$

$$l_2 = \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} x \oplus \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} y \oplus \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} z,$$

$$l_3 = \begin{vmatrix} y_3 & z_3 \\ y_4 & z_4 \end{vmatrix} x \oplus \begin{vmatrix} x_3 & z_3 \\ x_4 & z_4 \end{vmatrix} y \oplus \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} z,$$

$$l_4 = \begin{vmatrix} y_4 & z_4 \\ y_5 & z_5 \end{vmatrix} x \oplus \begin{vmatrix} x_4 & z_4 \\ x_5 & z_5 \end{vmatrix} y \oplus \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} z,$$

$$l_5 = \begin{vmatrix} y_5 & z_5 \\ y_6 & z_6 \end{vmatrix} x \oplus \begin{vmatrix} x_5 & z_5 \\ x_6 & z_6 \end{vmatrix} y \oplus \begin{vmatrix} x_5 & y_5 \\ x_6 & y_6 \end{vmatrix} z,$$

$$l_6 = \begin{vmatrix} y_6 & z_6 \\ y_1 & z_1 \end{vmatrix} x \oplus \begin{vmatrix} x_6 & z_6 \\ x_1 & z_1 \end{vmatrix} y \oplus \begin{vmatrix} x_6 & y_6 \\ x_1 & y_1 \end{vmatrix} z.$$

Now, the point of intersection of two lines can be found in the same way, where we think of the x, y, z as telling us which coordinate we have rather than giving us an equation. So, for our three points of intersection of opposite sides we have

$$q_1 = \left(\begin{vmatrix} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ \begin{vmatrix} x_4 & z_4 \\ x_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} \end{vmatrix}, \begin{vmatrix} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ \begin{vmatrix} y_4 & z_4 \\ y_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} \end{vmatrix}, \begin{vmatrix} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \\ \begin{vmatrix} y_4 & z_4 \\ y_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & z_4 \\ x_5 & z_5 \end{vmatrix} \end{vmatrix} \right),$$

$$q_2 = \left(\begin{vmatrix} \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \\ \begin{vmatrix} x_5 & z_5 \\ x_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & y_5 \\ x_6 & y_6 \end{vmatrix} \end{vmatrix}, \begin{vmatrix} \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \\ \begin{vmatrix} y_5 & z_5 \\ y_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & y_5 \\ x_6 & y_6 \end{vmatrix} \end{vmatrix}, \begin{vmatrix} \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} \\ \begin{vmatrix} y_5 & z_5 \\ y_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & z_5 \\ x_6 & z_6 \end{vmatrix} \end{vmatrix} \right),$$

$$q_3 = \left(\begin{vmatrix} \begin{vmatrix} x_3 & z_3 \\ x_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} \\ \begin{vmatrix} x_6 & z_6 \\ x_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & y_6 \\ x_1 & y_1 \end{vmatrix} \end{vmatrix}, \begin{vmatrix} \begin{vmatrix} y_3 & z_3 \\ y_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} \\ \begin{vmatrix} y_6 & z_6 \\ y_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & y_6 \\ x_1 & y_1 \end{vmatrix} \end{vmatrix}, \begin{vmatrix} \begin{vmatrix} y_3 & z_3 \\ y_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & z_3 \\ x_4 & z_4 \end{vmatrix} \\ \begin{vmatrix} y_6 & z_6 \\ y_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & z_6 \\ x_1 & z_1 \end{vmatrix} \end{vmatrix} \right).$$

Now, these three points will be collinear if the following matrix is singular.

