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# Unknotting Tunnels of Hyperbolic Tunnel Number n Manifolds

Stephan Daniel Burton
*Brigham Young University - Provo*

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Stephan D. Burton

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Jessica S. Purcell, Chair
James W. Cannon
Gregory R. Conner

Department of Mathematics

Brigham Young University

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## ABSTRACT

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Stephan D. Burton
Department of Mathematics, BYU
Master of Science

Adams conjectured that unknotting tunnels of tunnel number 1 manifolds are always isotopic to a geodesic. We generalize this question to tunnel number $n$ manifolds. We find that there exist complete hyperbolic structures and a choice of spine of a compression body with genus 1 negative boundary and genus $n \geq 3$ outer boundary for which $(n-2)$ edges of the spine self-intersect. We use this to show that there exist finite volume one-cusped hyperbolic manifolds with a system of $n$ tunnels for which $(n-1)$ of the tunnels are homotopic to geodesics arbitrarily close to self-intersecting. This gives evidence that the generalization of Adams's conjecture to tunnel number $n \geq 2$ manifolds may be false.

Keywords: Hyperbolic Geometry, Hyperbolic 3-manifolds, Unknotting Tunnel, Ford Domain, Knot Theory

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# Chapter 1. Introduction

A major task in the study of 3-manifolds is using geometry to understand topological spaces. One specific question that has arisen is how to identify arcs that are isotopic to a geodesic given only a topological description of a manifold. This work focuses on the special case where the arcs in question are unknotting tunnels. An *unknotting tunnel* $\tau$ of a 3-manifold $M$ with torus boundary components is an embedded arc with endpoints on $\partial M$ such that $M \backslash N(\tau)$ is a handlebody. A *system of unknotting tunnels* is a collection of arcs $\tau_1, \ldots, \tau_n$ such that $M \backslash N(\bigcup_{i=1}^{n} \tau_i)$ is a handlebody. Manifolds that admit a tunnel system consisting of $n$ arcs are called *tunnel number $n$ manifolds*, provided there is not a system of unknotting tunnels for $M$ consisting of fewer than $n$ tunnels.

Adams asked the question of whether an unknotting tunnel is always isotopic to a geodesic, and proved that in the case of tunnel number 1 manifolds, an unknotting tunnels with endpoints on different boundary components will be isotopic to a geodesic [1]. Adams and Reid showed that an unknotting tunnel in a two-bridg knots is always isotopic to a geodesic [2]. Cooper, Futer, and Purcell [6] recently showed that unknotting tunnels in tunnel number 1 manifolds are generically isotopic to geodesics, for a correct sense of the word "generic."

A natural generalization of Adams's question is to determine if unknotting tunnels of a tunnel number $n$ manifold will always be isotopic to geodesics. While there is mounting evidence that when $n = 1$, the tunnel will be isotopic to a geodesic, we will show that the generalization for $n > 1$ may be false. Specifically, we find a system of $n$ unknotting tunnels where $(n-1)$ tunnels are homotopic to geodesics arbitrarily close to having self-intersections, so these tunnels may not be isotopic to geodesics.

In order to understand the geometry of tunnel number $n$ manifolds, we study the geometry of $(1, n+1)$-compression bodies, i.e. compression bodies with genus 1 inner boundary and genus $(n+1)$ outer boundary. Cooper, Lackenby, and Purcell used the Ford domain of a $(1, 2)$-compression body to construct unknotting tunnels in finite volume manifolds with arbitrarily long length [7]. Viewing the $(1,2)$-compression body $C$ as $T^2 \times [0,1]$ with a 1-handle attached, the *core tunnel* $\tau$ of $C$ is the core of the 1-handle. The core tunnel $\tau$ corresponds to an unknotting tunnel in the manifold obtained by attaching a genus 2 handlebody to the outer boundary of $C$. Lackenby and Purcell used the Ford domain to study core tunnels of $(1,2)$-compression bodies, and showed that in many cases the core tunnel $\tau$ of a $(1,2)$-compression body $C$ is isotopic to a geodesic. They conjectured that the core tunnel will always be isotopic to a geodesic if $C$ is given a complete hyperbolic struture [13]. Their work is similar to previous work of Jørgensen who studied Ford domains of once punctured torus groups [12] and cyclic groups [17]. Akiyoshi, Sakuma, Wada, and Yamashita extended Jørgensen's work [3], and Wada [20] developed an algorithm to determine Ford domains of these manifolds. Lackenby and Purcell developed an algorithm for visualizing the Ford domain of a $(1,2)$-compression body [13].

We generalize the notion of a core tunnel of a compression body to the spine of a compression body. We show that in the case of $(1,n)$-compression bodies with $n \geq 3$, $(n-2)$ edges of the spine are not isotopic to geodesics. This shows that the generalization of Lackenby and Purcell's conjecture that the core tunnel will be isotopic to a geodesic is false for $(1,n)$-compression bodies when $n \geq 3$. The hyperbolic structures given in the proof give rise to cases where the generalization of Lackenby and Purcell's algorithm to visualize Ford domains fails. We then follow an argument similar to that of Cooper, Lackenby, and Purcell [7], to show the following theorem:

**Theorem 1.1.** *There exist finite volume one-cusped hyperbolic manifolds with a system of $n$ tunnels for which $(n-1)$ of the tunnels are homotopic to geodesics which are arbitrarily close to self-intersecting.*

The proof of this theorem does not guarantee that the geodesics will self-intersect, but shows that it is likely that there are finite volume tunnel number $n$ manifolds for which $(n-1)$ of the tunnels are not isotopic to a geodesic. The proof of this theorem relies upon a specific choice of the spine of a compression body $C$. By applying a topological move called a disk slide to the compression body $C$ obtained in the proof of theorem 1.1, we can obtain a new spine for $C$ whose edges are all isotopic to geodesics. This, and numerous computer examples lead us to the following conjecture:

**Conjecture 1.2.** *Given a $(1, n)$-compression body $C$ and a complete hyperbolic structure on $C$, there exists a spine for which all the edges are isotopic to a geodesic.*

# Chapter 2. Compression Bodies

In this section we give the definition of a compression body, and show how to describe a compression body in terms of boundary components. We will later consider geometrically finite hyperbolic structures of compression bodies and use these to construct unknotting tunnels. This section gives an overview of the basic topology of compression bodies. Much of this material is similar to Notes on Heegaard splittings by Johnson [11].

**Definition 2.1.** Let $C$ be the disjoint union of balls and manifolds of the form $S \times [0, 1]$ where $S$ is a closed surface. Let $D_1, D_2, \ldots, D_n, D'_1, D'_2, \ldots, D'_n$ be a collection of disks in $\partial C'$ with each disk either in $S \times \{1\}$ for some closed surface $S$, or in the boundary of some ball component. For each $i \leq n$ let $\varphi_i : D_i \to D'_i$ be a homeomorphism. The result of gluing $C$ by the maps $\varphi_1, \ldots, \varphi_n$ is a *compression body.*

Essentially, a compression body is the result of taking a surface $S$ cross $[0, 1]$ and attaching 1-handles. The boundary of the compression body $C$ consists of the negative boundary $\partial_- C = S \times \{0\}$, and the positive boundary $\partial_+ C = \partial C \backslash \partial_- C$. We will consider the specific case when the surface $S$ is a connected genus $m$ surface. In this case $\partial_- C$ will be a genus $m$ surface and $\partial_+ C$ will be a genus $n$ surface for some $n \geq m$. An $(m, n)$-compression body is one where $\partial_- C$ is a connected genus $m$ surface, and $\partial_+ C$ is a genus $n$ surface.

**Lemma 2.2.** *Let $F$ be a compact surface, let $D_1, \dots, D_k$ be a collection of pairwise-disjoint disks embedded in $F$ and let $D'_1, \dots, D'_k$ be a second collection of disjoint embedded disks. There is a homeomorphism, $\varphi : F \to F$, isotopic to the identity, such that $\varphi$ sends each disk $D_i$ onto the disk $D'_i$.*

*Proof.* See C. P. Rourke and B. J. Sanderson [18]. □

**Proposition 2.3.** *Any two $(m, n)$-compression bodies are homeomorphic.*

*Proof.* Suppose $C_1$ and $C_2$ are $(m, n)$-compression bodies. Then there is a a genus $m$ surface $S_1$ such that $C_1$ is constructed by taking $S_1 \times [0, 1]$ and attaching 1-handles. Similarly there is a genus $m$ surface $S_2$ such that $C_2$ is constructed by taking $S_2 \times [0, 1]$ and attaching 1-handles. Now there is a homeomorphism $\psi : S_1 \times [0, 1] \to S_2 \times [0, 1]$. Let $D_1, \dots, D_n, D'_1, \dots, D'_n$ be disks in $S_1 \times \{1\}$ so that attaching $D_i$ to $D'_i$ yields $C_1$. Let $E_1, \dots, E_n, E'_1, \dots, E'_n$ be disks in $S_2 \times \{1\}$ so that attaching $E_i$ to $E'_i$ yields $C_2$. By lemma 2.2 there is a homeomorphism $\varphi : S_1 \times \{1\} \to S_2 \times \{1\}$ such that $\varphi$ sends $\psi(D_i)$ to $E_i$ and $\psi(D'_i)$ to $E'_i$. This map extends to a homeomorphism $\hat{\varphi} : S_1 \times [0, 1] \to S_2 \times [0, 1]$ by taking $\varphi$ on each level set $S_1 \times \{t\}$. Composing $\hat{\varphi} \circ \psi$ gives a homeomorphism taking each $D_i$ to $E_i$ and each $D'_i$ to $E'_i$. Composing this with the quotients identifying each $D_i$ to $D'_i$ and each $E_i$ to $E'_i$ gives a homeomorphism from $C_1$ to $C_2$. □

## Chapter 3. The Topology of Compression Bodies

We now develop machinery that helps understand the topology of compression bodies. Many of the proofs are generalizations of Johnson's notes on Heegaard splittings, which proved the results in the case of handlebodies [11]. We develop the notion of a system of disks, and disk slides. We show that any two minimal systems of disks are slide equivalent. We also develop the notion of a spine, and show how a spine of a compression body relates to a system of disks.

## 3.1 Systems of Disks and Disk Slides

**Definition 3.1.** If $C$ is a compression body, a *system of disks* for $C$ is a collection $\{D_1, \ldots, D_n\}$ of properly embedded essential disks such that the complement of a regular neighborhood of $\bigcup_{i=1}^n D_i$ in $C$ is a collection of balls and the manifold $\partial_- C \times [0,1]$.

**Proposition 3.2.** *Given an $(m,n)$-compression body $C$, there is a system of disks $\mathbf{D}$ for $C$.*

*Proof.* Let $C$ be an $(m,n)$-compression body, $S$ be a closed genus $m$ surface, and $D_1, \ldots, D_n, D_1', \ldots, D_n'$ disks in $S \times \{1\} \subseteq S \times [0,1]$ so that gluing each $D_i$ to $D_i'$ yields $C$. We show that $D_1, \ldots, D_n$ forms a system of disks for $C$. Let $N$ be a regular neighborhood of $\bigcup_{i=1}^n D_i$. Then by the construction of $C$, it is clear that $C \backslash N$ is homeomorphic to $S \times [0,1]$. All that remains to be shown is that each $D_i$ is essential. If $D_i$ is not essential, then $D_i$ and a disk on $\partial C$ bound a ball in $C$. Thus $C \backslash N$ contains a ball component, yet it is homeomorphic to $S \times [0,1]$, a contradiction. Therefore $\{D_1, \ldots, D_n\}$ forms a system of disks for $C$. □

**Definition 3.3.** A system $\mathbf{D}$ of disks is *minimal* if the complement of a regular neighborhood of $\bigcup_{i=1}^n D_i$ in $C$ is homeomorphic to $\partial_- C \times [0,1]$.

**Lemma 3.4.** *If $C$ is an $(m,n)$-compression body with connected negative boundary, then a system of disks $\{D_1, \ldots, D_k\}$ for $C$ is minimal if and only if $k = n - m$.*

*Proof.* Suppose $\mathbf{D} = \{D_1, \ldots, D_k\}$ is a minimal system of disks. Then $C \backslash N$, where $N$ is a regular neighborhood of $\mathbf{D}$, is homeomorphic to $S \times [0,1]$ for some genus $m$ surface $S$. Note that $S \times \{1\}$ (which we identify with $S$) is a genus $m$ surface. This surface contains disks $E_i, E_i'$ parallel to each $D_i \in \mathbf{D}$. Becuase $S$ has genus $m$, we can take a collection of $m$ nontrivial simple closed curves $\alpha_1, \ldots, \alpha_m$ such that $S \backslash (\bigcup_{i=1}^m \alpha_i)$ is a planar surface. Since $\partial E_i$ and $\partial E_i'$ bound disks in $S$, we can isotope the curves $\alpha_1, \ldots, \alpha_m$ so that they are disjoint from each $E_i, E_i'$.

Suppose $k > n - m$. Then the collection of curves $\partial D_1, \ldots, \partial D_k, \alpha_1, \ldots, \alpha_m$ cut $\partial_+ C$ into a connected planar surface. However, this collection consists of $k + m > n - m + m = n$

simple closed curves. Since $\partial_+ C$ has genus $n$, cutting along any collection of $(n+1)$ or more curves must yield a disconnected surface, a contradiction.

Suppose $k < n - m$. Since $C \backslash N$ is homeomorphic to $S \times [0,1]$, which is connected, the result of removing a regular neighborhood of the curves $\partial D_1, \ldots, \partial D_k$ in $\partial_+ C$ is a connected genus $n-k$ surface with $2k$ punctures. Note that $n-k > n-n+m = m$, so the resulting surface has genus strictly greater than $m$. However, the result of removing a regular neighborhood of $\partial D_1, \ldots, \partial D_k$ in $\partial_+ C$ is homeomorphic to a punctured $S \times \{1\}$ which has genus $m$, a contradiction. Therefore $k = n - m$.

Conversely, suppose that $\mathbf{D} = \{D_1, \ldots, D_{n-m}\}$ is a system of disks for $C$, and that $N$ is a regular neighborhood of $\mathbf{D}$. If the complement $C \backslash N$ is connected, then $\mathbf{D}$ is a minimal system of disks. Assume by way of contradiction that $C \backslash N$ is not connected. Then $C \backslash N$ is homeomorphic to the disjoint union of $S \times [0,1]$ and a collection of balls $B_1, \ldots, B_k$, where $S$ is a genus $m$ surface and $k \geq 1$. Let $E_i$ and $E_i'$ be disks in the boundary of $C \backslash N$ parallel to $D_i$ for $i = 1, \ldots, n-m$. Without loss of generality, $E_1 \subseteq B_1$ and $E_1'$ is contained in $S \times [0,1]$ or $B_i$, where $i \neq 1$. Otherwise the result of identifying each pair $E_i, E_i'$ in $C \backslash N$, which is homeomorphic to $C$, contains a ball component and a component homeomorphic to $S \times [0,1]$, contradicting the fact that $C$ is connected. If $E_1' \subseteq B_i, i \neq 1$, then the result of gluing $B_1$ to $B_i$ along $D_1$ results in a ball. If $E_1' \subseteq S \times [0,1]$ then the result of gluing $B_1$ to $S \times [0,1]$ along $D_1$ is homeomorphic to $S \times [0,1]$. Therefore the complement of a regular neighborhood of the disks $D_2, \ldots, D_{n-m}$ in $C$ is homeomorphic to the disjoint union of $S \times [0,1]$ and balls $B_2, \ldots, B_k$. Repeating this process we see that the complement of the disks $D_{k+1}, \ldots, D_{n-m}$ in $C$ is homeomorphic to $S \times [0,1]$. Therefore removing a regular neighborhood in $\partial_+ C$ of $\partial D_{k+1}, \ldots, \partial D_{n-m}$ results in a punctured genus $m$ surface. However, removing a regular neighborhood in $\partial_+ C$ of $n-m-k$ nontrivial simple closed curves in such a way that the result is connected must be a genus $n-(n-m-k) = m+k$ surface with punctures. Since $k \geq 1$, this is a contradiction. Therefore $\mathbf{D}$ is minimal. □

Let $C$ be an $(m,n)$-compression body and $\mathbf{D} = \{D_1, \ldots, D_n\}$ be a system of disks for

$C$. Let $N$ be a regular neighborhood of $\mathbf{D}$. Then $C\backslash N$ is $\partial_- C \times [0,1]$ and a collection of balls. The boundary of $C\backslash N$ contains disks $E_i, E_i'$ parallel to $D_i$. Assume that two disks, $E_i$ and $E_j$ ($i \neq j$), are in the same component of $C\backslash N$. Let $\alpha$ be an arc from $E_i$ to $E_j$ whose interior is disjoint from $E_i \cup E_j$. Let $N'$ be a regular neighborhood in $C$ of $E_i \cup \alpha \cup E_j$. Then $\overline{N}'$ is a closed ball which intersects $\partial_+ C$ in a three-punctured sphere. The set $\partial N' \backslash \partial C$ consists of three disks: one parallel to $D_i$, one parallel to $D_j$, and another disk $D_i *_\alpha D_j$. Let $\mathbf{D}' = \{D_1, \ldots, \hat{D}_i, \ldots, D_n, D_i *_\alpha D_j\}$, where as usual $\hat{D}_i$ means remove $D_i$ from the collection. See figure 3.1.

We will show in lemma 3.6 that $\mathbf{D}'$ is a system of disks. This enables us to make the following definitions.

**Definition 3.5.** Two systems of disks are *isotopic* if there is an isotopy of $C$ (not necessarily fixing the boundary pointwise) that takes one system of disks to the other. If $\mathbf{D}$ and $\mathbf{D}'$ are as constructed above, then a system of disks isotopic to $\mathbf{D}'$ is said to be a *disk slide* of $\mathbf{D}$. Two systems of disks $\mathbf{D}$ and $\mathbf{D}'$ are said to be *slide equivalent* if there is a sequence of disk slides taking $\mathbf{D}$ to a system of disks isotopic to $\mathbf{D}'$.

It is not hard to see that slide equivalence is an equivalence relation, justifying the name.

**Lemma 3.6.** *If $C$ is an $(m,n)$-compression body, $\mathbf{D}$ is a system of disks and $\mathbf{D}'$ is the system of disks constructed as above, then $\mathbf{D}'$ is also a system of disks.*

*Proof.* The set $\partial N' \backslash \partial C$ constructed in the definition of the disk slide above consists of three disks: one parallel to $D_i$, one parallel to $D_j$ and the disk $D_i *_\alpha D_j$. Let $M$ be the result of removing a regular neighborhood of $D_i *_\alpha D_j$ from $C\backslash N$. Then $M$ is also the result of cutting off $N'$ from one of the components of $C\backslash N$. Since $N'$ is a ball, and $C\backslash N$ is a collection of balls and the manifold $S \times [0,1]$, the manifold $M$ is a collection of balls and the manifold $S \times [0,1]$. Since $M$ is also the result of removing a regular neighborhood of $\mathbf{D}'' = \mathbf{D}' \cup \{D_i *_\alpha D_j\}$ from $C$, we have that $\mathbf{D}''$ is a system of disks.

