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Making Sense of the Equal Sign in Middle School Mathematics

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ABSTRACT

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One of the main reasons that students struggle as they transition from arithmetic to algebra in the middle grades is that they fail to develop the appropriate understanding of the equal sign. Previous research has suggested that students need to move past an operational understanding and develop a relational understanding of the equal sign in order to work with algebraic equations successfully. Other research has suggested that the way that we interpret and utilize the equal sign is based on three main factors: multiple meanings of the equal sign, equation types, and structural conventions. This study extends both areas of research by analyzing two middle grade curricula and looking for what meanings, equation types, and structural conventions appear in both teacher and student materials. The study confirms that students are exposed to three main meanings of the equal sign in the middle grades. The study also describes which meanings of the equal sign are associated with particular equation types and the frequency with which these equation types appear throughout the 7th and 8th grade curricula. Study findings can be used to inform instruction, as they delineate the factors that are attended to while making sense of the equal sign in the middle grades.

Keywords: equal sign, equation types, structural conventions, algebra, middle school mathematics, interpretations of the equal sign

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CHAPTER 1: INTRODUCTION

Algebra has been a focus of mathematics education research for years as algebra is considered the "gatekeeper" to higher education and future career opportunities (Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005; Moses & Cobb, 2001). Although developing and mastering algebraic concepts is vital for many future opportunities, students continue to fail to make the connections and develop the necessary conceptual understandings that would facilitate such mastery. Much of the research that has been done on student failure in algebra shows that most students begin to struggle in their first year, as they transition from arithmetic to algebra (National Research Council, 1998).

As students begin learning algebra, they are introduced to new ways of thinking about some key mathematical concepts as they transition from arithmetic. One example of a key concept that students think about differently in algebra than they do in arithmetic is the equal sign (Carpenter, Franke, & Levi, 2003). Throughout arithmetic, students primarily see the equal sign used in many different iterations of the same problem type, e.g., $3 + 5 = [\quad]$, or $8 \times 2 = [\quad]$. As students see the equal sign used exclusively in this way, they develop an understanding that the equal sign means to perform the given operation and then put the answer to the right of the equal sign symbol. This understanding is commonly referred to as an operational understanding of the equal sign. Once students begin learning algebra, they can no longer rely solely on an operational understanding of the equal sign. In algebra, students are required to understand the equal sign as a symbol that relates the two expressions on either side of the equal sign as having the same value, e.g., $3x + 5x = 2x + 6x$, or $2x + 4 = 8$. In order to solve equations or make sense of algebraic situations, students must understand the equal sign as a symbol that indicates that the left side is the same value as the right. This understanding is referred to as a relational understanding of the equal sign. With a relational understanding of the equal sign, students can then understand why they are able to perform the same operations on both sides of the equal sign while maintaining an equivalent relationship.

Research shows that students often only have an operational understanding of the equal sign when they begin learning algebra. (Alibali,Knuth, Hattikudur, McNeil, & Stephens, 2007; Asquith, Stephens, Knuth, & Alibali, 2007; Byrd, & Matthews, 2015; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Knuth, et al., 2005; Molina & Ambrose, 2006). In a study done by Falkner et al. (1999), 145 sixth-grade mathematics students were asked to fill in the box of the statement: $8 + 4 = [\quad] + 5$, and every one of the 145 students responded with either 12 or 17. When thinking of the equal sign as an operational symbol, the students automatically thought that since $8 + 4 = 12$, 12 must be the number that belongs in the box. Some students then extended the equality strand and performed the next operation, which resulted in $8 + 4 = [12] + 5 = 17$. Some students just thought that they should perform all operations and then put their answer after the given equal sign resulting in $8 + 4 = [17] + 5$. Understanding the equal sign as a relational symbol is required for students to see that what goes in the box is the number that when added to five will give the same sum as $8 + 4$. This example from Falkner et al. shows how understanding the equal sign only as an operational symbol can be a major stumbling block for students transitioning into algebra.

Research also shows that a student's performance on problems involving equations is associated with how that student interprets the meaning of the equal sign. Knuth et al. (2005) interviewed 373 6^{th} through 8^{th} grade students to determine if there was a correlation between their understanding of the equal sign and their performance on an equivalent equations problem. They began the study by determining if the students understood the equal sign as an operational symbol or a relational symbol or both. They then compared the students' understanding of the equal sign to their ability to solve an equivalent equations problem that asked the students if the value that goes in the box for $2 \times [\quad] + 15 = 31$ is the same as the value that goes in the box for $2 \times [\quad] + 15 - 9 = 31 - 9$. From this study they found that the majority of middle school students interpreted the equal sign as an operational symbol rather than a relational symbol. They also found that the students' understanding of the meaning of the equal sign affected their success and approach to solving the equivalent equations problem. Students with an operational understanding were either unable to solve the problem correctly or used strategies that did not require them to recognize equivalence to solve it. Strategies that did not require a recognition of equivalence included the following: the "solve and compare" strategy where either the students would determine a solution for the first equation and then substitute that solution into the second equation and see if that value satisfied both equations, or they would determine a solution for each equation and then compare solutions; and the "after the equal sign" strategy where the students would determine that the two equations are the same because they both have 31 after the equal sign. Students with a relational understanding of the equal sign, however, were more likely to solve the problem correctly by simply recognizing equivalence—that by performing the same operation to both sides of the equation the second equation maintains the same relationship as the first.

From the studies by Falkner et al. (1999) and Knuth et al. (2005) we learn three important commonalities about students' understanding of the equal sign as they are introduced to algebra. First, we learn that when students begin learning algebra, their understanding of the equal sign is predominantly operational. Second, we learn that students will often adhere to an operational understanding of the equal sign in contexts that require a relational understanding to solve correctly. And lastly, we learn that there seems to be a connection between the students' ability to reason about the equal sign relationally and their success on an equivalence equation problem. These three commonalities suggest that student failure in learning algebra is caused, at least in part, by their lack of a relational understanding of the equal sign.

While many researchers recognize the operational and relational meanings of the equal sign as well as the importance of developing a relational understanding (Carpenter, Franke, & Levi, 2003; Falkner et al., 1999; Kieran, 1981; Knuth et al., 2005; Knuth et al., 2006), other researchers have suggested that how we interpret the equal sign is affected by more than just these two meanings (Matz, 1982; Prediger, 2010). While studying student errors in algebra, Matz recognized that there are multiple types of equations as well as structural conventions that affect the way a student reads and solves particular equations. Prediger also recognized that there are multiple possible meanings of the equal sign, not only throughout different contexts in algebra, but within a single problem. She recognized that the equal sign can be understood operationally, relationally (with four different relational interpretations of equality), and as a specification. I will go into more detail of Matz' and Prediger's findings in Chapter 2.

It is likely that proficient users of algebra use multiple meanings for the equal sign. The problem that Prediger (2010) recognizes is that many proficient users are able to navigate through the different meanings of the equal sign as needed completely unaware that they are doing so. The ambiguous change of meanings of the equal sign, especially in instruction, is likely the cause of many students' difficulties in grasping algebraic concepts. Prediger illustrates this problem with a girl named Emily who was being taught how to calculate 24×7 by decomposition. When shown $24 \times 7 = 20 \times 7 + 4 \times 7 = 140 + 28 = 168$, Emily, who had only seen the equal sign as an operational symbol, remarked that this was wrong because 24×7

does not equal 20. In this case, the teacher began using the equal sign in a way that Emily had never seen before yet was expected to understand its new meaning. To help students transition from arithmetic to algebra and develop the ability to switch the way they interpret the equal sign in different contexts, “it is crucial for teachers to be able to communicate explicitly on differences between meanings whenever the situation demands” (Prediger, 2010, pg. 84).

Previous research has elucidated the importance of understanding the equal sign correctly. It has shown us that for students to recognize equivalent equations, they need to have a relational understanding of the equal sign (Knuth et al., 2005). It has also shown us that there are likely more than one meaning of the equal sign that get used in algebra (Matz, 1982; Prediger, 2010). While we know that there are possible multiple meanings of the equal sign, research has not yet been done on which specific meanings get used in middle school mathematics courses where algebra is introduced. If teachers are to be explicit about the different meanings of the equal sign that they use in instruction, then they must first become overtly aware of these different meanings. For this reason, more research needs to be done on which meanings of the equal sign are used in middle school mathematics courses.

CHAPTER 2: BACKGROUND

Literature Review

The majority of research that has been done on the equal sign has been focused on the dichotomy of operational and relational interpretations, finding that students are more likely to be successful in algebra when they interpret the equal sign as a relational symbol (Carpenter, et al., 2003; Falkner, et al., 1999; Kieran, 1981; Knuth et al, 2005; Knuth et al., 2006). However, research also shows that it is not only the meaning of the equal sign that affects how one interprets how to use it. In this section, I argue that the way mathematicians interpret and utilize the equal sign appropriately in a given situation is affected by multiple factors To make this argument, I draw upon the contributions of Matz (1982), Prediger (2010), and Zwetzschler and Prediger (2013) to develop a framework for recognizing which meaning of the equal sign is being used in a given situation, as well as the other possible factors that affect the interpretation of the equal sign that are not limited to the operational and relational meanings.

From the literature on the equal sign, I identified three main categories of findings related to how we interpret and utilize the equal sign. These categories include meanings for the equal sign, equation types, and structural conventions. Determining how the equal sign is used is complex, and it is likely that people with a strong background in algebra unconsciously consider these three factors to determine how to interpret and utilize the equal sign in any given situation. I now discuss these three factors and illustrate how a sophisticated user of the equal sign might use them to determine how to correctly solve an algebra problem.

Meanings

The first factor that determines how the equal sign is utilized is the meaning which the reader assigns to the equal sign symbol. The three main meanings for the equal sign are operational, relational, and assignment. The *operational* meaning of the equal sign is well documented in the literature (Carpenter, et al., 2003; Falkner, et al., 1999; Kieran, 1981; Knuth et al, 2005, Prediger 2010). When reading the equal sign with an operational meaning, the reader is prompted to complete the given process or operation in order to transform the original expression and then to record the result immediately to the right of the equal sign. The language often associated with using the operational meaning for the equal sign is that the expression on the left, "goes to," "makes," "becomes," or "gives us" the expression on the right (Kieran, 1981; Matz 1982). For example, while working through the problem $3(x-2) = \_\_$ with an operational meaning, the user will think that by distributing the 3 among $(x-2)$, the expression on the left "becomes" $3x-6$, which will be written immediately to the right of the equal sign.

The second meaning of the equal sign is *relational* (Carpenter, et al., 2003; Falkner et al, 1999; Kieran, 1981; Knuth et al, 2005, Prediger 2010). When reading the equal sign with a relational meaning, the reader recognizes that the left and right expressions share the same numeric value or are equivalent in the sense that they could be transformed into each other using appropriate transformation rules. The language often associated with using a relational meaning for the equal sign is that the expression on the left "is the same as" the expression on the right (e.g. Kieran, 1981; Knuth et al., 2005). For example, if readers give the equal sign a relational meaning, then they would recognize that if $4 + x = 9$, then $4 + x - 2 = 9 - 2$. Because the value of the left expression is interpreted as being the same as the value of the right expression, subtracting 2 from both sides of the equal sign will result in the same value on both sides (Knuth et al, 2005).

Note that the relational meaning differs from the operational meaning in that relational reasoning focuses on the sameness of the two expressions rather than performing an operation to produce an answer. Thus, for example, when viewing the expression $(x + 3)(x - 3) = x^2 - 9$ from a relational perspective, the reader interprets the equation as a claim of "sameness" between the left and right expressions rather than a record of having applied a transformation or operation to the left expression to produce the right expression. If asked, the reader may nonetheless justify the sameness of the two expressions by reasoning operationally and identifying the transformations that could be performed on one of the expressions to produce the other; however, this operational reasoning about the equal sign is done to justify the sameness of the two expressions, and is different from the initial, relational meaning of the equation constructed by the reader.

The third meaning of the equal sign is specification (Cortes, Vergnaud, & Kavafian, 1990; Prediger, 2010), which I refer to as *assignment*. In the literature, researchers have used the term *specification* to indicate a particular meaning for the equal sign as well as a specific type of equation. To avoid confusion, I have chosen to distinguish between the two uses of *specification* by using the term *assignment* to refer to the specification meaning of the equal sign, and the term *specification* to refer only to a particular type of equation. When reading the equal sign as an assignment, the reader sees a value assigned to the unknown or a rule assigned to a function that is only true in this specific situation. The equal sign indicates that one of the expressions "means" or "is" the other expression. An assignment meaning is different from an operational meaning in that the reader is not trying to transform an expression. It is also different from a relational meaning in that the equal sign is interpreted as signifying the process of defining an object (e.g., variable, function) rather than presenting a claim of sameness (e.g., $2x - 5 = 13$) that might be manipulated, for example, to find the value for *x*. The reader recognizes that an assignment equation is to be used (e.g., to make a substitution) rather than to be worked on (e.g., to solve for a variable). For example, when read with an assignment meaning, the function $y = 2x + 52$, would be recognized as "letting" y be $2x + 52$, which would allow for substitution of the expression $2x + 52$ for the unknown y in a second equation, or for entering the expression $2x + 52$ as input for creating a graph on a graphing calculator. In contrast, the reader would need to interpret this equation relationally to manipulate the equation to solve for x in terms of y, or to determine the set of values that make the equation true.

Equation Types

Knowing the different possible meanings of the equal sign is an important step in interpreting and utilizing the equal sign correctly; however, which meaning one should use depends at least partially on the type of equation that is being considered. Therefore, the second factor that affects how we interpret and utilize the equal sign is equation types. In the literature, researchers have used two criteria to classify equations: the "domain over which they are true" (Matz, 1982, p. 40), and the way they are to be used by the reader. Based on these criteria, there are three main equation types that show up in the research, namely tautologies, constraint equations, and specifications.

Tautologies. Tautologies are common in both arithmetic and algebra. In arithmetic, tautologies are equations for which both sides of the equal sign have the same numeric value. In algebra, tautologies are equations with variables that are true for all values of those variables; often the expressions on each side can be transformed into the other using the transformation rules deemed valid by the mathematics community (Matz, 1982; Prediger, 2010; Zwetzschler & Prediger, 2013). It is important to recognize the different types of tautologies that are used in algebra because they can elicit different meanings of the equal sign. Three common tautologies are transformations, formal identities, and contextual identities.

Transformations, which Matz (1982) calls process-result actions, are tautologies in which an operation or procedure is performed on the expression on one side of the equal sign and the resulting expression is written on the other side. Transformation equations may consist of either a single transformation or a series of transformations, as seen in Figure 1. In each of these examples, it is possible for readers to think about the equal sign operationally or relationally. The operational meaning might first be used if the reader views an expression on one side of the equation as a procedure call and the expression on the other side as the result. Learners of mathematics may stop here, seeing the equal sign only as an indication of where to put the result of the procedure. Experienced mathematicians, however, might simultaneously see the equal sign as a prompt to perform a procedure as well as a symbol relating two expressions that represent the same value because of the mathematical properties involved in that procedure. The equation itself does not determine which meaning of the equal sign the reader may use to make sense of it.

$$5x + 2x = 7x$$

$$4(x + 3) = 4x + 12$$

$$(x - 3)(x + 4) = x^2 + 4x - 3x - 12 = x^2 + x - 12$$

Figure 1. Examples of transformation equations.

Another type of tautology that is common in algebra is a formal identity (Prediger, 2010). A formal identity is an equation that uses the equal sign to connect two symbolic expressions that are the same numeric value regardless of what values are assigned to the variable(s), as in the equation $(a - b)(a + b) = a^2 - b^2$. A formal identity differs from a transformation in terms of purpose. The purpose of a transformation is to show a record of a procedure call while an identity is meant by the author to express a mathematical property in terms of a relationship between expressions that always yield the same numeric value. For example, when reading the difference of squares equation $(a-b)(a+b)=a^2-b^2$ as a formal identity, the solver recognizes the relationship between the two expressions as yielding the same numeric value and does not need to perform a transformation on the first expression $(a-b)(a+b)$ to result in a^2-b^2. Note that the distinction between identities and transformation is determined by the (inferred) purpose the author intended the equation to serve.

The third type of tautology is contextual identities (Prediger, 2010). Contextual identities are similar to formal identities in that they are used to express a mathematical property rather than a record of a completed procedure; however, contextual identities are formulae in which the equality statement is true for all values of unknowns only when certain mathematical conditions are first met. For example, while working with a right triangle, it is true that the sum of the two side lengths squared is equal to the length of the hypotenuse squared, or $a^2+b^2=c^2$. This theorem is not true for oblique triangles, however, so the user must be careful to check that a triangle has a 90° angle before using this identity. Other examples of contextual identities include formulae for the volume of a cone or area of a triangle.

