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Limit cycles in a model of supply-side liquidity/profit-rate in the presence of a Phillips curve☆

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Abstract

In the present paper, we study how the dynamics of Foley's model may be affected by the introduction of a money wage Phillips curve with a perfect spill-over of price inflation on wage inflation. The upshot is a model with endogenous price and wage dynamics with unstable equilibrium, meaning that the integration of the Foley liquidity/profit rate cycle with the Goodwin cycle requires state intervention to stabilize the economy. With this approach, we show that Foley's initial insight is consistent with flexible prices and wages only in the presence of a rule for money supply. The stable equilibrium, in this case, may degenerate into a limit cycle if the growth rate of liquidity increases to a sufficient degree. To illustrate this result, we carry out the analysis both regarding general functions as well as in terms of a particular example.

JEL classifications: E32; E64

Keywords: Liquidity/profit-rate cycle; Stability conditions; Hopf-bifurcation

1. Introduction

Since Goodwin (1967), demand–distribution cycles in capitalist economies have been at the center of some studies in the Post-Keynesian tradition [see, e.g. Araujo et al. (2019), Barbosa-Filho and Taylor (2006) and Tavani and Zamparelli (2015)]. His growth-cycle model in which profits squeezed out of the income of the workforce are invested, thus determining the pace of capital accumulation, blends aspects of the Harrod–Domar growth set up with the Phillips curve. Such formulation allowed him to present a theory of economic fluctuations whereby the economic variables interact with each other in a cyclical and endogenous way. Flaschel has been engaged in a major research program on (disequilibrium) macroeconomic analysis that has resulted in a massive series of papers and books with several other

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contributors [see e.g. Flaschel (1993), Chiarella and Flaschel (2004), Flaschel and Krolzig (2006) and Franke et al. (2006)] on the extension of Goodwin's distributive cycle by introducing both effective demand forces and endogenous innovation, providing a platform of what Goodwin himself described as the Marx–Keynes–Schumpeter approach.

Within this tradition, Foley (1987) presented an important study on the existence of liquidity-profit rate cycles in capitalist economies and makes use of a model in the tradition of the nonlinear business cycle models developed by Goodwin (1982) and Hicks (1950). He analyses the dynamics of a capitalist economy populated by profit-seeking firms in which labour is surplus, and productivity makes the real wages consistent with given profit margins. In this vein, the model follows an explicitly disaggregated assessment of the economy and emphasizes money as a link amongst enterprises, with the growth rate of the economy adjusting to the growth rate of money through changes in the rate of capital turnover [see Foley (1987, p. 372)].

Moreira et al. (2014) have introduced a slight modification to this model by assuming that only the interest rate is a function of the amount of money per unit of capital, whereas Foley (1987, p. 370) assumes that profit/interest gap is a function of the amount of money per unit of capital. With this approach, the authors focused on the determinants of the rate of interest, and as a consequence, the role of money in the economy was emphasized. In the present paper, we thrust the investigation suggested by Moreira et al. a step further by introducing a money wage Phillips curve with a perfect spill-over of price inflation on wage inflation to the model [see Benassy (1986)]. The 'Phillips curve' is one of the most well-known concepts in Economics after the seminal work of Phillips (1958) on the relation between unemployment and the rate of change of money-wage rates in the UK.[1]

By following this approach, we offer an improvement over the original Foley's analysis insofar as we consider an explicit study of change in the wage and aggregate price. Arguably, by ignoring the dynamics of wage and price, the original framework overlooks an important dimension of the monetary side, namely the connection between prices and output, and in this vein, the dynamics of supply-side liquidity cannot be taken into full account in the original model. From this perspective, this paper extends Foley's model by highlighting the importance of monetary variables in the growth process. With this approach, the extended version of Foley's model becomes closer to Goodwin (1967) considering the dynamics of wage and price and the resulting variations in income distribution. This view is according to Franke et al. (2006, p.453) who consider that "'Goodwin' indicates that income distribution plays a crucial role in the dynamics of nominal and real variables. It is determined by the interplay of a wage as well as a price Phillips curve, and in turn, impacts positively on aggregate demand via workers' consumption and negatively via profit-oriented investment."

The dynamics of this extended system is then shown to be unstable, meaning that the integration of the liquidity profit rate cycle of Foley with the Goodwin employment rate profitability cycle requires state intervention to stabilize the economy. When a monetary authority with a money supply rule for the economy is introduced, the system becomes stable, meaning that the economy can be made viable through state intervention. But such equilibrium may degenerate into a limit cycle if the growth rate of liquidity increases to a sufficient degree, thus emphasizing the importance of proper management of the money supply in the economy. The generated cycle is then studied from the numerical point of view also. To illustrate this result, we carry on the analysis both regarding general functions as well as particular examples. In each case, it is possible to find a Hopf-bifurcation parameter where in general there happens the death of a stable corridor around the steady-state or the birth of a limit cycle around it.

This paper is structured as follows: the next section presents a modified version of the Foley's liquidity-profit rate cycle model. Section 3 studies the existence and stability of stationary points and shows the possibility of a Hopf bifurcation, which leads to the existence of a limit cycle in financial and production variables. Section 4 presents a numerical simulation by considering plausible assumptions on the reaction functions. Section 5 concludes.

2. The modified Foley model in the presence of a money wage Phillips curve

Foley (1987) considers two classes: capitalists and workers and follows the classical assumptions that all wages are consumed, and all profits are invested. By considering that v is the wage share, then $v = \frac{wL}{pY} = \frac{w}{px}$, where w is the nominal wage, L stands for actual employment, p refers to prices, Y is actual production and x represents the output-

[1] According to Flaschel and Krolzig (2006, p. 8), it played "an important role in macroeconomics during the 1960s and 1970s, and modified so as to incorporate inflation expectations, survived for much longer".

labour ratio. He assumes that prices, wage and the output–labour ratio are constant, meaning the presence of a fixed proportions technology without technical change that makes the wage share constant. By assuming that $w = p = 1$ and that there is no depreciation, the variation in the stock of capital, K, reflects the fact that all profit is invested, I, and is given by:

$$\dot{K} = I = rK = (1 - v)\,Y \tag{1}$$

where r is the profit rate. The output is by assumption either consumed or invested in this Marxian supply-side model. Then:

$$Y = wL + I = L + I \tag{2}$$

An important new variable considered in the model is firms' debt, denoted by D, which equals M, the monetary assets of the firms, plus its holdings of other enterprise debt, denoted by F, that is $D = M + F$. Money supply grows at the constant rate μ and is transferred as a subsidy to firms. Hence:

$$\dot{M} = \mu M \tag{3}$$

In what follows, let us consider real balances, money, and loans per units of capital, namely $= \frac{D}{K} = \frac{M}{K} + \frac{F}{K} = m + f$. Foley assumes that the variation of debt per unit of capital, namely $\frac{\dot{D}}{K}$, is a function of the difference between profit and interest rate, namely $r - i$. He also assumes that the difference between the profit and interest ratio depends only on the ratio $m = \frac{M}{K}$, namely $r - i = p(m)$, which makes:

$$\frac{\dot{D}}{K} = B\,(r - i) = B\left[p\,(m)\right] \quad \text{with} \quad B'\,[.] > 0 \tag{4}$$