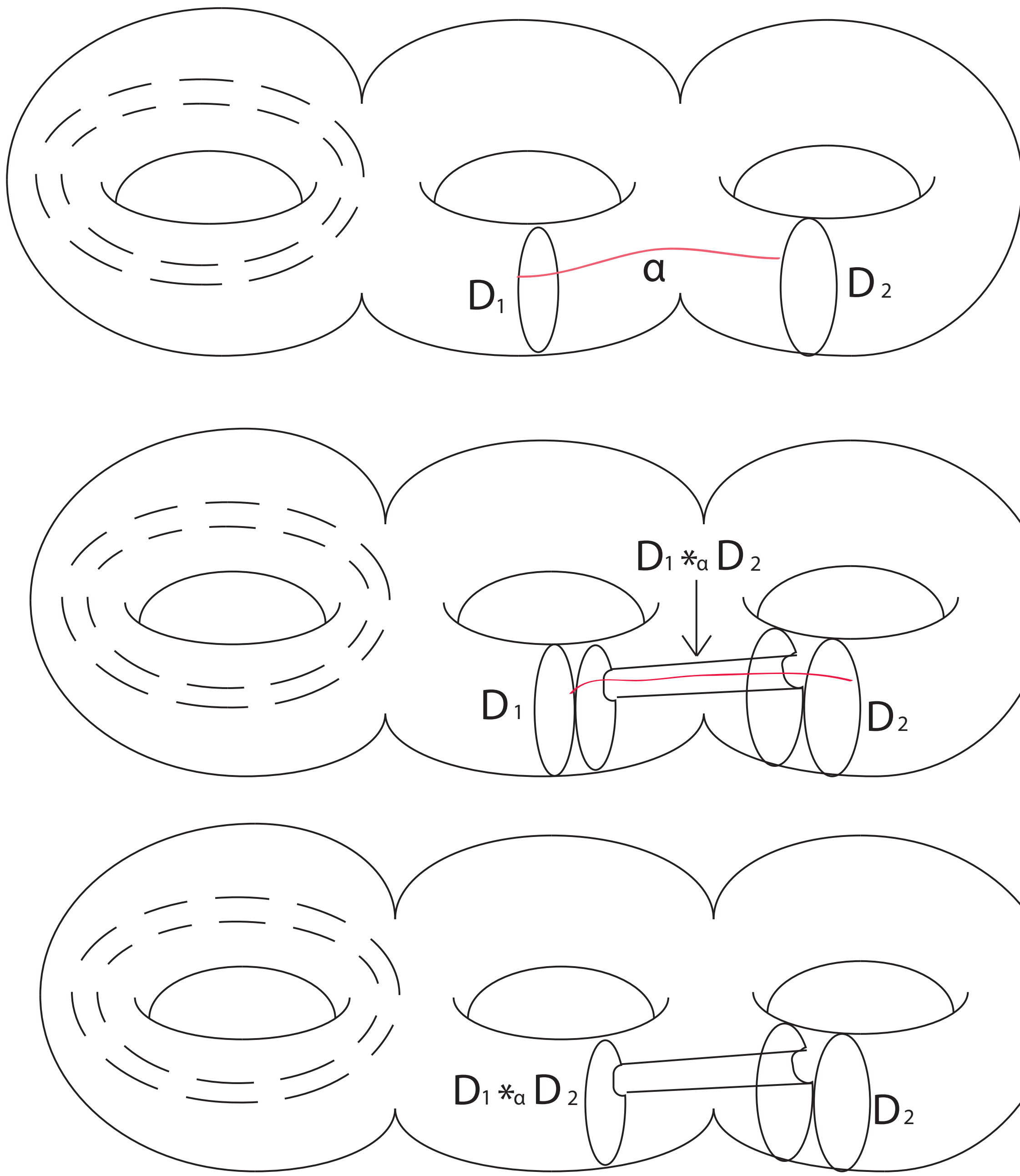


Figure 3.1: A disk slide in a (1, 3)-compression body.

Since $\mathbf{D}'$ is the result of removing the disk $D_i *_\alpha D_j$ from $\mathbf{D}''$, removing a regular neighborhood of $\mathbf{D}'$ from $C$ is also the result of attaching $\overline{N'}$ to one of the components of $M$ along the disks $E_i, E_i'$. Since $N'$ is a ball and $E_i$ and $E_i'$ are in different components, this results in a collection of balls and $S \times I$. Therefore $\mathbf{D}'$ is a system of disks. □

**Lemma 3.7.** *If two minimal systems of disks are disjoint, they are slide equivalent.*

*Proof.* Let $\mathbf{D} = \{D_1, \ldots, D_n\}$ and $\mathbf{D}' = \{D_1', \ldots, D_n'\}$ be disjoint minimal systems of disks for a compression body $C$. Because $\mathbf{D}$ is minimal, the complement of a neighborhood of $\mathbf{D}$ in $C$ is a manifold $M$ homeomorphic to $\partial_- C \times [0,1]$. The closure of a regular neighborhood of $D_i$ intersects $M$ in two disks $E_i, E_i'$ in the boundary of $M$. Since each disk of $\mathbf{D}'$ is properly embedded, we may assume that its boundary is disjoint from each $E_i$ and $E_i'$.

The disks of $\mathbf{D}'$ cut $M$ into $n+1$ components: $n$ balls and one component $M'$ homeomorphic to $M$. If one of the ball components does not contain some $E_i$ or $E_i'$, then a disk of $\mathbf{D}'$ is boundary parallel, which is impossible since the disks of $\mathbf{D}'$ are essential. Suppose $M'$ contains none of the disks $E_i$ or $E_i'$. Then the disks of $\mathbf{D}'$ cutting off $M'$ from $M$ cut off at least two components of $C$: a component containing $E_1$ and the component $M'$. Therefore the complement of a regular neighborhood of $\mathbf{D}'$ in $C$ is not connected, hence is not $\partial_- C \times [0,1]$, contradicting the fact that $\mathbf{D}'$ is minimal.

Because there are $n+1$ components of $M \backslash \mathbf{D}'$ and $2n$ disks $E_1, \ldots, E_n, E_1', \ldots, E_n'$ in the boundary of $M \backslash \mathbf{D}'$, and every component contains at least one such disk, there are at least two components of $M \backslash \mathbf{D}'$ that contain exactly one disk $E_i$ or $E_i'$. Let $B$ be one of these components. Since there are two choices for $B$, we may assume that $B$ does not contain the negative boundary, and is therefore a ball. Without loss of generality, assume that $B$ contains the disk $E_1$. If $B$ is cut off by a single disk $D_k'$ of $\mathbf{D}'$, then $D_k'$ is isotopic to $E_1$ which is isotopic to $D_1$. Otherwise assume that $D_1', D_2', \ldots, D_k'$ are the disks of $\mathbf{D}'$ which cut off $B$. Because $\partial B \backslash (D_1 \cup D_1' \cup D_2' \cup \ldots \cup D_k')$ is connected, there is an arc $\alpha_k$ from $\partial D_1'$ to $\partial D_k'$ that is disjoint from the other disks. Replace $D_1'$ with the disk $D_1' *_{\alpha_k} D_k'$. This new disk and the disks $D_2', \ldots, D_{k-1}'$ now cut off a component containing $E_1$, and none of the other disks $E_i, E_i'$. Continuing by finding an arc $\alpha_{k-1}$ connecting $D_1' *_{\alpha_k} D_k'$ to $D_{k-1}'$, *et cetera* we see that we can replace the disk $D_1'$ with the disk $D_1' *_{\alpha_k} D_k' *_{\alpha_{k-1}} D_{k-1}' *_{\alpha_{k-2}} \ldots *_{\alpha_2} D_2'$ which cuts off a component of $M$ containing $E_1$, and none of the other disks $E_i, E_i'$. This new disk is therefore isotopic to $E_1$ which is isotopic to $D_1$. Therefore $\mathbf{D}'$ is slide equivalent to the system of disks $\{D_1, D_2', \ldots, D_n'\}$. The disks $D_2', \ldots, D_n'$ cut $M$ into $n$ components and there are $2(n-1)$ disks $E_2, \ldots, E_n, E_2', \ldots, E_n'$ in the boundary. Thus we may continue in the same manner as above to show that $\mathbf{D}'$ is slide equivalent to $\{D_1, D_2, D_3', \ldots, D_n'\}$. Repeating the process $n$ times shows that $\mathbf{D}$ is slide equivalent to $\mathbf{D}'$. □

**Theorem 3.8.** *Any two minimal systems of disks for an $(m, n)$-compression body are slide equivalent.*

*Proof.* By lemma 3.7, it suffices to show that any two minimal systems of disks can be made disjoint by disk slides. We may assume that the disks are transverse, so $D_i \cap D_j'$ is a (possibly empty) collection of embedded arcs and simple closed curves. If a component of $D_i \cap D_j'$ is a closed loop, then this loop bounds a disk in $D_j'$. An *innermost* loop in $D_j'$ is a loop $\ell$ in $D_i \cap D_j'$ such that the interior of the disk in $D_j'$ bounded by $\ell$ is disjoint from the disks of $\mathbf{D}$. If $D_j'$ intersects a disk of $\mathbf{D}$ in a closed loop, then $D_j'$ contains an innermost loop $\ell \subset (D_i \cap D_j')$ for some $i$. Let $E$ be the disk in $D_j'$ bounded by $\ell$. Now $C \backslash \mathbf{D}$ is homeomorphic to $S \times [0,1]$ and $\ell$ is a simple closed curve in $S \times \{1\}$. The disk $E$ is properly embedded in $C \backslash \mathbf{D}$, so $(C \backslash \mathbf{D}) \backslash E$ consists of $S \times [0,1]$ and a ball $B$. The boundary of $B$ consists of $E$ and a portion of $D_i$, so we can isotope $E$ across $B$ into $D_i$. This induces an isotopy of $D_j'$ that removes the loop $\ell$.

Assume $D_i \cap D_j'$ consists of properly embedded arcs for each $i, j$. Define $I(\mathbf{D}, \mathbf{D}')$ to be the number of arcs of intersection over all disks in $\mathbf{D}$ and $\mathbf{D}'$. We show that there is some minimal system of disks $\mathbf{D}''$ that is slide equivalent to $\mathbf{D}$ and $I(\mathbf{D}', \mathbf{D}'') = 0$.

Let $\mathbf{D}'' = \{D_1'', \ldots, D_m''\}$ be a minimal system of disks slide equivalent to $\mathbf{D}$ such that $I(\mathbf{D}', \mathbf{D}'')$ is minimal. We show that $I(\mathbf{D}', \mathbf{D}'') = 0$. Suppose, by way of contradiction, that $I(\mathbf{D}', \mathbf{D}'') \neq 0$. Then for some $j$ the intersection $D_j' \cap (\bigcup D_i'')$ is nonempty, so we can assume it consists of a collection of arcs. Each of these arcs separates $D'_j$ into two disks. We say that an arc is *outermost* when the interior of one of these disks is disjoint from $\mathbf{D}''$. Note that $D'_j \cap (\bigcup D''_i)$ contains an outermost arc $\alpha$. Let $E \subseteq D'_j$ be the disk disjoint from $\mathbf{D}''$, and let $D''_i \in \mathbf{D}''$ be the disk such that $\alpha \subseteq D'_j \cap D''_i$.

Because $\mathbf{D}''$ is minimal, its complement in $C$ is homeomorphic to $S \times [0,1]$. Each disk $D''_i$ is parallel to two closed disks $F_i, F'_i$ in $S \times \{1\}$. The disk $E \cap (S \times [0,1])$ is properly embedded, and its boundary consists of an arc in a disk $F_i$ and an arc disjoint from all other $F_k, F'_k$.

Let $N$ be a regular neighborhood of $E \cup F_i$. The set $\partial N \backslash (S \times \{1\})$ consists of two disks $E_1$ and $E_2$, see figure 3.2. Every arc of intersection in $E_1 \cap \mathbf{D}'$ will be an arc parallel to $D''_i \cap \mathbf{D}'$. Since there is no arc of intersection parallel to $\alpha$, the number of arcs in $E_1$ is strictly less than the number of arks in $D''_i$. Similarly the number of arcs in $E_2$ is strictly less than the number of arcs in $D''_i$. We reduce the number $I(\mathbf{D}', \mathbf{D}'')$ by showing there is a sequence of disk slides that replaces $D''_i$ with $E_1$ or $E_2$.

The complement $(C \backslash \mathbf{D}'') \backslash (E_1 \cup E_2)$ consists of three components: two homeomorphic to balls $B_1, B_2$ and one homeomorphic to $S \times [0,1]$. Without loss of generality we will assume that $F_i \subseteq B_1$. Now $F'_i$ lies in either $B_2$ or $S \times [0,1]$. If $F'_i \subseteq B_2$ then let $B' = B_2$. Otherwise let $B'$ be the result of gluing $B_1$ to $B_2$ along the disk $E_1$ or $E_2$, where $E_k$ with $k = 1$ or 2 lies in the boundary of $B_1$ and $B_2$, viewing $B_1, B_2 \subseteq (S \times [0,1])$. Then the boundary of $B'$ contains exactly one of $F_i, F'_i$. Assume without loss of generality that $\partial B'$ contains the disk $E_1$. The boundary of $B'$ also contains other disks $F_k, F'_k$. Let $G_1, \ldots, G_m$ be all such disks in the boundary of $B'$, with $G_1 = F_i$.

Now $B' \backslash (E_1 \cup (\bigcup G_k))$ is path connected, so we can take an arc $\beta_1$ from $G_1$ to $G_2$ disjoint from $E_1, G_3, \ldots, G_m$. Let $N_1$ be a regular neighborhood of $G_1 \cup G_2 \cup \beta_1$. We then obtain a disk slide by replacing $F_i$ with $G_1 *_{\beta_1} G_2$. Now $G_1 *_{\beta_1} G_2$ separates $B'$ into two components: $B''_1$ containing $G_1, G_2$ and $B''_2$ containing $G_3, \ldots, G_m$. Since $\partial B''_2 \backslash (G_1 *_{\beta_1} G_2 \cup E_1 \cup (\bigcup_{k=3}^m G_k))$ is path connected, we can find an arc $\beta_2$ in $\partial B''_2$ from $G_1 *_{\beta_1} G_2$ to $\partial G_3$ disjoint from all the other $G_k$ and $E_1$. By replacing $G_1 *_{\beta_1} G_2$ with the boundary of a regular neighborhood of $G_1 *_{\beta_1} G_2 \cup \beta_2 \cup G_3$ will again define a disk slide, and cut $B_2''$ into two components: one containing $G_1 *_{\beta_1} G_2$ and $G_3$ and one containing $G_4, \ldots, G_m$. Repeat this process until the second component does not contain any $G_k$. Then $G_1 *_{\beta_1} G_2 *_{\beta_2} * \ldots * G_m$ cuts off a ball from $B'$ separating all the $G_k$ from $E_1$. Therefore $G_1 *_{\beta_1} G_2 *_{\beta_2} * \ldots * G_m$ is isotopic to $E_1$. We have therefore created a sequence of disk slides that replaces $D_i''$ with $E_1$.

The above construction shows that if $I(\mathbf{D}', \mathbf{D}'') > 0$, we can construct a system of disks $\mathbf{D}'''$ with $I(\mathbf{D}', \mathbf{D}''') < I(\mathbf{D}', \mathbf{D}'')$. It follows by the minimality of $I(\mathbf{D}', \mathbf{D}'')$ that $I(\mathbf{D}', \mathbf{D}'') = 0$. Now lemma 3.7 implies that $\mathbf{D}''$ is slide equivalent to $\mathbf{D}'$. Since $\mathbf{D}''$ is also slide equivalent to $\mathbf{D}$ then $\mathbf{D}$ is slide equivalent to $\mathbf{D}'$. □

### 3.1.1 Spines and Edge Slides.

Throughout this subsection we will consider $C$ to be an $(m, n)$-compression body. Recall that $\partial_- C$ is a connected genus $m$ surface.

**Definition 3.9.** Let $K$ be a graph embedded in $C$ with some valence-one vertices possibly embedded in $\partial_- C$. Let $N$ be a regular neighborhood of $K \cup \partial_- C$. If $C \backslash N$ is homeomorphic to $\partial_+ C \times [0, 1]$ then $K$ is a *spine* for $C$. (See figure 3.3.)

**Definition 3.10.** A spine $K$ is *dual* to a system of disks $\mathbf{D}$ if each edge of $K$ intersects a single disk of $\mathbf{D}$ exactly once, each disk in $\mathbf{D}$ intersects an edge of $K$, each ball component of $C \backslash \mathbf{D}$ contains exactly one vertex of $K$, and all vertices of $K$ in the $\partial_- C \times [0, 1]$ component of $C \backslash \mathbf{D}$ are contained in $\partial_- C$. (See figure 3.3.)

**Proposition 3.11.** *Given a system of disks* $\mathbf{D}$ *for a compression body* $C$*, there is a spine dual to* $\mathbf{D}$*. This spine is unique up to isotopy.*

*Proof.* Let $\mathbf{D} = \{D_1, \ldots, D_n\}$ and let $N$ be a regular neighborhood of $\mathbf{D}$. Then $C \backslash N$ is homeomorphic to $(S \times [0, 1]) \bigcup_{i=1}^m B_i$ where $S$ is the interior boundary, and $\bigcup_{i=1}^m B_i$ is a disjoint union of balls. Let $E_1, \ldots, E_{2n}$ be the disks in the boundary of $C \backslash N$ parallel to some $D_k$. For $i = 1, \ldots, m$ let $v_i$ be a point in the interior of $\partial B_i$, and let $v_{i,k}$ be a point

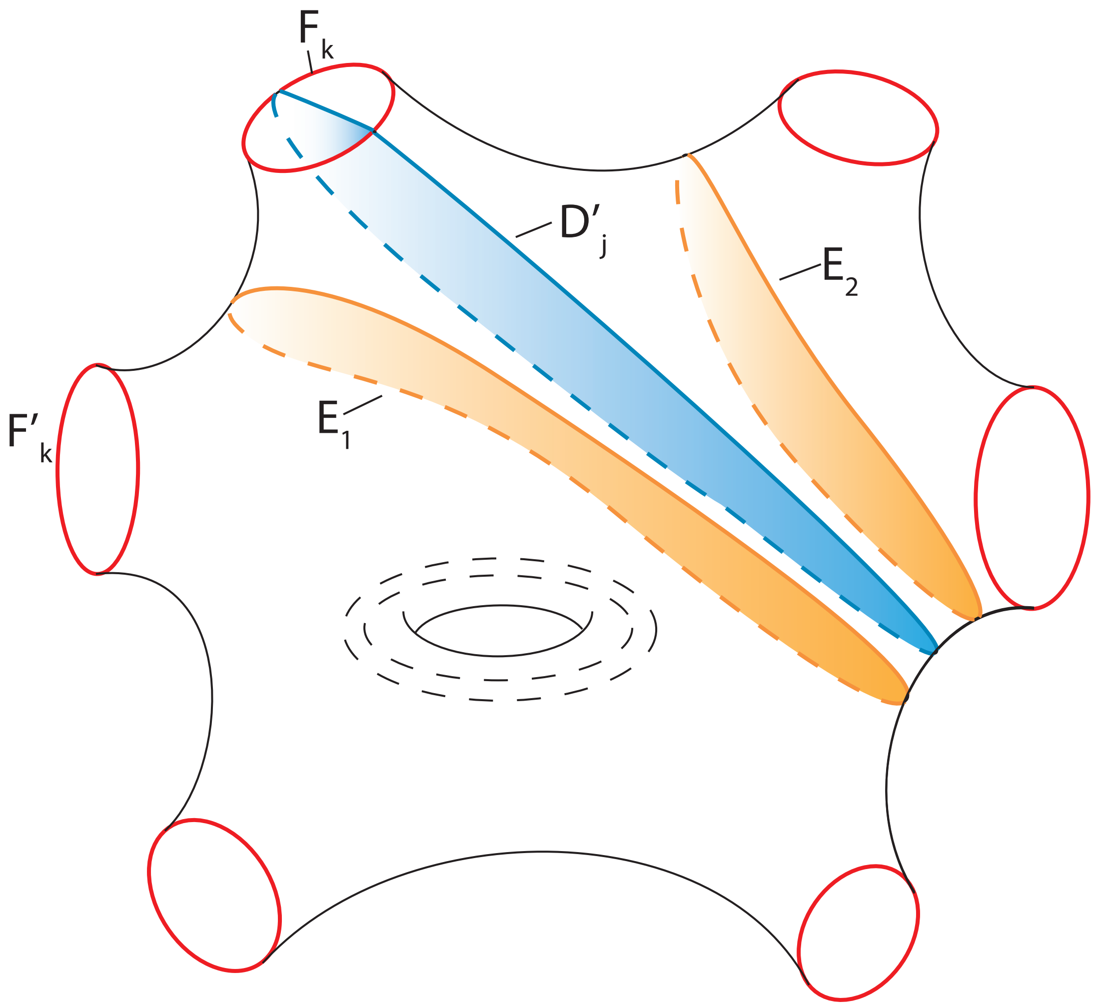


Figure 3.2: The disks $E_1, E_2$ in $S \times [0,1]$.