1. $3x+3=2x+7$
2. $3x+3-3=2x+7-3$
3. $3x=2x+4$
4. $3x-2x=2x-2x+4$
5. $x=4$

Figure 2. Deductions and reductions in a constraint equation.

Constraint equation. The second equation type is a constraint equation (Matz, 1982; Prediger, 2010). Constraint equations differ from tautologies in terms of the domain of values that make them true. In a constraint equation, the left expression and the right expression are not equivalent for all values of the unknown; instead, the value of the unknown is constrained to the set of numbers for which both expressions, when evaluated using a number from the set, yield the same numeric value (Matz, 1982). When solving a constraint equation, the solution, or the values of x which make the two expressions equivalent, is found by performing a series of deductions and reductions. Deductions are defined by Matz as the operations that are applied to both sides of the equations and reductions are defined as the transformations that are performed to simplify each side of the equation, often after deductions have been made. Deductions are followed by reductions on each side, resulting in a new constraint equation where the value of the unknown is maintained. This pattern continues until the unknown is isolated and the value becomes known, as is illustrated in Figure 2.

The deductions and reductions are made explicit in the example in figure 2. Line 1 is the original constraint equation, setting two non-equivalent expressions equal to each other. Line 2 illustrates a deduction that involves subtracting 3 from each side, which, while changing the values of the expressions from line 1, maintains the value of x. Line 3 is the result of reductions performed on each expression in line 2. From here, the cycle begins again, and in line 4, another deduction is performed by subtracting $2x$ from each side. Line 5 is the result of reductions performed on line 4 and is the end result. While solving a constraint equation, the user of the equal sign must go back and forth between different ways of thinking about equality, so it is important to make the distinction between a deduction and a reduction. Deductions do not preserve the equivalence of the two expressions on the same side of equal sign; for example, the expression $3x + 3$ on the left of the equal sign in line 1 is not equivalent to $3x + 3 - 3$ on the left of the equal sign in line 2. Reductions, however, do preserve equivalence between the expressions on the same side of the equal sign; for example, $3x + 3 - 3$ on the left of the equal sign in line 2 is equivalent to $3x$ on the left of the equal sign in line 3.

Specifications. The third equation type is a specification (Cortes et al., 1990; Prediger, 2010). A specification is a statement that defines the value of a variable or the rule for a function in a specific situation. Unlike an identity, a specification is only true for the context or problem for which the assignment was made. Examples of specifications include $x = 3$, $y = x^2 - 2x + 1$, or $f(x) = -x - 3$. Specifications are different from tautologies because they neither describe a transformation of one expression into another nor present a general mathematical claim. They differ from constraints in that the value of the variable or rule for a function is explicitly given without any need for solving the equation. Specifications are also unique because they are directly tied to the assignment meaning of the equal sign. In contrast, tautologies and constraints can possibly be thought of relationally or operationally, and therefore are not specifically tied to one specific meaning. For example, $x = 3$ is a specification equation only if we think of assigning the value 3 to the unknown x. While the relational meaning can be used to interpret the equation to mean x and 3 are the same number, the use of this alternate meaning results in $x = 3$ being classified as a constraint equation, not a specification.

The preceding discussion of $x = 3$ suggests that the classification of equation types depends on which meaning the reader chooses to use to interpret the equal sign. However, the research on equality has not explicitly addressed the connection between the inferred meaning of the equal sign and the classification given to equations. Thus, it is unknown whether there is a clear relationship between equation type and meaning of the equal sign. For example, while we have argued that a specification equation always involves the assignment meaning of the equal sign, this relationship is only tacitly addressed in the literature through the common practice of using the term *specification* to refer to both a meaning for the equal sign and an equation type.

More work needs to be done to explore the relationship between equation types and meanings of the equal sign and identify patterns.

Structural Conventions

The third factor that determines how we interpret the equal sign is structural conventions (Matz, 1982). The way that we read and write equations is an integral part in the way we interpret the equal sign. In arithmetic, problems are typically written and read one of two ways: from left to right (Figure 3a) or vertically top to bottom (Figure 3b). Because different types of equations abide by different reading and writing conventions (Matz, 1982), correct interpretations of the equal sign require familiarity with these conventions.

a) $1 + 2 = 3$

b) $$\begin{array}{r} 1 \\ +2 \\ \hline 3 \end{array}$$

Figure 3. Structural conventions in arithmetic: (a) vertical arithmetic problem, (b) horizontal arithmetic problem.

One of the main conventions is that of reading and writing from left to right (LTR). The LTR convention can be found in tautologies and specifications. For example, in a transformation the original expression is usually presented on the left side of the equal sign and the result of the transformation follows the equal sign on the right. This is extended to a chain of transformations when we begin with an algebraic object and operate on it, completing a series of transformations until we successfully arrive at the "simplest" form of the expression as in Figure 4. (Matz, 1982).

$$(x-2)(x+4) = x(x+4) + (-2)(x+4) = x^2 + 4x - 2x - 8 = x^2 + 2x - 8.$$

Figure 4. Example of a chain of transformations.

Although not as common, there are situations in which a tautology or specification might be written from right to left (RTL), e.g. $2 = x$ or $\_\_ = 2x + 3$, or in arithmetic, $\_\_= 5 + 3$. It is possible that because of habit and experience many people who see an equality that is written with the RTL convention simply switch the order of the expressions in their head and read it from LTR. On the other hand, it is also possible that after a reader has read a string of transformations from LTR, he or she may read those transformations from RTL as a description of how the last expression on the right can be transformed back into the first expression on the left.

Transformation chains can be written using the LTR convention, but they can also be written top to bottom in a single column (Matz, 1982). I will refer to this convention as the single column down (SCD) convention. In Figure 5a, the SCD convention begins with an algebraic object followed by subsequent reductions on each line forming one column. The SCD convention can also begin with an algebraic object on the left with one reduction to the right of the equal sign followed by each subsequent reduction that is written directly below the previous one continuing down the right column until arriving at the simplest form of the expression as in Figure 5b.

a) $$\begin{array}{l} (x-2)(x+4) = \\ x(x+4) + (-2)(x+4) = \\ x^2 + 4x - 2x - 8 = \\ x^2 + 2x - 8 \end{array}$$

b) $$\begin{aligned} (x-2)(x+4) &= x(x+4) + (-2)(x+4) \\ &= x^2 + 4x - 2x - 8 \\ &= x^2 + 2x - 8 \end{aligned}$$

Figure 5. Single Column Down (SCD) convention: (a) SCD with reduction expression on left, (b) SCD with reduction expression on right.

The third main convention for reading and writing equalities is both columns down (BCD). The BCD convention occurs in the solving of constraint equations as we apply deductions and reductions to each column separately (Matz, 1982). There are two common ways of writing the deduction step in a BCD while solving a constraint equation which are illustrated in figure 6. One way is to write the BCD with a horizontal deduction (BCD-HD) alongside the equation that the operation is being applied to (Figure 6a). The other way is to write the BCD with a vertical deduction (BCD-VD) written underneath the terms that the operation is being applied to, sometimes followed by a straight-line denoting that an operation has taken place and the resulting reduction is written on the next line (Figure 6b). Instead of reducing the constraint equation to its simplest form as we often do with tautologies, the BCD creates a new equation while maintaining the value of the unknown until we have isolated the unknown and determined its value.

a)

$$\begin{aligned} 3x + 3 &= 2x + 7 \\ 3x + 3 - 3 &= 2x + 7 - 3 \\ 3x &= 2x + 4 \\ 3x - 2x &= 2x - 2x + 4 \\ x &= 4 \end{aligned}$$

b)

$$\begin{aligned} 3x + 3 &= 2x + 7 \\ \underline{\quad -3} &\underline{= \quad\ -3} \\ 3x &= 2x + 4 \\ \underline{-2x} &\underline{= -2x} \\ x &= 4 \end{aligned}$$

Figure 6. Both Columns Down (BCD) convention: (a) with horizontal deduction (BCD-HD), (b) with vertical deduction (BCD-VD).

Research on the Use of the Equal Sign in Algebra Classrooms

The meanings, equation types, and structural conventions that I have discussed have surfaced through research that has been focused on student understanding of the equal sign. Whether inferred through student interviews (Falkner et al, 1999; Knuth et al, 2005), while analyzing student errors (Matz, 1982), or determining what student teachers need to understand so they can mitigate student misconceptions (Prediger, 2010), the information that we have gathered about the equal sign has been student centric. By looking at the research that has been done on student understanding of the equal sign, we are left only to infer that the above meanings, equation types, and conventions are likely used in middle grades mathematics classrooms where algebra is often first introduced. We do not have empirical research to confirm if all of the previously mentioned meanings, equation types, and structural conventions are used in instruction and curricula in middle grades mathematics, if they are an exhaustive list or if there are others that have not been discussed. To test the above framework for completeness and relevance while reasoning with the equal sign in middle school mathematics, we can examine the curricula used. Therefore, I ask the research question: Which meanings for the equal sign, equation types, and structural conventions are used in middle grades mathematics curricula?

Theoretical Framework

Students struggle with the equal sign because the meaning of the equal sign in a particular instance is not inherent in the symbol itself; instead, students must choose a meaning for the equal sign. From a constructivist point of view, meaning is constructed through the experiences of the reader (Simon, 1995; von Glasersfeld, 1983) and therefore each reader may give a different meaning to the same written material. For example, in the equation $a(b+c) = ab + ac$, readers might read it operationally if they interpret the equation as indicating that the operation of distributing the a among $(b+c)$ to produce $ab + ac$ is being performed, or they could read it relationally as an identity if they interpret the equation as a claim that $a(b+c)$ will yield the same value as $ab + ac$ for all choices of *a, b,* and *c*. Although it would probably be considered problematic, it is also possible that a reader might read the equal sign as an assignment and think that $a(b+c)$ *means* $ab + ac$. In short, there are multiple ways to interpret the equal sign when reading a given equation. The decision of which meaning to use is decided upon by the reader, and not intrinsic to the symbol itself.

Determining whether a meaning for the equal sign is appropriate in a particular instance is complex because the validity or correctness of an interpretation is based upon the taken-as-shared norms and practices of the mathematical community in which the equal sign is being used

(Cobb, Wood, & Yackel, 1993). With the idea that each individual constructs his or her own meaning, there are going to be times when a reader does not draw upon the correct meanings, where correctness is determined by taken-as-shared norms and practices. There are also times when a certain practice might be considered valid in one community and invalid in another. For example, in many communities it is unproblematic to use the equal sign in an equality chain where the user completes an operation and writes the result directly after the equal sign and then continues on the same chain with a new operation, e.g., $3 + 2 = 5 \times 3 = 15 + 5 = 20 \div 2 = 10$. The community of mathematics educators objects to this particular use of the equal sign because it does not conform to the norms and practices of the community concerning the use of the equal sign. However, among shop keepers, for example, this way of using the equal sign might be considered acceptable and even desirable when making quick calculations. This example shows that not all uses of the equal sign are equally valid, and that validity in use may depend heavily upon the mathematics community in which one is engaging in the practice.

The complexity of determining the meaning of the equal sign in a given equation does create a challenge to interpreting the meanings of the equal sign used in middle grades mathematics curricula. Because meanings are constructed by the individuals reading the equation, and not inherent in the symbols themselves, we cannot just look at an equation or someone's work and know which meaning of the equal sign was intended by the author. In order to determine the meaning of the equal sign that an author is using, it is necessary to look at the larger context for clues about what meaning and type of equation are being used. Furthermore, we can use our knowledge of the taken-as-shared norms and practices of teachers of mathematics to narrow down some of the possible interpretations—we can rule out some of them because they violate community norms and practices. Therefore, in order to interpret what meaning is being assigned to the equal sign in middle school mathematics curricula, I will be looking at the context in which the equal sign is used and considering the taken-as-shared norms and practices of the school mathematics community.

CHAPTER 3: METHODS

In this chapter I discuss the methods I used to answer the research question of which meanings for the equal sign, equation types, and structural conventions are used in middle grades mathematics curricula. In order to access a year's worth of teaching material in a shorter amount of time, I chose to analyze middle grades mathematics curricula rather than observe instruction in middle grades mathematics classrooms. I first discuss the two curricula I chose to analyze and why, followed by the chapters that I analyzed and why. I then discuss the analysis process, which includes my coding process as well as how I collected and organized the data.

Curricula

I analyzed two different curricula in this study. As the majority of the United States have chosen to adopt the common core as well as integrated mathematics for middle school mathematics, I chose curricula based on alignment to the Common Core State Standards for Mathematics (CCSSM) as ranked by edreports.org, as well as what curricula are commonly being used in the US as surveyed by a 2017 RAND report (Opfer, Kaufman, & Thompson, 2017). From the RAND report, the three most used curricula are Glencoe Math (McGraw Hill) (44%), Connected Math (34%), and Eureka Math (Engage NY) (34%). Of these three highly used curricula, Ed reports ranked Eureka Math the highest, with Connected Math Project 3 and Glencoe Math following behind respectively. Based on these rankings, the two curricula I chose to analyze were Connected Mathematics 3 (CMP3) and Eureka Math.

Even though there were many curricula from which I could choose, CMP3 and Eureka Math both stood out as reputable, effective, and widely used. The Connected Mathematics Project from Michigan State University has been in use for approximately 27 years. It was funded by the National Science Foundation and was developed by professors of mathematics education. It is used in all 50 United States as well as China and England. The CMP3 is the latest edition from the project and has been revised to align with the CCSSM. In a study of top instructional materials used for mathematics classroom lessons among teachers in states who have decided to use the common core, 34% of secondary mathematics teachers use CMP3. In the same study, 44% of teachers said that they use materials from Eureka Math/Engage New York (Opfer, et al., 2017). Eureka Math is an online curriculum developed by Engage NY through a collaboration of school teachers and scholars in the field of mathematics.

My goal for this study was to identify the different meanings of the equal sign, equation types, and structural conventions that appear throughout an entire 7th and 8th grade curriculum. In order to get a sufficient sample size of each curriculum, I analyzed three sections or modules from each year. The sections I analyzed include topics that cover three different CCSSM standards from each year. These standards are representative of at least two different branches of mathematics (algebra and geometry) that are integrated into 7th and 8th grade math. For 7th grade I analyzed the sections that predominantly cover the CCSSM standards of *ratios and proportions (*7.RP), *expressions and equations* (7.EE), and *geometry* (7.G). For 8th grade I analyzed the sections that predominantly cover the *expressions and equations* (8.EE), *functions* (8.F), and *geometry* (8.G) CCSSM standards. By choosing chapters that are representative of different branches of mathematics, I hoped to be able to find a more diverse representation of meanings, equation types and structural conventions that teachers encounter throughout the whole curriculum as well as investigate how the uses of the equal sign progress or mature through related ideas in the same year. Along with a broader picture of how the equal sign is used throughout the year, by continuing to investigate *expressions and equations* and *geometry* into the 8th grade year, I was able to explore the development of the use of the equal sign from one year to the next.

Each curriculum is set up differently, so I describe below exactly which chapters and lessons I analyzed from each year of each curriculum. I do this because both curricula use an integrated approach, so many chapters cover multiple standards from the CCSSM. The chapters I chose to analyze align most closely to the specific standards I wanted to look at. The CMP3 curriculum titles their chapters differently than the CCSSM, so I will give the title but also which standards from the CCSSM that chapter primarily covers. For 7th grade, I analyzed *Stretching and Shrinking* (7.RP), *Moving Straight Ahead* (7.EE), and *Filling and Wrapping* (7.G). For 8th grade, CMP3 split up the 8.EE standards among three different sections so I analyzed all three of those sections so that it would more closely match that of Eureka Math. The three sections for 8.EE were *Growing, Growing, Growing, Say it With Symbols, and It's in the System.* The other chapters I analyzed were *Thinking with Mathematical Models* (8.F), *Looking for Pythagoras* (8.G) and *Butterflies, Pinwheels, and Wallpaper* (8.G). A few lessons from *Thinking with Mathematical Models* were excluded as they cover other standards entirely (see Appendix A for a complete list of the lessons from each section and their corresponding CCSSM standards).