According to Foley (1987, p. 365) "the higher is m, the lower will be the interest rate relative to the profit rate, since with a higher m the enterprises are liquid and would have a high supply of loanable funds." Moreira et al. (2014) have introduced a slight change in the model by considering that the interest rate is a function of the amount of money per unit of capital according to:

$$\frac{\dot{D}}{K} = B\,[r - i\,(m)]\, with B^{'}\,[.] > 0,\; i^{'}\,(.) < 0 \tag{4.1}$$

In this case, $B\left[p\,(r, m)\right] = B[r - i\,(m)]$. According to this specification, firms increase their debt with increasing profitability spread by adding further loans to their liquidity position. In the original model, from the fact that $r = \frac{(1-v)Y}{K}$, we can conclude by taking logs and differentiating this expression that: $\hat{r} = \hat{Y} - \hat{K}$. It is then assumed that the output expansion that is given by an aggregate output-expansion function[2] :

$$\frac{\dot{Y}}{Y} = A\,(r, d) \quad \text{with} \quad A_d > 0,\; A_r > 0 \tag{5}$$

Since $\frac{\dot{K}}{K} = r$ and $\frac{\dot{Y}}{Y} = A\,(r, d)$, by taking logs and differentiating $y = \frac{Y}{K}$ it yields:

$$\hat{y} = A\,(r, d) - r \tag{6}$$

Hence, from expressions (1) and (5) we conclude that:

$$\hat{r} = A(r, d) - r \tag{7}$$

Let us change the focus from the profit rate to the growth rate of output. In order to accomplish that, consider that $y = \frac{Y}{K}$. By definition, we know that $r = \frac{(1-v)Y}{K}$, where v is the wage share. Then, by considering that $v = \frac{wL^d}{pY} = \frac{w}{px}$, where $x = \frac{Y}{L^d}$ is the output labour ratio, we obtain after some algebraic manipulation: $r = \left(1 - \frac{\omega}{x}\right) y$, where $\omega = \frac{w}{p}$. By also considering that $d = m + f$ expression (6) may be rewritten as:

$$\hat{y} = A\left[\left(1 - \frac{\omega}{x}\right) y, m + f\right] - \left(1 - \frac{\omega}{x}\right) y \tag{6.1}$$

[2] Skott (1989, p. 61) shows that the output-expansion function can be derived explicitly from microeconomic profit maximization firms.

From expressions (3) and (4) respectively, it is possible to obtain after some algebraic manipulation:

$$\dot{m} = m(\mu - r) \tag{8}$$

$$\dot{d} = B\left[r - i\left(m\right)\right] - rd \tag{9}$$

After some algebraic manipulation, expression (9) may be rewritten as:

$$\dot{f} = B\left[r - i\left(m\right)\right] - rf - \mu m \tag{9.1}$$

Moreira et al. (2014) have shown that the dynamic system formed by expressions (7) – (9) displays a stable equilibrium growth path that can be destabilized into a limit cycle due to an increase at the supply of money. One shortcoming of this model is that like in the original Foley's model, there also no specific study of aggregate price and wage changes. A more inclusive approach should consider an explicit treatment of the dynamics of these variables. To consider the dynamics of prices and wage, let us add a money wage Phillips curve with a perfect spill-over of price inflation on wage inflation [see Benassy (1986)] according to:

$$\hat{w} = h\left(e\right) + \hat{p},\ h^{'}\left(.\right) > 0 \tag{10}$$

where $e = \frac{L}{\bar{L}}$ denotes the employment rate.[3] According to this specification the growth rate of wage is a positive function of the rate of employment and inflation. Expression (10) is akin to a Goodwin's (1967) real wage Phillips curve. By using the definition of real wage, namely $\omega = \frac{w}{p}$, we can rewrite this expression as:

$$\hat{\omega} = h\left(e\right) = h\left(\frac{yx^{-1}}{L/K}\right) \tag{11}$$

where L/K is a given magnitude now since it is assumed that labor supply grows as the capital stock.[4] By normalizing L/K and x to 1, we obtain:

$$\hat{\omega} = h\left(e\right) = h\left(y\right),\ h^{'}\left(.\right) > 0 \tag{11.1}$$

Consider now the system formed by Eqs. (6.1), (8), (9) and (11.1). This is a non-linear dynamical system in four variables: (y, d, m, ω). In the next section, we show that this dynamical system is unstable, meaning that the integration of the liquidity profit rate cycle of Foley with the Goodwin employment rate profitability requires a monetary rule. To demonstrate the workability under state intervention, the central bank is thus assumed to know the structure of the model and to pursue a money supply rule for the stabilization of the real-financial market interaction. We assume the following simple money supply rule of the central bank:

$$\hat{M} = \mu + (r - r_0) \tag{12}$$

where r_0 is the steady-state interest rate. According to this rule, money supply is growing at a target rate μ if the interest rate is equal to r_0, namely, $r = r_0$. An attempt of the system to grow faster than the steady-state money supply implies that $r > r_0$ which would imply that the monetary authority will increase the money supply in order to avoid a liquidity shortage.[5] As will be shown shortly, such a mechanism will restore the stability to the system. By dividing both sides of Eq. (12) by K, we obtain after some algebraic manipulation that:

$$\dot{m} = m(\mu - r_0) \tag{12.1}$$

Note now that the growth rate of money per unit of capital, namely $\hat{m}$, becomes exogenously determined since m is fixed at its steady-state value m_o. Then the system formed by expressions (6.1), (8), (9) and (11.1) may be reduced to three equations, namely:

$$\dot{y} = Ay\left[\left(1-\omega\right)y, m_o + f\right] - \left(1-\omega\right)y^2 \tag{13}$$

$$\dot{\omega} = \omega h\left(y\right),\ h^{'}\left(.\right) > 0 \tag{14}$$

[3] Note that $\bar{L}$ stands for the available workforce while L is the employed labour.
[4] It is a reasonable hypothesis under a fixed proportion technology.
[5] In the symmetrical case, namely $r < r_0$, the monetary authority decreases the money supply in order to avoid liquidity surplus.

$$\dot{f} = B\left[(1-\omega)\,y - i\,(m_0)\right] - (1-\omega)\,fy - \mu m_0 \tag{15}$$

In the next section we show that in the presence of a monetary rule such as (12), it is possible to find a stable solution for the system. However, it will be shown that the system loses stability upon variations of a parameter, namely the growth rate of the money supply, via a Hopf bifurcation.