$$L = \begin{pmatrix} \begin{vmatrix} \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ \begin{vmatrix} x_4 & z_4 \\ x_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \\ \begin{vmatrix} y_4 & z_4 \\ y_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & y_4 \\ x_5 & y_5 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} & \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \\ \begin{vmatrix} y_4 & z_4 \\ y_5 & z_5 \end{vmatrix} & \begin{vmatrix} x_4 & z_4 \\ x_5 & z_5 \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \\ \begin{vmatrix} x_5 & z_5 \\ x_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & y_5 \\ x_6 & y_6 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} \\ \begin{vmatrix} y_5 & z_5 \\ y_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & y_5 \\ x_6 & y_6 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} y_2 & z_2 \\ y_3 & z_3 \end{vmatrix} & \begin{vmatrix} x_2 & z_2 \\ x_3 & z_3 \end{vmatrix} \\ \begin{vmatrix} y_5 & z_5 \\ y_6 & z_6 \end{vmatrix} & \begin{vmatrix} x_5 & z_5 \\ x_6 & z_6 \end{vmatrix} \end{vmatrix} \\ \begin{vmatrix} \begin{vmatrix} x_3 & z_3 \\ x_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} \\ \begin{vmatrix} x_6 & z_6 \\ x_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & y_6 \\ x_1 & y_1 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} y_3 & z_3 \\ y_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & y_3 \\ x_4 & y_4 \end{vmatrix} \\ \begin{vmatrix} y_6 & z_6 \\ y_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & y_6 \\ x_1 & y_1 \end{vmatrix} \end{vmatrix} & \begin{vmatrix} \begin{vmatrix} y_3 & z_3 \\ y_4 & z_4 \end{vmatrix} & \begin{vmatrix} x_3 & z_3 \\ x_4 & z_4 \end{vmatrix} \\ \begin{vmatrix} y_6 & z_6 \\ y_1 & z_1 \end{vmatrix} & \begin{vmatrix} x_6 & z_6 \\ x_1 & z_1 \end{vmatrix} \end{vmatrix} \end{pmatrix}.$$

But we know that the six points are on a conic so the matrix Q from Lemma 55 is singular, so L is singular too, and our points are collinear.

Example 33. In Figure 31 we see an example of Pascal's Tropical Hexagon. We note that in the hexagon, we have colored opposite side the same color. So, in

Figure 31(c), the three points of intersection of opposite sides are where the lines of the same color intersect each other.

6.2 Tropical Elliptic Curves and the Group Law

In the following section we discuss briefly tropical elliptic curves that the group law that can be defined on them as discussed in [19] and [20]. We give a few definitions and a small amount of background information, and then give a few results concerning the group law that follow from the complete tropical Bézout's theorem.