The chapters I analyzed from Eureka Math for 7th grade were Module 1: *Ratios and Proportions,* Module 3: *Expressions and Equations,* and Module 6: *Geometry.* Because these chapters address the same standards as their titles, I looked at them in their entirety. For 8th grade, I analyzed all of Module 4: *Linear Equations*, all of Module 5: *Examples of Functions from Geometry*, the first 5 lessons from Module 6: *Linear Functions*, and lessons 1-5 and 15-23 of Module 7: *Introduction to Irrational Numbers Using Geometry.*

For the analysis of these aforementioned sections, I looked at each instance of the equal sign as it appeared in explanations and examples in the teacher instructional material, and explanations, examples, and problem sets from the student material. It is likely that teachers develop their examples and language used for instruction from the instructional materials that they use. I also looked at the meanings, equation types, and structural conventions used by the authors in the explanations, examples, and problem sets in the student manuals because these also reflect how teachers might teach and use the equal sign. By combining the context observed from the teacher instructional material as well as the student manual I was better able to infer which meanings, and equation types show up in middle school mathematics. I also looked at which structural conventions were used throughout the curricula in instructional materials as well as explanations and examples in the student material. I did recognize, though, that structural conventions used in written curricula may be what is expected that the teachers will use in instruction, or it may be chosen because of ease in typing it up. Since I was looking at how the equal sign was used in instruction and in curricular materials in 7^{th} and 8^{th} grade mathematics and not how students use the equal sign, I did not look at the solutions for the problem sets, nor try to imagine how the students would work out their solutions.

Analysis

In the initial phase of data analysis, another researcher and myself referred to a preliminary coding chart I had created that included the different meanings, equation types, and structural conventions that were discussed in Chapter 2. We began by coding three lessons from each curriculum individually. Because we looked at each individual equation (two expressions connected by an equal sign) as the unit of analysis, we first went through each section and highlighted each equation, then went back and coded each equation directly on the page where the equation occurred. We used the definitions from the coding sheet along with our expert fluency with mathematical texts to make sense of each equation based upon its context, and then decided what meaning of the equal sign we used, what equation type we read it as, and what direction it was read. Then we would consider whether there were other viable interpretations of the equation that also fit the context. Interpretations that were not viable were ruled out. We then looked at situations where two or more subsequent equations were written horizontally or vertically without any words between them and coded each arrangement for the type of structure it represented. Situations where only one equation was involved were recognized as stand-alone equations that did not follow the listed structural conventions; therefore, only the direction in which the equation was read was coded.

Sometimes we encountered equations with which we struggled to determine the meaning, equation type, and/or structural conventions. After coding the lessons individually, we would meet to discuss these difficult equations as well as the equations on which our coding did not agree. We were able to resolve these conflicts by developing more detailed descriptions of our codes and if we felt like a certain equation could not be categorized by one of the codes already listed, we worked together to come up with a new equation type and definition under which it could fit. Through this process, we determined that there were multiple equation types and different structural conventions that were not listed in the original framework, so those equation types and conventions were added to the coding list.

Perhaps the most significant changes were made to the codes for structural conventions. After a few attempts to code sections and compare codes, we realized that we often disagreed about or could not determine the direction of reading. We found that for some equations, we were not sure in which direction we had actually read it. For example, with the equation

$2(2l + 2w) = 2 \times 2(l + w)$ (CMP3-7, Stretching and Shrinking, p.103), we realized that at times we read it LTR and at other times RTL, each time going back and forth between right and left to make sense of the procedure that took place. In this situation it was difficult for us to determine which direction we initially read the equation, let alone determine the direction intended by the author. We determined that the actual direction in which we read particular equations may not be consciously accessible to us and may require eye movement tracking equipment to accurately judge. Because of this difficulty, we decided not to code direction as we could not make a valid determination and therefore it would not add anything to the study.

Along with direction, we revised both the BCD and SCD conventions. Throughout multiple phases of coding we encountered situations where equations were arranged according to the BCD convention but were not constraint equations, so to include situations beyond constraint equations we redefined the structural convention as lists of equations (LOE). This change also led us to track if and which equation operations were being used between subsequent equations in the list, which caused us to create a set of codes for equation operations. As we identified SCDs, defined as expressions or equations that continued vertically with multiple subsequent expressions each connected by an equal sign written on the right or left, we realized that there were many instances where the same process occurred, but in one long horizontal string rather than in an SCD. Because both vertical and horizontal instances represented the same situation of subsequent expressions being connected by an equal sign, we decided to rename the actual object or process as a string of equalities (SOE). After identifying an SOE, we then noted whether it was vertical or horizontal. We then unpacked each string into a set of single equations. For example, we unpacked the SOE $75\%\ of\ d^2 = \frac{3}{4}(2r)^2 = \frac{3}{4}(4r^2) = 3r^2$ into three different equations, the first being $75\%\ of\ d^2 = \frac{3}{4}(2r)^2$, the second equation would be $\frac{3}{4}(2r)^2 = \frac{3}{4}(4r^2)$, and then the last equation would be $\frac{3}{4}(4r^2) = 3r^2$. Unpacking each SOE into individual equations allowed us to track whether or not the meaning of the equal sign or the equation type changed as the string continued. These structural revisions not only better fit the data that we analyzed, but also allowed us to gain further insight into how structural conventions affect the way the equal sign is utilized.

We iterated the process of coding the same material separately then comparing and refining our codes multiple times until we began to get rates of interrater reliability of 80% or higher. After the 4^{th} phase of data analysis, we reached an interrater reliability of 93% for the Eureka Math lessons that were analyzed but needed to complete one more set of revisions of the codes and one more phase of data analysis to reach an interrater reliability of 88% for the CMP3 curriculum. To maintain a high level of consistency, after reaching a high level of agreement, we continued to share and compare our analysis for most of the remaining chapters, completing about 90% of the analysis together.

After completing the analysis of each grade and curriculum, I organized the data in a way that allowed me to answer the research questions, as well as find any other interesting patterns. I first organized the data by structural convention for a couple of reasons. First, it allowed me to group the equations included in the structure together to help me recognize patterns between equations within a structure. I also chose to organize it this way because I had already recognized that, on multiple occasions, equation types and meanings of the equal sign changed within the same structure, and by organizing the data by structure first, it was easier to keep track of this phenomenon. I then recorded each equation with the meaning of the equal sign followed by the equation type. This was helpful in that it allowed me to sort through the data by meanings and notice which equation types were connected with each meaning. I was then able to complete a count of the different types, meanings and conventions that showed up in each grade, curriculum, and unit. After completing counts and finding connections between meanings of the equal sign, equation types, and structural conventions, I then identified typical and atypical examples of each code. This allowed me to identify characteristics of each code that we had only tacitly attended to in our coding, resulting in descriptions that more clearly and accurately reflected the characteristics of the codes we used in our analysis. These examples also help set the bounds for each equation type, which we use in the results section to give the reader a better understanding of the categories we have developed. By completing this process of analysis, I was able to determine which meanings, equation types, and structural conventions occurred in three standards of the CCSSM in the CMP3 and Eureka Math 7th and 8th grade curricula. Results of the counts as well as the refined definitions and examples are discussed in the results section of chapter 4.

CHAPTER 4: RESULTS

In this section I discuss three main results that were found from analyzing the Eureka Math and Connected Math Project 3 (CMP3) curricula. The first main result is the types of equations that occurred in the 7th and 8th grade curricula as well as the meanings of the equal sign that are associated with them. In this result I include refined definitions of the equation types as well as typical and atypical examples of each type. The second main result is which structural conventions occurred in the curricula as well the equation types they more commonly were associated with. The final main result that I discuss is the frequency in which equations with the different meanings of the equal sign appear throughout the two curricula in both 7th and 8th grade.

Equation Types

After analyzing the two curricula I was able to organize the equation types and their purpose based on the meanings that are associated with each equation type as seen in Table 1. I was able to organize the equation types this way because when each equation type appeared, it was consistently associated with one specific meaning every time, with one exception that shared two different meanings. The three meanings that appeared in the research include the operational meaning, assignment meaning, and relational meaning as discussed in the literature review. I discuss each meaning of the equal sign and the equation types that are associated with them below.

Table 1

Summary of equation types and their purpose organized by the meanings that they are associated with.

Meanings of the Equal Sign	Equation Types	Purpose
Operational	Transformation Equations	to perform an operation and record the result after the equal sign
Assignment	Specification equations	to specify or assign a specific value to a variable or unknown so that the reader can use that value as an input later in the problem
	Restriction specification equations	to restrict the values that can be assigned to a variable in a problem
Relational	Constraint equations with parameters	to serve as a general form for a family of functions or to provide a template
	Constraint equations with variables	to find one or more corresponding pairs from the solution set of the equation
	Constraint equations with one unknown	to determine which value(s) of the unknown would make the situation true
	Formal Identity	to express a general mathematical property or principle
	Contextual Identity	to model the relationship between quantities in a specific context
	Numeric Identity	to justify why a result is correct
	Numeric Non-identity	to justify that a number was in fact *not* a solution
	Transformation Equations	to prove or justify rather than produce a result
	Unit Identity	to set one unit equal to a different unit and provide information
	Result Equations	to declare a resulting value or solution that has been found by completing a computation or measurement

Equation Types with Operational Meanings

The first equation type I discuss is that which is associated with the operational meaning of the equal sign. As a reminder, the operational meaning is assigned when the intent of the equal sign is to prompt the reader to find the result of an operation or procedure and then record the answer to the right of the equal sign. Consequently, all equations associated with the operational meaning shared this particular structure: an expression on the left that was transformed via a procedure to yield the expression on the right. Because of the focus on transforming one expression into the other, I refer to this type of an equation as a *transformation equation*. An equation was coded as a transformation when the main purpose of the equation appeared to be to perform a given procedure. Some transformations contained only one equal sign, while others contained a string of equalities. However, regardless of how many equal signs were used in the equation, the focus was the same: performing operations and recording the results.

Two examples of a transformation equation are illustrated in Figure 7. The first example is the most common type of transformation equation that occurred in the two curricula. It is purely arithmetic and requires the reader to perform the stated operation in the expression on the left and record the answer to the right of the equal sign. In this specific example the equation comes from trying to find Sam's time in a relay race with Fred. The second example is also a very common type of transformation equation that occurred in the two curricula. This one is algebraic and is used to transform the left equation into a more simplified expression. The reader is still required to perform the given operation of distributing the 2 across the expression in parenthesis, but rather than finding the solution as a single value, as in the first example, the resulting expression is considered the final solution. During the analysis of the curricula, we found that no other equation types shared the focus on the operation and resulting transformation which is why there is only one equation type in the operational meaning category.

a)

Total Time (hours)	Fred's Time (hours)	Sam's Time (hours)
18.35	8	18.35-8=10.35

b) $$\mathbf{2(3x+2)} = 6x+4$$

Figure 7. Transformation equations with operational meaning: (a) Arithmetic example (Eureka Math 7, Module 3, pg. 134), (b) Algebraic example (Eureka Math 8, Module 4, pg. 45)

Equation Types with an Assignment Meaning

The next equation type I discuss is that which uses the assignment meaning. As a reminder, the assignment meaning is given to the equal sign when the equal sign is used to label or assign a value to a variable or object. Consequently, all equations associated with the assignment meaning achieved one of the following two purposes: to specify or assign a specific value to a variable or unknown so that it can be used as an input later in the problem, or to restrict the values that can be assigned to a variable in a problem. These two purposes gave rise to two different equation types: *specification equations* and *restriction specification equations*.

Specification equations achieved the first purpose, namely, to assign specific values or expressions to mathematical objects. There were three common uses for specification equations that appeared in the curricula. The most common use for specification equations was to state the value of one unknown so that the corresponding value of a second unknown could be determined. The following prompt illustrates this purpose: "If $y = 2x^2 + 8x$ find the values of x

when $y = 0$" (CMP3-8, Say It with Symbols, pg. 25). In this example, the specification equation $y = 0$ is used to assign a value to y in the given equation so that the corresponding value of x, the second unknown, can be found.

A second common use for specification equations that appeared in the curricula was to provide a numeric value that the reader should substitute in for an unknown to test if that value could be a solution to the given constraint equation. The following example illustrates this use: "Is the equation a true statement when $x = -3$? In other words, is -3 a solution to the equation $6x + 5 = 5x + 8 + 2x$? Explain" (Eureka Math 8, Module 4, pg.33). In this example, the specification equation $x = -3$ is used to assign the value -3 to the variable x in the given equation and substitute it in to determine if -3 is a solution or not.

A third common use for specification equations, especially in the geometry chapters, was to assign values to the measures of line segments, sides, or an angle so that the reader could either create a shape with those measures or determine values for the measures of other parts of the shape. For example, students were prompted to do the following: "Use your tools to draw ΔABC in the space below, provided $AB = 5cm, BC = 3\ cm, and\ \angle A = 30°$" (Eureka Math 7, Module 6, pg.128). In this example, and the many like it in the curricula, each specification equation is used to specify or assign a specific value for the measure of an unknown side or angle for the purpose of drawing the shape or calculating the shape's other lengths and angle measures.

While the first three examples that I mentioned were the most common uses of specification equations, there was one less common use that is worth mentioning. This use consisted of specifying the rule for a function so that the function could be entered into the graphing calculator and graphed. For example, the text stated that " $y_1 = 3(2x - 5)$ and $y_2 = 2(3x - 1) + x$" (CMP3-8, Say It With Symbols, pg. 23), which was followed by the instructions to enter these functions into the graphing calculator along with a screen shot of the plot 1 screen and then the resulting graph. I considered these two function examples as specification equations because the reader is expected to take what has been assigned to y_1 and y_2 and type that assignment as an input into their corresponding place on the calculator. Because these equations were explicitly meant to be used as calculator inputs, I decided to code them as specification equations. However, if there was no expectation to use the function equation as an input in either the calculator or another function then I did not consider it as a specification but rather a different equation type that I will discuss in the next category.

The second equation type associated with the specification meaning of the equal sign is the *restriction specification equation*. A restriction specification equation restricts the values that can be assigned to a mathematical object without specifying an exact value. An example of restriction specification equations is given in the following excerpt: "Let $\frac{a}{b}$ and $\frac{c}{d}$ be rational numbers, where $a, b, c,$ and d are integers, $b \neq 0, and\ d \neq 0$" (CMP3-8, Say It With Symbols, pg. 273). In this example, the equations $b \neq 0,$ and $\mathrm{d} \neq 0$ are specifying what values must be restricted from being substituted in for the variables b and d. Because the equations $b \neq 0$ and $\mathrm{d} \neq 0$ specify which values *cannot* be assigned to the variables b and d, they are restriction specifications equations. However, because the purpose of these equations is to provide information about the values of the variables, they fall under the assignment meaning of the equal sign.

Equation Types with a Relational Meaning

The final set of equation types I discuss are those associated with the relational meaning of the equal sign. As a reminder, the relational meaning is assigned when the equal sign is used to relate the expressions on both sides as sharing the same numeric value or being equivalent expressions in the sense that they could be transformed into each other using appropriate transformation rules. The remainder of equation types that occurred in the research seemed to require the use of the relational meaning and therefore fell into this set. Below, I describe each equation type in this set, and like I have done previously, provide one or two examples from the curricula to explain these equation types coincide with the relational meaning of the equal sign.

Constraint Equations. A *constraint equation* comprises two expressions, set equal to each other, that are not equivalent for every value of the unknown(s). Instead, the value of the unknown is constrained to the set of numbers for which both expressions, when evaluated using a number from the set, yield the same numeric value. As I coded the data, I found it useful to organize constraint equations into three different categories: constraint equations with parameters, constraint equations with variables, and constraint equations with one unknown. These three equation types differ in number of unknowns, uses, and strategies for solving for the unknown variables. Consequently, students need to be able to distinguish between these three types of equations.

Constraint equations with parameters are equations where both variables and parameters are unspecified within the context of the equation. Parameters are constant quantities in a model on which the varying quantities depend. They do not vary within a specific model or setting but can vary across a general family of functions they help to define. Variables are the varying quantities in the model that can assume multiple values dependent on the conditions set by the parameters. Equations that we coded as constraint equations with parameters most often served as a general form for a family of functions or relations such as $y = mx + b$, $x^n = p$, or $x^2 + y^2 = r^2$ where the numeric values of the parameters are unspecified. As in the example of the y-intercept form of a linear equation, $y = mx + b$, m and b are the parameters that determine the set of (x, y) ordered pairs that makes the linear equation true.