3. Existence and stability of equilibrium solutions

To study the stability of the extended Foley's model in the presence of a money wage, Phillips curve with a perfect spillover of price inflation on wage inflation let us consider first the system formed by expressions (6.1), (8), (9) and (11.1). We get for the determinant of the Jacobian of the dynamics at the steady-state (where growth rate formulations do not matter for the distribution of signs) the sign distribution:

$$detJ = \begin{vmatrix} 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ + & 0 & 0 & \end{vmatrix} < 0 \tag{16}$$

The sign structure in the above Jacobian implies instability of the steady-state solution. The extended system is therefore always unstable, meaning that the integration of the liquidity profit rate cycle of Foley with the Goodwin employment rate profitability cycle does not work without a monetary rule. Flexible wages and prices are therefore endangering the stability of the economy, and hence government intervention is needed to make this economic system viable. Consider now the reformulated system formed by expressions (13) – (15). At equilibrium $P^* = \left(y^*, f^*, \omega^*\right)$, the following equations hold:

$$A\left[\left(1-\omega^*\right)y^*, m_o + f^*\right] = \left(1-\omega^*\right)y^* \tag{17}$$

$$h(y^*) = 0 \tag{18}$$

$$B\left[\left(1-\omega^*\right)y^* - i\,(m_o)\right] = \left(1-\omega^*\right)y^* f^* + \mu m_o \tag{19}$$

Once y^* is determined from expression (18), the values of f^* and ω^* are determined simultaneously from expressions (17) and (15). Computations of the boundary equilibria and the analysis of the existence of positive equilibrium points and their stability for the system, provide the information needed to determine the coexistence or extinction of $y\,(t)$, $f\,(t)$ and $\omega(t)$. To do so, we compute the Jacobian matrix $J(P^*)$ of Eqs. (13)–(15). The characteristic equation of Jacobian matrix is given by

$$\lambda^3 + s_1\lambda^2 + s_2\lambda + s_3 = 0, \tag{20}$$

where: $s_1 = -trace\,J\,(P^*) = j_{11} + j_{22} + j_{33}, s_2 = j_{11}j_{22} - j_{12}j_{21} + j_{11}j_{33}$ and $s_3 = -\det J\,(P^*) = -j_{31}(j_{12}j_{23} - j_{13}j_{22})$. The signs of the real parts of the eigenvalues of $J(P^*)$ evaluated at a given positive equilibrium point $P^* = \left(y^*, f^*, \omega^*\right)$ determine its stability. From the Routh-Hurwitz criteria[6], the interior $P^* = \left(y^*, f^*, \omega^*\right)$ solution for the system (6)", (9)" and (11)" is locally asymptotically stable if the following conditions hold: $s_1 > 0,\ s_2 > 0,\ s_3 > 0,\ s_1s_2 - s_3 > 0$.

Proposition 1 ([7]). The interior solution $P^* = (y^*, \omega^*, f^*) > 0$ for the system of Eqs. (13)–(15) is locally asymptotically stable if $h^{'}(y^*)$ is sufficiently small and the following conditions hold: $(i)\,1 < A_y < 1 + y^*and\,(ii)\;B'\,(.) < f^*$.

If $A_y > 1$ and $B^{'}\,(.) < f$ then the output expansion is sufficiently strong with respect to the impact of the profit rate and borrowing is sufficiently weak with respect the to interest rate differential. Assuming that $h^{'}(.)$ is sufficiently small – in a certain corridor – may work here, but also $A_y < 1 + y$ is needed. Let us now study the possibility of the

[6] The Routh–Hurwitz conditions for local asymptotic stability [see Lorenz (1993)] requires that the eigenvalues of the Jacobian matrix all have negative real parts.

[7] See Appendix A for the proof.

existence of a limit cycle in the system (Eqs. (13)–(15)) by using the Hopf bifurcation analysis. In contrast to Foley (1987), we choose the growth rate of the money supply μ as a bifurcation parameter.[8] Departing from the fact that the steady-state $P^* = (y^*, \omega^*, f^*)$ is asymptotically stable, we would like to know if P^* will lose its stability when the parameter μ changes.

The matrix $J(P^*)$ shows that $E = 0$ holds for $\mu = 0$ and $E > 0$ at time zero. Since $s_1(\mu)s_2(\mu)$ is a quadratic function of μ times a term that depends on the moving equilibrium, while for $s_3(\mu)$ this only holds in a linear fashion (if the dependence on the remaining coefficients of the matrix $J(P^*)$ is ignored) we can assume that the functions $A(.)$, $B(.)$ and $h(.)$ can easily be chosen such that the value of E becomes negative if the growth rate of money supply μ increases. For $\mu = \mu^*$ the Jacobian matrix $J(P^*)$ has a pair of complex eigenvalues with zero real part if and only if $s_1(\mu^*)\, s_2(\mu^*) - s_3(\mu^*) = 0$ or, equivalently:

$$\lambda^2(\mu^*) + s_2(\mu^*)[(\mu^*) + s_1(\mu^*)] = 0 \tag{21}$$

which has three roots: $\lambda_1(\mu^*) = i\sqrt{s_2(\mu^*)}$, $\lambda_2(\mu^*) = -i\sqrt{s_2(\mu^*)}$, and $\lambda_3(\mu^*) = -s_1(\mu^*) < 0$. For all μ, the roots are in general of the form:

$$\lambda_1(\mu) = u(\mu) + iv(\mu), \tag{22}$$

$$\lambda_2(\mu) = u(\mu) - iv(\mu) \tag{23}$$

$$\lambda_3(\mu) = -s_1(\mu) \tag{24}$$

To apply the Hopf bifurcation theorem at $\mu = \mu^*$, we need to verify if the transversality condition holds [see Chiarella et al. (2005, p. 99)], namely:

$$\left[\frac{d\, Re(\lambda_i)(\mu)}{d\mu}\right]_{\mu=\mu^*} \neq 0, \; i = 1, 2. \tag{25}$$

The existence of the parameter μ^* considered above is not difficult to prove. This is the content of:

Proposition 2 ([9]). There exists a one-parameter family of periodic solutions at $\mu = \mu^*$, bifurcating from the equilibrium point $P^* = (y^*, \omega^*, f^*)$ with period T, where $T \to T_0$ as $\mu \to \mu^*$, where $T_0 = 2\pi/\sqrt{s_2}$, with s_2 given by $s_1(\mu^*)\, s_2(\mu^*) - s_3(\mu^*) = 0$.

According to proposition two, the trajectories of the system move away from the equilibrium point and converge to the corresponding periodic Hopf orbit. If this periodic orbit exists for $\mu \to \mu^*$ from above, the Hopf-bifurcation is called subcritical while the opposite is called supercritical. In the latter case a stable limit cycle is generated when μ passes μ^* from below, while a stable 'corridor' – bounded by a periodic orbit – gets lost in the first case as the parameter μ reaches the bifurcation point μ^* from below. The character of the occurring Hopf-bifurcation is however difficult to determine in the third dimension and thus must be a matter of numerical simulations of the model.[10] In the next section, we present a numerical simulation that shows the working of propositions 1 and 2. To sum up, we conclude that the introduction of a money wage Phillips curve produces local instability into the Foley's dynamics of supply-side liquidity/profit-rate, and requires the existence of a monetary rule to establish stability. However, if the monetary authority increases the money supply growth rate continuously, the system may be led to a Hopf bifurcation.

