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A string number-line lesson sequence

to promote students' relative thinking and understanding of scale, key elements of proportional reasoning


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Using a string number line teachers can support the development of relative thinking and the understanding of linear scale contributing to the development of the complex concept of proportional reasoning.

Introduction

Proportional reasoning involves relative thinking: the ability to think about multiple quantities simultaneously, and in relative terms as opposed to absolute terms (Ontario Ministry of Education, 2012). Concepts related to relative and absolute thinking and the ability to think relatively and multiplicatively are essential to understanding proportionality (Hilton, Hilton, Dole, & Goos, 2015). It takes time for students to develop proportional reasoning and without targeted teaching, many students fail to develop the important skills and conceptual understanding that underpins it (Bangert-Drowns, Hurley, & Wilkinson, 2004; Kastberg, D'Ambrosio, & Lynch-Davis, 2012; Lamon, 2012). Proportional reasoning is essential for students to succeed in many mathematical areas, including ratio and proportion, measurement and unit conversions, geometry, and probability. It is also necessary in other subjects, such as geography and science (Akatugba & Wallace, 2009). In fact, scale and proportion have been identified as crosscutting concepts fundamental to understanding and reasoning in science (National Research Council, 2012).

An important application of relative thinking involves the ability to use, interpret and create linear scales. This requires students to understand range, scale intervals, and the relative positioning of numbers, particularly in situations where the intervals are not indicated or where they are not in increments of one. It is likely that students who have difficulties in this area may also encounter challenges in reading and using scales (e.g., on measuring devices or graph axes); in interpreting or representing data using graphs; or in creating subject-specific representations, such as time-scales in science or history. Research has identified possible reasons behind students' difficulties in using and interpreting linear scales. These include difficulties identifying relative situations; using absolute thinking (not considering the value of a quantity in relation to other quantities); and ignoring relevant data (Lamon, 1993; Misailidou & Williams, 2003; Van de Walle, Karp, & Bay-Williams, 2010).

A challenge for teachers is that proportional reasoning and its underpinning concepts are not made explicit in the curriculum. A search of the *Australian Curriculum: Mathematics* (ACARA, 2017) revealed no instances where proportional reasoning is mentioned and no explanation of relative thinking. While these aspects may not be explicit in the curriculum, it is clear that the concepts are pervasive. A search for 'scale' in the mathematics curriculum revealed many areas and concepts that require students to think relatively, including time-scales, scale factors, using scale in graphing and mapping, and reading scaled instruments. It is also clear that students must be able to use these forms of reasoning from the early years of primary school. For example, the mathematics curriculum requires children in Year 1 to locate numbers one to 100 on a number line; Year 2 children to understand representations of objects and their positions and to represent patterns on a number-line; Year 3 children to use appropriate scale to place four-digit numbers on a number line; and children in Year 4 to read and interpret graduated scales and represent familiar fractions on a number line. The demands increase through the year levels with students in Year 9 expected to be able to apply relative thinking and scale in areas as diverse as geometry, functions on the Cartesian plane, scale factors, very small and very large time-scales, and algebra. This situation has important implications for teachers because scale and scaling are central to the development of students' mathematical reasoning in

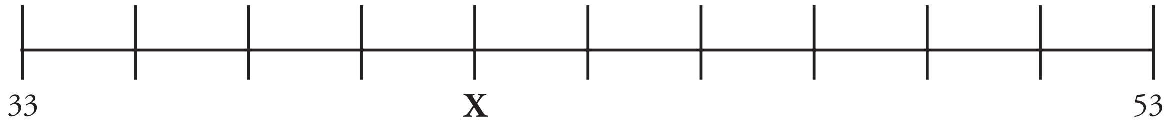

Figure 1. Item in which students were required to identify X.

these areas (Booth & Siegler, 2006; Saxe, Shaughnessy, Gearhart, & Chopra Haldar, 2013).