There are two common uses for a constraint equation with parameters in the curricula as illustrated in Figure 8. One use is to give a general description of a family of equations as illustrated in Figure 8a, and the other is to present a template for inserting information and generating a particular equation or result as illustrated in Figure 8b. In both examples the equation $y = kx$ is shown as it is used in the student material. In Figure 8a, the equation is used to help students know how to identify a proportional relationship and recognize the multiplicative relationship between two quantities. Students are not yet asked to use it to find specific values for x and y, but to understand that $y = kx$ means that the value of y will always be k times the value of x. Once students are familiar with the general forms of equations, then they can identify what type of function they are working with, (e.g., whether it be a linear or exponential function). In Figure 8b, the equation $y = kx$ is used as a template for finding either the constant of proportionality, or corresponding values for x and y. The example specifically states that the reader should use this equation and then substitute in the value for k if it is known. Then the reader is told how to find the values for the variables x and y through substitution into the given equation.

Constraint equations with variables are equations that involve two or more variables (i.e., two or more quantities that covary) but no parameters. Constraint equations with variables describe a particular function or relation. An example of a constraint equation is "$2x + 3y = 9$" (Eureka Math 8, Module 4, pg. 308). Note that this constraint equation can be generated by replacing the parameters a, b, and c in the constraint equation $ax + by = c$ with 2, 3, and 9, respectively. This example illustrates that constraint equations with variables can be created from constraint equations with parameters by replacing the parameters with specific numbers.

a) Vocabulary

If a proportional relationship is described by the set of ordered pairs (x, y) that satisfies the equation $y = kx$ for some number k, then k is called the *constant of proportionality*. It is the number that describes the multiplicative relationship between measures, x $a$and y, of two types of quantities. The (x, y) pairs represent all the pairs of numbers that make the equation true.

b) Lesson Summary

How do you find the constant of proportionality? Divide to find the unit rate, $\frac{y}{x} = k$.

How do you write an equation for a proportional relationship? $y = kx$, substituting the value of the constant of proportionality in place of k.

What is the structure of proportional relationship equations, and how do we use them? x and y values are always left as variables, and when one of them is known, they are substituted into $y = kx$ to find the unknown using algebra.

Figure 8. Constraint equation with parameters: (a) as a general description (Eureka Math 7, Module 1, pg. 62), (b) as a template (Eureka Math 7, Module 1, pg. 81)

There are a few different purposes for constraint equations with variables. The most common purpose in the curricula was to find one or more corresponding pairs from the solution set of the equation. An example of this use is illustrated in Figure 9. In this example from the student materials, the authors demonstrate how the points of intercept are found by replacing x and y with zero and then calculating the corresponding value of the ordered pair. In this specific example, these ordered pair solutions are used to graph the linear relation, but in many examples, the end goal was just to find one or more ordered pair.

Lesson Summary

The graph of a linear equation is a line. A linear equation can by graphed using two points: the x-intercept point and the y-intercept point.

Example:

Graph the equation: $2x + 3y = 9$

Replace x with zero and solve for y to determine the y-intercept point.

$$2(0) + 3y = 9$$

$$3y = 9$$

$$y = 3$$

The y-intercept point is at (0,3).

Replace y with zero and solve for x to determine the x-intercept point.

$$2x + 3(0) = 9$$

$$2x = 9$$

$$x = \frac{9}{2}$$

The x-intercept point is at $\left(\frac{9}{2}, 0\right)$.

Figure 9. Finding x and y intercepts of constraint equation with variables. (Eureka Math 8, Module 4, pg. 308)

Another way the authors often used this equation type was to check if a known property held true. Sometimes this property could be tested by substituting values into a constraint equation with variables to see if the resultant statement was true. For example, in a chapter on the Pythagorean Theorem, the text noted, “The converse of the Pythagorean Theorem states that if a triangle with side lengths $a, b\ and\ c$ satisfies $a^2 + b^2 = c^2$, then the triangle is a right triangle”, (Eureka Math 8, Module 7, pg. 224). In this case, the equation $a^2 + b^2 = c^2$ is used as a constraint equation that must be met for a triangle to be right and becomes a test for the existence of a relationship between the sides that is necessary for the triangle to be right. Whereas, if we already know that the triangle is right, then we can use this equation as a contextual identity that allows us to determine an unknown side length of a right triangle.

Constraint equations with one unknown are equations with only one letter whose value is unspecified. Generally, this type of equation can be used to check if a particular value of the unknown is a solution and can often be manipulated to find solutions to the equation by applying a set of operations to both sides of the equation until the value of the unknown is found. There may be zero, one, or multiple solutions for these constraint equations. The main difference between constraint equations with variables and those with one unknown is that in the later, the value of the unknown quantity is not influenced or determined by the value of another varying quantity in the equation.

One of the main uses for constraint equations with one unknown is to model a situation so that one can determine which value(s) of the unknown would make the situation true. One example from the text that illustrates this use is given in Figure 10. In this example, the expression on the left is a result of adding the expression of Bonnie's age in 5 years to Shelby's age in 5 years. Since the question is asking to find Bonnie's age, her age is represented by the letter x, which is the unknown. These combined expressions are set equal to the sum of their ages, which is 98. The reader is now able to find the value of x that makes this equation true by completing a set of operations on both sides of the equation. Note that there is no other quantity in the equation that varies besides x, which enables the reader to determine which values of x make the equation true.

Shelby is seven times as old as Bonnie. If in 5 years, the sum of Bonnie's and Shelby's ages is 98, find Bonnie's present age. Use an algebraic approach.

$$x + 5 + 7x + 5 = 98$$

Figure 10. Constraint equation with one unknown (Eureka Math 7, Module 3, pg. 135)

Formal identity. A *formal identity* is an equation that is intended to express a general mathematical property or principle. The equal sign in the equation is used to indicate that the two algebraic expressions on either side of the equal sign will produce the same numeric value regardless of what values are assigned to the variable(s). One example of a formal identity would be the distributive property $a(b + c) = ab + ac$, where the two symbolic expressions on opposite sides of the equal sign produce the same numeric value regardless of the values of a, b, and c. Another example includes "$-\left(\frac{p}{q}\right) = \frac{-p}{q} = \frac{p}{-q}$" (Eureka Math 7, Module 3, p.8) where, again, each symbolic expression produces the same numeric value regardless of the values substituted into the variables. In the curricula that I analyzed, formal identities were often used to describe a general mathematical property, and then illustrated immediately afterward using a numeric example. For example, after the rules $a^m \times a^n = a^{m+n}$ and $\frac{a^m}{a^n} = a^{m-n}$ for multiplying and dividing exponents were presented, students were asked to simplify $\frac{2^5 \times 2^6}{2^9}$ to check their understanding of the rules (CMP3-8, Growing, Growing, Growing, pg.232).

Contextual identity. A *contextual identity* is an equation that models the relationship between quantities in a specific context. Contextual identities typically involve letters that are used as variables and parameters and are usually formulaic in nature. Contextual identities are true for all values of the variable(s) as long as the equation is used in the specific context. Like formal identities, contextual identities are usually presented as a mathematical property within a specific context and then illustrated using numerical examples. For example, both curricula offered a list of formulae for finding the area or volume of multiple shapes (e.g., $V_{cone} = \frac{1}{3}\pi r^2 h$, $A_{square} = lw$) and then followed with numerical examples that required the use of one or more of these formulae.

Numeric identity. A *numeric identity* is an equation of equivalent numeric expressions that is used to justify why a result is correct. The following is an example of a numeric identity: "$7\frac{1}{2}$ is a solution to $2n = 15$ because $2\left(7\frac{1}{2}\right) = 15$" (Eureka Math 7, Module 3, pg. 106). In this example, the author is using $2\left(7\frac{1}{2}\right) = 15$ as a numerical fact to justify why $7\frac{1}{2}$ must be the solution. This is different than an equation transformation where the purpose of the equation is to find the answer to 2 times 7 ½; instead, the purpose of this equation is to justify the claim that 7 ½ is a solution to the equation. The authors' use of the word "because" suggests that they are engaged in justification and not computation. I coded equations as numeric identities if they were used to justify a result rather than to produce a result. Often these justifications were marked with words such as "since" or "because."

In contrast, if the equation was not given as a justification for the result, but rather was just to produce the result, than it was coded as an equation transformation. For example, "if the sponsor pledges to donate $4 for each kilometer walked, and the student plans to walk 5 kilometers, then you can estimate that the student will collect $4 \times 5 = \$20$ from each sponsor" (CMP3-7, Moving Straight Ahead, pg. 61). In this case, I coded the equation as an equation transformation because 4×5 was used to compute the result. However, if the authors had written, "the answer is $20 since $4 \times 5 = 20$," then this equation would have been coded as a numeric identity because it was being used to justify the correctness of the result ($20).

Occasionally, equations were used similarly to numeric identities but rather than justifying that a number was a solution, they were used to justify that a number was in fact *not* a solution. I chose to code this type of equation as a *numeric non-identity*. For example, students were asked to come up with different solution sets for the equation $10a + 5s = 400$ and after determining that two possible solutions were $(10{,}60)$ and $(15{,}50)$ they were then asked if the solutions in part (1) could be a solution to the equation $a + s = 50$ in a later part of the same problem, the response was, "No, because $10 + 60 \neq 50$ and $15 + 50 \neq 50$" (CMP3-8, It's in the System, pg. 51). In this example, the author substituted the supposed solutions into the equation, and justified why these numbers were not solutions by showing that the expressions on either side of the equal sign were not equivalent.

One of the most common cases of numeric non-identities was found when working through a solution to test if it was correct or not. The example in Figure 11 illustrates this use of a numeric non-identity. In this example, the first equation was set up to assume that the expressions on both sides of the equal sign share the same value. However, as the author worked through reducing both sides of the equation, the last line resulted in a numeric non-identity because the two sides were not of the same value. This practice of using the equal sign when the authors did not yet know if two expressions are in fact the same value may actually be considered inappropriate in some mathematics communities, and thus may be problematic for students' development of the meaning of the equal sign. Sometimes the authors avoided this practice by placing a question mark on top of the equal signs of all but the final equation as they tested a potential solution.

Is the triangle with leg lengths of 9 in. and 9 in. and hypotenuse of length $\sqrt{175}$ in. a right triangle? Show your work, and answer in a complete sentence.

$$9^2 + 9^2 = (\sqrt{175}^2)$$

$$81 + 81 = 175$$

$$162 \neq 175$$

Figure 11. Example of numeric non-identity. (Eureka Math 8, Module 7, pg. 223).

Transformation equations. *Transformation equations* were the only type of equation that was used with more than one meaning of the equal sign. As noted above, transformation equations were used with the operational meaning of the equal sign to record the results of performing procedures. However, transformation equations were also used with a relational meaning, usually occurring in a proof or justification of a solution. This different use is illustrated in the example in Figure 12. In this example, the author's purpose was to use the distributive property to explain and justify why increasing a figure by 25% is the same as multiplying the original number by 125%. In cases like this one, the authors seemed to be more focused on the truthfulness of the transformations applied than on the results of the transformations, because they were trying to show why the end expression was equivalent to the original expression in the string of equalities. While the resulting expression was important because that was what they were trying to prove, the purpose of these transformation equations was to *justify* rather than *produce* the end result.

"To increase a figure by 25% means you multiply the original number by 25% and then add it to the original number. This is the same as multiplying the original number by 125% because $x + 0.25x = 1x + 0.25x = (1 + 0.25)x = 1.25x$."

Figure 12. Equation transformation with relational meaning. (CMP3-7 Stretching and Shrinking, p.58)

Subsequent expressions in transformation equations that used a relational meaning of the equal sign were not always a simplification of the previous expression. Because the purpose was not only to produce a result, but to justify the final result, sometimes, subsequent expressions actually added complexity or length to the previous expression. However, this added complexity was strategically inserted to enable the authors to create a string of transformations that yielded and justified the desired result. The examples in Figure 13 illustrate these "strategic" transformations. In Figure 13a, the authors' purpose was to justify the numeric equivalency of $\sqrt{8}$ and $2\sqrt{2}$ by creatively rewriting the previous expression into an equivalent expression that would help the reader understand how to arrive at the given solution. Each subsequent expression is not necessarily a direct result of an operation from the previous expression. In Figure 13b, the authors' purpose was to prove why $A = \pi r^2$ is the area of a circle. This example used a combination of transformations to give a step by step explanation of how the formula for the area of a circle was derived. By using "creative" transformations to help justify the final result, both of these examples help illustrate why a relational meaning is required to make sense of the transformation equation.

a) "[Students] will see that the length of $\sqrt{8}$ units is twice the length of $\sqrt{2}$ units and may generalize the algebraic method they learned in Problem 2.2 to find that $\sqrt{8} = \sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} = 2\sqrt{2}$"

b) "$Area\ (circle) = \frac{1}{2} \cdot circumference \cdot r = \frac{1}{2}(2\pi r) \cdot r = \pi \cdot r \cdot r = \pi r^2$"

Figure 13. Equation transformations with a relational meaning: (a) as justification for result. (CMP3-7 Looking for Pythagoras p.89), (b) used to prove a formula. (Eureka Math 7, Module 3, p.258).

Unit identity. A *unit identity* is an equation that sets one unit equal to a different unit. Some examples from the text include equations like $1ft = 12in$, or $1yd = 3ft$. Unit identities, like numeric identities, state a known fact and do not provide an explanation or show the operation of how the author arrived at the conclusion that these expressions are equivalent. However, unlike numeric identities, unit identities were not commonly used as a justification for the conclusion, but as information that was needed to perform unit conversions.

Result equations. The last equation type that falls under the relational meaning category is the *result equation*. A result equation is a declaration of a resulting value or solution that has been found by completing a computation or measurement and is usually written in the form of an unknown set equal to the resulting value. Some common examples found in the texts were equations like "$p = 90$", which the authors used to state the result of having added up the sides of a rectangle to find the perimeter, or "$\overline{AB} = 4$," which was the result of having counted the squares on a grid to measure the length of $\overline{AB}$. Another example includes "$4^0 = 1$" (CMP3-8, Growing, Growing, Growing, pg. 72) where 4^0 is treated as an unknown in the question, "In the last problem, you saw that $2^0 = 1$. Do 3^0 and 4^0 also equal 1?" After completing some operations, using the equation $r = 3^{n-1}$ and finding that 3^0 does, in fact, equal 1, the authors conclude that it must be that $4^0 = 1$, because similar computations would lead to this conclusion. Note that the result equation does not report the operations that took place to find the result, so therefore the purpose of the equation is to declare the result rather than to compute or justify the result. In each of these examples, a key characteristic of the results equations was that they were not written as part of the transformations or measurements that took place to compute the results.

While most of the result equations were easy to recognize and code, one particular use of result-like equations caused difficulty in the coding. An example of this context—a result reported at the end of a series of equations used to solve an equation--is shown in Figure 14. In this example, $x = 4$ could be interpreted three ways: as the result of a transformation applied to the equation above it (and thus as a constraint equation with one unknown, just like the equation above it); a declaration that x must have the value of 4 because of all of the computations performed on this equation (making $x = 4$ a result equation); or an assignment of 4 for the variable x, allowing 4 to be substituted for any occurrence of x (making $x = 4$ a specification equation). It is likely that the authors, as sophisticated users of the equal sign, read this equation *all three ways*. My own experience with students, however, suggests that these three ways of reading $x = 4$ must be learned and are neither automatic nor simultaneous. Because the first reading of this equation by experts is likely to be as a result of a transformation performed on the preceding constraint equation with one unknown, I chose to code this instance of $x = 4$ as a constraint equation with one unknown. I coded other cases of solving equations similarly.

$$7x - 3 = 5x + 5$$
$$7x - 3 + 3 = 5x + 5 + 3$$
$$7x = 5x + 8$$
$$7x - 5x = 5x - 5x + 8$$
$$2x = 8$$
$$\frac{2x}{2} = \frac{8}{2}$$
$$x = 4$$

Figure 14. Solution embedded in the computations (Eureka Math 8, Module 7, pg. 77).

A second, less confusing difficulty in coding result-like equations consisted of contexts where a numeric value was both assigned to a variable and treated as if it might be the result of an equation solving process. An example of this type of equation is found in the following: "Angel transformed the following equation from $6x + 4 - x = 2(x + 1)$ to $10 = 2(x + 1)$. He then stated that the solution to the equation is $\boldsymbol{x = 4}$**.** Is he correct? Explain" (Eureka Math 8, Module 4, pg. 35). In this example, the reader might interpret the equation $x = 4$ as either an assignment of the value 4 to x (so that it could be substituted into the equation), or as a claim that if the equation were solved, the computations would yield that x must be 4. In these contexts, I coded the equation as a specification equation because it seemed that the authors intended that it be read as an assignment and not as a result of solving an equation.