4. Numerical results and discussion

We now present two numerical simulations to verify the local asymptotically stability of positive equilibrium point and to describe how a phase portrait changes as a parameter changes, then it is said to have undergone a Hopf

[8] This differs – despite the use of the same symbol – from the parameter μ which Foley uses in his paper.

[9] See Appendix A for the proof.

[10] This approach is according to Chiarella et al. (2005, p. 99) who consider that "the local analysis – along with a Hopf bifurcation result – serves us as a first piece of information about the economy's general potential for oscillatory behaviour. It thus provides a firm basis for a subsequent global analysis, which from three dimensions on will soon have to resort to computer simulations."

bifurcation of the system. By considering a slight change of coordinates, namely $f \leftrightarrow m_o + f$, let us assume the following functional forms for the relevant functions, namely $A(r, f)$, $b(r - i(m_o))$ and $h(y)$:

$$A(r, m_o + f) = A(r, f) = r\left(\frac{2+r}{3-4r+r^2}\right)\left(\frac{1+3f}{1+2f}\right), \tag{26}$$

$$b(r - i(m_o)) = \frac{\left[(2 + 0.01 i(m_o)\right] + 0.01[r - i(m_o)]}{\left[(3 - 4i(m_o)\right] - 4[r - i(m_o)]} = \frac{2 + 0.01r}{3 - 4r}, \ \forall r \tag{27}$$

$$h(y) = (y - 0.6), \tag{28}$$

With such specifications it is possible to guarantee that: $\frac{\partial A}{\partial r} = D_1 a = \frac{\left(-6r^2+6r+6\right)(1+3y)}{(3-4r+r^2)^2(1+2y)} > 0$ and $\frac{\partial A}{\partial(m_o+f)} = D_2 a = \left(\frac{2+r}{3-4r+r^2}\right)\left(\frac{r}{(1+2y)^2}\right) > 0$. Besides $0 < r < 0.5 + 0.5\sqrt{5}$, $y > 0$, $b'(r - i(m_o) = b'(r) = \frac{8.03}{(3-4r)^2} > 0$, $h'(y) = 1 > 0$ with $x = \frac{1}{2}$, $m_o = 0.01$, $\delta = \mu m_o = 0.01\mu$. By substituting Eqs. (26), (27) and (28) into Eqs. (13), (14) and (15), we obtain the following dynamical model into three autonomous nonlinear differential equations:

$$y' = y^2(1 - 2\omega)\left(-1 + \frac{(2 + y(1 - 2\omega))(1 + 3y)}{\left(3 - 4y(1 - 2\omega) + y^2(1 - 2\omega)^2\right)(1 + 2y)}\right) \tag{13.1}$$

$$f' = \frac{[(2 + 0.01y(1 - 2\omega))]}{[(3 - 4y(1 - 2\omega))]} - y(1 - 2\omega)f + 0.01y(1 - 2\omega) - \delta \tag{14.1}$$

$$\omega' = \omega(y - 0.6) \tag{15.1}$$

We now present a numerical simulation to verify the situation of a Hopf bifurcation for the system (13.1)–(15.1). In what follows let us focus our attention on the value of δ as our Hopf parameter of bifurcation. But it is important to note that $\delta = \mu m_o$, where $m_o = 0.01$. So, in fact, μ, namely the growth rate of money supply, is our Hopf parameter of bifurcation as it is in Moreira et al. (2014). Now consider that: $\delta\ \varepsilon\ [\delta_1, \delta_2] = [0.652570405982870376981, 0.652570405982870376982]$. If we increase the values of δ, by the Intermediate Value Theorem, applied to function $E(\delta)$, there is at least one $\delta = \delta^*\ \varepsilon[\delta_1, \delta_2]$ such that $E(\delta^*) = 0$, that is, the complex eigenvalues are purely imaginary. So it is clear that the system enters into a Hopf bifurcation for increasing δ, and $y(t)$, $f(t)$ and $\omega(t)$ show oscillations when $\delta = \delta^*$. To see this, let us consider:

(i) If $\delta_1 = 0.652570405982870376981$, the system has a positive equilibrium $P^* = (0.6,\ 1.230799848,\ 0.4545871276)$ which is stable, with eigenvalues: $\lambda_{2,3} = -1.4313744634009\,10^{-11} \pm 0.182566061902310\,i$, $\lambda_1 = -0.0489008041603725$. Here: $\lambda_1 < 0$, $Re\,\lambda_{2,3} < 0$; with characteristic polynomial $(J, y) = y^3 + s_1 y^2 + s_2 y + s_3$; $s_1 = 0.04890080419 > 0$, $s_2 = 0.03333036696 > 0$, $s_3 = 0.001629881747 > 0$; $E(\delta_1) = 1.10^{-12} > 0$.

(ii) If $\delta_2 = 0.652570405982870376982$, the system has a positive equilibrium $P^* = (0.6,\ 1.230799847,\ 0.4545871276)$ which is a saddle spiral with unstable focus, with eigenvalues $\lambda_{2,3} = 1.4408482687622\,10^{-11} \pm 0.1825660619086571\,i$, $\lambda_1 = -0.0489008042118817$. Here: $\lambda_1 < 0$, $Re\,\lambda_{2,3} > 0$ with characteristic polynomial: $(J, y) = y^3 + s_1 y^2 + s_2 y + s_3$ where $s_1 = 0.04890080419 > 0$, $s_2 = 0.03333036696 > 0$, $s_3 = 0.001629881749 > 0$; $E(\delta_2) = -1.10^{-12} < 0$.

As δ increases in the interval [0.65, 0.66], the positive equilibrium $P^* = \left(y^*(\delta),\ f^*(\delta), \varpi^*(\delta)\right)$ is unique and changes from a stable focus to a saddle spiral with unstable focus. We also note that $r^* = y^*(\delta)(1 - 2\varpi^*(\delta)) > 0$ in this interval where $\varpi^*(\delta) < 0.5$. The periodic solutions of $y(t)$, $f(t)$ and $\omega(t)$ for $\delta^* \cong \delta_1$ are illustrated in Figs.1–5.[11] This simulation shows that instability arises from an increase in the money supply being sufficiently high to trigger Hopf bifurcations and thus yielding endogenous cyclical behaviour. This illustration confirms numerically what was

[11] See Appendix B.

shown theoretically in Propositions 1 and 2. While the monetary authority may have a stabilizing effect on the economy by using a monetary rule, if it pursues a policy regarding continuously increasing the growth rate of money supply to spur the steady-state growth rate of the economy, the system loses stability and undergoes a Hopf bifurcation.

5. Concluding remarks

In this paper, we have considered the implications of considering the dynamics of wage and price for the dynamics of supply-side liquidity/profit rate cycles by introducing a money wage Phillips curve with a perfect spill-over of price inflation on wage inflation in the Foley (1987) model. With this approach, we have made Foley's approach more inclusive of the monetary side and closer to Goodwin's model. The obtained model was shown to be unstable, thus requiring the existence of monetary authority to provide a money supply rule for the economy. In the presence of a monetary authority providing a money supply rule for the stabilization of the real-financial market interaction, the model displays a stable equilibrium growth path that can be destabilized into a limit cycle due to an increase in the growth rate of the supply of money.