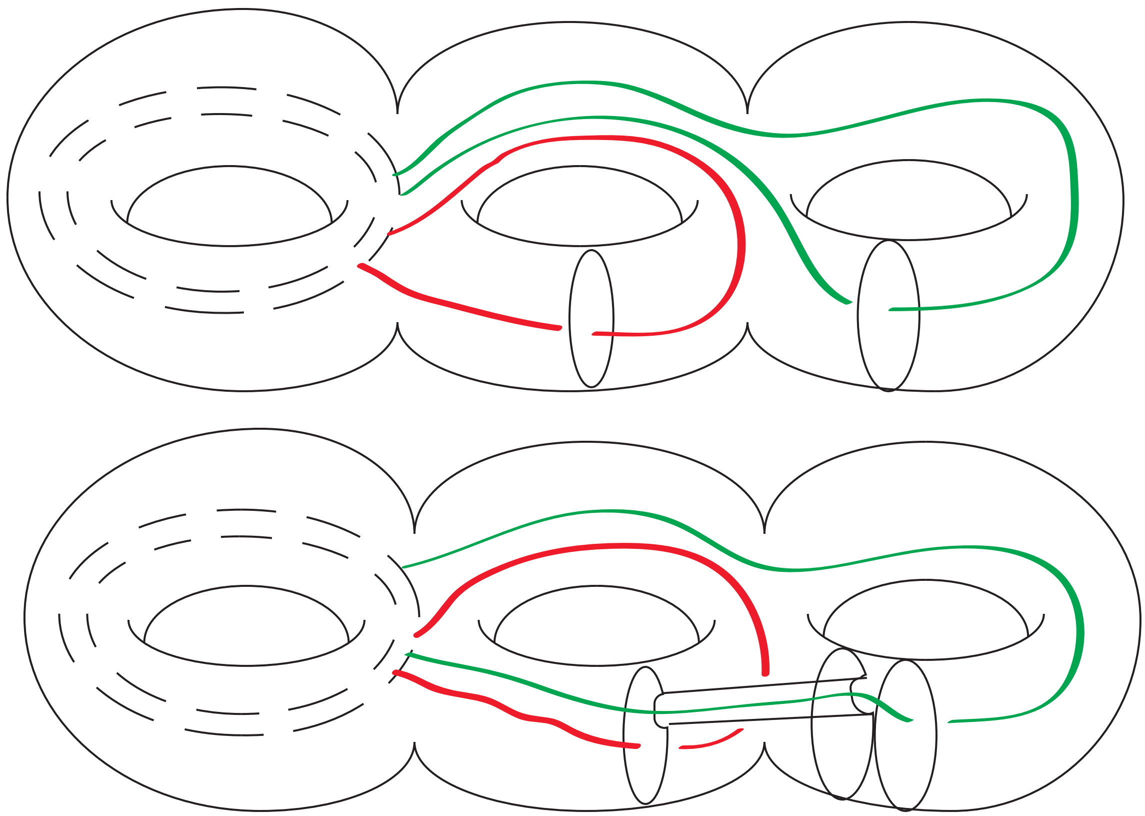

Figure 3.3: Two examples of spines dual to systems of disks in a $(1, 3)$-compression body.

on $\partial E_k$ for each $E_k \subseteq B_i$. Let $G_i$ be the graph consisting of one edge $e_{i,k}$ connecting $v_i$ to $v_{i,k}$ for each $k$. Isotope each $G_i$ so that $v_i$ lies in the interior of $B_i$ and each $v_{i,k}$ lies in the interior of $E_k$. Extend each $G_i$ in $C$ so that the endpoints $v_{i,k}$ lie in some $D_j$ and so that each edge intersects exactly one $D_j$ exactly once.

Assume without loss of generality that $E_1, \ldots, E_\ell$ lie in $S \times [0, 1]$. Choose points $(w_i, 1)$ in the interior of each $E_i$. Let $G_0$ be the graph consisting of the vertices $(w_i, 0), (w_i, 1)$ for each $i$ and the edges $e_i$, where $e_i$ is the straight line between $(w_i, 0)$ and $(w_i, 1)$. Extend $G_0$ in $C$ so that the endpoints of each $e_i$ not lying in $S$ lie in the disk $D_j$ parallel to $E_i$.

Isotope each $G_i$ slightly so that if $G_i$ and $G_j$ have vertices in the same $D_k$ then the vertices agree. Let $K = \bigcup_{i=0}^m G_i$. Then $K$ is dual to the system $D_k$ and forms a spine of $C$.

Suppose that $K$ and $K'$ are spines dual to the system of disks $\mathbf{D}$. Let $B_i$ for $1 \leq i \leq m$ be defined as above. The graph $K \cap B_i$ consists of a single vertex $v_i$ in the interior of $B$ and vertices $v_{i,j} \in E_j$ for each $E_j \subseteq B_i$, and edges $e_{i,j}$ between $v_i$ and $v_{i,j}$. The graph $K' \cap B$ consists of similar vertices $v'_i$ and $v'_{i,j}$ and edges $e'_{i,j}$. Let $N_i$ and $N'_i$ be regular neighborhoods of $B_i \cap K$ and $B_i \cap K'$ respectively. Then for each $E_j \subseteq B_i$ we have that $E_j \backslash N_i$ is an annulus embedded in the closure of a regular neighborhood of $\partial_+ C$ homeomorphic to $\partial_+ C \times [0, 1]$. This annulus can be isotoped so that it intersects each level surface $\partial_+ C \times \{t\}$ once. Therefore $B_i \backslash N_i$ is homeomorphic to $R \times [0, 1]$ where $R$ is a punctured sphere. Similarly $B_i \backslash N'_i$ is homeomorphic to $R \times [0, 1]$. Let $f : B_i \backslash N_i \to R \times [0, 1]$ and $g : R \times [0, 1] \to B_i \backslash N'_i$ be homeomorphisms. Then $g \circ f : B_i \backslash N_i \to B_i \backslash N'$ is a homeomorphism. Because $B_i \cap K$ and $B_i \cap K'$ are trees, the neighborhoods $N_i$ and $N'_i$ are balls. Therefore the homeomorphism $g \circ f$ extends to a homeomorphism $h_i$ of $B_i$. Because $N_i$ and $N'_i$ are regular neighborhoods, this map can be chosen to send $K$ to $K'$.

We now consider the component $S \times [0, 1]$ of $C \backslash N$. By doing a small isotopy, we may assume that the edges of $K'$ and $K$ meet at the same points on $S \times \{1\}$, since the spines $K$ and $K'$ are dual to the system of disks $\mathbf{D}$. We will now construct an isotopy $H_t$ of $S \times [0, 1]$ fixing the outer boundary. Let $\mathbb{E}$ be the collection of the disks in $S \times [0, 1]$ which are parallel to some $D_i \in \mathbf{D}$. Since the endpoints of $K$ on the outer boundary of $S \times [0,1]$ are the same as the endpoints of $K'$, then for some $\epsilon > 0$ we have that $K \cap (S \times [1-\epsilon, 1])$ and $K' \cap S \times [1-\epsilon, 1]$ are contained in $\mathbb{E} \times [1-\epsilon, 1]$. Since $\mathbb{E} \times [1-\epsilon, 1]$ is a collection of balls, we may isotope $\mathbb{E} \times [1-\epsilon, 1]$ so that $K$ is sent to $K'$. This induces an isotopy $G_t$ in $S \times [1-\epsilon, 1]$. We will now shrink $S \times [0,1]$ to $S \times [1-\epsilon, 1]$ in a way that will send $K$ homeomorphically to the portion of $K$ in $S \times [1-\epsilon, 1]$ as follows. By applying an isotopy if necessary, we can assume that the spine $K$ consists of vertical lines in $S \times [0,1]$. Define the map $s : S \times [0,1] \to S \times [1-\epsilon, 1]$ by $s(x,t) = (x, 1-\epsilon+\epsilon t)$. Define a similar map $s'$ shrinking $S \times [0,1]$ to $S \times [1-\epsilon]$ and sending $K'$ homeomorphically to the portion of $K'$ in $S \times [1-\epsilon, 1]$. Consider the ambient isotopy $H_t = (s')^{-1} \circ G_t \circ s$ of $S \times [0,1]$. It sends $K$ to $K'$ and fixes the endpoints of $K$ and $K'$.

$\square$

Let $e_1$ and $e_2$ be edges of a spine $K$ of a compression body $C$. Suppose each edge is parametrized by the closed interval $[0,1]$. Let $\alpha$ be a loop consisting of three smaller arcs: $\alpha_1$ the segment of $e_1$ from 1/3 to 0, $\alpha_2$ the edge $e_2$, and $\alpha_3$ some arc in the interior of $C$ connecting the final point of $\alpha_2$ to the initial point of $\alpha_1$ such that the loop $\alpha$ bounds a disk $D$ in $C$. Let $e'$ be the arc $\alpha_3$ followed by the segment of $e_1$ from 1/3 to 1. An example of an edge slide is shown in figure 3.4.

**Definition 3.12.** The graph $K'$ formed by replacing the edge $e_1$ in $K$ with $e'$ in the construction above is called an *edge slide* of $K$. We write $e' = e_1 *_D e_2$.

**Lemma 3.13.** *Let $K$ be a spine for a compression body $C$ and $K'$ an edge slide of $K$. Then $K'$ is a spine of $C$.*

*Proof.* Let $K$ be a spine and $K'$ an edge slide of $K$ defined by a disk $D$. Then the boundary of $D$ consists of three subarcs: $\alpha_1 \subseteq e_1, \alpha_2 = e_2$, and an arc $\alpha_3 \subset C$, where $e_1$ and $e_2$ are edges of $K$. Let $K'' = K \cup D$, let $N$ be a regular neighborhood of $K$ and let $N''$ be a neighborhood of $K''$. The closure of $K'' \backslash N$ is a disk which intersects $N$ in a single arc.

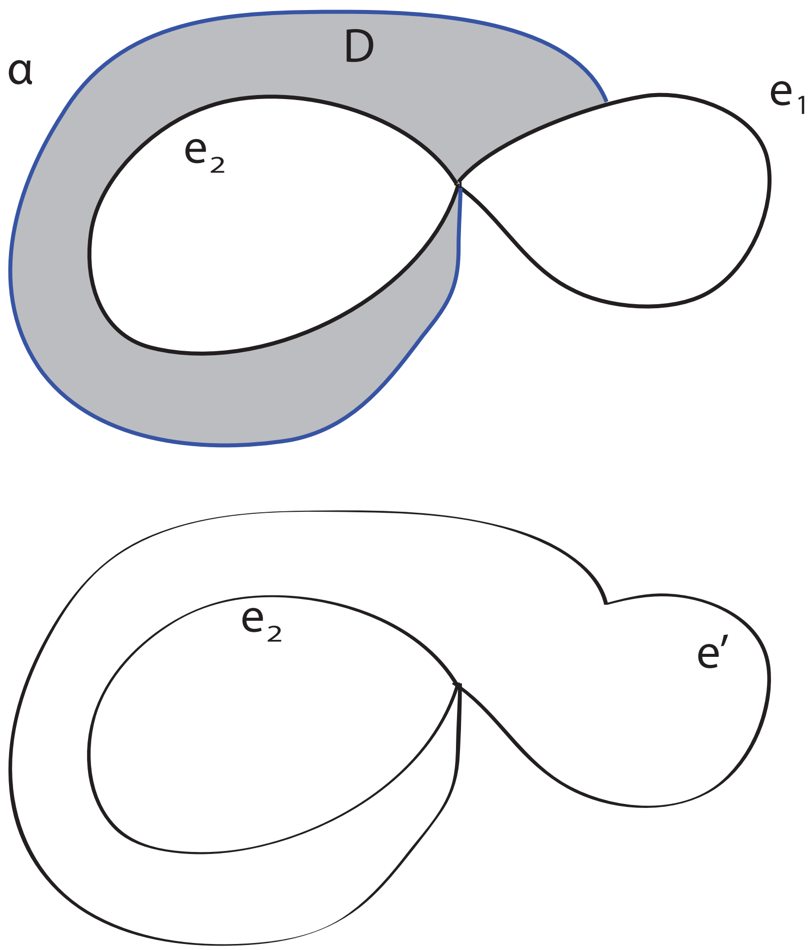


Figure 3.4: An edge slide of a spine.

Therefore $N''\backslash N$ is a ball whose closure intersects $N$ in a disk. This ball may be isotoped into $N$ which then induces an isotopy of $C$ sending $N''$ into $N$. Since $C\backslash N$ is homeomorphic to $\partial_+ C \times [0,1]$, we have that $C\backslash N''$ is homeomorphic to $\partial_+ C \times [0,1]$. Let $N'$ be a regular neighborhood of $K'$. In a similar manner as above, $N''\backslash N'$ is a ball whose closure intersects $N'$ in a single disk, hence $N''$ is ambient isotopic to $N'$. Therefore $C\backslash N'$ is homeomorphic to $\partial_+ C \times [0,1]$ implying $K'$ is a spine. □

# Chapter 4. The Geometry of Compression Bodies

In this and the remaining chapters, we will consider $C$ to be a $(1, n)$-compression body. We will consider hyperbolic structures on the compression body $C$. Throughout we will use the upper halfspace model of hyperbolic geometry. The geodesics consist of Euclidean semicircles orthogonal to the plane $z = 0$. We identify the plane $z = 0$ with $\mathbb{C}$. Geodesic planes in $\mathbb{H}^3$ are Euclidean hemispheres and vertical planes. A horosphere is a Euclidean hemisphere tangent to a point on the extended complex plane $\mathbb{C} \cup \{\infty\}$. The orientation preserving isometries of $\mathbb{H}^3$ correspond to elements of $\mathrm{PSL}(2, \mathbb{C})$ via Möbius transformations. A hyperbolic structure on $C$ is obtained by taking a discrete faithful representation $\rho : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$ and forming the quotient $M = \mathbb{H}^3/\rho(\pi_1(C))$. Much of the terminology we use in this chapter comes from [13].

## 4.1 Isometric Spheres and the Ford Domain

**Definition 4.1.** A discrete subgroup $\Gamma \leq \mathrm{PSL}(2, \mathbb{C})$ is *geometrically finite* if $\mathbb{H}^3/\Gamma$ admits a convex, finite sided fundamental domain. If $\Gamma$ is geometrically finite, we say the manifold $\mathbb{H}^3/\Gamma$ is *geometrically finite.*

**Definition 4.2.** A discrete subgroup $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is *minimally parabolic* if it has no rank one parabolic subgroups.

For a discrete, faithful representation $\rho : \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ of a $(1,n)$-compression body $C$, the image $\rho(\pi_1(C))$ will be minimally parabolic if for all $g \in \pi_1(C)$ we have the following property: $\rho(g)$ is parabolic if and only if $g$ is conjugate to an element of the fundamental group of the torus boundary component of $C$.

**Definition 4.3.** A discrete, faithful representation $\rho : \pi_1(C) \to PSL(2,\mathbb{C})$ is a *minimally parabolic geometrically finite uniformization of* $C$ if $\rho(\pi_1(C))$ is minimally parabolic and geometrically finite, and if $\mathbb{H}^3/\rho(\pi_1(C))$ is homeomorphic to $C$.

**4.1.1 Ford Domains.** Throughout this subsection, we will assume that $C = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold with a single rank 2 cusp. We are particularly interested in the case that $C$ is a $(1,n)$-compression body. We will view $\mathbb{H}^3$ with the upper half space model. We assume that the point at infinity projects to the cusp. If $H$ is a horosphere about infinity, we define the subgroup $\Gamma_\infty \leq \Gamma$ to be the subgroup fixing $H$. Since $\Gamma$ is minimally parabolic, we have $\Gamma_\infty \cong \mathbb{Z} \times \mathbb{Z}$.

**Definition 4.4.** Let $g \in \Gamma \backslash \Gamma_\infty$. Then $g^{-1}(H)$ is a horosphere centered at a point of $\mathbb{C}$, viewing $\partial\mathbb{H}^3$ as the extended complex plane. The *isometric sphere* of $g$ is the set $S_g$ of points in $\mathbb{H}^3$ equidistant from $H$ and $g^{-1}(H)$.

Isometric spheres in $\mathbb{H}^3$ are Euclidean hemispheres orthogonal to $\mathbb{C}$. The isometric sphere $S_g$ is well-defined, even if $H$ and $g^{-1}(H)$ intersect.

**Definition 4.5.** A *vertical fundamental domain* for $\Gamma_\infty$ is a fundamental domain for the action of $\Gamma_\infty$ which is cut out by finitely many vertical geodesic planes in $\mathbb{H}^3$.

**Definition 4.6.** Let $g \in \Gamma \backslash \Gamma_\infty$. The closure of the isometric sphere $S_g$ in $\mathbb{H}^3 \cup \mathbb{C}$ divides $\mathbb{H}^3 \cup \mathbb{C}$ into two components. Let $B_g$ be the interior of the ball component containing $g^{-1}(\infty)$. Define $\mathcal{F}$ to be

$$\mathcal{F} = \mathbb{H}^3 \backslash \bigcup_{g \in \Gamma \backslash \Gamma_\infty} B_g$$

We call $\mathcal{F}$ the *equivariant Ford domain.* The intersection of $\mathcal{F}$ with a vertical fundamental domain for $\Gamma_\infty$ is called a *Ford domain.*

The Ford domain of a manifold is not canonical because the choice of vertical fundamental domain is not canonical. However, the equivariant Ford domain is canonical.