Frequency of Equations with Different Meanings of the Equal Sign in the Curricula

In this section I discuss how frequently the different meanings of the equal sign appeared throughout the equations analyzed in the two curricula and in the two grades. I first discuss the frequency of the different meanings of the equal sign as they occurred throughout all equations that were analyzed regardless of grade or curriculum. I then discuss the distribution of equations with the different meanings as they occurred, not only in each grade and curriculum, but also between teacher and student materials. Finally, I discuss how the use of the equal sign develops over units and from 7th to 8th grade.

Table 2

Table of number of equations analyzed and the percentage equations with each meaning of the equal sign. (Eureka Math 7th and 8th grade, CMP3 7 and 8)

Total number of equations analyzed	11,149
Total percentage of equations with assignment meaning	6.7%
Total percentage of equations with operational meaning	14%
Total percentage of equations with relational meaning.	79.3%

By looking at Table 2, we can see the overall percentage of the different meanings of the equal sign as they appeared in all the equations that were analyzed. From this table, the relational meaning is the predominant meaning given to the equal sign in both 7th and 8th grade and in both curricula. As was mentioned in Chapter 2, the literature indicates that students in 7th and 8th grades should be using the equal sign as a relational symbol, so it is significant that the majority of equations that were analyzed in the 7th and 8th grade curricula used the equal sign as a relational symbol. While this supports the idea that students in grades 7-8 should be using the equal sign with a relational meaning (Carpenter, Franke, & Levi, 2003; Falkner et al., 1999), the equal sign as an operational symbol still plays an important role in these curricula, because 14% of the equations used the operational meaning of the equal sign. The same could also be said about the assignment meaning of the equal sign, as it occurred in 6.7% of the equations analyzed. While the assignment and operational meanings are used less frequently than the relational meaning, they are both used often enough that students need to be aware of them and know how to read and use equations that use any of these three meanings of the equal sign.

By looking at the distribution of equations across grade level, curriculum, and teacher-student materials (see Table 3), we can get a sense of how the meanings of the equal sign progress from 7th to 8th grade. One important trend to note is that the frequency of the use of the relational meaning of the equal sign did go up across each column in the Eureka Math curriculum from 7th (65.7%) to 8th grade (83.8%). As for the CMP3 curriculum, the frequency of equations that used the relational meaning of the equal sign remained consistently high from 7th to 8th grade, occurring over 80% of the time in both grades.

Table 3

Breakdown of frequency of equal sign meanings among grade, and teacher and student material. (Eureka Math 7, Eureka Math 8, CMP3-7, and CMP3-8)

Meaning of the equal sign	Eureka Math 7	Eureka Math 7 Teacher	Eureka Math 7 Student	CMP3-7	CMP3-7 Teacher	CMP3-7 Students
Assignment	7.5%	2.9%	54.5%	8.6%	9.3%	5.3%
Operational	26.8%	28.9%	5.2%	8.2%	9.6%	2%
Relational	65.7%	68.2%	40.3%	83.2%	81.1%	92.7%

Meaning of the equal sign	Eureka Math 8	Eureka Math 8 Teacher	Eureka Math 8 Student	CMP3-8	CMP3-8 Teacher	CMP3-8 Students
Assignment	5%	4.1%	12.2%	9.1%	10.3%	3.9%
Operational	11.2%	12.3%	2.5%	8.4%	9.1%	5.6%
Relational	83.8%	83.6%	85.3%	82.5%	80.6%	90.5%

Another trend that is important to note is that in each grade and curriculum, the frequency of equations that use the operational meaning of the equal sign is higher in the teacher material than the student material. This is most likely the case because in the teacher material, explanations and solutions were included with all problems and examples. In contrast, student material, with a few exceptions in each curriculum, did not provide explanations and solutions, but only problems for students to solve. It is important to note the lack of explanations and solutions provided in the analyzed student material, because recognizing this lack helps us understand that a low frequency of the operational meaning of the equal sign in the student materials is likely an inaccurate estimation of the prevalence with which the students might be using the operational meaning of the equal sign.

The last important datum to note from Table 3 is the high frequency (54.5.%) of equations that use the assignment meaning of the equal sign in the Eureka Math 7 student material. This is especially interesting since the frequency of the same type of equation in the teacher material is only 2.9%. A look at the distribution of different meanings of the equal sign over units within the year can bring further understanding to this piece of data. Because of its size, I have included the table that illustrates the breakdown of different meanings of the equal sign and distribution of equation types over grade, curriculum, teacher material, student material, and unit in Appendix B. The data in this table explains that 80.4% of the equations in the student material of Eureka Math 7 Module 6, the Geometry module, were specification equations that were used to assign values to measures of objects. In the Eureka Math 7 student material, 187 equations were analyzed, 22 (12%) of which came from the Ratios and Proportions module, 81 (43%) came from the Expressions and Equations module, and 84 (45%) came from the Geometry module. In the student material from the other Eureka Math 7 modules, equations that used the assignment meaning occurred with a frequency of 3.8% in the Ratios and Proportions unit, and 39.8% in the Expressions and Equations unit. Because the small amount of equations from the student material does not represent each module proportionally, we cannot conclude that the assignment meaning of the equal sign is actually a predominant use of the equal sign in the Eureka Math 7 curriculum.

To gain a better understanding of how the meaning of the equal sign progresses throughout the year we can look at Tables 4 and 5. Because understanding the relational meaning of the equal sign is an important goal in 7th and 8th grades, we might expect to see the occurrence of the relational meaning to increase proportionally as the students progress through the curriculum. In the table below, I have listed the modules in the suggested order that they should appear throughout the school year. These tables illustrate a breakdown of the frequency of meanings of the equal sign across units in each grade and curriculum. By looking at these two tables we can see that there is not a steady increase in the use of any of the meanings of the equal sign across a year of instruction. It seems, rather, that the frequency of meanings of the equal sign depends on the unit that is being taught and not on the time of the school year.

Table 4

Frequency of each meaning of the equal sign in the modules covering Ratio and Proportions (RP), Expressions and Equations (EE), and Geometry (G) in the order that they appear throughout the school year. (Eureka Math 7, and CMP3 7)

	Eureka Math (7.RP)	Eureka Math (7.EE)	Eureka Math (7.G)	CMP3 (7.RP)	CMP3 (7.EE)	CMP3 (7.G)
Assignment	1.9%	6.7%	11.8%	0%	8.7%	11.2%
Operational	46.8%	15%	32.4%	40%	2.9%	22.4%
Relational	51.4%	78.3%	55.7%	60%	88.4%	66.4%

Table 5

Frequency of each meaning of the equal sign in the modules covering Expressions and Equations (EE), Functions (F), and Geometry (G) (in CPM3-8, two books covered both EE and F modules which makes up the 8.EE&F column) in the order that they appear throughout the school year. (Eureka Math 8 and CMP3-8)

	Eureka Math (8.EE)	Eureka Math (8.F)	Eureka Math (8.G)	CMP3-8 (8.EE)	CMP3-8 (8.EE&F)	CMP3-8 (8.F)	CMP3-8 (8.G)
Assignment	5.4%	6%	3.2%	11.6%	10.1%	5.6%	5.8%
Operational	7%	26%	14.6%	0.6%	8.5%	16.8%	12%
Relational	87.6%	68%	82.2%	87.9%	81.3%	77.5%	82.2%

Another conclusion that we can make from this data is that all three meanings of the equal sign are used significantly throughout the three units that were analyzed. Even though the most frequent meaning applied to the equal sign in every unit, curricula, and grade was the relational meaning, there was not one unit in either curriculum or grade where the relational meaning was exclusively used. While it seems that by the time students are in 7th grade, the relational meaning is important regardless of unit content, we cannot conclude that it is the only meaning of the equal sign that students should develop. If we look at any of the columns from either Table 4 or Table 5, we can see that the relational meaning of the equal sign, while being used the majority of the time, is not the only meaning that is used. With the exception of the CMP3 7.RP unit, which uses only two meanings of the equal sign, each unit in both grades and curricula uses all three meanings of the equal sign. For example, in the CMP3, 7.G unit, the assignment meaning is used in 11.2% of equations, the operational meaning is used in 22.4% of equations, and the relational meaning is used in 66.4% of the equations. Since this data is not unique to this unit, it gives us reason to believe that students are likely being exposed to and required to use all three meanings of the equal sign in every content area, and therefore all three meanings need to be developed by students in the 7th and 8th grade.

By looking at how frequently the different equations that require the relational meaning of the equal sign appeared in the two curricula, we can gain a better understanding of what students are most frequently presented with and how teachers can help them be more successful. Table 6 lays out the different equation types and their frequency across each grade and curriculum. The main points of data that are worth discussing include the frequency of the different types of constraint equations, the different purposes of equations and their frequencies, and simply which equation types appear with the most and least frequencies.

Table 6

Frequency of equation types with the relational meaning across grade and curriculum (Eureka Math 7, Eureka Math 8, CMP3-7, & CMP3-8)

Equation Type	Eureka Math 7	CMP3-7	Eureka Math 8	CMP3-8
Constraint Equation with Parameters	2.9%	14.9%	6.8%	17.2%
Constraint Equation with Variables	10.1%	29.7%	29.2%	40.9%
Constraint Equation with One Unknown	63.3%	26.7%	45.3%	12.8%
Formal Identity	0.4%	0.3%	0.4%	2.5%
Contextual Identity	9.5%	10.6%	3.0%	5.4%
Numeric Identity	3.4%	6.3%	5.2%	7.0%
Numeric Non-Identity	0.0%	0.1%	0.6%	0.2%
Transformation Equation	4.1%	2.7%	3.1%	5.3%
Unit Identity	0.7%	0.3%	0.2%	0.0%
Result Equations	4.8%	4.2%	3.5%	5.9%

Looking at the top three rows of Table 6, we can see which types of constraint equations were more common among grades and curricula. Eureka Math 7 had a high frequency of constraint equations with one unknown (63.3%) which went down to 45.3% in 8th grade. While Eureka Math 8 increased the amount of more generalized constraint equations with an increase of constraint equations with parameters (from 2.9% to 6.8%) and those with variables (from 10.1% to 29.2%) the majority of constraint equations contained only one unknown. In both grades, Eureka Math used more constraint equations with one unknown than any other equation type. However, in the CMP3 curricula, this was not the case. In 7th grade, with the frequency of constraint equation with variables at 29.7%, it was only 3% higher than the frequency of those with one unknown. CMP3-7 had a more equal frequency of constraint equations with one unknown and with variables than either grade of Eureka Math. From the rise in frequency of constraint equations with variables from 29.7% in 7th grade to 40.9% in 8th grade, we can see that

CMP3 used more generality in its presentation of constraint equations. This is not a judgement of which curriculum had a better frequency, but rather we can use this data to inform us of what types of constraint equations we should expect students to see in the 7th and 8th grades. We cannot assume that one type of constraint equation will appear more frequently in every curriculum; therefore, it is good for us to be aware of different possible mixtures of constraint equations to better prepare our students. Even though we cannot predict the frequency of each type of constraint equation in a given curriculum, the data suggests that in the 7th and 8th grade, constraint equations, regardless of type, will be the main types of equations that will appear.

If we divide the equation types into purposes, after constraint equations with the purpose of finding a solution set, the next two most common equation types are those with the purpose to define a formula with which to find another value, and those with the purpose to justify a result. In both curricula for the 7th grade, the second most frequent equation types are formal and contextual identities. Combining them, the Eureka Math 7 curricula uses these types of equations with 9.9% frequency and CMP3-7 uses them with a 10.9% frequency. The third most frequent equation types in both 7th grade curricula include numeric identities and transformation equations which are both used for justification. Eureka Math 7 uses justification equation types 7.5% of the time and CMP3-7 uses them 9% of the time. However, in 8th grade, there is a shift in frequencies and the second most frequent equation types are those that are used for justification, with Eureka Math 8 using the justification equation types 8.3% of the time and CMP3-8 using them 12.3% of the time. This is in contrast to the frequency of identities that are used for formulas to determine new values which occurred 3.4% in Eureka Math 8 and 7.9% of the time in CMP3-8. This gives us insight into the possible progression of equation types from 7th to 8th grade and the increase in complexity from using equations as formulas to follow to using equations to justify results and even justify the validity of the formulas that we use.

Lastly, we can look at Table 6, to identify what equation types students are less likely to encounter in 7th and 8th grade. We can see that students were less likely to encounter numeric non-identities as well as unit identities in 7th and 8th grade. Using the data from this table helps us to identify which equation types are most common and therefore imperative for students to develop the ability to recognize and work with.

Structural Conventions

While many of the equations in the data were written as single, stand-alone equations, others were grouped together using one or more structural conventions, often to indicate a relationship between the equations. The two structural conventions used by the authors in both texts was *lists of equations* (LOEs) and *strings of equalities* (SOEs). I include an analysis here of the specific equation types associated with the use of each structural convention, because a knowledge of how types of equations and structural conventions are related seemed to be important in making sense of the equal signs in the grouped equations.

Lists of Equations

The first structural convention I discuss is *lists of equations* (LOEs). LOEs are sequences of equations that appear vertically or horizontally right after each other, not separated by words or space; horizontal LOEs are separated, however, by commas or semicolons. We found that LOEs were almost exclusively associated with equation types that involve a relational meaning. Usually, the equations in both vertical and horizontal LOEs were connected through equation operations on each equation, and when they were, the LOEs were comprised solely of equations that used the relational meaning of the equal sign. Occasionally, LOEs contained equations that were not connected by equation operations, and in these cases, the equations in the LOEs used either the operational or assignment meaning of the equal sign. I decided to code these cases as LOEs even though the equations in them were not all connected by equation operations (EOs) because they used the same structure as defined in the beginning of this section: a series of equations listed right after one another not separated by words or space. I discovered that while some of these LOEs with unconnected equations involved the relational meaning of the equal sign, many did not. Furthermore, I noticed that in all cases of LOEs that consisted solely of equations that used the operational and/or specification meanings of the equal sign, these equations were not connected by EOs.

The examples in Figure 15 illustrate some unconnected LOEs. These LOEs have the same structure as the LOEs in Figures 16 and 17, but they are actually quite different. In all three examples, the second equation in the list is not a result of an EO applied to the previous equation. In Figure 15a, the first equation in the list is a transformation equation with an operational meaning of the equal sign, because the author simply performed the given operation and wrote the result to the right of the equal sign. The second equation in the list uses the result of the first equation to calculate a new result and is also a transformation equation that applies the operational meaning to the equal sign. Figure 15b is similar to Figure 15a in that the first equation is a transformation equation that applies an operational meaning to the equal sign, with the result written to the right of the equal sign; however, the second equation is a constraint equation with variables that applies a relational meaning and is created by using the result of the first equation. Figure 15c is simply a short list of specification equations that apply the assignment meaning to the equal sign and are not created by moves or operations performed on the previous equation in the list.

a)

$$\frac{0.16}{3}\pi \approx 0.167$$

$$\frac{0.167}{6} \approx 0.028$$

b) "The average swimming pool holds about 17,300 gallons of water. Suppose such a pool has already been filled one quarter of its volume. Write an equation that describes the volume of water in the pool if, at time 0 minutes, we use the hose described above to start filling the pool.

$$\frac{1}{4}(17300) = 4325$$

$$y = 4.4x + 4325$$

c) Graph the linear equation $ax + by = c$, where $a = 0, b = 1$, and $c = 1.5$.

Figure 15. Three examples of LOEs: (a) Vertical LOE using the result of previous equation to find the result of a new equation. (Eureka Math 8, Module 7, p.307), (b) Vertical LOE using the result of the first equation to write the second equation. (Eureka Math 8, Module 5, p. 43), (c) Horizontal LOE with no connection between equations. (Eureka Math 8, Module 4, pg. 185).

To have a better understanding of how often students might encounter equations in LOEs, we can look at how frequently LOEs appeared in the two curricula in the 7th and 8th grade as shown in Table 7. In the Eureka Math curricula, 41.5% of the equations from the 7th grade curriculum and 42.3% of the equations from the 8th grade curriculum were contained in an LOE. This is very different from the CMP3 curricula in which 11.1% of equations from 7th grade and 5.9% of equations in 8th grade were contained in an LOE. We can see that Eureka Math tended to organize equations much more frequently in LOEs than CMP3 did. While it is important to note how frequently LOEs occurred in a curriculum, it is also helpful to look at what percentage of those LOEs were connected by EOs or not. From the table below, we can see that while the CMP3 curriculum did not organize a large number of equations into LOEs, of the small amount of LOEs that were found, a larger portion of LOEs were either unconnected, or only partially connected (where a connected LOE might follow a list of unconnected LOEs without a space in between, making it look like one LOE) by EOs than the Eureka Math LOEs (45.7% and 29.5% compared to 20% and 5.6% respectively). It is important to note that even though the majority of LOEs in both curricula and in both grades were connected by at least one EO, each grade in each curriculum had a significant amount of their LOEs either unconnected or partially connected.