Appendix A.

Proof of Proposition 1

From Routh–Hurwitz criteria,[12] the system (6.1), (9.1) and (11.1) is stable around the positive equilibrium point $P^* = (r^*, m^*, d^*)$ if the conditions $s_1 > 0, s_2 > 0, s_3 > 0, s_1 - s_2 s_3 > 0$ are satisfied. But we know that: $s_1 = -trJ(P^*) = g\left[2 - \frac{\partial A}{\partial r}(g, d^*)\right]$, since it is assumed that hence $\frac{\partial A}{\partial r}(g, d^*) < 2$, which yields $s_1 > 0$. The value of s_2 is given by: $s_2 = j_{11}j_{22} - j_{12}j_{21} + j_{11}j_{33} = -g\left\{g\left[\frac{\partial A}{\partial r}(g, d^*) - 1\right] + \frac{\partial A}{\partial d}(g, d^*)\left[B^{'}(p(g, m^*)) - -d^*\right]\right\}$. Then, if $\frac{\partial A}{\partial r}(g, d^*) < 1$ and $B^{'}(p(g, m^*)) < d^*$, we get: $g\frac{\partial A}{\partial r}(\mu, d^*) - g + \frac{\partial A}{\partial d}(g, d^*)\left[B^{'}(p(g, m^*)) - d^*\right] < 0$, which yields: $s_2 > 0$. The value of s_3 is given by: $s_3 = detJ(P^*) = \mu m^* \frac{\partial A}{\partial d}\left(g, d_3^*\right) B^{'}(p(g, m^*)) \frac{\partial p}{\partial m}(g, m^*)$. Note this condition is easily satisfied, since: $\frac{\partial A}{\partial d}(r, d) > 0$, $B^{'}[.] > 0$ and $\frac{\partial p}{\partial m} = -i^{'}(m) > 0$, because of $i^{'}(m) < 0$. The value of $E = s_1 s_2 - s_3$ is given by:

$$E = s_1 s_2 - s_3 = g\frac{\partial A}{\partial d}\left(g, d^*\right)\left\{g\left[\frac{\partial A}{\partial r}\left(g, d^*\right) - 1\right]\left[B^{'}\left(p\left(g, m^*\right)\right) - d^*\right] - -m^* B^{'}\left(p\left(g, m^*\right)\right)\frac{\partial p}{\partial m}\left(m^*\right)\right\}.$$

Note that if $\frac{\partial A}{\partial r}(g, d^*) < 1$ and $B^{'}(p(g, m^*)) < d^*$, then $E = s_1 - s_2 s_3 > 0$, since $B^{'}(p(g, m^*))\frac{\partial p}{\partial m}(m^*) < 0$. Hence, from the Routh–Hurwitz criteria, the system (6.1), (9.1) and (11.1) is stable around the positive equilibrium point $P^* = (r^*, m^*, d^*)$.

Proof of Proposition 2

According to Chiarella et al. (2005, p. 99), to apply the Hopf bifurcation theorem at $\mu = \mu^*$, we need to verify the transversality condition (23). Substituting $\lambda_i(\mu) = u(\mu) \pm iv(\mu)$ into (23) and calculating the derivative, we have:

$$A(\mu)u^{'}(\mu) - B(\mu)v^{'}(\mu) = -C(\mu), \tag{27a}$$

$$C(\mu)u^{'}(\mu) + A(\mu)v^{'}(\mu) = -D(\mu), \tag{28a}$$

where:

$$A(\mu) = 3u^2(\mu) + 2s_1(\mu)u(\mu) + s_2(\mu) - 3v^2(\mu), \tag{29}$$

$$B(\mu) = 6u(\mu)v(\mu) + 2s_1(\mu)v(\mu), \tag{30}$$

$$C(\mu) = s_1^{'}(\mu)u^2(\mu) + s_2^{'}(\mu)u(\mu) + s_3^{'}(\mu) - s_1^{'}(\mu)v^2(\mu), \tag{31}$$

[12] See Lorenz (1993) for the Routh–Hurwitz conditions for local asymptotic stability.

$$D(\mu) = 2s_1^{'}(\mu)\,u(\mu)\,v(\mu) + s_2^{'}(\mu)\,v(\mu)\,. \tag{32}$$

Since $u(\mu^*) = 0$, $v(\mu^*) = \sqrt{s_2(\mu^*)}$, we have:

$$A\left(\mu^*\right) = -2s_2(\mu^*) \tag{29.1}$$

$$B\left(\mu^*\right) = 2s_1(\mu^*)\sqrt{s_2(\mu^*)} \tag{30.1}$$

$$C\left(\mu^*\right) = s_3^{'}\left(\mu^*\right) - s_1^{'}\left(\mu^*\right)s_2\left(\mu^*\right), \tag{31.1}$$

$$D\left(\mu^*\right) = s_2^{'}\left(\mu^*\right)\sqrt{s_2(\mu^*)}. \tag{32.1}$$

Solving for $u^{'}(\mu^*)$ from Eqs. (27)–(28), we obtain

$$\left[\frac{d\,Re(\lambda_i)(\mu)}{d\mu}\right]_{\mu=\mu^*} = u^{'}\left(\mu^*\right) = -\left[\frac{A(\mu^*)\,C(\mu^*) + B(\mu^*)D(\mu^*)}{A^2(\mu^*) + B^2(\mu^*)}\right] \tag{33}$$

By substituting Eqs. (26)–(29) into Eq. (30), we obtain:

$$\left[\frac{d\,Re(\lambda_i)(\mu)}{d\mu}\right]_{\mu=\mu^*} = \frac{s_3^{'}(\mu^*) - s_1(\mu^*)\,s_2^{'}(\mu^*) - s_1^{'}(\mu^*)\,s_2(\mu^*)}{2\left[s_1^2(\mu^*) + s_2(\mu^*)\right]} = \frac{s_3^{'}(\mu^*) - \left[s_1(\mu)\,s_2(\mu)\right]_{\mu=\mu^*}^{'}}{2\left[s_1^2(\mu^*) + s_2(\mu^*)\right]} > 0 \tag{33.1}$$

where:

$$s_3^{'}\left(\mu^*\right) > \left[s_1(\mu)\,s_2(\mu)\right]_{\mu=\mu^*}^{'}, \tag{34}$$

$$s_1\left(\mu^*\right)s_2\left(\mu^*\right) - s_3\left(\mu^*\right) = 0 \tag{35}$$

$$\lambda_3\left(\mu^*\right) = -s_1\left(\mu^*\right) < 0, \tag{36}$$

$$s_1\left(\mu^*\right) > 0 \tag{37}$$

Therefore, from expressions (34)–(37), the transversality condition holds. This implies that a Hopf-bifurcation occurs at $\mu = \mu^*$ and is non-degenerate.