**Lemma 4.7.** *Suppose that*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in PSL(2, \mathbb{C}).$$

*Then $S_{g^{-1}}$ is a Euclidean hemisphere centered at $g(\infty) = a/c$, and $S_{g^{-1}}$ has Euclidean radius $|c|^{-1}$.*

This lemma is well known and follows from a straightforward computation, so we will omit the proof. The lemma will help us to concretely visualize the Ford domain of a manifold.

It is well known (Proposition 5.7 [4]) that if $\Gamma < \mathrm{PSL}(2, \mathbb{C})$ is geometrically finite, then every convex fundamental domain for $\mathbb{H}^3/\Gamma$ has finitely many faces. Since Ford domains are convex fundamental domains, it follows that $C = \mathbb{H}^3/\Gamma$ is geometrically finite if and only if a Ford domain for $C$ has a finite number of faces.

**Example 4.8.** Let $C$ be a $(1, 3)$-compression body. Then $\pi_1(C) \cong (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z} * \mathbb{Z}$. We will choose generators $\alpha, \beta$ for $\mathbb{Z} \times \mathbb{Z}$ and let $\gamma, \delta$ be the generators of the other $\mathbb{Z}$ terms of the free product. Consider the representation

$$\rho(\alpha) = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} \qquad \rho(\beta) = \begin{bmatrix} 1 & 100i \\ 0 & 1 \end{bmatrix}$$
$$\rho(\gamma) = \begin{bmatrix} 0 & 1 \\ -1 & -5i \end{bmatrix} \qquad \rho(\delta) = \begin{bmatrix} -5 - 5i & -26 - 25i \\ 1 & 5 \end{bmatrix}$$

Set $\Gamma = \rho(\pi_1(C))$ and let $\Gamma_\infty \leq \Gamma$ be the subgroup of parabolics fixing $\infty$. Here we have chosen $\rho(\alpha)$ and $\rho(\beta)$ somewhat arbitrarily so that they give a very large parabolic translation length. Drawing the isometric spheres $S_{\gamma^{\pm 1}}, S_{\delta^{\pm 1}}$ gives us the picture in figure

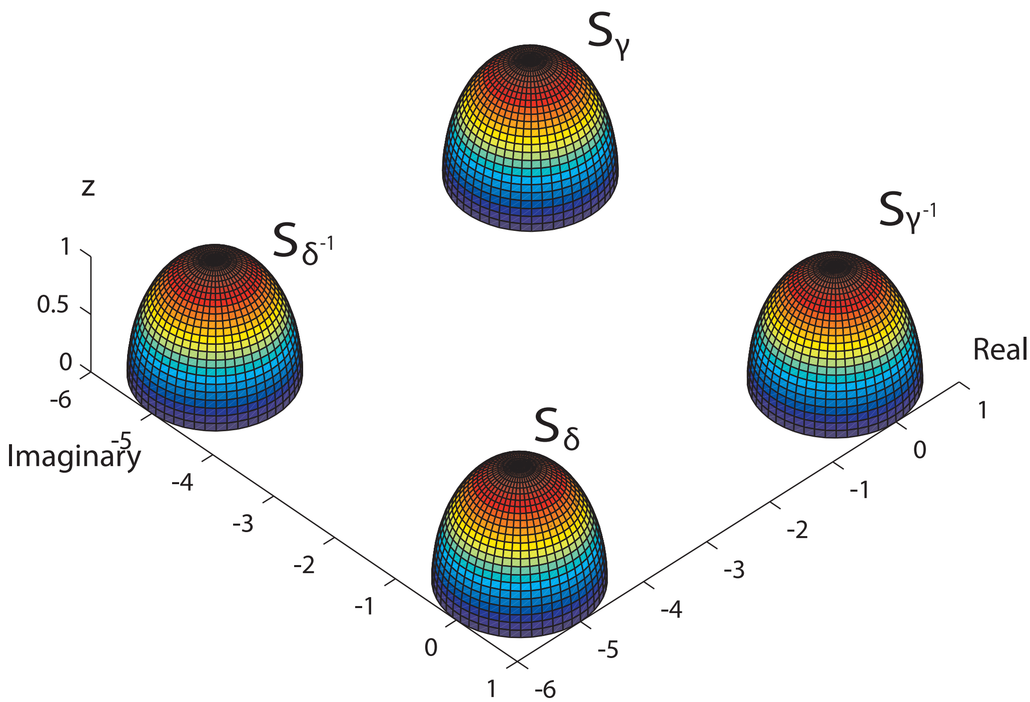


Figure 4.1: Part of one translate of the Ford Domain

4.1. If we draw other isometric spheres that are not parabolic translates of $S_{\gamma^{\pm 1}}, S_{\delta^{\pm 1}}$, these spheres will be hidden underneath other isomeetric spheres. We will make this notion more precise in definition 4.9. For example, if we draw $S_{\gamma\delta^{-1}}$ and $S_{\delta\gamma^{-1}}$ and look at the intersection of the isometric spheres with $\mathbb{C}$, we obtain the picture in figure 4.2. In fact, we will later show that if we drew every isometric sphere $S_g$, where $g \in \Gamma \backslash \Gamma_\infty$, the only isometric spheres that are"visible from $\infty$" will be $S_{\gamma^{\pm 1}}, S_{\delta^{\pm 1}}$ and their translates by elements of $\Gamma_\infty$.

We make precise the notion of visible isometric spheres.

**Definition 4.9.** Let $g \in \Gamma \backslash \Gamma_\infty$. The isometric sphere $S_g$ is *visible* if there exists an open set $U \subseteq \mathbb{H}^3$ such that $U \cap S_g \neq \emptyset$, and the hyperbolic distances satisfy

$$d(x, h^{-1}(H)) \geq d(x, H) = d(x, g^{-1}H)$$

for every $x \in U \cap S_g$ and $h \in \Gamma \backslash \Gamma_\infty$, where $H$ is some horosphere about infinity.

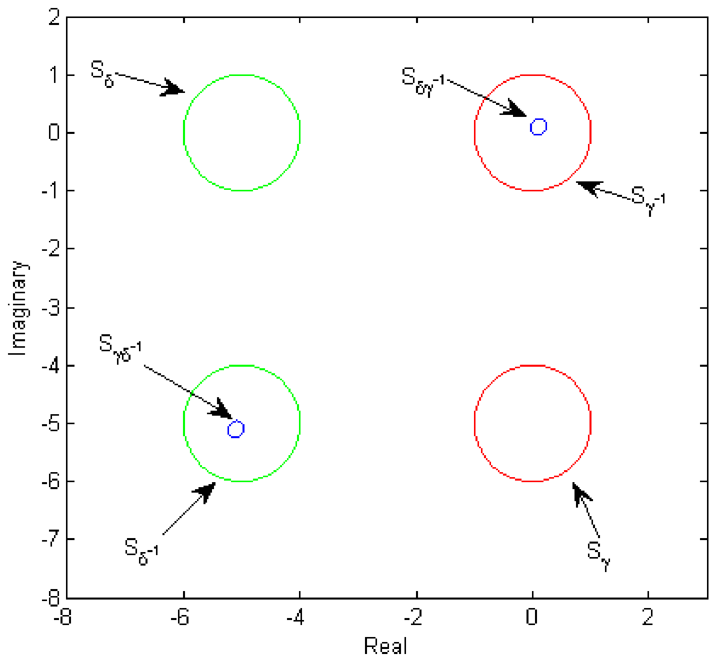


Figure 4.2: The intersection of part the isometric spheres $S_{\gamma^{\pm 1}}, S_{\delta^{\pm 1}}, S_{(\gamma\delta^{-1})^{\pm 1}}$ with $\mathbb{C}$

We say that the intersection $S_g \cap S_h$ of isometric spheres is *visible* if there exists an open $U \subseteq \mathbb{H}^3$ such that

$$d(x, f^{-1}H) \geq d(x, H) = d(x, g^{-1}H) = d(x, h^{-1}H)$$

for every $x \in U \cap S_g \cap S_h$ and $f \in \Gamma \backslash \Gamma_\infty$.

Intuitively, this definition means that for each $x \in U \cap S_g$ and each $h \in \Gamma \backslash \Gamma_\infty$, the point $x$ is not contained in the hemisphere bounded by $S_h$ and containing $h^{-1}H$.

The following fact may be found in [13].

**Lemma 4.10.** *For $\Gamma$ discrete, the following are equivalent.*

*(i) The isometric sphere $S_g$ is visible.*

*(ii) There exists a two dimensional cell of the sell structure on $\mathcal{F}$ contained in $S_g$. Additionally, if $\Gamma$ is geometrically finite, $S_g$ is visible if and only if $S_g \not\subseteq \bigcup_{h \in \Gamma \backslash (\Gamma_\infty \cup \Gamma_\infty g)} \bar{B}_h$.*

To visualize Ford domains, we will draw isometric spheres one by one. Because we are examining geometrically finite structures, we will only need to draw finitely many isometric spheres. Therefore we need a tool that will tell us when we have drawn all the isometric spheres in the Ford domain. To do this we use the Poincaré Polyhedron Theorem.

**Theorem 4.11** (Poincaré Polyhedron Theorem)**.** *Let $g_1, \dots, g_n \in PSL(2,\mathbb{C})$ and $\Gamma_\infty \cong \mathbb{Z} \times \mathbb{Z}$ be a subgroup of $PSL(2,\mathbb{C})$ consisting of parabolics fixing the point at infinity. Let $P$ be the polyhedron cut out by the isometric spheres corresponding to the $g_i^{\pm 1}$ and a vertical fundamental domain. Let $M$ be the object obtained from $P$ by gluing the isometric spheres corresponding to $g_i^{\pm 1}$ by the face pairing isometries $g_i$ for each $i$, and then gluing the faces of the vertical fundamental domain by elements of $\Gamma_\infty$. Assume that for each edge $e$ of $M$ (that is, for each equivalence class of intersections of isometric spheres under the equivalence given by gluing), the sum of the dihedral angles about $e$ is $2\pi$. Assume that the monodromy about $e$ is the identity. Then $M$ is a smooth hyperbolic manifold with $\pi_1(M) \cong \Gamma = \langle g_1, \dots, g_n \rangle$ and $\Gamma$ is discrete.*

The proof of this theorem can be found in (Theorem 2.21, [13]) as a result of Epstein and Petronio [10].

**Lemma 4.12.** *Let $\Gamma$ be a subgroup of $PSL(2,\mathbb{C})$ with rank 2 parabolic subgroup $\Gamma_\infty$ fixing the point at infinity. Suppose the isometric spheres corresponding to a finite set of elements of $\Gamma$, as well as a vertical fundamental domain for $\Gamma_\infty$, cut out a polyhedron $P$, so that face pairings given by the isometries corresponding to isometric spheres and to elements of $\Gamma_\infty$ yield a manifold with fundamental group $\Gamma$. Then $\Gamma$ is discrete and geometrically finite, and $P$ must be a Ford domain of $\mathbb{H}^3/\Gamma$.*

The proof of this result can be found in (Theorem 2.22, [13]).

Throughout this paper, we will draw numerous examples of Ford domains. In all cases, we may apply lemma 4.12 to show that the finite number of isometric spheres we draw cut out the entire Ford domain. As an example of how this lemma is applied, we show that the isometric spheres drawn in example 4.8 form a complete Ford domain.

**Proposition 4.13.** *The representation given in example 4.8 is a discrete, geometrically finite uniformization of a* $(1,3)$*-compression body, and its Ford domain is given by intersecting a vertical fundamental domain with the exterior of* $S_{\gamma^{\pm 1}}, S_{\delta^{\pm 1}}$.

*Proof.* Select a vertical fundamental domain containing the isometric spheres $S_{\gamma^{\pm 1}}$ and $S_{\delta^{\pm 1}}$. This is possible because $\rho(\alpha)$ and $\rho(\beta)$ have sufficiently large translation lengths. Let $P$ be the intersection of this fundamental domain with the exterior of the isometric spheres $S_{\gamma^{\pm 1}}$ and $S_{\delta^{\pm 1}}$. Identify the vertical sides of $P$ by elements of $\Gamma_\infty$. Then glue $S_\gamma$ to $S_{\gamma^{-1}}$ and $S_\delta$ to $S_{\delta^{-1}}$ by the maps $\rho(\gamma^{-1})$ and $\rho(\delta^{-1})$ respectively. Since $P$ has no edges, the Poincare polyhedron theorem implies that the result of applying these gluings to $P$ is a smooth manifold $M$. Then lemma 4.12 implies that $M$ is homeomorphic to $\mathbb{H}^3/\Gamma$.

Now viewing the manifold topologically, we see that the result of gluing together the faces of the vertical fundamental is $T^2 \times [0,1]$. The isometric spheres $S_{\gamma^{\pm 1}}$ and $S_{\delta^{\pm 1}}$ are then identified, which is equivalent to attaching one-handles. The result is then homeomorphic to the interior of a $(1,3)$-compression body. Therefore $\rho$ is a discrete geometrically finite uniformization of a $(1,3)$-compression body. □

Throughout this paper we will consider smooth paths of Ford domains. A smooth path of a Ford domain is a family of representations $\rho_t : \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ such that $\rho_t(x)$ varies smoothly for each generator of $\rho_t$. By taking a suitable path, we can cause isometric spheres to intersect. When isometric spheres intersect, new isometric spheres become visible as proved in the lemma below.

**Lemma 4.14.** *Let* $\Gamma$ *be a discrete torsion free subgroup of* $PSL(2,\mathbb{C})$*, with* $\Gamma_\infty \leq \Gamma$ *a rank two parabolic subgroup fixing* $\infty$*. Let* $\gamma, \delta \in \Gamma \backslash \Gamma_\infty$*, and assume* $S_\gamma, S_\delta$ *and* $S_\gamma \cap S_\delta$ *are visible. Then* $S_{\gamma\delta^{-1}} \cap S_{\delta^{-1}}$ *is visible and* $\delta$ *maps the visible portion of* $S_\gamma \cap S_\delta$ *isometrically to the visible portion of* $S_{\gamma\delta^{-1}} \cap S_{\delta^{-1}}$*. Additionally there exists some visible isometric sphere* $S_\eta$ *with* $S_\eta \neq S_{\delta^{-1}}$ *such that* $S_\eta \cap S_{\delta^{-1}} = S_{\gamma\delta^{-1}} \cap S_{\delta^{-1}}$.

*Proof.* Choose a horosphere $H$ about $\infty$ such that the horoball bounded by $H$ projects to an embedded neighborhood of the cusp in $M$. Since $S_\gamma \cap S_\delta$ is visible, there is an open set

$U \subseteq \mathbb{H}^3$ such that

$$d(x, \varphi^{-1}H) \geq d(x, H) = d(d, \gamma^{-1}H) = d(x, \delta^{-1}H) \tag{4.1.1}$$

for every $x \in U \cap S_\gamma \cap S_\delta$ and $\varphi \in \Gamma \backslash \Gamma_\infty$. Apply the isometry $\delta$ to $\mathbb{H}^3$. We then obtain

$$d(\delta(x), \delta\varphi^{-1}H) \geq d(\delta(x), \delta H) = d(\delta(x), \delta\gamma^{-1}H) = d(\delta(x), H)$$

for every $\varphi \in \Gamma \backslash \Gamma_\infty$. Therefore each $y = \delta(x) \in \delta(u) \cap S_{\gamma\delta^{-1}} \cap S_{\delta^{-1}}$ satisfies the inequality of definition 4.9, hence $S_{\gamma\delta^{-1}} \cap S_{\delta^{-1}}$ is visible.

Cover the 1-cell of the Ford domain containing $S_\gamma \cap S_\delta$ by open sets satisfying equation 4.1.1. Since the above argument applies for each $U_\alpha$, we see that $\delta$ maps the visible portion of $S_\gamma \cap S_\delta$ isometrically to the visible portion of $S_{\gamma\delta^{-1}} \cap S_{\delta^{-1}}$.

Since $S_{\gamma\delta^{-1}} \cap S_{\delta^{-1}}$ is visible, it contains a one dimensional cell of the Ford domain, hence there is some two dimensional cell of the Ford domain adjacent to $S_{\gamma\delta^{-1}} \cap S_{\delta^{-1}}$. Since $S_\delta$ is visible, so is $S_{\delta^{-1}}$. Therefore one of these 2-cells is contained in $S_{\delta^{-1}}$. The other 2-cell is contained in some $S_\eta$, hence $S_\eta$ is visible for some $\eta$. □

Note that in lemma 4.14, the isometric sphere $S_\eta$ may equal $S_{\gamma\delta^{-1}}$, but this is not always the case, as shown in the following example.

**Example 4.15.** Consider the family of representations $\rho_t : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$ where $C$ is a $(1,3)$-compression body

$$\rho_t(\alpha) = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} \quad \rho_t(\beta) = \begin{bmatrix} 1 & 100i \\ 0 & 1 \end{bmatrix}$$
$$\rho_t(\gamma) = \begin{bmatrix} 0 & -1 \\ 1 & it \end{bmatrix} \quad \rho_t(\delta) = \begin{bmatrix} 1.05 + 2.5i & 6.3525 \\ 1 & 1.05 - 2.5i \end{bmatrix}$$

Here have chosen $\rho(\alpha)$ and $\rho(\beta)$ so that it is easy to choose a vertical fundamental domain.

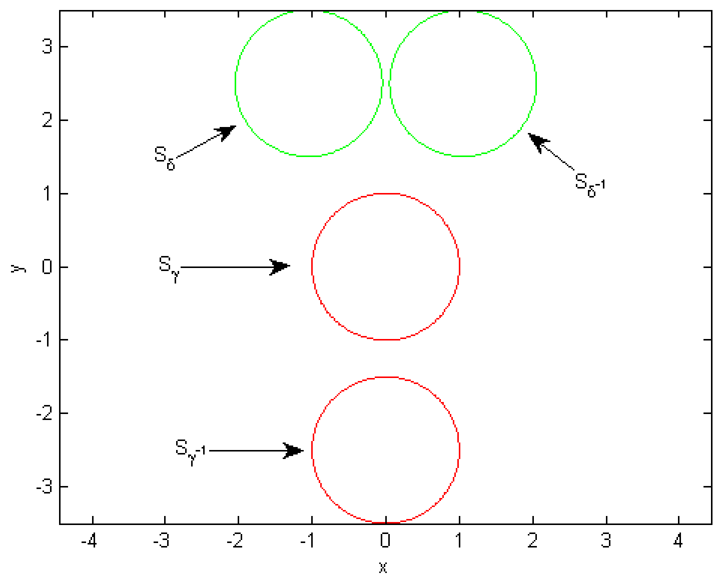


Figure 4.3: When $t = 2.5$ the isometric spheres do not intersect.