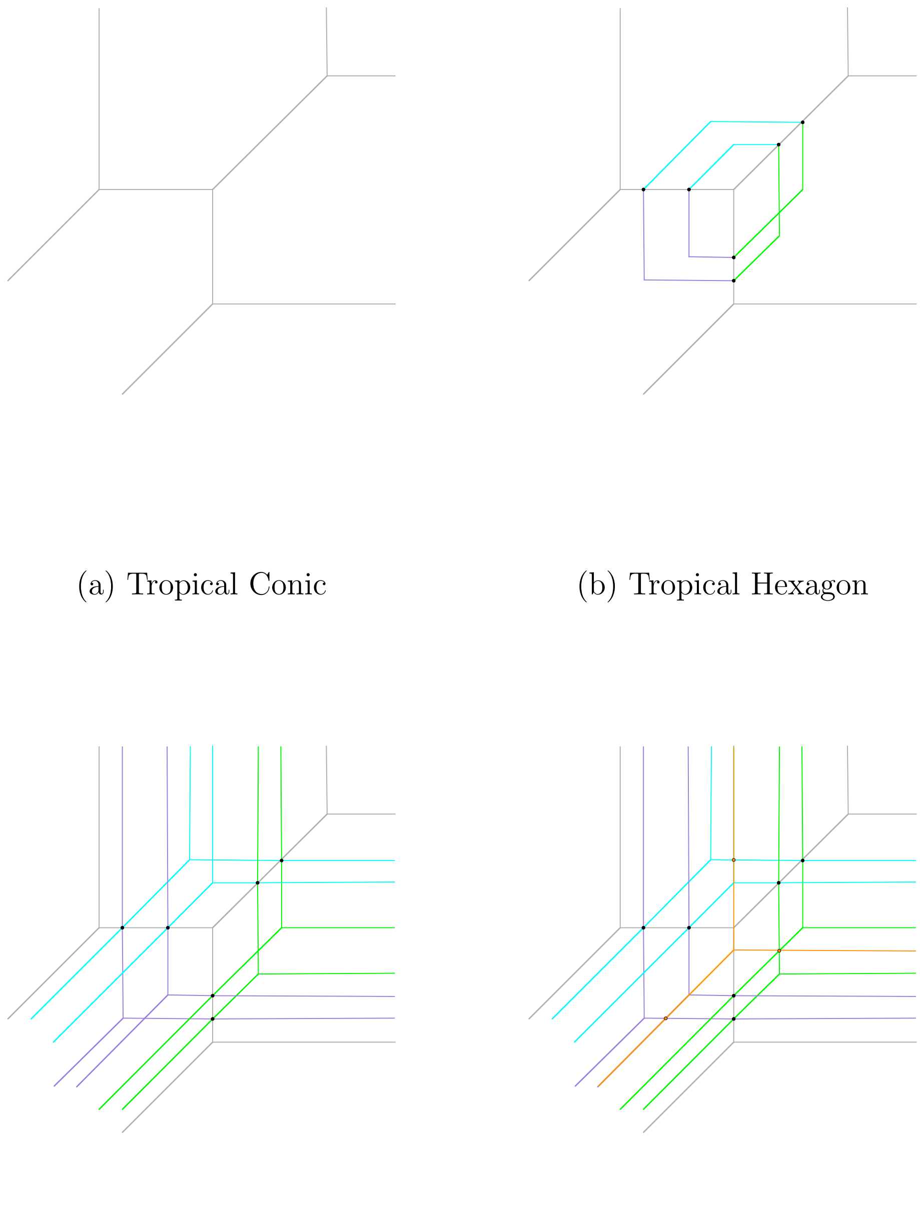
Definition 32. Let $f(x, y, z)$ be a homogenous tropical polynomial of degree n. The *genus* of $\mathcal{Z}(f)$ is defined to be number of interior lattice points of Δ_f. That is to say, the number of points of the lattice that are interior to ∂^n which are vertices of Δ_f.

As shown in [15] and [20] the genus the same number as the number of cycles that the curve $\mathcal{Z}(f)$ contains.

Definition 33. A *tropical elliptic curve* is a tropical plane curve of genus one.

In both [19] and [20], only cubic tropical elliptic curves are considered, so those are the ones we will restrict our attention to as well. To define the group law on cubic tropical elliptic, or any elliptic curve, we need to define divisors.

Definition 34. Set $Div(\mathcal{C})$ to be the free Abelian group generated by the points of a tropical plane curve $\mathcal{C}$. A *divisor* D is an element of $Div\mathcal{C}$, that is $D = \sum_{P\in\mathcal{C}} \mu_P P$ where $\mu_P \in \mathbb{Z}$. The sum $\sum_{P\in\mathcal{C}} \mu_P$ is the *degree* of the divisor D, denoted $\deg(D)$. The set of degree zero divisors in denoted $Div^0(\mathcal{C})$. If $f(x, y, z)$ is a homogeneous



(a) Tropical Conic (b) Tropical Hexagon

(c) Points of Intersection (d) Line

Figure 31: Pascal's Tropical Hexagon

polynomial then the *divisor of f $divf$* is defined to be the formal sum of the stable intersection points with of $\mathcal{Z}(f)$ and $\mathcal{C}$ with their intersection multiplicities. That is, $divf = \sum_{P \in \mathcal{Z}(f) \cap \mathcal{C}} i_P P$ where i_P is the intersection multiplicity of P. A tropical divisor D is *principal* if there are homogeneous tropical polynomials f and g of the same degree such that $D = divf - divg$.

In [19], the polynomials f and g needed to form a principal divisor must have the same Newton polygon, which means that Δ_f and Δ_g have the same boundary. In [20], the polynomials must both be products of lines. These conventions are necessary to prove the group law is well defined, although we will discuss that in detail. The result that in Lemma 57, which follows from the complete tropical Bézout's theorem, does not depend on either of those restrictions to f and g.