Appendix B.

The periodic solutions of $y(t)$, $f(t)$ and $\omega(t)$ for $\delta^* \cong \delta_1$ are illustrated in Figs. 1–5.

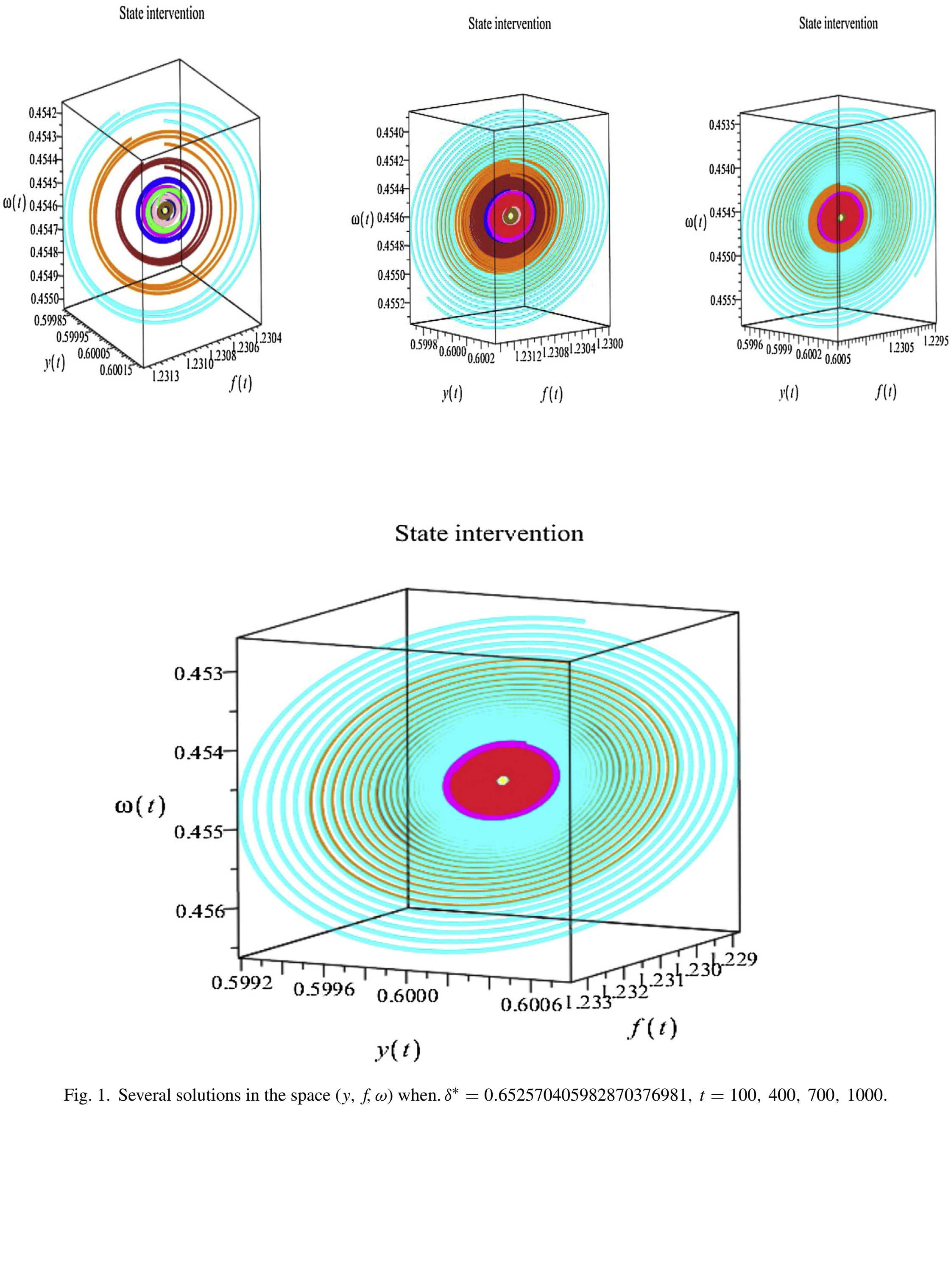

Fig. 1. Several solutions in the space $(y,\ f, \omega)$ when. $\delta^* = 0.652570405982870376981,\ t = 100,\ 400,\ 700,\ 1000.$