We will consider the representations where $t \in [1, 2.5]$. When $t = 2.5$ the Ford domain consists of the four visible isometric spheres in figure 4.3. When $t = 2.0$, the isometric spheres $S_\gamma$ and $S_{\gamma^{-1}}$ intersect and the isometric spheres $S_{\gamma^{\pm 2}}$ become visible, as predicted by lemma 4.14. Figure 4.4 below shows the Ford domain when $t = 1.9$ to give the picture of what happens when $S_\gamma$ and $S_{\gamma^{-1}}$ intersect. When $t = 400/363$ the isometric sphere $S_{\gamma^2}$ intersects the isometric spheres $S_{\delta^{\pm 1}}$ simultaneously, see figure 4.5. Lemma 4.14 tells us that the intersections $S_{\gamma^{-2}\delta} \cap S_{\gamma^2}, S_{\gamma^{-2}\delta^{-1}} \cap S_{\gamma^2}, S_{\gamma^{-2}} \cap S_{\delta^{-1}\gamma^2}$, and $S_{\gamma^{-2}} \cap S_{\delta\gamma^2}$ will be visible. However, the isometric spheres $S_{(\gamma^{-2}\delta)^{\pm 1}}$ and $S_{(\delta\gamma^2)^{\pm 1}}$ are not visible as they are hidden behind the isometric spheres $S_{\delta^{\pm 1}}$ and $S_{(\gamma^{-2}\delta\gamma^2)^{\pm 1}}$. These last two spheres became visible as a result of the intersection of $S_{\gamma^2}$ intersecting $S_{\gamma^{-2}\delta}$ and $S_{\gamma^{-2}\delta^{-1}}$. This becomes more apparent as we continue to decrease $t$ to $400/363 - .2$, as in figure 4.6.

While lemma 4.14 does not specify which isometric sphere becomes visible, it does guarantee that if isometric spheres begin to intersect along a path of Ford domains, then new isometric spheres will become visible. The work of Lackenby and Purcell [13] shows that this is the only way that an isometric sphere may become visible. In other words, if a path

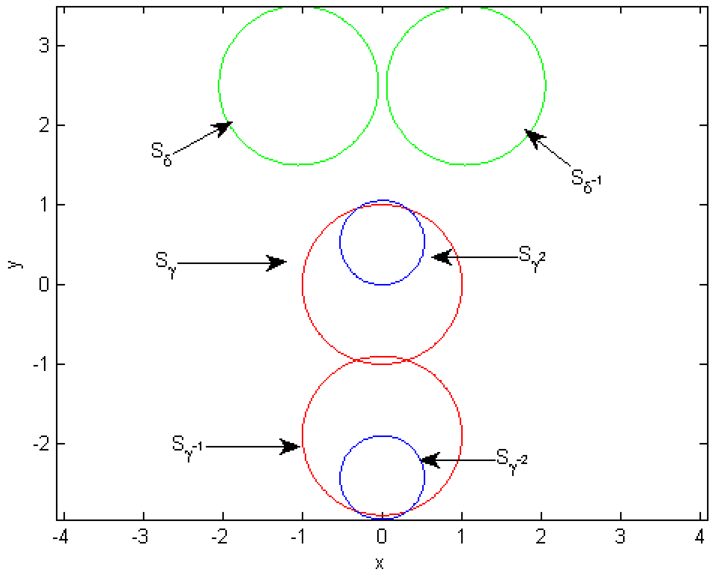


Figure 4.4: When $t = 1.9$, $S_\gamma$ and $S_{\gamma^{-1}}$ intersect and $S_{\gamma^{\pm 2}}$ become visible.

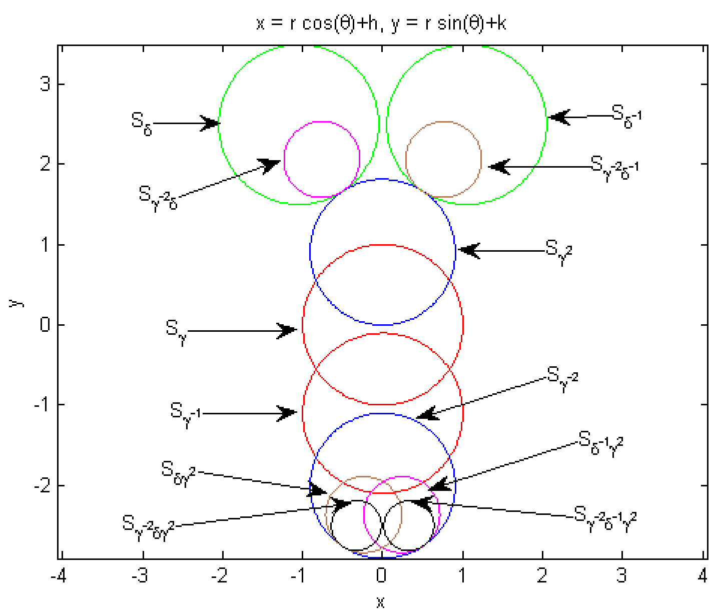


Figure 4.5: When $t = 400/363$, $S_{\gamma^2}$ intersects the isometric spheres $S_{\delta^{\pm 1}}$ simultaneously

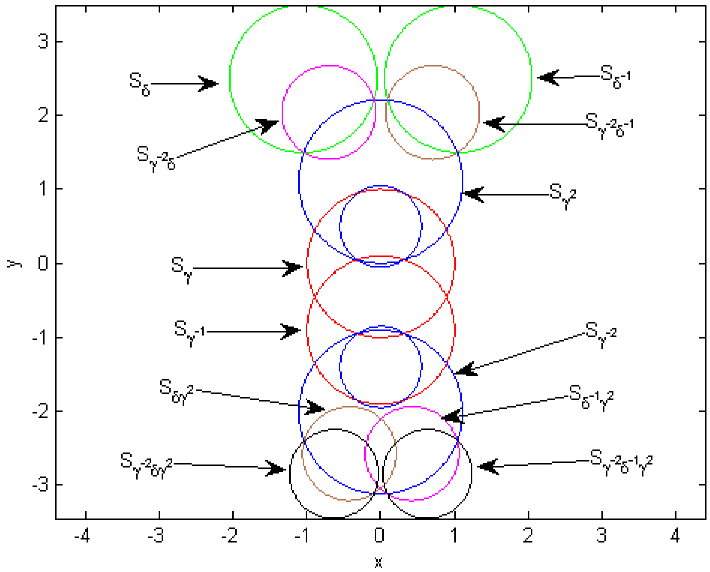


Figure 4.6: The isometric spheres $S_{(\gamma^{-2}\delta)^{\pm 1}}$ and $S_{(\delta\gamma^2)^{\pm 1}}$ remain invisible as $t$ decreases.

of Ford domains introduces no new intersections of isometric spheres, then no new isometric spheres will become visible.

By considering the intersection of visible isometric spheres, Lackenby and Purcell developed an algorithm to draw Ford domains [13].

**Algorithm 4.16** (Lackenby-Purcell Algorithm)**.** Begin with a choice of loxodromic generators $\gamma_1, \ldots, \gamma_n$ for $\Gamma$. Let $L_0$ and $L_1$ be lists. The list $L_0$ will consist of drawn isometric spheres, while the list $L_1$ will consist of isometric spheres to be drawn. Then perform the following steps:

(i) Draw the isometric spheres $S_{\gamma_1^{\pm 1}}, S_{\gamma_2^{\pm 1}}, \ldots, S_{\gamma_n^{\pm 1}}$ and add these isometric spheres to the list $L_0$.

(ii) For each pair of isometric spheres $S_\gamma$ and $S_\delta$ drawn in step (i) that intersect, add $S_{\gamma\delta^{-1}}$ and $S_{\delta\gamma^{-1}}$ to the list $L_1$.

(iii) Draw the first isometric sphere $S_\xi$ in the list $L_1$.

(iv) Add $S_\xi$ to $L_0$, and remove $S_\xi$ from $L_1$.

(v) If $S_\xi$ intersects an isometric sphere $S_\gamma$, add each $S_{\gamma\xi^{-1}}, S_{\xi\gamma^{-1}}$ not in $L_0$ or $L_1$ to the list $L_1$.

(vi) Repeat steps (iii) through (v) until the list $L_1$ is empty.

Lackenby and Purcell conjectured that this algorithm would eventually terminate when drawing Ford domains of $(1,2)$-compression bodies. In the case of $(1,n)$-compression bodies with $n \geq 3$, the algorithm does not always draw the Ford domain. In fact, algorithm 4.16 fails to draw the Ford domain in the final step of example 5.7, and the Ford domains in the proof of theorem 6.2. In all of these examples, an isometric sphere corresponding to one of the generators is not visible in the Ford domain. However, in these examples, there is a choice of generators such that the isometric spheres corresponding to these generators are visible in the Ford domain, and algorithm 4.16 draws the complete Ford domain. It is still open whether algorithm 4.16 will draw the Ford domain for some choice of generators.

We are interested in studying a tunnel system for a manifold. We can often identify the tunnel system with geometric duals of the Ford spine, which we now describe. The dual as described here is similar to the canonical polyhedral decompositions for finite volume manifolds described by Epstein and Penner [9]. Let $C$ be a $(1,n)$-compression body and assume $\pi_1(C) \cong \Gamma \leq \mathrm{PSL}(2,\mathbb{C})$. For each $S_\gamma$ where $\gamma \in \Gamma \backslash \Gamma_\infty$, there is an edge $e_\gamma$ which runs from the center of $S_\gamma$ to the point at infinity in $\mathbb{H}^3$. The edge $e_\gamma$ is called the dual to $S_\gamma$. Suppose that $\gamma_1, \ldots, \gamma_{n-1}$ are the loxodromic generators of $\Gamma$. In the next chapter we will show that in some cases collection of duals to the isometric spheres $S_{\gamma_1}, \ldots, S_{\gamma_n}$ correspond to a spine of the compression body $C$. In all cases, the collection of duals is homotopic to a spine of $C$.

# Chapter 5. A Geometric View of the Topology

In this chapter we discuss how we can use the geometric tools of chapter 4 to understand the topological tools of chapter 3 for $(1,n)$-compression bodies.

**Definition 5.1.** Let $C$ be a $(1,n)$-compression body, and let $\gamma_1, \ldots, \gamma_{n-1}$ be a minimal set of loxodromic generators of $\Gamma = \rho(\pi_1(C))$, where as usual $\rho : \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ is a discrete, faithful representation. A Ford domain $\mathcal{F}$ is called *simple* if $\{S_{\gamma_i}, S_{\gamma_i^{-1}} : 1 \leq i \leq n-1\}$ is the set of visible isometric spheres in $\mathcal{F}$, and none of the visible isometric spheres intersect.

If $\mathcal{F}$ is a simple Ford domain, then $\mathcal{F}$ is easy to understand. In particular, we show that in this case, the pairs of visible isometric spheres correspond to disks in a minimal system of disks for $C$.

**Theorem 5.2.** *If $\mathcal{F}$ is a simple Ford domain of a $(1,n)$-compression body $C$, with visible faces $\{S_{\gamma_i}, S_{\gamma_i^{-1}} : 1 \leq i \leq n-1\}$, then the closure of the image of the disks $S_{\gamma_1}, \ldots, S_{\gamma_{n-1}}$ under the action of $\Gamma$ forms a minimal system of disks for $C$.*

*Proof.* The action of the parabolic generators of $\Gamma$ glues up $\mathcal{F}$ to form a manifold homeomorphic to $T^2 \times (0,1)$. Since the disks $S_{\gamma_i}, S_{\gamma_i^{-1}}$ are disjoint, the action of $\gamma_i$ identifies $S_{\gamma_i}$ and $S_{\gamma_i^{-1}}$ , which is topologically equivalent to attaching a 1-handle. Similarly, the complement of a regular neighborhood of the $S_{\gamma_i}, S_{\gamma_i^{-1}}$ in $\mathcal{F}$ glues up to form a manifold homeomorphic to $T^2 \times (0,1)$. Therefore closure of the images of the isometric spheres $S_{\gamma_1}, \ldots, S_{\gamma_{n-1}}$ form a system of disks for $C$. The fact that the system is minimal follows from lemma 3.4. $\square$

**Theorem 5.3.** *Let $\rho : \pi_1(C) \to PSL(2,\mathbb{C})$ be a minimally parabolic geometrically finite uniformization of the $(1,n+1)$-compression body $C$. Suppose $\gamma_1, \ldots, \gamma_n$ are loxodromic generators of $\rho(\pi_i(C)) = \Gamma$. Let $\tilde{d}_i$ be the geodesic dual to $S_{\gamma_i^{-1}}$. Then under the quotient action of $\Gamma$, the dual edges $\tilde{d}_i$ are homotopic to a spine of $C$. If the Ford domain is simple and each $S_{\gamma_i}$ is visible, the edges $\tilde{d}_i$ form a spine of $C$.*

*Proof.* Take the closure a regular neighborhood $N$ of $\partial_- C$ so that the closure $\overline{N}$ is homeomorphic to $\partial_- C \times [0,1]$. Choose $p = (p', 1) \in \partial_- C \times \{1\}$ and let $q = (p', 0) \in \partial_- C \times \{0\}$. Let $f : [0,1] \to C$ be the straight line from $p$ to $q$.

In the universal cover $\mathbb{H}^3$, choose a vertical fundamental domain $D$ for $\Gamma$. We may take $D$ to contain $\gamma_i(\infty)$ for all $i = 1, 2, \ldots, n$. The lift $\tilde{p}$ of $p$ into $D$ is a point on a horoball $H$ about $\infty$. For each loxodromic generator $\gamma_i$ define $\tilde{p}_i = \gamma_i(\tilde{p})$. The point $\tilde{p}_i$ lies on a horosphere centered at $\gamma_i(\infty)$. For each $i = 1, 2, \ldots, n$, let $\tilde{g}_i$ be a geodesic arc in $D$ from $\tilde{p}$ to $\tilde{p}_i$. Under the action of $\Gamma$, the arc $\tilde{g}_i$ becomes a loop in the homotopy class of $\gamma_i$.

Let $\tilde{f}_i$ be a geodesic arc in $D$ from $\tilde{p}_i$ to $\gamma_i(\infty)$, and let $\tilde{f}'_i$ be a geodesic arc from $\infty$ to $\tilde{p}$. Under the action of $\Gamma$, the closure of the quotient of the arcs $\tilde{f}_i$ and $\tilde{f}'_i$ in $C$ become an arcs from $p$ to points on $\partial_- C$, which are homotopic to $f$ rel $p$, and the homotopy may be taken to keep an endpoint of each of the arcs on $\partial_- C$. Set $\tilde{h}_i$ to be the arc $\tilde{f}'_i$ followed by $\tilde{g}_i$ followed by$\tilde{f}_i$. Then $\tilde{h}_i$ runs from $\infty$ to $\gamma_i(\infty)$. Therefore $\tilde{d}_i \simeq \tilde{h}_i$. Since under the action of $\Gamma$, the arcs $\tilde{h}_i$ together with $\partial_- C$ form a spine of $C$, the edges $\tilde{d}_i$ form a spine of $C$.

We now consider the case where the Ford domain $\mathcal{F}$ is simple. We follow an argument similar to the proof of lemma 3.11 in [13] to show that $\mathcal{F}$ deformation retracts to the union of the geodesic duals to the visible isometric spheres, and a horoball $H$ about infinity disjoint from the isometric spheres of the Ford domain. We will construct this deformation retractso that it descends to a deformation retract of $C$, hence the dual edges of the Ford domain glue up to form a spine for $C$.

Since there are finitely many visible faces in $\mathcal{F}$, we may choose some $\epsilon > 0$ such that the Euclidean cylinders $C_i, C'_i$ of radius $\epsilon$ centered at $\gamma_i^{-1}(\infty)$ and $\gamma_i(\infty)$ respectively, do not intersect the geodesic duals. We may also take $\epsilon$ to be strictly less than the minimal radius of any visible face of the Ford domain. For $i = 1, \ldots, n$, let $D_i$ be the disk in $S_{\gamma_i}$ bounded by $S_{\gamma_i} \cap C_i$, and let $D'_i$ be the disk in $S_{\gamma_i^{-1}}$ bounded by $S_{\gamma_i^{-1}} \cap C'_i$. Let $H_1$ be the boundary of the horoball $H$, and define

$$H'_1 = \{p \in H_1 : \text{ the vertical line through } p \text{ does not intersect any isometric sphere of } \mathcal{F}\}.$$

The set $(H_1 \backslash H'_1) \cap \mathcal{F}$ consists of disks $E_i, E'_i$ corresponding to the isometric spheres $S_{\gamma_i}$ and $S_{\gamma_i^{-1}}$ respectively. Let $S$ be the result of isotoping $H_1$ in $\mathcal{F}$ so that $H'_1$ remains fixed, and so that each $E_i$ and $E_i'$ is isotoped to $D_i$ and $D_i'$ respectively.

Let $x \in \mathbb{H}^3 \cap \mathcal{F}$. The nearest point on $H_1$ to $x$ lies on a vertical line through $x$ to $\infty$. These vertical lines give a foliation of $\mathcal{F}$. We may construct $S$ above so that it meets each vertical line of this foliation exactly once. Let $f_1$ be the retraction of $\mathcal{F}$ to the union of $H$ and the region $R_1$ bounded by $H_1$ and $S$, by mapping each $x \in \mathcal{F} \backslash R_1$ to the intersection of $S$ with the vertical line through $x$. By constructing $S$ carefully, we may ensure that $f_1$ is equivariant with respect to the action of $\Gamma$ on $\mathcal{F}$. The horosphere $H_1$ can be given by the equation $z = c$ for some $c > 0$. For $t > 0$, let $H_t$ be the plane $z = ct$. To each $x_H \in H_t$ there corresponds a point $x_S \in S$ such that $x_H$ and $x_S$ lie on a vertical line. Let $\pi_3$ be the projection of $\mathbb{H}^3$ onto the $z$-coordinate. Let $S_t = \{x_H \in H_t : \pi_3(x_H) \leq \pi_3(x_S)\} \bigcup \{x_S \in S : \pi_3(x_S) \leq \pi_3(x_H)\}$. Let $R_t$ be the union of $H$ and the region bounded by $H_1$ and $S_t$.

We are now ready to define the deformation retraction of $\mathcal{F}$ to $S$. Let $f_0$ be the identity, and $f_1$ be as above. For $t \in (0, 1)$, define $f_t$ to be the retraction keeping $R_t$ fixed, and for each $x \in \mathcal{F} \backslash R_t$ let $f_t(x)$ be the intersection of $S_t$ with the vertical line through $x$. Note that the geodesic duals remain fixed for all maps $f_t$, therefore $f_t$ is a deformation retract of $\mathcal{F}$ to a regular neighborhood of the goedesic duals and the horoball $H$. We can extend this to a deformation retract of $\mathcal{F}$ to the union of the geodesic duals and $H$. Since each $f_t$ is equivariant with respect to the action of $\Gamma$, this descends to a deformation retract of $C$ to a neighborhood of $\partial_- C$ and the image of the geodesic duals under the quotient.

□

Given a minimal system of disks $\mathbf{D}$ for a compression body $C$ and a spine $K$, we can associate each edge $e_i$ of $K$ to a generator $\gamma_i$ of the fundamental group $\pi_1(C)$. After performing a disk slide, we obtain a new system of disks $\mathbf{D}'$ for $C$ and a spine $K'$ dual to $\mathbf{D}'$. We would like to understand how the edges of $K'$ relate to the edges of $K$ in terms of the fundamental group.