Table 7

Total number of LOEs, percentage of LOEs combined with SOEs, total number of Equations found in LOEs, and percent of connected LOEs vs. unconnected or partial LOEs (Eureka Math 7, CMP3-7, Eureka Math 8, & CMP3-8)

	Eureka Math 7	**CMP3-7**	**Eureka Math 8**	**CMP3-8**
Total number of Equations	2566	838	5283	2462
% of Equations in LOEs	41.5%	11.1%	42.3%	5.9%
Total number of LOEs	304	35	595	44
% of LOE/SOE Combinations	2.0%	5.7%	0.7%	0.0%
% of Connected LOEs	80%	54.3%	94.4%	70.5%
% of unconnected/ partial LOEs	20%	45.7%	5.6%	29.5%

It is important to note that all cases of LOEs in the analyzed curricula that were unconnected or partially connected occurred only in the teacher instruction materials. While the authors can probably assume that teachers are able to notice the different types of LOEs without too much work, using these examples in instruction could still be problematic. Because these LOEs that are not connected by EOs are treated as unproblematic in the text, they may not draw teachers' attention to the difficulties that students might face if teachers use this type of LOE when teaching. If students have been learning to solve equations, it is likely that they will read

LOEs with the intent of identifying the equation operations that connect one equation to the next; consequently, they will be confused by LOEs whose equations are not connected by EOs.

Operations on Equations

There were six different equation operations (EO) that were used to connect the equations in a LOE: deductions and reductions as mentioned by Matz (1982), substitutions, cross-multiplication, reflections, and equation combinations. Table 8 provides definitions and examples of each equation operation. I discuss each EO in more detail below.

Deductions and reductions were the most common EOs that connected equations in an LOE and were used mainly to solve equations. A deduction is an operation or procedure applied to both sides of the equation. In contrast, a reduction is the simplification of one or more sides of the equation. This simplification was often necessary because of the operation or procedure that had been applied to the equation through a deduction. Note that one of the major differences between deductions and reductions is the relationship between the expressions on each side of the equal sign with the corresponding expressions in the previous equation. Corresponding expressions in the two equations are unequal if a deduction has been applied, and equal if a reduction has been applied.

Table 8

Equation operations, definitions, and examples (Eureka Math 7, CMP3-7, Eureka Math 8, CMP3-8)

Equation Operation	Definition	Example from Curricula
Deduction	The operations applied to both sides of the equation	$x + 132 = 180$ $x + 132 - 132 = 180 - 132$ (Eureka Math 7, Module 3, p. 152).
Reduction	Applying the operations from the deduction or simplifying the previous equation in the list by performing written procedures.	$x + 132 - 132 = 180 - 132$ $x = 48$ (Eureka Math 7, Module 3, p. 152).
Substitution	Replacing one or more unknowns in an equation with a number or expression to create a new equation.	$S = 0.11H + b$ $85 = 0.11(225) + b$ (Eureka Math 8, Module 6, pg. 114)
Cross-Multiplication	Multiplying each numerator by the other fraction's denominator.	$\frac{x}{5} = \frac{6}{12}$ $12x = 6(5)$ (Eureka Math 8, Module 4, pg. 85)
Reflection	Flipping the expressions so that they are on the opposite side of the equal sign than the original equation.	$ms_1 + b = s_2$ $s_2 = ms_1 + b$ (Eureka Math 8, Module 4, pg. 299)
Equation Combination	Adding or subtracting multiples of two whole equations to create a new equation.	$100x = 12.121212\ldots$ $-\quad x = 0.121212\ldots$ $99x = \boxed{}$ (CMP3-8, Looking for Pythagoras, pg. 201)

Often, the authors made it clear what deduction was applied to the equation by writing out the specific procedure on both sides of the equation, as in the deduction example in Table 8, where the authors specified that 132 should be subtracted from both sides of the equal sign. By recording the deduction within the equation, it is clear to the reader what arithmetic operations were performed in the deduction. However, in both curricula, there were many times when the authors did not specify in an equation what deduction was applied to arrive at the resulting, simplified equation; rather, the deduction and the subsequent reduction were applied to the equation mentally by the author before the next equation was written. In these cases, I chose to code the EO as a deduction/reduction combination because it was obvious that the subsequent equation was more than just the result of a reduction of the previous equation, i.e., the expressions on either side of the equation were not equal to the expressions on the same side of the previous equation. An example of a deduction/reduction combination EO is illustrated in Figure 16. In this example, because the expressions in the second equation in the LOE are not equivalent to the expressions on the same side in the first equation, it was clear that some deduction operation must have taken place to arrive at the resulting equation. Because I was able to determine one possible way to arrive at the second equation using only one deduction of subtracting 45 from both sides of the original equation and one reduction, I assumed that the author probably only used one deduction/reduction combination EO between the first and second equations in the LOE. We can assume the same when going from the second to the third equation as well, since it is possible to arrive at the third equation by performing one deduction/reduction of dividing both sides by 0.1.

$$75 = 0.1x + 45$$
$$30 = 0.1x$$
$$300 = x$$

Figure 16. Vertical LOE with deduction/reduction combination. Eureka Math 8, Module 5, pg. 92)

In some cases, to get from one equation to the next in an LOE required more than one deduction/reduction combination, yet none were specifically written in the equations. While it is possible to create a series of deduction/reduction combinations to arrive at the same resulting equation, one might do so using a different series of EOs than the author did; therefore, I coded EOs as *unspecified transformations* when there seemed to be multiple deduction/reduction combinations that had been applied to an equation to yield the next equation. An example of an unspecified transformation is illustrated in Figure 17. In this example, the second and third equations in the LOE are results of a reduction only since the expressions on each side of the equation are equivalent to the expressions on the same side of each previous equation. However, to get from the third equation in the LOE to the fourth, multiple EOs are required and it is unclear what series of EOs the authors applied to arrive at the final equation. Note that I chose to code these EOs consisting of multiple deduction/reduction operations as unspecified transformations rather than as deduction/reduction combinations because I anticipated that these complex EOs would be more difficult for novices to read, and I wanted to keep track of how often they appeared in the curricula.

$$
\begin{aligned}
3(2x-5) &= 2(3x-1)+x \\
6x-15 &= 6x-2+x \\
6x-15 &= 7x-2 \\
-13 &= x
\end{aligned}
$$

Figure 17. Vertical LOE with unspecified transformations. (CMP3-8, Say It with Symbols, pg. 23).

The other four EOs were coded as such as long as they seemed to be the only EO performed from one equation to the next (and thus did not become one of several EOs applied to a single equation, yielding an unspecified transformation). Of these additional four EOs, only one of them was shared among both grades and curricula: substitutions. *Substitutions* were identified when one equation that contained one or more letters resulted in a subsequent equation that remained the same as the first equation but with one or more letters having been changed to either numbers or expressions. The EO *cross multiplication* was unique to Eureka Math and was introduced as a single EO that was identified when the original equation consisted of a fraction expression on both sides of the equal sign and the resulting equation contained no fractions and consisted of the denominator of each fraction being multiplied by the opposite numerator. As long as the reduction had not been performed before the subsequent equation was written, then it was easy to identify when cross-multiplication had been used. *Reflections* were also unique to Eureka Math and were identified when the expression on the right of the equal sign was replaced with the expression on the left of the equal sign, and vice versa. *Equation combinations* were unique to CMP3 and were identified when multiples of two equations were added or subtracted to create a new equation. Equation combinations were the least common EO that I saw in the chapters I chose to analyze. As can be seen in the equation combination example in Table 8, the LOE looks like a normal arithmetic subtraction problem but with equations instead of just numbers. The EO consists of the act of writing the second equation directly below the first, and then subtracting corresponding sides to yield the third equation. Equation combinations differ from the other types of EOs in that they yield LOEs where the second equation in the LOE is not connected to the first equation by an EO.

Note that LOEs involving EOs present a unique challenge to readers, because readers are required to think relationally and operationally simultaneously. When reading a constraint equation by itself, the reader needs to be reading it with a relational meaning of the equal sign. However, when the author performs the equation operations, there is a transformation taking place that requires an operational understanding, where the author performs the written operation and writes the resulting equation after performing a reduction. To read the LOE, the reader must identify how equations in the LOE have been modified and infer the equation operations that were used to yield these modifications. Thus, as readers make sense of LOEs, they must reason both relationally and operationally; they must continue to keep in mind that each equation expresses a relation, often a constraint, between the two expressions, but also interpret the changes from one equation to another as a result of performing equation operations.

To better understand which EOs are more likely to be used in 7^{th} and 8^{th} grade, we can look at the data in Table 9. From this table we can see that the most frequently used EO in each grade and curriculum is either a reduction or a combination of a deduction and a reduction. This is not surprising since reductions can occur following a deduction, or simply while simplifying an expression in a previous equation. In the Eureka Math curriculum, it seems as if there is an increased expectation of sophistication from 7^{th} grade to 8^{th} grade because of the increase in frequency of deduction/reduction combinations going from 5.5% to 19.5%. Because of the lower amount of LOEs used in CMP3, we would expect a lower amount of EOs; however, in both 7^{th} and 8^{th} grade, the most frequent EO was the deduction/reduction combination, and while the frequency does decrease from 50% to 36.3%, the amount of EOs increased from 30 to 80 from 7^{th} to 8^{th} grade. Even though the frequency of deduction/reduction combinations decreases, the amount of LOEs increased, which could also imply an increase in sophistication expected in 8^{th} grade. Substitutions occurred more frequently in both 7^{th} grade curricula than in 8^{th} grade, which could imply an increase in sophistication of solving equations by algebraic manipulation rather than substituting in values to check if they could be a solution. As for the remaining EO types, all three occurred significantly less than any series of deductions or reductions. From this data, we can see that while the less significant EOs are useful for working with LOEs, the deduction and reduction EOs are essential for success in working with LOEs.

Table 9

Frequency of different EOs found in LOEs (Eureka Math 7, CMP3-7, Eureka Math 8, CMP3-8)

	Eureka Math 7	**CMP3-7**	**Eureka Math 8**	**CMP3-8**
Total # of EOs	694	30	1606	80
Deductions	18.3%	3.3%	15.9%	11.3%
Reductions	56.3%	23.3%	59.6%	35.0%
Deduction/Reduction Combo	5.5%	50.0%	19.5%	36.3%
Unspecified Transformations	3.7%	6.7%	1.1%	12.5%
Substitutions	16.1%	16.7%	2.1%	2.5%
Cross-Multiplication	0.0%	0.0%	1.6%	0.0%
Reflections	0.0%	0.0%	0.2%	0.0%
Equation Combinations	0.0%	0.0%	0.0%	2.5%

Strings of Equalities

Another type of structure that often occurred in the data was strings of equalities (SOEs). We defined SOEs as a series of expressions that are connected by two or more equal signs. SOEs, like LOEs, appear frequently as either vertical or horizontal strings. This is most commonly seen in examples like the ones in Figure 18.

a)

$$75\%\ of\ d^2 = \frac{3}{4}(2r)^2 = \frac{3}{4}(4r^2) = 3r^2$$

b)

$$\begin{aligned} V &= \frac{4}{3}\pi r^3 \\ &= \frac{4}{3}\pi(7^3) \\ &= \frac{4}{3}\pi(343) \\ &= \frac{1372}{3}\pi \\ &= 457\frac{1}{3}\pi \end{aligned}$$

Figure 18. Examples of SOEs: (a) Horizontal SOE with one meaning of the equal sign. (CMP3-7, Filling and Wrapping, p.159), (b) Vertical SOE with multiple meanings of the equal sign. (Eureka Math 8, Module 5, p.146)

I chose to use the examples in Figure 18 to illustrate the different levels of complexity that occur in SOEs. As one works through an SOE, there may be different equation types, and therefore, there could also be changes of meaning of the equal sign throughout. SOEs can be thought of as a simplified form, or shorthand version, of an LOE as each equal sign and the expressions on either side form an equation. If these equations were written out, we would notice that the expression on the right side of each equation would be the same as the expression on the left side of each subsequent equation. SOEs save time and space by recording the equations in a continuous string rather than rewriting each equation separately. For example, in Figure 18a, the first equation is created by the first equal sign: $75\%\ of\ d^2 = \frac{3}{4}(2r)^2$. If this were a stand-alone equation, it would be coded as a transformation equation with an operational meaning of the equal sign. The equal sign in this example is prompting the user to put the transformed and simplified expression to the right of it. If the author had not used a SOE, the next equation would be $\frac{3}{4}(2r)^2 = \frac{3}{4}(4r^2)$, which is a transformation equation with an operational meaning of the equal sign. The final equation would then be $\frac{3}{4}(4r^2) = 3r^2$, which maintains the same equation type and meaning as the previous equations. Note that this SOE contains 3 equations—one for each equal sign—but only requires the author to write four expressions rather than the six expressions she would need to write if she wrote the equations separately. This example illustrates how SOEs save space and time and reduce redundancy. While SOEs make it so that the user does not have to rewrite multiple equations, we decided to analyze them by the equations that were made by each single equal sign.

The example in Figure 18b is much more complex as it uses multiple different equation types requiring switches between meanings of the equal sign. The first equation, made by the first equal sign, is a contextual identity, and the second equation that is linked by the second equal sign would be considered a constraint equation with one unknown as it would look like $\frac{4}{3}\pi r^3 = \frac{4}{3}\pi(7^3)$ if it were a stand-alone equation. Each expression after $\frac{4}{3}\pi(7^3)$ is a result of completing an operation on the previous expression, so therefore, the last three equations in the string, using the last three equal signs, would be considered equation transformations. This example is complex as it not only works through three different equation types, but it requires a meaning shift from relational to operational within the same SOE. Occasionally, I encountered SOEs that would start horizontally and end vertically or vice versa, as in Figure 19. Because of their rarity, occurring in 2.4% of the SOEs, it is likely that the combining of horizontal and vertical SOEs was simply because of space limits in the text and not a preferred structural representation.

$$(365) \times (1.28 \times 10^{11}) = (3.65 \times 10^{2})(1.28 \times 10^{11})$$
$$= 3.65 \times 1.28 \times 10^{2} \times 10^{11} = 4.672 \times 10\text{^}13$$

Figure 19. SOE that is horizontal and vertical. (CMP3-8, Growing, Growing, Growing, p.240)

To better understand how often SOEs are likely to occur in 7th and 8th grade, as well as which meanings of the equal sign are likely to be given in SOEs, we can look at Table 10 below. From the data in this table, we can see that the majority of SOEs in either grade or curriculum are constructed of two or three strings. Any SOEs that are longer than five strings only occur in Eureka Math 8, suggesting an expectation of higher sophistication in 8th grade compared to 7th grade. From this data we also find that the majority of SOEs that do not require a change in meaning of the equal sign in the Eureka Math curriculum for both grades use the operational meaning (22.9% and 33.5% respectively). We find that in the CMP3 curriculum for 7th and 8th grade, the majority of SOEs that do not require a change in meaning are equations that use the relational meaning (47.4% and 57.3% respectively). We also find that while not every grade and curriculum use SOEs that require a change between meanings, the amount that do is significant in each grade and curriculum. For example, 67.7% of SOEs in Eureka Math 7 require a change in meaning of the equal sign, 36.8% of SOEs in CMP3-7, 46.1% of SOEs in Eureka Math 8, and 18.8% of SOEs in CMP3-8 require a change in meaning of the equal sign. If students are to fully understand the steps that were taken to get to the final expression in the SOE, they need to be aware of the flexibility needed to make sense of the equal sign while reading a SOE.