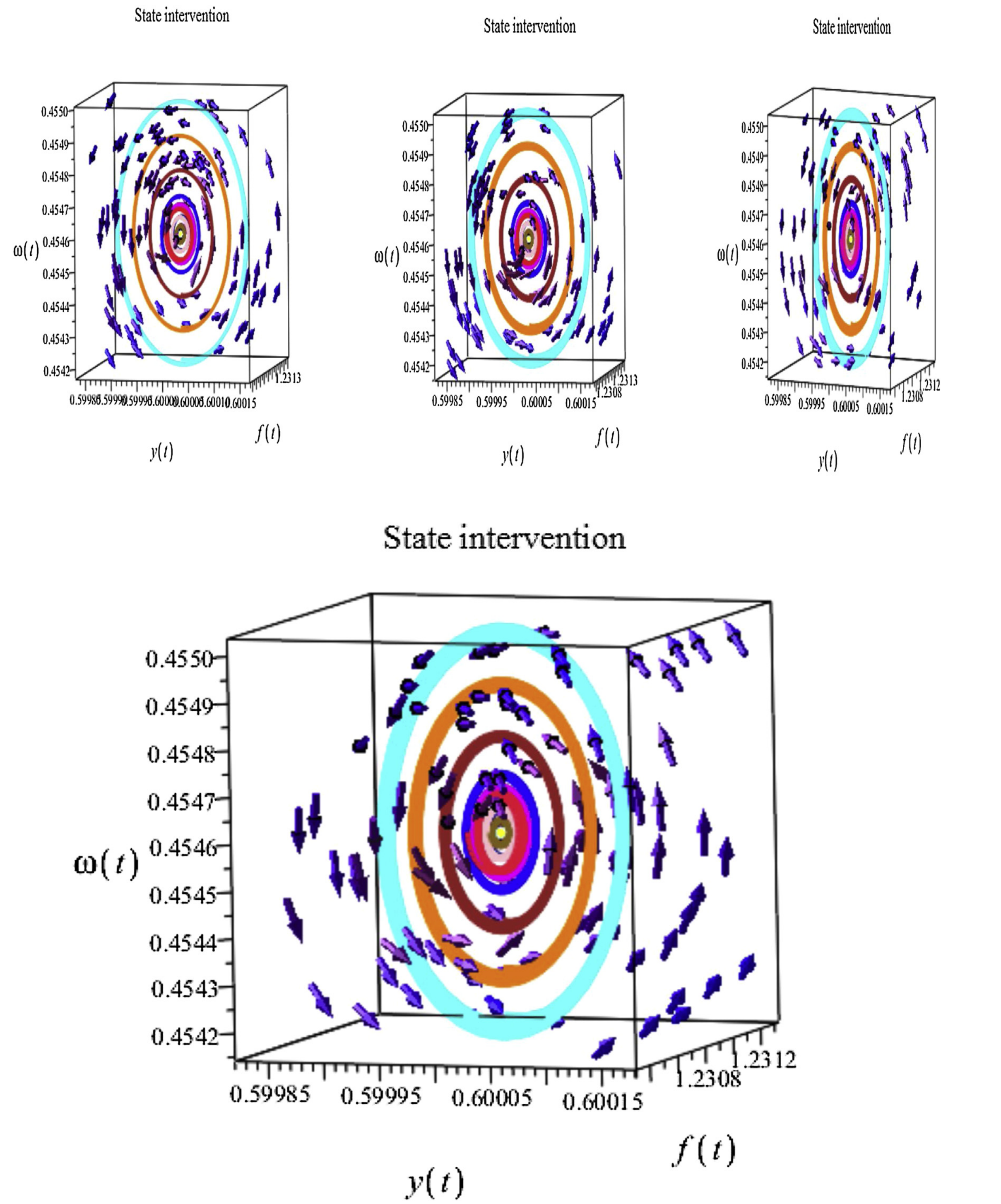

Fig. 2. Vector field for several solutions in the space (y, f, ω) when. $\delta^* = 0.652570405982870376981$. *Here* $t = 0..100$, $t = 0..400$, $t = 0..700$, $t = 0..1000$.

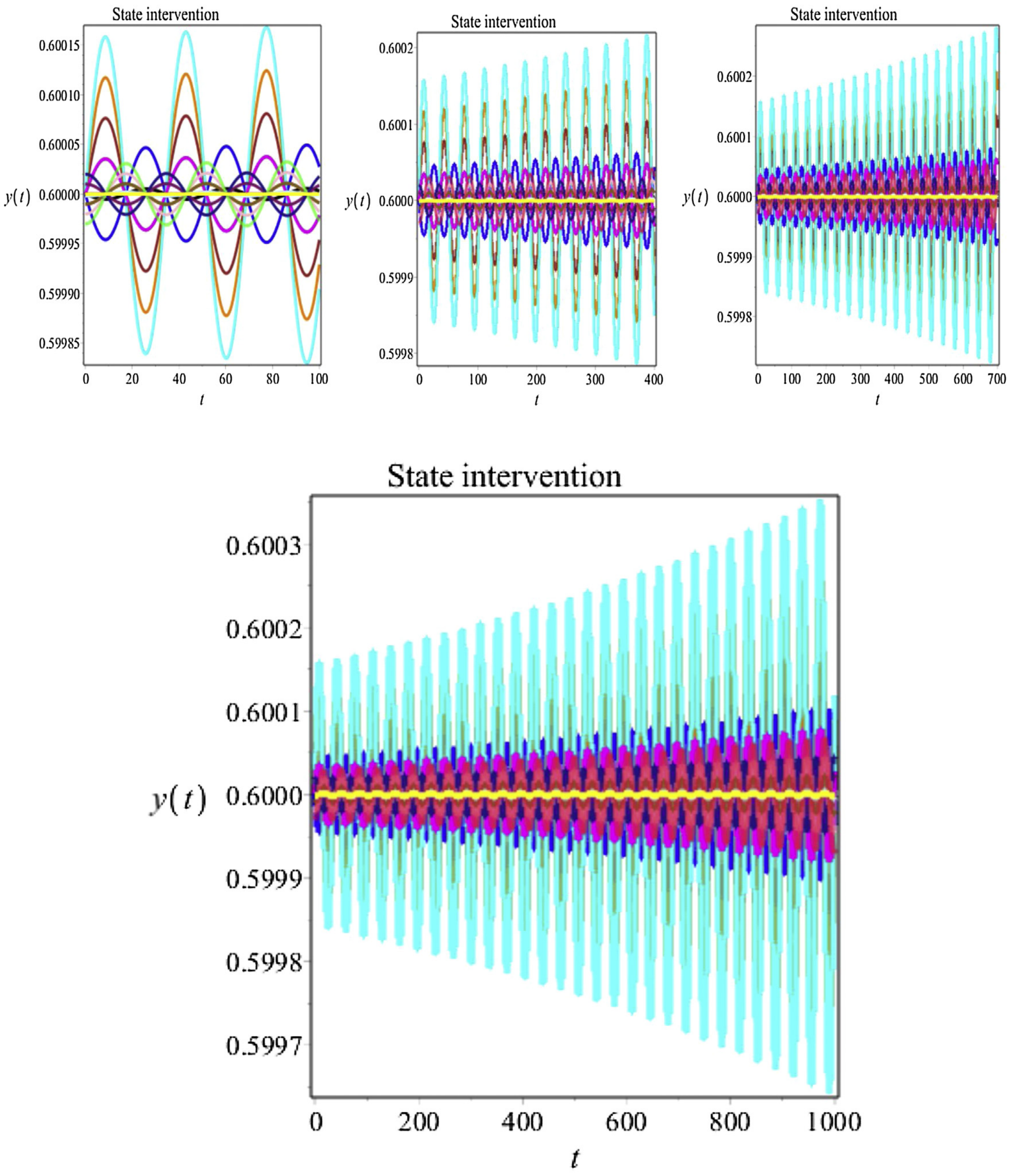

Fig. 3. Time series solutions in the plane $(t, y(t))$, when $\delta^* = 0.652570405982870376981$, with 25 Initial conditions.$(y(0), f(0), \omega(0))$.*Here* $t = 0..100$, $t = 0..400$, $t = 0..700$, $t = 0..1000$.