**Lemma 5.4.** *Let $\mathbf{D} = \{D_1, D_2, \ldots, D_{n-m}\}$ be a system of disks for an $(m, n)$-compression body. Let $K$ be the spine dual to $\mathbf{D}$ consisting of the edges $e_1, \ldots, e_{m-n}$ where each $e_i$ is dual* *to the disk* $D_i$*. For each* $e_i$*, let* $\gamma_i$ *be the corresponding generator in* $\pi_1(C)$*. Let* $S \times [0,1]$ *be the result of removing a regular neighborhood of* $\mathbf{D}$ *from* $C$*. Let* $E_k, E'_k$ *be the disks in* $S \times [0,1]$ *which are parallel to* $D_k$*. For each edge* $e_k$ *of* $K$*,* $e_k \cap (S \times [0,1])$ *consists of arcs* $d_k$ *and* $d'_k$ *running from a vertex of the spine to* $E_k$ *and* $E'_k$ *respectively. Fix* $i, j$ *with* $i \neq j$*. Let* $\omega$ *be a loop in* $S \times [0,1]$ *consisting of the following six subarcs:*

- $\omega_1 = d_j$
- $\omega_2$ *in* $E_j$ *from the endpoint of* $\omega_1$ *to* $\partial_+ C$
- $\omega_3$ *in* $\partial_+ C$ *from the endpoint of* $\omega_2$ *to* $E_i$*, disjoint from all* $E_k, E'_k$ *except* $E_i$ *and* $E_j$*.*
- $\omega_4$ *in* $E_i$ *from the endpoint of* $\omega_3$ *to the endpoint of* $d_i$
- $\omega_5 = d_i$
- $\omega_6$ *in* $\partial_- C$ *connecting the endpoints of* $d_j$ *and* $d_i$ *in such a way that* $\omega$ *will be homotopically trivial in* $S \times [0,1]$*.*

*Let* $e'$ *be the loop in* $C$ *consisting of the following subarcs:*

- $e'_1 = d'_j$
- $e'_2 = \omega_2$
- $e'_3 = \omega_3$
- $e'_4 = \omega_4$
- $e'_5 = d'_i$
- $e'_6 = \omega_6$

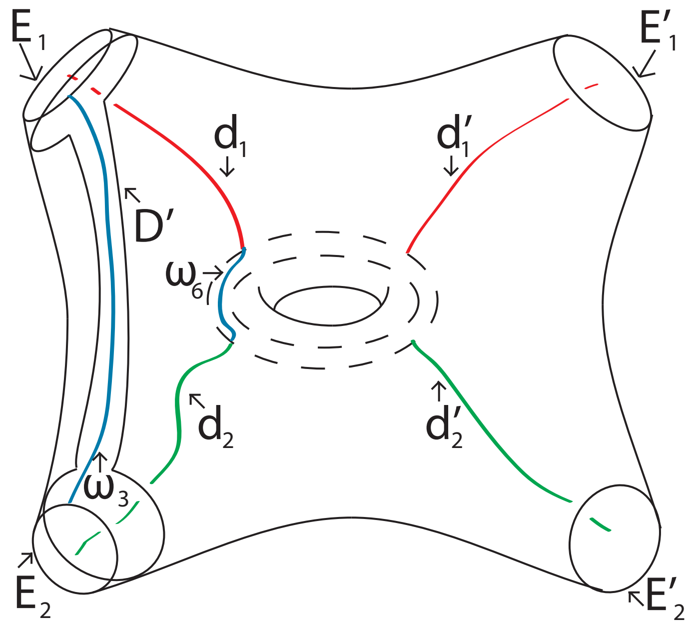


Figure 5.1: Disks and edges in the construction of lemma 5.4

*Set* $D' = D_i *_{\omega_3} D_j$ *and* $\mathbf{D'} = \{D_1, D_2, \ldots, \hat{D}_i, \ldots, D_{n-m}, D'\}$ *(where* $\hat{D}_i$ *means omission). Then the graph* $K'$ *consisting of the edges* $e_1, \ldots, \hat{e}_j, \ldots, e_{m-n}, e'$ *forms a spine dual to* $\mathbf{D'}$*. Here* $e'$ *is dual to* $D_j$ *and* $e_i$ *is dual to* $D'$*, and* $e_k$ *is dual to* $D_k$ *for all* $k \neq i, j$*. Moreover,* $e' \simeq \gamma_j^{-1}\omega\gamma_i$ *rel* $\partial\_C$*. See figure 5.1*

*Proof.* We need to show that $e' \simeq \gamma_j^{-1}\omega\gamma_i$, that each edge of $K'$ intersects exactly one disk of $\mathbf{D'}$ exactly once, that each edge of $K'$ does not intersect any other edge of $K'$, and that $K'$ is isotopic to a spine.

*Step 1*: $e' \simeq \gamma_j^{-1}\omega\gamma_i$

We will consider the arcs $d_k, d'_k$ to be oriented so that they run from a point on $S \times [0,1]$ to

$E_k, E'_k$. Then

$$\begin{aligned} e' &\simeq d'_j * \omega_2 * \omega_3 * \omega_4 * \overline{d'_i} \\ &\simeq d'_j * \overline{d_j} * d_j * \omega_2 * \omega_3 * \omega_4 * \overline{d_i} * d_i * \overline{d'_i} \\ &\simeq d'_j * \overline{d_j} * \omega * d_i * \overline{d'_i} \\ &\simeq \gamma_j^{-1} * \omega * \gamma_i \end{aligned}$$

where $*$ denotes concatination of paths as in the study of aglebraic topology, and as in the definition of the product in $\pi_1(C)$.

*Step 2*: Each edge intersects exactly one disk exactly once.

The edges $e_k$ intersect the disk $D_k$ exactly once, and if $k \neq i, j$ they remain disjoint from any $D_\ell$ for $\ell \neq k$. We need to show that $e_k$ is disjoint from $D'$ if $k \neq i, j$. Suppose $e_k$ intersects $D'$. If the intersection is not transverse, then a small isotopy of $e_k$ will make $e_k$ disjoint from $D'$. Suppose the intersection is transverse, so $e_k$ intersects $D'$ at a point $p$. Let $B$ be the ball in $C$ bounded by $E_i, E_j, D'$ and $\partial_+ C$. Then a portion of $e_k$ lies in $B$. Since $e_k$ must intersect the negative boundary and $E_i, E_j$ and $D'$ cut out a ball from $S \times [0, 1]$, $e_k$ must intersect $E_i, E_j$, or $D'$. However, $e_k$ cannot intersect $E_i$ or $E_j$ because it is dual to $D_k$. Therefore there must be some other point of intersection of $D'$ and $e_k$. Let $q$ be the point of intersection such that the portion of $e_k$ from $p$ to $q$ lies entirely in $B$. This defines an arc in $B$ with endpoints on $D'$, which may be isotoped to lie in $D'$, and then isotoped off of $D'$.

We show that the edge $e_i$ intersects $D'$ exactly once. Consider the ball $B$ constructed above. The edge $e_i$ must intersect $D_i$ exactly once, at the point $p_1$. Therefore a portion of $e_i$ must lie in $B$. Since $e_i$ has endpoints on the negative boundary and cannot intersect $D_j$ or $\partial_+ C$, there is some point $p_2$ where $e_i$ and $D'$ intersect, and the portion of $e_i$ between $p_1$ and $p_2$ lies entirely inside $B$. Suppose that $e_i \cap D' \neq \{p_2\}$. We may assume that $e_i$ intersects $D'$ transversely, so the intersection consists of a finite, discrete set of points. Suppose $q_1$ is a point in $e_i \cap D'$ and $q_1 \neq p_2$. Then a portion $\beta$ of $e_i$ (besides the arc between $p_1$ and $p_2$) lies inside $B$. Now $\beta$ cannot intersect $D_j, \partial_+ C$ or $D_i$, so since the endpoints of $d_i$ lie on $\partial_- C$, $\beta$ must intersect $D'$ at some other point $q_2$. We can then isotope $d_i$ so that we eliminate the points $q_1$ and $q_2$ from the intersection.

The edge $e'$ does not intersect $D_k$ for $k \neq i, j$ since it is the concatenation of arcs $d'_j * \omega_2 * \omega_3 * \omega_4 * d'_i * \omega_6$, all of which are disjoint from any $D_k$ with $k \neq i, j$. Because $e'$ meets $E_j$ and $E'_j$, it must intersect $D_j$. By isotoping $e'$ within the ball in $C$ bounded by $E_j$ and $E'_j$, we can ensure that $e'$ meets $D_j$ exactly once.

*Step 3*: None of the edges of $K'$ intersect, except possibly at the vertex on $S \times \{0\} \cong \partial_- C$. If there are any intersections, since $C$ is a 3-manifold and the edges are embedded 1-manifolds, a small isotopy will make the edges disjoint.

*Step 4*: We show that $K'$ is isotopic to a spine.

Let $N$ be the union of a regular neighborhood of $K$ and $\partial_- C$ in $C$. Then $C \backslash N$ is homeomorphic to $\partial_+ C \times [0,1]$. Therefore there is a retraction of $C \backslash N$ onto $\partial_+(C) \times \{1\}$ which we identify with $\partial_+(C)$. Let $N_k$ be a regular neighborhood of $D_k$ in $C$. Then $\partial N_k \backslash \partial_+ C$ consists of disks $E_k$ and $E'_k$ parallel to $D_k$. The intersections $A_k = E_k \cap (C \backslash N)$ and $A'_k = E'_k \cap (C \backslash N)$ are annuli, each with one boundary component on $\partial_+ C \times \{1\}$ and the other boundary component on $\partial_+ C \times \{0\}$. We may isotope these annuli so that each level surface $\partial_+ C \times \{t\}$ intersects each $A_k$ and $A'_k$ in a single essential loop. Thus the result of removing $\bigcup_{k=1}^{n-m} N_k$ from $C \backslash N$ is $S \times [0,1]$ where $S$ is a genus $n$ surface with $2(n-m)$ punctures. By taking $N$ sufficiently small, we can ensure that $S \cap \partial_- C$ is path connected. Now we may isotope $\omega$ through $N$ to lie in the boundary of $S \times [0,1]$, with its endpoints on $\partial_- C$. Since $\omega$ is homotopically trivial in $S \times [0,1]$, a homotopy of $\omega$ to a point bounds an immersed disk in $S \times [0,1]$. Since $\omega$ lies in the boundary of $S \times [0,1]$, Dehn's lemma guarantees the existence of an embedded disk $D$ in $S \times [0,1]$ bounded by $\omega$. Extend $D$ in $C$ so that it includes a disk between $d_i'$, an arc parallel to $d_i'$ and an arc in $E_i$. Then $D$ defines an edge slide of $K$, and the resulting spine is isotopic to $K'$. $\square$

## 5.1 Geometric Disk Slides

We now consider paths of Ford domains. Let $\rho : \pi_1(C) \to PSL(2,\mathbb{C})$ be a geometrically finite minimally parabolic representation of a $(1,n)$-compression body. Let $\alpha, \beta$ be parabolic generators and $\gamma_1, \ldots, \gamma_{n-1}$ loxodromic generators. By changing the images of the generators continuously, while keeping $\alpha, \beta$ parabolic and the $\gamma_i$ loxodromic, we may obtain a continuous path of Ford domains.

Let $C$ be a $(1,n)$-compression body. We now describe a way to smoothly transition from one simple Ford domain to another called a geometric disk slide. We do this by taking a path of representations $\rho_t : \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ where $t \in [0,1]$, and the representations $\rho_0$ and $\rho_1$ have simple Ford domains. Let $\gamma_1(t), \ldots, \gamma_{n-1}(t)$ be loxodromic generators of $\rho_t(\pi_1(C))$. Let $\alpha(t)$ and $\beta(t)$ be parabolic generators of $\rho_t(\pi_1(C))$. By varying the matrices $\gamma_i(t), \alpha(t)$ and $\beta(t)$ smoothly, we may obtain a smooth path of representations. If these representations are geometrically finite uniformizations of $C$, we obtain a smooth path of Ford domains.

**Definition 5.5.** A *geometric disk slide* is a smooth path of Ford domains consisting of the following steps:

(i) Fix $i \neq j$.

(ii) For $t \in [0,1]$, vary $\gamma_i(t)$ smoothly in a way that $S_{\gamma_i(t)}$ moves along a path so that it becomes visibly tangent to $S_{\gamma_j(t)}$ and no other isometric sphere. We require that for $t \in [0,1)$ the isometric spheres $S_{\gamma_k^{\pm 1}(t)}$ remain disjoint, and that when $t = 1$ only the isometric spheres $S_{\gamma_i(t)}$ and $S_{\gamma_j(t)}$ intersect.

(iii) For $t \in [0,2]$, push $S_{\gamma_i(t)}$ toward the center of $S_{\gamma_j(t)}$ in such a way that only the isometric spheres $S_{\gamma_1^{\pm 1}(t)}, \ldots, S_{\gamma_{n-1}^{\pm 1}(t)}, S_{(\gamma_j\gamma_i^{-1})^{\pm 1}(t)}$ are visible. Throughout this portion of the path, we require all of that these isometric spheres all remain disjoint, except the following pairs of isometric spheres may intersect:

- $S_{\gamma_i(t)}$ and $S_{\gamma_j(t)}$
- $S_{\gamma_i\gamma_j^{-1}(t)}$ and $S_{\gamma_j^{-1}(t)}$
- $S_{\gamma_j\gamma_i^{-1}(t)}$ and $S_{\gamma_i^{-1}(t)}$

(iv) Choose the path in step (iii) so that when $t = 2$, the isometric spheres $S_{(\gamma_j\gamma_i^{-1})^{\pm 1}}$ have radius 1.

(v) When $t = 2$, write $A(t) = \gamma_i\gamma_j^{-1}(t)$ and $\gamma_i(t) = A\gamma_j(t)$. We now consider the group $\rho_t(\pi_1(C))$ to be generated by $\alpha(t), \beta(t), \gamma_1(t), \ldots, \hat{\gamma_i(t)}, \ldots, \gamma_{n-1}(t), A(t)$ where $\hat{}$ indicates omission.

(vi) For $t \in [2, 3]$, move $S_{A(t)}$ away from the center of $S_{\gamma_j}$ until the Ford domain is simple. Throughout this process we require that only the isometric spheres $S_{\gamma_1^{\pm 1}(t)}, \ldots, S_{\gamma_{n-1}^{\pm 1}(t)}, S_{A^{\pm 1}(t)}$ are visible, and that all of these isometric spheres remain disjoint, except the following pairs of isometric spheres may intersect:

- $S_{\gamma_i(t)}$ and $S_{\gamma_j(t)}$
- $S_{\gamma_i\gamma_j^{-1}(t)}$ and $S_{\gamma_j^{-1}(t)}$
- $S_{\gamma_j\gamma_i^{-1}(t)}$ and $S_{\gamma_i^{-1}(t)}$

Note that during a geometric disk slide, the isometric spheres $S_{\gamma_i(t)}$ and $S_{\gamma_j(t)}$ will intersect, hence lemma 4.14 indicates that $S_{\gamma_i(t)\gamma_j^{-1}(t)} \cap S_{\gamma_j^{-1}}$ and $S_{\gamma_i(t)\gamma_j^{-1}(t)} \cap S_{\gamma_i^{-1}(t)}$ will be visible. There is no guarantee that $S_{(\gamma_i(t)\gamma_j(t))^{\pm 1}}$ will be visible, but in many cases this will be true.

A possible way to ensure that step (iv) of the disk slide is satisfied, is make $S_{\gamma_i(t)}$ and $S_{\gamma_j(t)}$ have radius 1, and push the centers of these two isometric spheres toward each other until the centers of these isometric spheres are separated by a Euclidean distance of 1. This is shown by the following lemma.

**Lemma 5.6.** *Suppose the isometric spheres $S_{\gamma_1}$ and $S_{\gamma_2}$ have radius 1. Then the radius of $S_{\gamma_1\gamma_2^{-1}}$ is the inverse of the distance in $\mathbb{C}$ between the center of $S_{\gamma_1}$ and $S_{\gamma_2}$.*

*Proof.* Let

$$\gamma_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \qquad \gamma_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

be elements of $\mathrm{PSL}(2,\mathbb{C})$. Then the $(2,1)$ entry of $\gamma_2\gamma_1^{-1}$ is $c_2d_1 - d_2c_1$, so lemma 4.7 implies that $S_{\gamma_1\gamma_2^{-1}}$ has radius $|c_2d_1 - d_2c_1|^{-1}$. On the other hand, lemma 4.7 indicates that the distance between the centers of the isometric spheres $S_{\gamma_1}$ and $S_{\gamma_2}$ is given by

$$\left|\frac{d_1}{c_1} - \frac{d_2}{c_2}\right| = \left|\frac{c_2d_1 - d_2c_1}{c_1c_2}\right| = |c_2d_1 - d_2c_1|$$

The last step follows because $S_{\gamma_1}$ and $S_{\gamma_2}$ have radius 1, implying $|c_1| = |c_2| = 1$. □

A possible way to satisfy the final step of the geometric disk slide, is ensure that all the isometric spheres $S_{\gamma_1^{\pm 1}(3)}, \ldots, \hat{S_{\gamma_i(3)^{\pm 1}}}, \ldots, S_{\gamma_{n-1}^{\pm 1}(3)}, S_{A^{\pm 1}(3)}$ (where $\hat{}$ indicates omission) have radius 1, and that no two centers of these isometric spheres are separated by Euclidean distance $\leq 2$. In this case, the Poincaré Polyhedron theorem may be applied in a similar manner to the proof of proposition 4.13 to show that the only visible isometric spheres are $S_{\gamma_1^{\pm 1}(3)}, \ldots, \hat{S_{\gamma_i(3)^{\pm 1}}}, \ldots, S_{\gamma_{n-1}^{\pm 1}(3)}, S_{A^{\pm 1}(3)}$.

The following example shows that there exists a geometric disk slide.