Lemma 57. *The tropical principal divisors are divisors of degree zero.*

Proof. Suppose D is a principal divisor, and suppose the $\mathcal{C}$ is a curve of degree n. Then $D = divf - divg$ for polynomials f, g of the same degree, say d. But, by Bézout's theorem, $\mathcal{Z}(f) \cap \mathcal{C}$ has $n \cdot d$ stable points of intersection, counting multiplicity. So, the $\deg(divf) = n \cdot d$. Similarly, $\deg(divg) = n \cdot d$. Thus $\deg(D) = \deg(divf - divg) = \deg(divf) - \deg(divg) = n \cdot d - n \cdot d = 0$. Thus, all principal divisors are divisors of degree zero. ☙

In [19] and [20] it is shown how to define the group law on cubic tropical elliptic curves, which is analogous to the classical group law on elliptic curves, and that the group is isomorphic to S^1. It is also shown in [19] that if all of the points that we are considering for a sum are in *general enough position*, then we have a geometric method of adding the points which is similar to that in the classical case. The points are in general enough position if all of the points need to make lines are in general position to each other, and if all of the lines are in general position with respect to the the curve $\mathcal{C}$. We use the geometric method picking an origin $\mathcal{O}$, and then for two points P and Q, find the line $\mathcal{L}$ that passes through P and Q, and define $P \star Q$ to be the third point of intersection of $\mathcal{L}$ and $\mathcal{C}$. By Bézout's theorem we know that there are three well defined points of intersection in $\mathcal{L} \cap \mathcal{C}$, so we can find this third point. Now we find the line $\mathcal{L}_{\mathcal{O}}$ between $P \star Q$ and $\mathcal{O}$. Again, $\mathcal{L}_{\mathcal{O}} \cap \mathcal{C}$ consists of three points, so we have a third point of intersection. This point of intersection is $P + Q$. It is quite possible that the points we need are not all in general enough position to use this geometric approach. However, in [19], it is shown that for points that are not in general enough position, the pairs of points that are in general enough position can be found which pairs are linearly equivalent to the original pairs of points. All of this is explained sufficiently in [19], so we will simply use the fact that the addition can be done with lines for the following lemmas, which follow from the Fundamental Theorem of Tropical Algebra and Bézout's theorem.

Lemma 58. *Let $f(x) = a_n x^n \oplus a_{n-1} x^{n-1} \oplus \cdots \oplus a_1 x + a_0$ be a least coefficients polynomial, and let $\alpha_1, \ldots, \alpha_n$ be the roots of $f(x)$. Then $\bigodot_{i=1}^{n} \alpha_i = a_0 \oslash a_n$.*

Proof. Since $f(x)$ is a least coefficients polynomial, it can be factored completely as $a_n(x \oplus \alpha_n) \cdots (x \oplus \alpha_1)$. But this means that $a_n \alpha_n \cdots \alpha_1 = a_0$. Thus, $\bigodot_{i=1}^{n} \alpha_i = a_0 \oslash a_n$, our desired result. ✩

Lemma 59. *If $f(x) = a_n x^n \oplus \cdots \oplus a_1 x \oplus a_0 \in \mathcal{Z}\langle x \rangle$ where f is a least-coefficients polynomial, and $\alpha_1, \ldots, \alpha_n$ are the roots of $f(x)$ where $\alpha_i \neq \infty$ for any i, and $n-1$ of these roots are in $\mathcal{Z}$ then the remaining root is also in $\mathcal{Z}$.*

Proof. By Lemma 58 we know that $\bigodot_{i=1}^{n} \alpha_i = a_0 \oslash a_n$. Suppose that $\alpha_i \in \mathcal{Z}$ for every $i \neq j$ for some j. Then, since $\mathcal{Z}$ is a semi-field, it follows that $a_0 \oslash (a_n \alpha_1 \cdots \hat{\alpha_j} \cdots \alpha_n) \in \mathcal{Z}$. But $a_j = a_0 \oslash (a_n \alpha_1 \cdots \hat{\alpha_j} \cdots \alpha_n)$, so the final root is also in $\mathcal{Z}$.