Table 10

Number of SOEs, the number of equations in a string, and the meanings of the equal sign that are used in each equation found in an SOE (Eureka Math 7, CMP3-7, Eureka Math 8, CMP3-8)

	Eureka Math 7	CMP3-7	Eureka Math 8	CMP3-8
# of SOEs	266	19	284	96
% of SOE/LOE	0	0	2.1%	0
Strings of 2	80.1%	36.8%	60.9%	72.9%
Strings of 3	16.2%	52.6%	23.9%	20.8%
Strings of 4	3.0%	5.3%	13.7%	5.2%
Strings of 5-8	0.8%	5.3%	4.2%	1.0%
Assignment Only	0.8%	0.0%	0.0%	3.1%
Operational Only	22.9%	15.8%	33.5%	20.8%
Relational Only	8.6%	47.4%	23.2%	57.3%
Switch between relational and assignment	42.1%	10.5%	0.7%	2.1%
Switch between relational and operational	25.6%	26.3%	45.4%	16.7%

Lists of Equations Mixed with Strings of Equalities

Though not extremely common, there were a few instances where a LOE would end in a SOE or vice versa. Figure 20 illustrates examples of this combination of LOE and SOE. Like the horizontal/vertical SOE hybrids, instances of a combined LOE/SOE were very rare, occurring in less than 2% of LOEs, and were therefore likely written that way because of space limitation in printing the text and not as preferred structural choice. The example found in Figure 20 begins with a constraint equation with one unknown, which requires a relational meaning to solve, and continues the LOE until the last line. Then the author switches to a horizontal SOE beginning with a constraint equation switching to a string of equation transformations. This switch from LOE to SOE requires a switch from relational to operational thinking in the same problem. If the authors of the curricula seamlessly switch meanings while working one problem, we can assume that teachers are also likely to do so. This could cause confusion to students if teachers do not call attention to when they are switching between meanings of the equal sign.

$$\frac{y}{5\frac{1}{3}} = \frac{1}{4}$$

$$4y = 5\frac{1}{3}$$

$$y = \frac{5\frac{1}{3}}{4} = \frac{16}{3} \cdot \frac{1}{4} = \frac{4}{3}$$

Figure 20. Vertical LOE that switches to horizontal SOE. (Eureka Math 7, Module 1, p.124)

Discussion

In this section I discuss the findings from this study and compare these findings to previous research as discussed in chapter 2. I first discuss the meanings that were given to the equal sign and the equation types that were found during the analysis and compare them to previous research discussed in chapter 2. I then discuss and compare the structural conventions that were found in the data to those found in previous research. Lastly, I discuss other items that were found in the data that have not been discussed in previous research.

Meanings Given to the Equal Sign as Found in the Analysis

As discussed in chapter 2, the three main aspects that affect how one interprets and works with the equal sign are meaning, equation type and spatial conventions. Previous research informed us about the meanings that are given to the equal sign: operational, relational and assignment. After analyzing the two curricula in both the 7th and 8th grades, all three meanings of the equal sign were found. We did not discover any meanings that had not already been mentioned in previous research. the data suggests that the meanings for the equal sign in grades 7th and 8th are limited to these three meanings.

In previous research, there has been a claim that in arithmetic, students learn that the equal sign is an operational symbol, and that in order for them to successfully transition into algebra they need to leave that meaning behind while adopting the relational meaning of the equal sign (Carraher, Schliemann, Brizuela, & Earnest, 2006). The results from this study show that rather than dropping the operational meaning of the equal sign from their understanding, students need to add to it. Students need to add the relational meaning as well as the assignment meaning to their understanding of the equal sign. Teachers need to help students develop all three meanings of the equal sign well enough that they are able to switch back and forth between the three, sometimes within the same problem.

Equation Types as Found in the Analysis

In chapter 2, three equation types were mentioned: tautologies, constraint equations, and specifications. Tautologies are equations that are true for any value substituted in for the unknowns or variables. Equation types that were considered tautologies included transformations, identities, and contextual identities. Constraint equations were described as an equation with two expressions on either side of the equal sign that is only true for specific values of the unknown that are constrained to the set of numbers for which both expressions, when evaluated using a number from the set, yield the same numeric value. And specifications were described as an equation that defines the value of a variable or the rule for a function in a specific situation.

In this study, multiple equation types were found that extend the list that was accumulated from previous research. In addition to the equation types already listed under tautologies, three new equation types were included from this study: formal identity, numeric identity, and unit identity. While these equations could be considered tautologies because of the domain over which they are true, I chose not to code them as tautologies but to recognize them as individual equation types. I chose to do this because the equation types listed above do not all share the same purpose, and the purpose of each equation type is not determined by the domain over which they are true. For example, numeric identities and transformation equation when using the relational meaning of the equal sign primarily share the purpose of justification, whereas formal identities and contextual identities are primarily formulas with the purpose of finding a new value. Transformation equations that use the operational meaning of the equal sign are also used to complete some operation or procedure to find a result. And lastly, unit identities primarily are used to prompt the reader to substitute one value in for another. Knowing the purpose of an equation greatly affects the reader's ability to successfully use and work with it. By organizing equations by meaning for the equal sign and purpose, rather than by domain over which they are true, I make explicit the purpose of each equation type.

Extensions were also added to the constraint equation category. As mentioned in chapter 2, constraint equations as a whole were discussed, but through this study, three different types of constraint equations were found: constraint equations with parameters, constraint equations with variables, and constraint equations with one unknown. Because these different types of constraint equations are worked through differently, and have different types of solutions, as discussed in the last chapter, they merit individual notice. For example, constraint equations with parameters are primarily used to describe a general form or family of an equation as in the general slope-intercept equation of a line, $y = mx + b$. In contrast, constraint equations with variables are used to describe a specific situation, and the purpose of this type of constraint equation is to determine at least one solution set that makes the equation true where one of the values in the set is dependent on the other value. And the purpose of a constraint equation with one unknown is to find the value(s) for the unknown that makes the equation true. Each type of constraint equation has a unique purpose and being able to recognize the different types of constraint equations and their purpose is valuable as that ability will allow the reader to know what their goal is, how to use the equation or which strategies to use to solve it.

Specification equations, as discussed in chapter 2, also appeared in the data from this study and were also added onto. Originally, specification equations were in a category by themselves, but through this study, restriction specifications were added to the list. As discussed above, restriction specifications are used to restrict the values that can be assigned to an unknown without specifying the values that can be assigned. The addition of restriction specifications increases our awareness of the different understandings that students need to develop, namely, that the assignment meaning of the equal sign can be used two ways: to identify values that *can* be given to a mathematical object (i.e., through specification equations), and values that *cannot* (i.e., through restriction specification equations).

Connecting Meanings of the Equal Sign to Equation Types

Previous research delineated the different meanings associated with the equal sign as well as many of the different equation types; however, it did not make a connection between the two. While the previous research made it possible to organize equation types by the domain of their solutions, this study makes it possible to organize equation types by the meaning that is assigned to the equal sign. Recognizing which meaning is associated with each equation type allows the reader to have a better idea of how to proceed with using or solving the equation. Initially, when coding each equation, I defined each one by its purpose—how it was meant to be used or solved.

This separation by purpose led to each individual equation type, and then eventually to the finding that all but one equation type (transformation equations) was associated with a single meaning of the equal sign. The reason that transformation equations match up with more than one meaning of the equal sign is because these equations can have two different purposes: to determine a result, or to justify a result. By identifying the purpose of the equation, I was able to determine which meaning was being assigned to the equal sign. Knowing which equation types use which meanings of the equal sign will allow teachers to be more aware and explicit of what meanings students need to be applying when working with certain equations. And as students make the connection between equation type and meaning, they will develop the understandings and flexibility needed to successfully work through the many different equations they will face in the 7th and 8th grade.

Structural Conventions as Found in the Analysis

When discussing structural conventions, previous researchers discussed them in terms of the direction in which the structures should be read and suggested which equation types might be used with each structural convention. They claimed that transformation chains were worked through uniformly and read either in a single line from left to right, or from top to bottom in a single column down, with each reduction usually following the equal sign. In contrast, the execution of constraint equations consists of applying operations to both sides of the equal sign, requiring the reader to notice both columns down, or both sides of the equal sign from top to bottom (Matz, 1982). In this study, I found that it was impossible to consistently determine whether an equation was intended to be read left-to-right or right-to-left. Because of this difficulty, I decided not to analyze in this study the direction in which equations should be read.

The two main structural conventions that were found in this study were lists of equations and strings of equalities. While these structural conventions are not too different from conventions discussed in chapter 2, they are more inclusive of the structures that were found in the analyzed data. Lists of equations (LOEs) are sequences of equations that appear vertically or horizontally right after each other. LOEs resemble the structure of both columns down but are not limited to constraint equations with one unknown, as Matz (1982) suggested. LOEs can also include equations of any type as long as they appear vertically or horizontally right after each other. Recognizing that other types of equations can be found in LOEs is important, so that teachers can draw attention to the fact that just because some equations are listed in an LOE does not imply that they are connected by equation operations. By being aware of this, teachers can help mitigate some possible confusion for 7th and 8th grade students who might be looking for equation operations in LOEs that are not, in fact, connected by any equation operations.

The second structural convention that appeared in this study is strings of equalities (SOEs). SOEs are expressions connected by two or more equal signs and appear vertically, horizontally, or as a combination of both. While SOEs resemble the transformation chains (as discussed in chapter 2), we gained new information from the analysis of these two curricula. We learned that SOEs are not limited to transformation equations, even though transformation equations are the primary equation type that utilizes the SOE structure. And because SOEs are not limited to one type of equation, we learn that they are also not limited to one meaning of the equal sign. As a reader works through an SOE, she is often required to switch between meanings of the equal sign as she progresses through each individual equation. While, the idea of SOEs is not completely new, the fact that we are often required to switch between meanings of the equal sign within one SOE is. Again, being aware of the possibility as well as when we do, in fact, switch between meanings is important so that teachers can help students develop this ability.

Equation Operations as Found in the Analysis

One of the findings from this study that was not discussed in previous research is the equation operations (EOs) that take place within an LOE. Previously, the only EOs that were mentioned were deductions and reductions (Matz, 1982); however, there are several more EOs that were used in the curricula. In addition to deductions and reductions, the EOs that surfaced through this study include substitutions, cross-multiplication, reflections, and equation combinations. I also found that many LOEs were simply lists of equations that were not connected by any EO. The appearance of additional EOs, as well as recognizing the absence of EOs in some LOEs helps us realize the complexity of reading LOEs, suggesting that they may be more difficult to read than previously thought. By recognizing this complexity, teachers can be more aware of how they present LOEs to their students as well as be more explicit when using EOs during instruction

In summary, this study shows that users of algebra interact with multiple types of equations that use all three meanings of the equal sign and are often required to switch between meanings while working through one problem. With the addition of more equation types, delineation of structural conventions, and equation operations, we gain insight into the complexity of reading and working with algebraic equations. By being aware of all of the complex aspects that affect how we interpret and determine how to proceed with an equation, teachers can be explicit in their instruction and help students develop these understandings in a much deeper way and likely be more successful as they transition into algebra.

CHAPTER 5: CONCLUSION

As students transition from arithmetic to algebra, many struggle to interpret and, therefore, utilize the equal sign appropriately, which would allow them to work with algebraic equations more successfully. Previous research has focused on the need for students to move beyond an operational understanding and develop a relational and assignment understanding of the equal sign. Because the meaning of the equal sign is not inherent in the symbol alone, to gain a better understanding of how we interpret and utilize the equal sign in middle school mathematics, I analyzed the meanings of the equal sign, equation types, and structural conventions used in instructor and student materials of two middle school curricula. As a result of this analysis, I was able to identify which meanings, equation types, and structural conventions students are likely to be exposed to. I was also able to identify which meanings of the equal sign are associated with each equation type and reveal the complexity of reasoning associated with the use of structural conventions in introductory algebra and geometry topics in the 7^{th} and 8^{th} grade.

Contributions

The results of this study provide at least three main contributions to research on our understanding of how we interpret and utilize the equal sign. The first contribution deals with the meanings that we assign to the equal sign. Research on this subject has already identified three meanings of the equal sign: operational, relational, and assignment. Based on the data provided from two middle school curricula, these three meanings seem to create an exhaustive list of the meanings that appear in the 7^{th} and 8^{th} grade mathematics material. Not only does the data show that students are exposed to only three meanings of the equal sign, but it also confirms that all three of these meanings are significantly present in 7^{th} and 8^{th} grade mathematics across multiple topics, and not just those typically associated with algebra. This contribution helps us to taper future research to these three different meanings, as well as to recognize that it is not enough to focus on a relational understanding of the equal sign. Students need to be able to understand the equal sign as an operational symbol, as well as a relational and assignment symbol if they are to be successful in 7^{th} and 8^{th} grade mathematics. This contribution also informs teachers on what meanings of the equal sign need to be more explicitly modeled and discussed during instruction.

The second main contribution that comes from this study is the extension of our understanding of different equation types. Not only were more equation types identified from this study than previous research had discussed, but the data from this study suggests that equation types can be grouped or organized by meaning and purpose. Earlier research previously organized equation types by the domain over which they are true. However, after looking at the data, it seems as though the previous organization was overly simplified. By taking into account the purpose of the equation type we were able to identify which meanings of the equal sign are associated with each equation type. These data shows that how a specific equation affects the way we interpret and utilize the equal sign is far more complex than simply looking at the domain of the solution sets. Being able to group equation types by their purpose and their associated meanings will allow teachers to be more aware of opportunities in which they can be explicit in their discussions of how they determine how to make sense of different equations and why.

The last main set of contributions that comes from this study is related to the data on structural conventions, including lists of equations and strings of equalities. By extending the *both columns down* structure to the *list of equations*, the data showed that there are times when the list of equations structure is used, but the subsequent equations in a given list are not connected by an equation operation. This is significant because now teachers can be aware of possible points of confusion for students if these examples are not discussed and explicitly talked about differently than a list of equations that consist solely of constraint equations that are connected by equation operations. Also, a major part of this contribution is the equation operations that were identified throughout the curricula. By knowing that different equation operations occur specifically with list of equations containing constraint equations, teachers can again be more explicit in their instruction and teach students how to read the changes in equations to infer the equation operations that have been performed.

The data from this study also contributes much to our understanding of strings of equalities. The main contribution about this structure is the realization that when equations are strung together, a level of complexity is added to how we interpret and utilize the equal sign. While advanced users of the equal sign are able to read through a string of equalities and make sense of the processes that occur from one expression to the next, they are likely unaware of how the equation types and meanings of each equal sign in the string may differ from those that preceded it. The data from this analysis informs us that as we move from one expression to the next in a string of equalities, often the equation type can change from one equation to the next, sometimes resulting in a change in the meaning that is assigned to subsequent equal signs. This helps us recognize yet another area where students might be getting confused if these changes in equation type and meaning of the equal sign are not being made explicit. All of the data on how different equation types and structures affect the way that we utilize the equal sign shows us that interpretation of the equal sign is a much more complex task than previous research has suggested.

Implications

The results of this study leave us with two types of implications: implications for teaching, and implications for research. This study has shown that reading the equal sign is a complex practice, which leaves implications for teachers if they are to help students develop the ability to read and use the equal sign appropriately in the middle grades. The first implication is that teachers must become aware of the different complexities of making sense of the equal sign. The awareness of how equation types are related to specific meanings of the equal sign, and especially of how equation types can change throughout an SOE can be very valuable for a mathematics teacher. Not only does this awareness allow the teacher to see how complicated it is working with equations, it also helps teachers in recognizing reasons for student misunderstandings and errors. Teachers who are aware of these complexities can be more helpful in identifying reasons for student mistakes as well as be more proactive in helping students avoid these mistakes.

One area where teachers can be more proactive in preventing student mistakes is working with LOEs. From this study we have learned that not all LOEs are connected by equation operations, yet appear in the same way as LOEs that do. For students who are just being introduced to LOEs, seeing some that are not connected by EOs can be confusing as they may interpret the equation type incorrectly, thinking that it must be a constraint equation because of how it is written. Teachers who are aware of the possibility of seeing LOEs that are not connected by EOs can anticipate some of these confusions and discuss the EOs or lack of EOs with their students and help them see that without EOs, the LOE is not likely an example of a constraint equation. Although, if teachers want to prevent the possible confusions of deciphering between connected and unconnected LOEs, I suggest that teachers try to avoid using LOEs that do not use EOs in middle school mathematics.

After becoming aware of the different complexities of making sense of the equal sign, teachers then need to recognize how they use these factors to make sense of the equal sign. It is likely that teachers unknowingly use equation types and structural conventions as “context clues” as they switch back and forth between meanings of the equal sign unconsciously. Once teachers recognize how they makes sense of the equal sign, they can then begin to model different equation types, structural conventions and equation operations to their students. This will allow the students to discover different ways of thinking about the equal sign as well as make connections between the different factors of making sense of the equal sign.

Lastly, once teachers recognize the roles that meanings of the equal sign, equation types, and structural conventions play in determining how they read and use the equal sign, teachers may want to be explicit during instruction about how each part affects how to proceed with each equation. For example, after working through a problem or series of problems that involve solving equations, a teacher could talk about the individual parts of each equation that led them to select the appropriate equation transformation. By incorporating discussion about how the teacher makes sense of equations, teachers should have many opportunities to bring explicit attention to the meanings of the equal sign, the different types of equations, and the structural conventions that are used when working with equations, and to model how their interpretations of these elements support their work with equations.