**Example 5.7.** Let $C$ be a $(1,3)$-compression body. Then $\pi_1(C) \cong (\mathbb{Z} \times \mathbb{Z}) * \mathbb{Z} * \mathbb{Z}$. Let $\alpha$ and $\beta$ be generators of $\mathbb{Z} \times \mathbb{Z}$ and let $\gamma$ and $\delta$ generate the other $\mathbb{Z}$ factors of $\pi_1(C)$. Consider the family of representations $\rho_t(\pi_1(C))$ given by:

$$\rho_t(\alpha) = \begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix} \qquad \rho_t(\beta) = \begin{bmatrix} 1 & 100i \\ 0 & 1 \end{bmatrix}$$

$$\rho_t(\gamma) = \begin{bmatrix} 0 & 1 \\ -1 & -5i + (-1+i)t \end{bmatrix} \qquad \rho_t(\delta) = \begin{bmatrix} -5-5i & -26-25i \\ 1 & 5 \end{bmatrix}$$

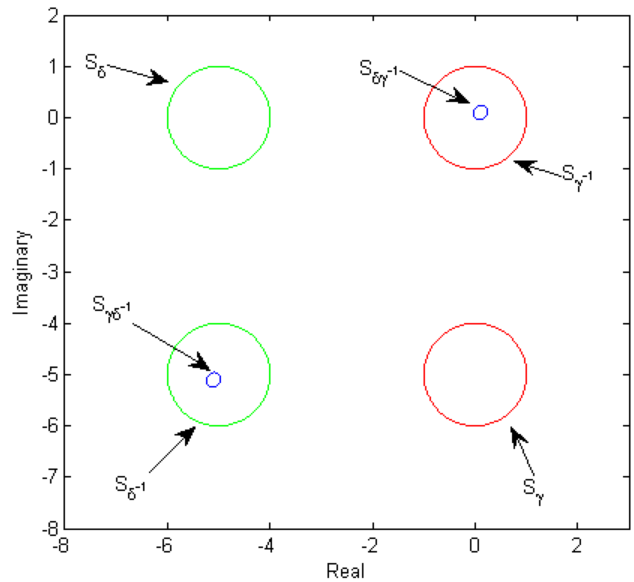


Figure 5.2: The Ford domain for the representation $\rho_0$

When $t = 0$ we obtain the simple Ford domain in figure 5.2. The parabolic elements were chosen so that the translation lengths would be large, thus preventing intersections of isometric spheres corresponding to the loxodromic generators and the parabolic subgroup $\Gamma_\infty$. When $t = 5 - \sqrt{2}$ the isometric spheres $S_\gamma$ and $S_\delta$ intersect, and the isometric spheres $S_{\gamma\delta^{-1}}$ and $S_{\delta\gamma^{-1}}$ begin to emerge (see figure 5.3).

When $t = 5 - \dfrac{\sqrt{2}}{2}$ the radius of the spheres $S_{\gamma\delta^{-1}}, S_{\delta\gamma^{-1}}$ is 1 (see figure 5.4). Now we begin to change the representation of $\pi_1(C)$ in a different way. We will vary the image of $\gamma\delta^{-1}$ while fixing the image of $\delta$. Let $t_0 = 5 - \sqrt{2}/2$. Let $a_{1,1}$ and $a_{2,1}$ be the $(1,1)$ and $(2,1)$ entries of $\rho_{t_0}(\gamma)$ respectively. Define $m = -\dfrac{a_{1,1}}{a_{2,1}}$ and

$$M = \begin{bmatrix} 1 & -m \\ 0 & 1 \end{bmatrix}, \quad Z(s) = \begin{bmatrix} 0 & 0 \\ 0 & -(1+i)s \end{bmatrix}$$

We now define the representation $\rho'_s : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$ by:

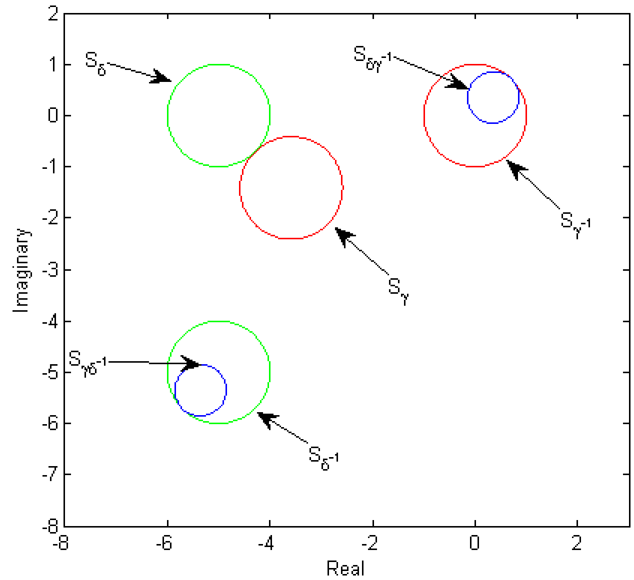


Figure 5.3: At $t = 5 - \sqrt{2}$ the isometric spheres intersect

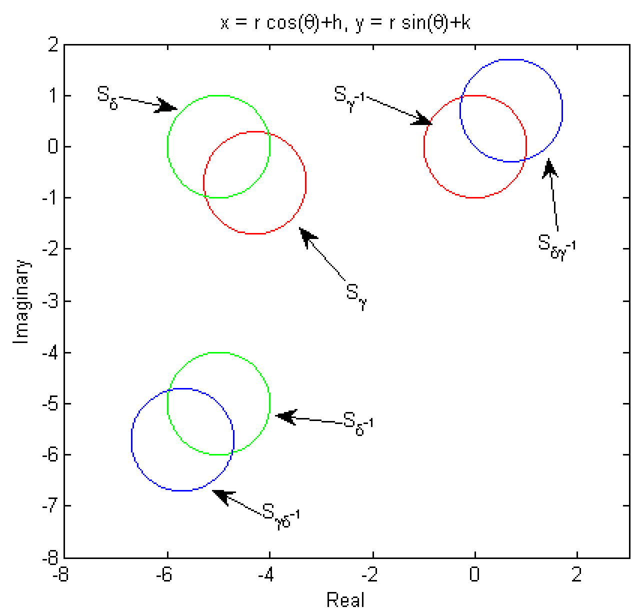


Figure 5.4: The radius of the new sphere is 1

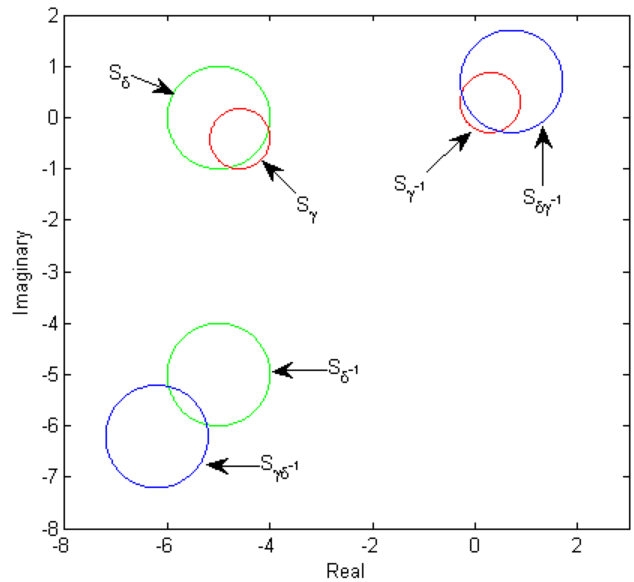


Figure 5.5: The isometric spheres $S_\gamma$ and $S_{\gamma^{-1}}$ begin to disappear

$$\rho'_s(\gamma\delta^{-1}) = M^{-1}(M\rho_{t_0}(\gamma\delta^{-1})^{-1}M^{-1} + Z(s))^{-1}M$$

$$\rho'_s(\delta) = \rho_{t_0}(\delta)$$

$$\rho'_s(\alpha) = \rho_{t_0}(\alpha)$$

$$\rho'_s(\beta) = \rho_{t_0}(\beta)$$

One may easily check that this actually defines a representation into $PSL(2,\mathbb{C})$, i.e. that $\rho'_s(\gamma\delta^{-1}) \in \mathrm{PSL}(2,\mathbb{C})$. When $s = 0$, $\rho'_s = \rho_{t_0}$. As $s$ increases, the isometric sphere $S_{\gamma\delta^{-1}}$ pulls away from $S_{\delta^{-1}}$. On the other hand, the radius of $S_{\gamma^{-1}}$ decreases, and this isometric sphere begins to hide behind $S_{\delta\gamma^{-1}}$. The radius of $S_{\gamma^{-1}}$ also decreases, and the isometric sphere begins to hide behind $S_\delta$ (see figure 5.5). After increasing $s$ even more, we eventually obtain a simple Ford domain, where the isometric sphere $S_{\delta^{\pm 1}}$ and $S_{(\gamma\delta^{-1})^{\pm 1}}$ are visible (see figure 5.6).

Recall that Lackenby and Purcell developed algorithm 4.16 to visualize the Ford domain

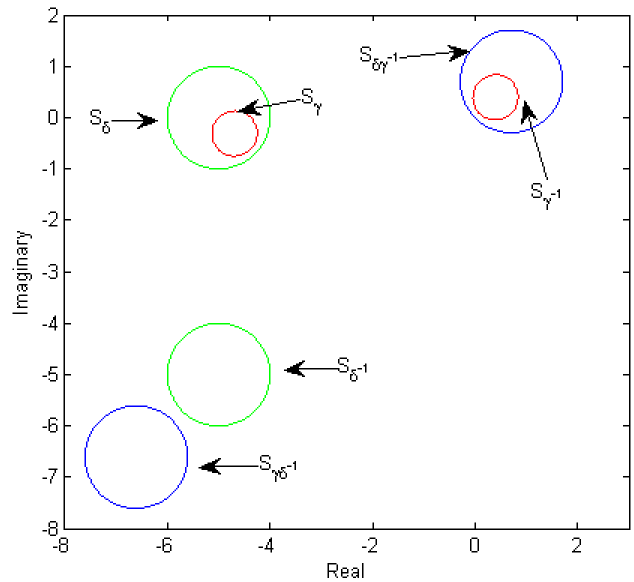


Figure 5.6: The final result of a geometric disk slide

of a $(1, 2)$-compression body. The generalization of this algorithm to $(1, n)$-compression bodies fails to draw the Ford domain of the endpoint of the path $\rho'_s$ if we choose the loxodromic generators $\gamma$ and $\delta$. This is because $S_{\gamma^{\pm 1}}$ and $S_{\delta^{\pm 1}}$ are all disjoint, so the algorithm will tell us only to draw these isometric spheres and then stop. The algorithm would not instruct us to draw the visible isometric spheres $S_{(\gamma\delta^{-1})^{\pm 1}}$. When the algorithm terminates, $S_{\gamma^{-1}}$ is visible while $S_\gamma$ is not visible, hence the resulting picture cannot be the Ford domain. This proves the following proposition.

**Proposition 5.8.** *There exist hyperbolic structures on the $(1, n)$-compression body with $n \geq 3$, and a choice of generators of the fundamental group $\Gamma$ of $C$ such that algorithm 4.16 fails to draw the Ford domain.*

Notice that the geometric disk slide gives a way of transitioning from one simple Ford domain to another. Since a simple Ford domain corresponds to a minimal system of disks for a compression body $C$, and all minimal systems of disks are slide equivalent, the geometric disk slide must correspond to some sequence of disk slides.

**Theorem 5.9.** *Suppose $\mathcal{F}$ is a simple Ford domain of a $(1, n+1)$-compression body con-*

*taining the isometric spheres $S_{\gamma_1^{\pm 1}}, \ldots, S_{\gamma_n^{\pm 1}}$. Let $\mathbf{D} = \{D_1, \ldots, D_n\}$ be a minimal system of disks for $C$, where each $D_k$ is the image of $S_{\gamma_k}$ under the quotient. Let $\mathbf{D}'$ be the system of disks in $C$ corresponding to a geometric disk slide $\mathcal{F}'$ of $\mathcal{F}$ sending $S_{\gamma_i}$ underneath $S_{\gamma_j}$. Then $\mathbf{D}'$ is isotopic to $\{D_1, \ldots, \hat{D}_i, \ldots, D_n, D_i *_\alpha D_j\}$ for an appropriate choice of $\alpha$. If $\omega$ is constructed as in theorem 5.4 with $\omega_3 = \alpha$, then $\omega$ is trivial in $C$.*

*Proof.* Let $K$ be the spine dual to $\mathbf{D}$ consisting of the edges $e_k$ where each $e_k$ is the image (under the quotient map) of the geodesic dual to $S_{\gamma_k^{\pm 1}}$. We may identify each $e_k$ with a loxodromic generator $\gamma_k$ of $\Gamma = \rho(\pi_1(C))$. Let $E_k, E'_k$ be the disks constructed in the statement of lemma 5.4. Let $\alpha$ be an arc in $\partial_+ C$ running from $E'_i$ to $E'_j$, remaining disjoint from the other disks $E_k, E'_k$. Construct $\alpha$ in such a way that if $\omega$ is constructed as in lemma 5.4 with $\omega_3 = \alpha$ then $\omega$ is trivial. Let $K'$ be the spine dual to $\mathbf{D}'$, constructed in a similar manner as $K$. The visible isometric spheres of $\mathcal{F}'$ are $S_{\gamma_1^{\pm 1}}, \ldots, \hat{S}_{\gamma_i^{\pm 1}}, \ldots, S_{\gamma_n^{\pm 1}}, S_{(\gamma_i \gamma_j^{-1})^{\pm 1}}$. Since the Ford domain is simple, the edges of $K'$ are isotopic to the image of the duals corresponding to the loxodromic generators $\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_n, \gamma_i \gamma_j^{-1}$. Let $K''$ be the spine dual to the disk slide $\mathbf{D}'' = \{D_1, \ldots, \hat{D}_i, \ldots, D_n, D_i *_\alpha D_j\}$. By lemma 5.4, edges of $K''$ are also isotopic to the image of the duals corresponding to the loxodromic generators $\gamma_1, \ldots, \hat{\gamma}_i, \ldots, \gamma_n, \gamma_i \gamma_j^{-1}$. Therefore $K'$ and $K''$ are isotopic. Since the system of disks dual to a spine is unique up to isotopy, this implies $\mathbf{D}'$ is isotopic to $\mathbf{D}''$, hence $\mathbf{D}'$ is given by the disk slide $\mathbf{D}''$ of $\mathbf{D}$.

□

# Chapter 6. Tunnel Systems with Intersecting Geodesic Representatives

In this section we show that the geodesic duals in the Ford domain may be made to intersect while retaining a geometrically finite structure. We then prove that there exist finite volume one-cusped hyperbolic manifolds with a system of $n$ tunnels for which the geodesic representative of $(n-1)$ of the tunnels are arbitrarily close to self-intersecting. Since a tunnel homotopic to a self-intersecting geodesic cannot be isotopic to a geodesic, this gives evidence that tunnels may not always be isotopic to geodesics.

Suppose we are given a $(1,3)$-compression body $C$ with the representation $\rho : \pi_1(C) \to \mathrm{PSL}(2,\mathbb{C})$ giving a simple Ford domain, for example the representation of example 4.8. The geodesic dual to this picture consists of four arcs which are each vertical lines running from the center of one of the isometric spheres $S_{\gamma^{\pm 1}}, S_{\delta^{\pm 1}}$ to the point at infinity.

Now consider what happens to the geometric dual when we perform a geometric disk slide sending $S_\gamma$ underneath $S_\delta$ as in example 5.7. At some point in time, as we pull apart the isometric spheres $S_{\gamma\delta^{-1}}$ and $S_{\delta^{-1}}$, the center of $S_\gamma$ intersects $S_\delta$ at a point $p$. Under the image of the quotient, this point is identified with the point $q$ corresponding the intersection of the center of $S_{\gamma^{-1}}$ and the isometric sphere $S_{\delta\gamma^{-1}}$.

**Lemma 6.1.** *Let $\gamma$ and $\delta$ be loxodromic generators of a $(1,n)$-compression body $C$. Suppose that the faces $S_{\delta^{\pm 1}}, S_{(\delta\gamma^{-1})^{\pm 1}}$ of the Ford domain of $C$ are visible. Assume that the center of the isometric sphere $S_\gamma$ is contained in the interior of the Euclidean half-ball bounded by $S_\delta$. Then the geometric dual $\tilde{d}$ to $S_\gamma$ is mapped to a geodesic $d$ under the action of $\Gamma$ that lifts to three visible arcs in the Ford domain:*

*(i) A geodesic arc $\alpha_1$ from $\infty$ to a point on $S_\delta$*

*(ii) A geodesic arc $\alpha_2$ from a point on $S_{\delta^{-1}}$ to a point on $S_{\gamma\delta^{-1}}$ (provided that this arc does not intersect any other visible isometric spheres)*

*(iii) A geodesic arc $\alpha_3$ from $\infty$ to a point on $S_{\delta\gamma^{-1}}$*

*See figure 6.1.*

*Proof.* Choose a horosphere $H$ about $\infty$. Let $S$ be the set of points in $\mathbb{H}^3$ equidistant from $\delta^{-1}(H)$ and $\gamma^{-1}(H)$. Let $p_1$ be the intersection of $S_\delta$ and $\tilde{d}$, and let $p_2$ be the intersection of $S$ and $\tilde{d}$. Note that $p_2$ is contained inside the Euclidean half-ball bounded by $S_\delta$ and containing $\delta^{-1}(H)$. By applying $\delta$ to $\mathbb{H}^3$, $\delta^{-1}(H)$ is mapped to $H$, and $H$ is mapped to

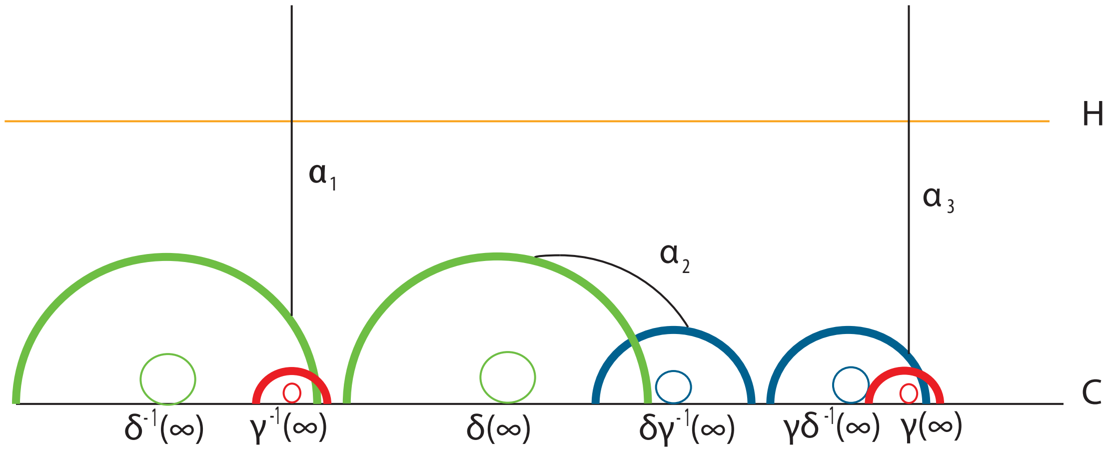


Figure 6.1: Lift of $d$ in the Ford domain consists of the arcs $\alpha_1, \alpha_2$, and $\alpha_3$.

$\delta(H)$. Therefore the isometric sphere $S_\delta$ maps to $S_{\delta^{-1}}$ isometrically. Likewise $S$ gets mapped isometrically to $S_{\gamma\delta^{-1}}$. The geodesic dual $\tilde{d}$ gets mapped to the geodesic running from $\delta(\infty)$ to $\delta\gamma^{-1}(\infty)$. Now $\delta(\tilde{d})$ is a geodesic which passes through $\delta(p_1) \in S_{\delta^{-1}}$ and $\delta(p_2) \in S_{\gamma\delta^{-1}}$.

In a similar manner as above, apply $\gamma$ to $\mathbb{H}^3$. The isometric sphere $S_\gamma$ is mapped isometrically to $S_{\gamma^{-1}}$, and $S$ is mapped to $S_{\delta\gamma^{-1}}$. The geodesic dual $\tilde{d}$ gets mapped to the geodesic dual to $S_{\gamma^{-1}}$. Therefore $\tilde{d}$ gets mapped to an arc containing vertical line from a point on $S_{\gamma^{-1}}$ to $\infty$.