Note that if $f(x)$ is a polynomial such that ∞ is a root of $f(x)$, then we can factor x^k out of the whole polynomial for some $k \in \mathbb{N}$, and the remainder satisfies the desired condition for the lemma.

We note that if the polynomial is a least-coefficient polynomial, with all of its coefficients in any tropical semi-field, then all of the roots are in that semi-field. For example, if the coefficients are all in $\frac{1}{2} \cdot \mathcal{Z}$ when the polynomial is in least-coefficient form, then all of the roots are in $\frac{1}{2}\mathcal{Z} = \{\frac{p}{2} | p \in \mathbb{Z}\} \cup \{\infty\}$.

Lemma 60. *Let P and Q be two integer points. Then the line that passes through them is given by an equation $b_0 x \oplus b_1 y \oplus b_2 z$ where $b_0, b_1, b_2 \in \mathcal{Z}$.*

Proof. The line that passes through $P = (p_0, p_1, p_2)$ and $Q = (q_0, q_1, q_2)$ is given by

$$\begin{vmatrix} p_1 & p_2 \\ q_1 & q_2 \end{vmatrix} x \oplus \begin{vmatrix} p_0 & p_2 \\ q_0 & p_2 \end{vmatrix} y \oplus \begin{vmatrix} p_0 & p_1 \\ q_0 & q_1 \end{vmatrix} z.$$

But each $p_i, q_j \in \mathcal{Z}$ so, each of the above coefficients above are also in $\mathcal{Z}$.

Lemma 61. *The integer points of elliptic curves are closed under addition.*

Proof. Let $f(x, y, z) = a_0 x^3 \oplus a_1 x^2 y \oplus a_2 x y^2 \oplus a_3 y^3 \oplus a_4 x^2 z \oplus a_5 xyz \oplus a_6 y^2 z \oplus a_7 x z^2 \oplus a_8 y z^2 \oplus a_9 z^3 \in \mathcal{Z}\langle x, y, z \rangle$. Let P and Q be two integer points of $\mathcal{Z}(f)$. Then by Lemma 60 the line through P and Q is given by an equation of the form $b_0 x \oplus b_1 y \oplus b_2 z$ where $b_i \in \mathcal{Z}$ for each i. The resultant tells us where the points of intersection of the cubic and the line are. Since all of the entries of the Sylvester matrix are polynomials with integer coefficients, the determinant $\mathcal{R}_{f,g}$ is a polynomial with integer coefficients, which factors completely. But we already know that two of the three roots, P and Q of $\mathcal{R}_{f,g}$ are integers, so by Lemma 59, the remaining root $P \star Q$ is also an integer point. Thus, if we pick $\mathcal{O}$ to be an integer point, then the third point of intersection of the line through $\mathcal{O}$ and $P \star Q$ is also an integer point. Thus the integer points of elliptic curves are closed under addition.

In the classical case, the associativity of the group law is also a consequence of Bézout's theorem. However, due to the interesting intersections of tropical curves, it is not a straightforward consequence tropically.

6.3 Conclusion

In conclusion we note that although when we first consider the affine version of tropical Bézout's theorem, it seems like an almost trivial fact. But the proof of it, even in the affine case is far from trivial, as shown in [16]. But then when we look at examples in the tropical projective plane, we are led to wonder if the theorem is even true. And surprisingly enough, it is, which we showed using a tropical resultant. But, we also noted that tropically linear algebra, although having similar results, could not be built up in quite the same way as its classical analogues, and almost none of the proofs followed proofs similar to those of their analogues theorems. And yet, they all worked out in the end. But then the resultant also did not behave in quite the same way, but the way that it did behave was sufficient to prove our desired results. Finally we saw that although Pascal's Tropical Hexagon and the group law on tropical elliptic curves still work, they don't follow in the same way from Bézout's theorem as they do classically. And so we see that with many tropical theorems, although the theorems are analogues of classical theorems, the proofs are definitely not analogues of classical proofs.

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