The results of this study also have at least one implication for research, and that is that to understand the real meaning of a symbol, such as the equal sign, we need to look into actual data to see how it is used in practice. By reflecting on their own understandings and analyzing student errors, mathematics education researchers have been able to suggest lists of meanings, equation types, and structural conventions. However, the question remains as to whether this list is complete. This can only be confirmed by gathering data from contexts in which the equal sign is being used. The resulting data from this study extends the previous findings that were based upon thought experiments and students' errors, not only by adding new components, but also suggesting that expert accounts are not always complete. Previous expert accounts left out important factors, such as interpreting equations based on purpose and meaning for the equal sign instead of solely on domain. By looking at the data on how the equal sign is used in middle school curricula, we can make better sense of the equal sign and its complexity. This suggests that research on the sense making of other algebraic symbols may benefit from collecting and analyzing data from contexts in which the symbols are used.

Limitations and Directions for Future Research

I recognize that no study is perfect and therefore each has its limitations, as is the case with this study. This study was limited by the restriction of data collection and analysis to written texts from two curricula for grades 7-8. Analyzing only written texts was limiting because the lack of explicit verbal instruction for many examples left us to infer the authors' intended use of the equal sign, and because we could not be completely sure if a structural convention was used because of preference or ease of printing. Studying only two curricula was limiting because we cannot assume that the frequency of certain equation types, or meanings of the equal sign reported in this study is an actual representation of all middle school curricula. And lastly, focusing on only two grades was limiting because we cannot assume that every equation type, meaning, or structural convention possibly used in other grades or branches of mathematics was observed. It is possible that there might be additional meanings, equation types, and structural conventions in mathematics for the equal sign.

Although this study has its limitations, these limitations suggest directions for future research. To better understand how the equal sign is used in instruction, future research could look at how the equal sign, equation types, and structural conventions are used and discussed in live classroom discourse. To address the limitations put on the study by only analyzing two curricula for two grades, future research could extend to more curricula covering different areas or levels of mathematics, such as arithmetic, secondary math, or even upper level college math. Doing so would allow us to gain greater insight into what equation types, structural conventions, and possibly meanings were not observable in the middle grades as well as gain a broader understanding of how the equal sign is utilized in each level of mathematics. These extensions would allow us to see more explicitly how the equal sign is used in practice as well as be aware of any new meanings, equation types or structural conventions that we need to prepare students for. Another way this study could be extended is by having teachers' explicitly model and discuss their use of the equal sign, and then examine how this instruction influences students' use of the equal sign.

Another direction for future research is to develop a learning trajectory to scaffold student thinking. It is clear from the data that students are expected to understand the equal sign in three different ways. Future research could focus on developing a learning trajectory to help students in elementary school simultaneously develop these three meanings. Rather than exclusively developing an operational understanding for many years and then leaving that understanding behind for a relational one, researchers could focus on how to develop the relational understanding and introduce the assignment understanding, all while continuing to use the operational understanding. Researchers could look into what order these meanings should be developed in and when. By developing an effective way to scaffold student understanding of the equal sign in elementary mathematics, students could make the transition from arithmetic to algebra much more successfully.

One last direction where we can further this research is in observing the direction in which experts read equations. After a short while of coding, we realized that it was impossible to determine the authors' intended direction or order in which an equation should be read; we also realize that as readers we may not be aware of the order in which we, ourselves, read things. Eye-tracking software could be used to determine the order in which equations are read by experts. This type of study could inform us on how experts piece together information to make sense of equations, which could be very helpful in instruction.

Conclusion

The ability to make sense of the equal sign is imperative for student success in middle school mathematics. Unfortunately, the complexity of the equal sign makes it difficult for students to develop the correct understandings of the equal sign to do so. This study was done primarily to find what types of equations, meanings of the equal sign, and structural conventions appear in 7th and 8th grade curricula to gain a better sense of what uses of the equal sign students are likely to encounter in the middle grades. Not only did we find what types of equations, meanings of the equal sign, and structural conventions appear in 7th and 8th grade, but this study demonstrates that there are complex patterns in the way these three elements are related in the equations found in middle grades mathematics textbooks. This study implies that it is imperative that teachers recognize this complexity and teach more explicitly the role these three components play in making sense of how the equal sign is used in the middle grades. With the finding from this study, we have greater direction in how to more effectively help students develop the skills to interpret and utilize the equal sign successfully.

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Appendix A

Table 11

List of lessons that were analyzed from each year and curriculum.

Connected Math 3 7th Grade	Eureka Math 7th Grade

• Stretching and Shrinking: Understanding Similarity (7.RP)
  • Enlarging and Reducing Shapes
  • Similar Figures
  • Scaling Perimeter and Area
  • Similarity and Ratios
• Moving Straight Ahead: Linear Relationships (7.EE)
  • Walking Rates
  • Exploring Linear Relationships with Graphs and Tables
  • Solving Equations
  • Exploring Slope: Connecting Rates and Ratios
• Filling and Wrapping: Three-Dimensional Measurement
  • Building Smart Boxes: Rectangular Prisms
  • Polygonal Prisms
  • Area and Circumference of Circles
  • Cylinders, Cones, and Spheres
	• Ratios and Proportions (7.RP)
	  • Topic A: Proportional Relationships Lessons 1-5
	  • Topic B: Unit Rate and Constant of Proportionality Lessons 6-10
	  • Topic C: Ratios and Rates Involving Fractions Lessons 11-15
	  • Topic D: Ratios of Scale Drawings Lessons 16-22
	• Expressions and Equations (7.EE)
	  • Topic A: Use Properties of Operations to Generate Equivalent Expressions Lessons 1-6
	  • Topic B: Solve Problems Using Expressions, Equations, and Inequalities Lessons 7-15
	  • Topic C: Use Equations and Inequalities to Solve Geometry Problems Lessons 16-26
	• Geometry (7.G)
	  • Topic A: Unknown Angles Lessons 1-4
	  • Topic B: Constructing Triangles Lessons 5-15
	  • Topic C: Slicing Solids Lessons 16-19
	  • Topic D: Problems Involving Area and Surface Area Lessons 20-24
	  • Topic E: Problems Involving Volume Lessons 25-27

Connected Mathematics 3 8th Grade	Eureka Math 8th Grade

• Growing, Growing, Growing: Exponential Functions (8.EE, and 8.F)
  • Exponential Growth Patterns: Non-linear Functions
  • Examining Growth Patterns: Exponential Functions
  • Growth Factors and Growth Rates
  • Exponential Decay Functions
  • Patterns with Exponents
• Say It With Symbols: Making Sense of Symbols (8.EE and 8.F)
• Equivalent Expressions
• Combining Expressions
• Solving Equations
• Looking Back at Functions
• Reasoning with Symbols
• It's In the System: Systems of Linear Equations and Inequalities (8.EE)
  • Linear Equations With Two Variables
  • Solving Linear Systems Algebraically
• Thinking With Mathematical Models: Linear and Inverse Variation (8.F)
  • Exploring Data Patterns
  • Linear Models and Equations
  • Inverse Variation
• Looking for Pythagoras: The Pythagorean Theorem (8.G)
  • Coordinate Grids
  • Squaring Off
  • The Pythagorean Theorem
  • Using the Pythagorean Theorem: Analyzing Triangles and Circles
• Butterflies, Pinwheels, and Wallpaper: Symmetry and Transformations (8.G)
  • Symmetry and Transformations
  • Transformations and Congruence
  • Transforming Coordinates
  • Dilations and Similar Figures
	• Linear Equations (8.EE)
	  • Topic A: Writing and Solving Linear Equations
	  Lessons 1-9
	  • Topic B: Linear Equations in Two Variables and Their Graphs
	  Lessons 10-14
	  • Topic C: Slope and Equations of Lines
	  Lessons 15-23
	  • Topic D: Systems of Linear Equations and Their Solutions
	  Lessons 24-30
	  • Topic E: Pythagorean Theorem
	  Lesson 31
	• Examples of Functions from Geometry (8.F)
	  • Topic A: Functions
	  Lessons 1-8
	  • Topic B: Volume
	  Lessons 9-11
	• Linear Functions (8.F)
	  • Topic A: Linear Functions
	  Lessons 1-5
	• Introduction to Irrational Numbers Using Geometry (8.G)
	  • Topic A: Square and Cube Roots
	  Lessons 1-5
	  • Topic C: The Pythagorean Theorem
	  Lessons 15-18
	  • Topic D: Applications of Radicals and Roots
	  Lessons 19-23

Appendix B

Table 12

Breakdown of frequency of each equation type and its corresponding meaning of the equal sign as found in each module, separated by teacher and student material (Eureka Math 7)

	EM7-1 (7.RP) Teacher	EM7-1 (7.RP) Student	EM7-3 (7.EE) Teacher	EM7-3 (7.EE) Student	EM7-6 (7.G) Teacher	EM7-6 (7.G) Student
Operational	48.1%	23.0%	15.8%	5.1%	36.9%	0.9%
Transform. Equations	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Assignment	1.7%	3.8%	3.8%	39.8%	2.1%	80.4%
Specification	100.0%	0.0%	95.3%	100.0%	100.0%	100.0%
Restriction Specification	0.0%	0.0%	4.7%	0.0%	0.0%	0.0%
Relational	50.1%	73%	80.4%	55.1%	61.0%	18.7%
Constraint w/ Parameters	11.5%	31.6%	1.9%	0.0%	0.0%	0.0%
Constraint w/ Variables	25.1%	21.1%	5.7%	0.0%	10.8%	40.0%
Constraint w/ Unknown	45.4%	0.0%	64.8%	61.1%	74.6%	10.0%
Formal Identity	0.0%	0.0%	0.6%	1.9%	0.0%	0.0%
Contextual Identity	7.0%	15.8%	10.8%	11.1%	7.2%	20.0%
Numeric Identity	4.8%	0.0%	4.9%	5.6%	0.0%	0.0%
Numeric Non-Identity	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Transform. Equations	0.0%	0.0%	5.3%	5.6%	2.6%	30.0%
Unit Identity	0.1%	0.0%	0.7%	7.4%	0.0%	0.0%
Equation Result	5.3%	31.6%	4.5%	1.9%	4.6%	0.0%

Table 13

Breakdown of frequency of each equation type and its corresponding meaning of the equal sign as found in each module, separated by teacher and student material (CMP3-7)

	CMP3 (7.RP) Teacher	CMP3 (7.RP) Student	CMP3 (7.EE) Teacher	CMP3 (7.EE) Student	CMP3 (7.G) Teacher	CMP3 (7.G) Student
Operational	43.9%	22.2%	3.5%	0.7%	23.4%	0.0%
Transform. Equations	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Assignment	0.0%	0.0%	9.5%	5.9%	11.7%	0.0%
Specification	0.0%	0.0%	100.0%	88.9%	100.0%	0.0%
Restriction Specification	0.0%	0.0%	0.0%	11.1%	0.0%	0.0%
Relational	56.1%	77.3%	87.0%	93.4%	64.8%	100.0%
Constraint w/ Parameters	8.7%	0.0%	21.1%	5.5%	0.0%	0.0%
Constraint w/ Variables	0.0%	0.0%	32.6%	43.3%	3.6%	33.3%
Constraint w/ Unknown	8.7%	0.0%	30.4%	35.4%	2.4%	0.0%
Formal Identity	8.7%	0.0%	0.0%	0.0%	0.0%	0.0%
Contextual Identity	8.7%	0.0%	2.9%	0.8%	66.3%	50.0%
Numeric Identity	39.1%	71.4%	5.5%	1.6%	2.4%	16.7%
Numeric Non-Identity	0.0%	0.0%	0.0%	0.0%	0.0%	0.0%
Transform. Equations	17.4%	0.0%	0.2%	0.8%	15.7%	0.0%
Unit Identity	0.0%	28.6%	0.0%	0.0%	0.0%	0.0%
Equation Result	8.7%	0.0%	4.9%	0.0%	6.0%	0.0%

Table 14

Breakdown of frequency of each equation type and its corresponding meaning of the equal sign as found in each module, separated by teacher and student material (Eureka Math 8)

	EM8 (8.EE) Teacher	EM8 (8.EE) Student	EM8 (8.F) Teacher	EM8 (8.F) Student	EM8 (8.G) Teacher	EM8 (8.G) Student
Operational	7.9%	1.1%	27.7%	10.1%	15.1%	5.1%
Transform. Equations	100.0%	100.0%	100.0%	100.0%	100.0%	100.0%
Assignment	4.2%	12.7%	5.0%	16.0%	3.2%	3.4%
Specification	91.2%	98.3%	100.0%	100.0%	100.0%	100.0%
Restriction Specification	8.8%	1.7%	0.0%	0.0%	0.0%	0.0%
Relational	87.8%	86.2%	67.3%	73.9%	81.7%	91.5%
Constraint w/ Parameters	8.2%	12.3%	7.2%	15.7%	0.0%	0.0%
Constraint w/ Variables	46.8%	19.3%	32.2%	35.3%	54.3%	55.6%
Constraint w/ Unknown	30.1%	63.8%	41.7%	35.3%	10.3%	11.1%
Formal Identity	0.2%	0.0%	0.0%	0.0%	1.0%	3.7%
Contextual Identity	1.9%	2.3%	7.4%	13.7%	3.7%	1.9%
Numeric Identity	4.8%	1.3%	2.3%	0.0%	10.1%	1.9%
Numeric Non-Identity	0.9%	0.3%	0.0%	0.0%	0.4%	0.0%
Transform. Equations	1.5%	0.0%	0.0%	0.0%	10.4%	9.3%
Unit Identity	0.2%	0.0%	0.9%	0.0%	0.0%	0.0%
Equation Result	3.7%	0.5%	8.3%	0.0%	2.0%	0.0%

Table 15

Breakdown of frequency of each equation type and its corresponding meaning of the equal sign as found in each module, separated by teacher and student material (CMP3-8)

	CMP3- (8.EE&F) Teacher	CMP3 (8.EE&F) Student	CMP3 (8.EE) Teacher	CMP3 (8.EE) Student
Operational	8.5%	8.8%	0.8%	0.0%
Transform. Equations	100.0%	100.0%	100.0%	100.0%
Assignment	11.1%	6.0%	15.4%	0.0%
Specification	91.6%	69.2%	98.4%	0.0%
Restriction Specification	8.4%	30.8%	1.6%	0.0%
Relational	80.5%	85.3%	83.8%	100.0%
Constraint w/ Parameters	13.2%	8.6%	37.2%	16.7%
Constraint w/ Variables	41.3%	38.9%	15.4%	3.8%
Constraint w/ Unknown	12.3%	17.3%	36.3%	78.0%
Formal Identity	3.5%	11.9%	0.0%	0.0%
Contextual Identity	4.1%	5.4%	0.0%	0.0%
Numeric Identity	6.2%	5.4%	1.2%	0.0%
Numeric Non-Identity	0.0%	0.0%	0.9%	0.0%
Transform. Equations	8.0%	11.4%	0.0%	0.0%
Unit Identity	0.0%	0.0%	0.0%	0.0%
Equation Result	6.0%	1.0%	6.6%	0.0%

Breakdown of frequency of each equation type and its corresponding meaning of the equal sign as found in each module, separated by teacher and student material (CMP3-8) (*continued)*

	CMP3 (8.F) Teacher	CMP3 (8.F) Student	CMP3 (8.G) Teacher	CMP3 (8.G) Student
Operational	18.9%	4.9%	12.9%	7.0%
Transform. Equations	100.0%	100.0%	100.0%	100.0%
Assignment	6.1%	2.4%	5.8%	5.6%
Specification	93.3%	100.0%	91.3%	100.0%
Restriction Specification	6.7%	0.0%	8.7%	0.0%
Relational	75.0%	92.7%	81.3%	87.3%
Constraint w/ Parameters	34.4%	23.7%	3.4%	3.2%
Constraint w/ Variables	7.7%	26.3%	35.8%	27.4%
Constraint w/ Unknown	36.6%	42.1%	10.9%	25.8%
Formal Identity	0.0%	0.0%	0.3%	0.0%
Contextual Identity	10.4%	2.6%	11.2%	17.7%
Numeric Identity	4.9%	0.0%	18.1%	22.6%
Numeric Non-Identity	0.0%	0.0%	0.6%	0.0%
Transform. Equations	1.6%	0.0%	6.5%	0.0%
Unit Identity	0.0%	0.0%	0.0%	0.0%
Equation Result	3.8%	5.3%	11.8%	3.2%