Now $\gamma(\tilde{d})$, $\delta(\tilde{d})$, and $\tilde{d}$ are mapped to a geodesic $d$ in the quotient $\mathbb{H}^3/\Gamma$. Therefore the portions of these arcs which are in the Ford domain are lifts of $d$ to the Ford domain. Note that $\tilde{d}$, $\delta(\tilde{d})$, and $\gamma(\tilde{d})$ contain the arcs $\alpha_1, \alpha_2$ and $\alpha_3$ respectively.

□

**Theorem 6.2.** *There exists a geometrically finite, minimally parabolic uniformization $\Gamma$ of a $(1,3)$-compression body, and a loxodromic generator $\gamma \in \Gamma$ such that the image of the geometric dual to $S_{\gamma^{-1}}$ under the action of $\Gamma$ has a self-intersection.*

*Proof.* We prove this by giving a specific example. Consider the family of representations

$$\rho_t(\alpha) = \begin{bmatrix} 1 & 20 \\ 0 & 1 \end{bmatrix} \qquad \rho_t(\beta) = \begin{bmatrix} 20i & 1 \\ 0 & 1 \end{bmatrix}$$

$$\rho_t(\gamma) = \begin{bmatrix} -49+20i-10it & 700-400i+(20+151i)t \\ -10 & 151-20i \end{bmatrix} \qquad \rho_t(\delta) = \begin{bmatrix} -10 & 151-20i \\ -1 & 15-2i \end{bmatrix}$$

Notice that whenever $t \in [0,4]$, the isometric spheres $S_{\gamma^{\pm 1}}$ are invisible, with $S_\gamma$ covered by $S_{\delta^{-1}}$ and $S_{\gamma^{-1}}$ covered by $S_{\delta\gamma^{-1}}$. By lemma 6.1, under the action of $\rho(\pi_1(M))$, a portion of the dual to $S_\gamma$ is mapped to a geodesic running from a point $p_{\gamma\delta^{-1}}(t)$ on $S_{\gamma\delta^{-1}}$ to a point $p_{\delta^{-1}}(t)$ on $S_{\delta^{-1}}$. Define $p_{\gamma^{-1}}(t)$ to be the intersection of the geodesic dual to $S_{\gamma^{-1}}$ with $S_{\delta\gamma^{-1}}$. For each $t$ define a Euclidean triangle $T_t$ with the edges $e_1(t), e_2(t)$ and $e_3(t)$ being the projections of the geodesic segments $[p_{\gamma^{-1}}(t), p_{\gamma\delta^{-1}}(t)]$, $[p_{\delta^{-1}}(t), p_{\gamma^{-1}}(t)]$ and $[p_{\delta^{-1}}(t), p_{\gamma\delta^{-1}}(t)]$ onto $\mathbb{C}$ respectively. For $i = 1,2,3$, let $m_i(t)$ be the slope of $e_i(t)$. Define a function $f : [0,4] \to \mathbb{R}$ by

$$f(t) = \begin{cases} -\text{Area}(T_t) & \text{if } m_1 < m_2 \\ \text{Area}(T_t) & \text{if } m_1 \geq m_2 \end{cases}$$

Intuitively, $f(t)$ is negative when $p_{\gamma^{-1}}(t)$ is below $e_2(t)$, positive when $p_{\gamma^{-1}}(t)$ is above $e_2(t)$, and zero when the points $p_{\gamma^{-1}}(t), p_{\delta^{-1}}(t), p_{\gamma\delta^{-1}}(t)$ are colinear. Because the points $p_{\gamma^{-1}}(t), p_{\delta^{-1}}(t)$ and $p_{\gamma\delta^{-1}}(t)$ vary continuously with $t$, and $f(t) = 0$ when these points are colinear, $f(t)$ defines a continuous function. As can be seen in figure 6.2, when $t = 0$ we have $f(t) < 0$ since $p_{\gamma^{-1}}(t)$ must be below the line segment $e_1(t)$. Similarly, when $t = 4$ we obtain $f(t) > 0$ (see figure 6.2). The intermediate value theorem guarantees that there is some $t_0 \in [0,4]$ for which $f(t_0) = 0$, i.e. $p_{\gamma^{-1}}(t_0), p_{\delta^{-1}}(t_0)$ and $p_{\gamma\delta^{-1}}(t_0)$ are colinear. Hence when $t = t_0$ the image of the geodesic dual to $S_{\gamma^{-1}}$ under the action of $\Gamma$ self-intersects.

□

**Theorem 6.3.** *There exists a geometrically finite, minimally parabolic uniformization* $\Gamma$ *of*

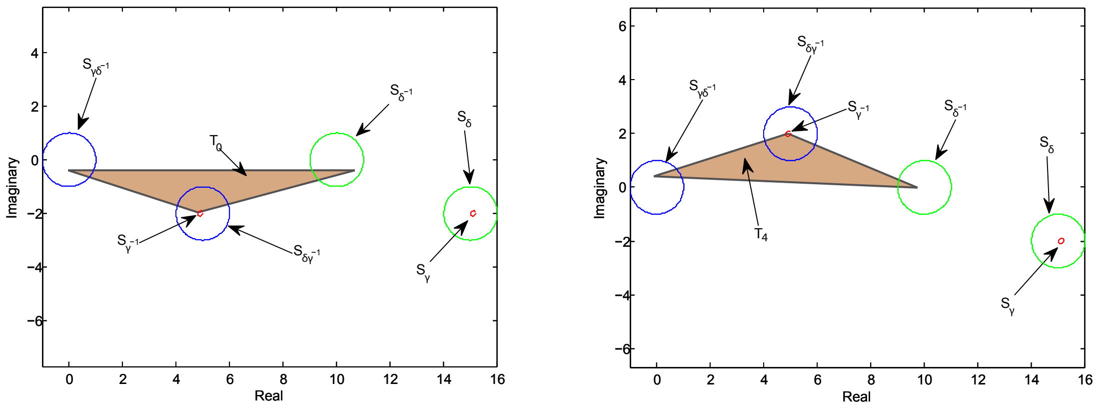


Figure 6.2: When $t = 0$, the Ford domain is as pictured on the left, and $f(0) < 0$. When $t = 4$, the Ford domain is as pictured on the right and $f(4) > 0$.

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*a $(1, n+1)$-compression body and a choice of loxodromic generators $\delta_1, \ldots, \delta_n$ of $\Gamma$ for which the image of the geometric dual to the isometric spheres $S_{\delta_1}, \ldots, S_{\delta_{n-1}}$ under the action of $\Gamma$ each self-intersect.*

*Proof.* Set

$$A = \begin{bmatrix} 1 & 10 \\ 0 & 1 \end{bmatrix}$$

Consider the $n$-parameter family of representations

$$\gamma_k(t_k) = A^{k-1} \begin{bmatrix} 0 & 1 \\ -1 & 5 + (t_k - 2)i \end{bmatrix} A^{-(k-1)}, 1 \le k \le n$$

$$\alpha = \begin{bmatrix} 1 & 11n \\ 0 & 1 \end{bmatrix}$$

$$\beta = \begin{bmatrix} 1 & 10i \\ 0 & 1 \end{bmatrix}$$

The elements $\alpha, \beta, \gamma_1, \ldots, \gamma_n$ generate a discrete subgroup $\Gamma$ of $\mathrm{PSL}(2, \mathbb{C})$, and the resulting

Ford domain is simple. The result of gluing the faces of the Ford domain is a $(1, n+1)$-compression body. By applying a change of generators corresponding to a disk slide, we maintain a similar geometric picture, but make $n-1$ of the pairs of isometric spheres corresponding to generators invisible as follows. Set $\delta_k = \gamma_k^{-1}\gamma_n$ for $1 \leq k < n$ and $\delta_n = \gamma_n$. The elements $\delta_1, \ldots, \delta_n, \alpha, \beta$ still generate $\Gamma$, but $S_{\delta_k^{\pm 1}}$ are invisible for $k < n$. Part of the geometric dual to $\delta_k$ when $k \neq n$ is mapped to a geodesic running from $S_{\delta_k}$ to $S_{\delta_n}$. By applying a similar argument to that in 6.2 we see that by varying $t_k$ for $k < n$ we can obtain a structure where the geodesic dual to $S_{\delta_k}$ self intersects. Since varying $t_k$ has no effect on the elements $\delta_i, i \neq k$, varying $t_k$ only affects the image of the geodesic dual to the isometric sphere $S_{\delta_k}$. Therefore by performing the above procedure for each $k = 1, \ldots, n-1$, one at a time, we obtain a geometric structure where the geodesic duals to $S_{\delta_1}, \ldots, S_{\delta_{n-1}}$ self-intersect. □

The following lemma can be found in [7] and is useful for obtaining an indiscrete representation $\rho : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$ of a compression body from a discrete representation.

**Lemma 6.4.** *Let $\Gamma$ be a discrete torsion free subgroup of PSL$(2, \mathbb{C})$ such that $M = \mathbb{H}^3/\Gamma$ has a rank two cusp. Suppose the point at $\infty$ projects to the cusp, and $\Gamma_\infty \leq \Gamma$ is the subgroup of parabolics fixing $\infty$. Then for every $\gamma \in \Gamma \backslash \Gamma_\infty$ the isometric sphere $S_\gamma$ has radius at most $T$, where $T$ is the minimal Euclidean translation length of all elements of $\Gamma_\infty$.*

*Proof.* Choose an embedded horoball $H$ about $\infty$ which bounds a horoball neighborhood of the rank-two cusp. Such a choice of $H$ is possible by the Margulis lemma. Let $\alpha \in \Gamma_\infty$ be an element whose translation length is $T$. Suppose $S_\gamma$ has radius $R_0 > T$. If $S_\gamma$ is not visible, we can replace it with an isometric sphere $S_{\gamma'}$ which covers the highest point of $S_\gamma$, so we may assume $S_\gamma$ is visible and has radius greater than $R_0$.

Since $\alpha$ has translation length $T$, the isometric sphere $S_{\gamma\alpha^{-1}}$ will have its center a Euclidean length of $T$ away from the center of $S_\gamma$. Up to conjugation of $\Gamma$, we may assume that $S_\gamma$ is centered at 0 and that the center of $S_{\alpha\gamma^{-1}}$ is real. Since $\gamma^{-1}(H)$ and $\alpha\gamma^{-1}(H)$ are horoballs of equal Euclidean radius, the set of points $P$ equidistant from these horoballs is a vertical plane perpendicular to the real axis.

Apply the isometry $\gamma$ to $\mathbb{H}^3$. We will compute the radius $R_1$ of $S_{\gamma\alpha^{-1}\gamma^{-1}}$. Note that $S_{\gamma\alpha^{-1}\gamma^{-1}} = \gamma(P)$. The isometry $\gamma$ is the same as applying an inversion of $S_\gamma$ followed by a Euclidean isometry. The Euclidean isometry will not affect the radius of $S_{\gamma\alpha^{-1}\gamma^{-1}}$. Since $S_\gamma$ is centered at 0, the inversion will send the point $T \in \mathbb{C} \cap \mathbb{R}$ to $R_0^2/T$. The inversion also sends the point $T/2 \in \mathbb{C} \cap P$ to $(2R_0^2)/T$. Therefore the inversion induced by $\gamma$ sends $P$ to a hemisphere of radius $R_1 = |R_0^2/T - (2R_0^2)/T| = R_0^2/T$. Since $R_0 > T$ we have that $R_1 = R_0^2/T > R_0$.

We can now apply the same argument as above, replacing $\gamma$ with $\gamma\alpha^{-1}\gamma^{-1}$ to find another isometric sphere of radius $R_2 > R_1$, and so on, and continue this process infinitely many times to obtain isometric spheres of radius $R_0 < R_1 < R_2 < \ldots$. This gives an infinite collection of distinct isometric spheres of increasing radii, all of which must fit inside a vertical fundamental domain for $\Gamma_\infty$, which is impossible since $\Gamma$ is discrete. □

Before we prove the main theorem, we need to introduce two more definitions.

**Definition 6.5.** The *representation variety* $V(C)$ of a compression body $C$ is the space of conjugacy classes of representations $\rho : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$, where $\rho$ sends elements of $\pi_1(\partial_- C)$ to parabolics. This definition is similar to one given by Marden in [15], and is more restrictive than one found in [8]. Convergence in $V(C)$ is defined by algebraic convergence. We denote the subset of conjugacy classes of minimally parabolic geometrically finite uniformizations of $C$ by $GF_0(C) \subseteq V(C)$. We will give $GF_0(C)$ the algebraic topology. Marden [14] showed that $GF_0(C)$ is open in $V(C)$.

**Definition 6.6.** A *maximally cusped structure for* $C$ is a geometrically finite uniformization $\rho : \pi_1(C) \to \mathrm{PSL}(2, \mathbb{C})$ of $C$ such that every component of the boundary of the convex core of $\mathbb{H}^3/\rho(\pi_1(C))$ is a 3-punctured sphere.

In a maximally cusped structure for $C$, a full pants decomposition of $\partial_+ C$ is pinched to parabolic elements. A theorem of Canary, Culler, Hersonsky, and Shalen [5] shows that the conjugacy classes of maximally cusped structures for $C$ are dense on the boundary of $GF_0(C)$ in $V(C)$. This theorem is an extension of work by McMullen [16], and plays an important role in proving theorem 6.7.

Recall that Cooper, Lackenby, and Purcell used the Ford domain of compression bodies to show that unknotting tunnels may have arbitrarily long length [7]. We will follow their work to prove that there are systems of unknotting tunnels that are homotopic to geodesics which are arbitrarily close to self-intersecting. The method of this proof does not guarantee that the geodesics will self-intersect, but it does show that we can find structures for which the geodesics are arbitrarily close to self-intersecting.

**Theorem 6.7.** *There exists a hyperbolic manifold with a tunnel system consisting of $n$ tunnels for which $(n-1)$ of the tunnels are homotopic to geodesics that are arbitrarily close to self-intersecting.*

*Proof.* We begin with the geometrically finite representation $\rho_0$of the $(1, n+1)$-compression body constructed in theorem 6.3, with generators $\alpha, \beta, \delta_1, \ldots, \delta_n$, and where the geodesic duals to $S_{\delta_1}, \ldots, S_{\delta_{n-1}}$ glue up to self-intersect. The translation lengths of $\rho_0(\alpha)$ and $\rho_0(\beta)$ are bounded by some number $L$. We can consider $\rho_0$ as an element of $V(C)$. Recall that $\pi_1(C) \cong (\mathbb{Z} \times \mathbb{Z}) * F_{n-1}$ where $F_{n-1}$ is free on $(n-1)$ generators. Let $\alpha, \beta$ be generators of $\mathbb{Z} \times \mathbb{Z}$ and $\gamma_1, \ldots, \gamma_{n-1}$ be generators of $F_{n-1}$. Let $\mathcal{R}$ be the set of all representations $\rho$ of $\pi_1(C)$ where $\rho(\alpha), \rho(\beta)$ are parabolics fixing infinity with translation length bounded by $L$, and $\rho(\gamma_i) = \rho_0(\delta_i)$. By suitably normalizing $\rho(\alpha), \rho(\beta)$ to avoid conjugation, we can view $\mathcal{R}$ as a subset of $V(C)$. Note that $\rho_0 \in \mathcal{R}$.

If by shrinking the parabolic translation lengths any intersection found in the proof of theorem 6.3 becomes invisible, the fact that the Ford domain is a fundamental domain for the action of $\rho(\pi_1(C))$ guarantees that the intersection will occur elsewhere in the Ford domain. Since the representation $\rho_0$ has self-intersecting geodesic duals, this implies that all representations in $\mathcal{R}$ will have self-intersecting geodesic duals.

Consider a path in $\mathcal{R}$ from $\rho_0$ to some indiscrete representation. Such a path is obtained by decreasing the minimal translation length of $\rho(\alpha)$ or $\rho(\beta)$ so that it becomes smaller than the radius of some isometric sphere. Such structures are indiscrete by lemma 6.4. This path intersects $\partial GF_0(C)$ at some point, $\rho_\infty$. Since maximally cusped structures are dense in $\partial GF_0(C)$, we can construct a sequence of geometrically finite representations $\rho_k$ of $\pi_1(C)$ such that the conformal boundaries of the manifolds $C_k = \mathbb{H}^3/\rho_k(\pi_1(C))$ are maximally cusped genus $(n+1)$ surfaces, $C_k$ are homeomorphic to the interior of $C$, and the algebraic limit of the manifolds $C_k$ is $M = \mathbb{H}^3/\rho_\infty(\pi_1(C))$. For any $\epsilon > 0$, when $k$ is sufficiently large, $(n-1)$ tunnels will be within $\epsilon$ of self-intersecting.

The work of Canary, Culler, Hersonsky, and Shalen [5] shows that maximally cusped hyperbolic structures on the genus $(n+1)$ handlebody are dense in the boundary of geometrically finite structures on handlebodies. Therefore there is some hyperbolic manifold $\mathbb{H}^3/\Gamma_1$ homeomorphic to the interior of a genus $(n+1)$ handlebody $H_{\text{top}}$, such that every component of the boundary of the convex core of $\mathbb{H}^3/\Gamma_1$ is a 3-punctured sphere. We will denote the hyperbolic manifold $\mathbb{H}^3/\Gamma_1$ by $H_{\text{hyp}}$.

The boundary of the convex core $\mathcal{C}(C_k)$ of $C_k$ consists of three-punctured spheres, as does $\partial\mathcal{C}(H_{\text{hyp}})$. Since there is only one hyperbolic structure on three-punctured spheres, we can obtain an isometry $\varphi_k$ gluing $\mathcal{C}(C_k)$ to $\mathcal{C}(H_{\text{hyp}})$ to obtain a manifold $M_k$ with $3n+1$ rank two cusps. One of these cusps comes from $\partial_- C_k$. The other $3n$ cusps come from the boundary curves corresponding to some pants decomposition of $C_k$. Now we can glue $C$ to $H_{\text{top}}$ by extending the isometry $\varphi_k$ to a homeomorphism from $\partial_+ C$ to $H_{\text{top}}$ to obtain the manifold $M_k'$. By drilling out $3n$ boundary curves corresponding to a pants decomposition of the Heegaard surface of $M_k'$ we obtain the manifold $M_k$.

Select Dehn filling slopes $s^1, s^2, \ldots s^{3n}$ for the torus boundary components of $M_k$ corresponding to a pants decomposition of the Heegaard surface. These slopes must be taken so that the Heegaard surface of $M_k'$ is preserved. This can be done by taking slopes of the form $1/m$, since these will act the same as gluing $\partial H$ to $\partial_+ C$ by a high power Dehn twist. The result is a manifold with a tunnel system consisting of $n$ unknotting tunnels. By taking the slopes $s^i = 1/m_i$ to have $m_i$ sufficiently large, the work of Thurston [19] shows that the Dehn filled manifold $M_k^{\text{filled}}$ approaches $M_k$ in the geometric topology. Therefore given $\epsilon > 0$ we can take the $m_i$ sufficiently large to ensure that $(n-1)$ of the unknotting tunnels of $M_k^{\text{filled}}$ are within $\epsilon$ of self-intersecting.

□

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