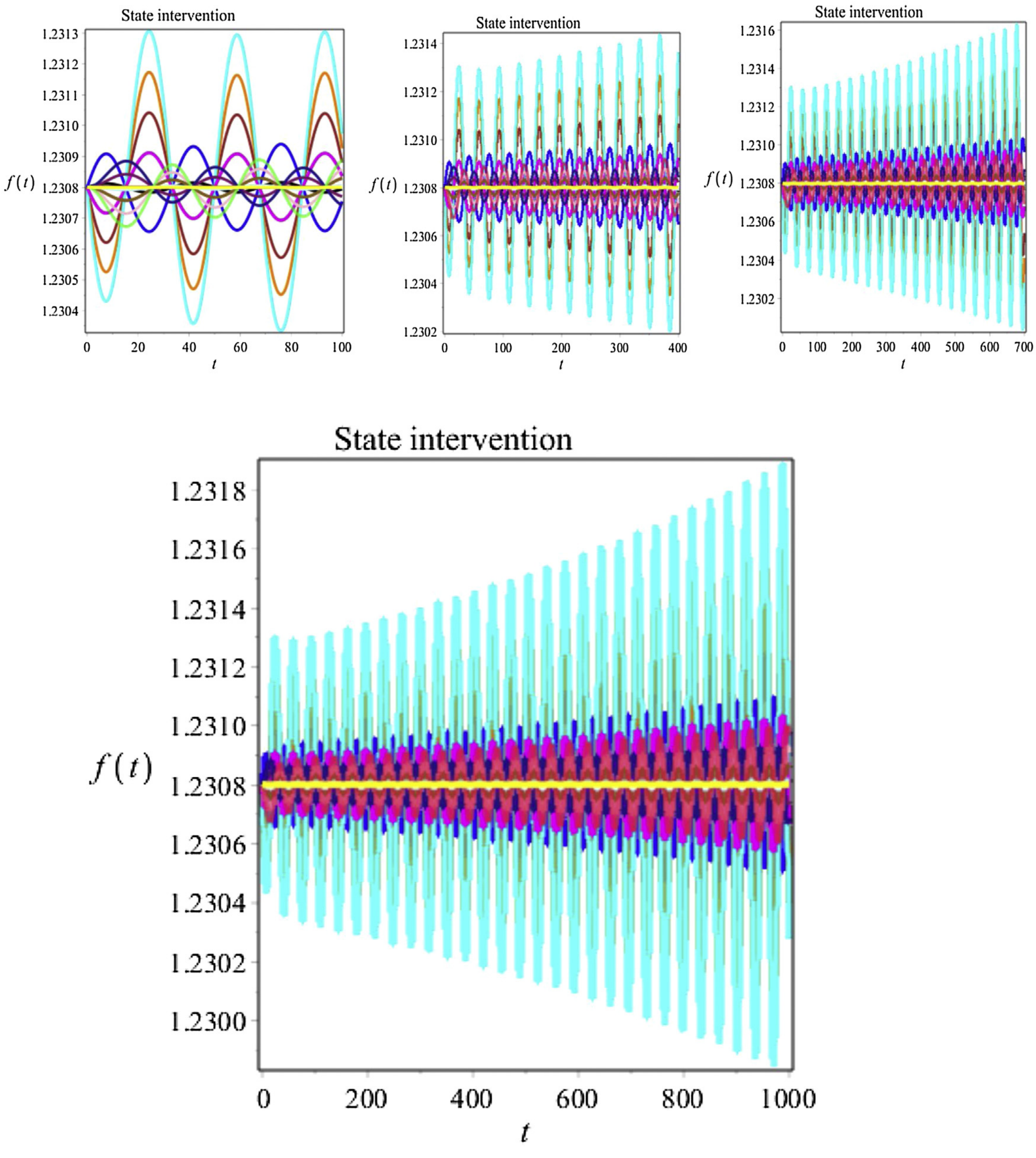

Fig. 4. Time series solutions in the plane $(t,\ f(t))$, when $\delta^{*} = 0.652570405982870376981$, with 25 Initial conditions.$(y(0),\ f(0),\ \omega(0))$ *.Here* $t = 0..100,\ t = 0..400, t = 0..700, t = 0..1000$.

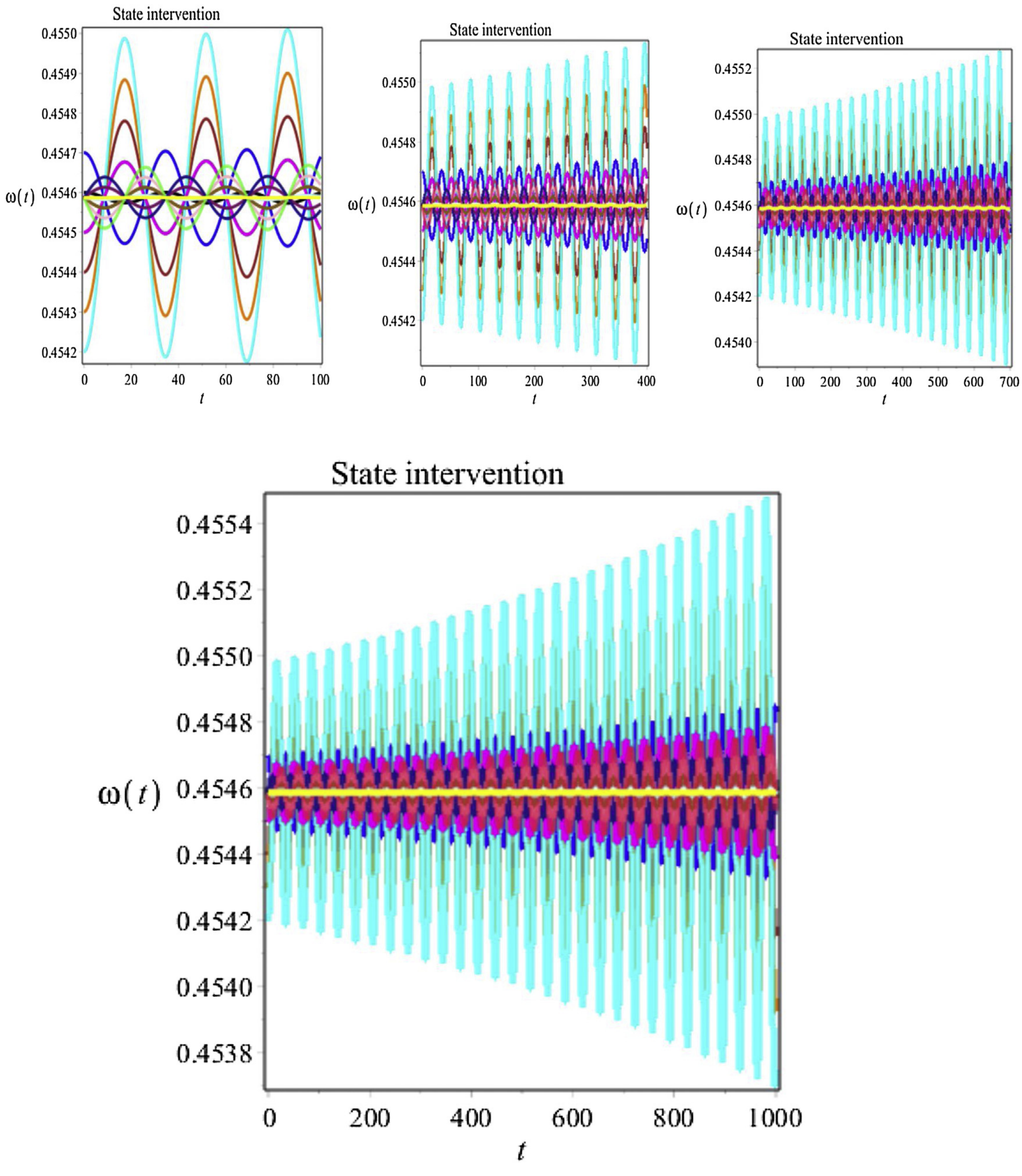

Fig. 5. Time series solutions in the plane $(t, \omega(t))$, when $\delta^* = 0.652570405982870376981$, with 25 Initial conditions.$(y(0)\,,\ f(0)\,,\omega(0))$.*Here* $t =$ 0..100, $t =$ 0..400, $t =$ 0..700, $t =$ 0..1000.

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