In primary school, one of the first encounters children have with formal scales occurs when they learn about numbers using a number line. The number line has been identified as a core mathematical tool used to support children's development of relative thinking (e.g., scale, magnitude, relative positioning of whole numbers and fractions); understanding of whole numbers, sequencing, and number relationships; and their understanding of equivalence (e.g., fractions, decimals, and percentages) (Geary, Hoard, Nugent & Byrd-Craven, 2008; National Council of Mathematics Teachers, 2006).

The string number line is an example of an empty number line. The empty number line has been found to be useful for developing children's number sense and their confidence and ability to use numbers flexibly and is typically used as a support for counting, addition, and subtraction, often with no requirement for the lengths or distances between numbers to indicate relative value (Bobis, 2009). In this article, the use of the string number line is different; there is a definite requirement for the distances between numbers to be relative to their values because this is essential to developing an understanding of linear scale. Being a physical representation, the string number line allows students to manipulate the numbers and the distances between them, which makes it useful for developing students' understanding of scale and allows teachers to focus on developing children's understanding of the relative positioning of different numbers. It also allows children to reposition numbers and to use relative thinking when rescaling is required.

This article describes part of a study in which researchers designed lesson sequences based around using a string number line to help teachers support children's development of relative thinking and understanding of linear scale. This approach proved effective for developing students' understanding of concepts related to scale and relative thinking as well as their mathematical language, which is essential in the development of conceptual understanding (Hilton et al., 2015). It was also useful for teachers' own professional learning and as a means of assessing and monitoring their students' understanding and progress.

Impetus for the study

Our previous research, which involved 2500 middle years students, revealed that many students from Year 5 to Year 9 have difficulties in a range of applications of proportional reasoning, including scale (see Hilton, Hilton, Dole, & Goos, 2016). The data collected about students' understanding of linear scale showed that a very high proportion of students had difficulties in identifying a missing value on a number line with 10 intervals between two known numbers, which were 20 units apart, as shown in Figure 1.

Only a small percentage of primary school students (6.6% Year 5 and 10.4% Year 6), used the correct relative thinking to answer the question. The most commonly used erroneous reasoning involved absolute thinking in which the students counted on from the first number, ignoring the last number thereby ignoring the scale of the number line. The low percentages of students who were able to answer this question accurately suggest a need to develop ways of targeting these concepts to strengthen students' ability to deal with linear scale, and the relative thinking and understanding of scaling and re-scaling this requires.

In the current study, we are working with a small group of teachers, which allows us to collaborate with the teachers to develop and trial short lesson sequences (comprised of 8–10 lessons) to help us to learn more about how to support the teaching and learning of proportional reasoning. In its first year, eight teachers of Years 3–5 participated in four one-day professional development workshops (one per term), with implementation of the lesson sequences between each. The goal was to design and trial activities that could be easily implemented by teachers using simple resources and to determine their impact on students' ability to use proportional reasoning. Two of the lesson sequences used the string number line to promote children's ability to use relative thinking and scaling with whole numbers (Term 2) and fractions (Term 3). This article focuses on the use of the string number line lesson sequence involving whole numbers and its impact on children's ability to solve problems involving linear scale, relative thinking, and re-scaling.

The string number line lesson sequence

The lessons

The teachers were given an instructional script that contained a structured series of discussion prompts to be used on each of four days per week for two weeks (a total of 8 lessons). It was expected that each day's activities would require around ten minutes, although the teachers initially spent 15-20 minutes per day. As they became more familiar with the activities, the language and the children's abilities, the time became closer to the anticipated ten minutes per day. In addition to discussion prompts and suggested questions, the teachers used a glossary of terms to ensure that they used accurate and consistent mathematical language when conducting their lessons. The main reasons for providing the script and glossary for the teachers were (1) to help teachers foreground and model the mathematical language associated with scale; (2) to support teachers who felt unsure about how they could promote their students' understanding of scale and rescaling on the number line; and (3) to provide an opportunity for teachers to enact learning from the workshops in their classes. Table 1 shows the main ideas associated with each of the lessons. Appendix 1 has the script and instructions for the first two lessons.

Figure 2 shows a string number line as it was used in this study. The first photo illustrates an introductory step in Lesson 1. The second photo shows students determining the place of the midpoint and placing the missing number.

The pre- and post- tests

The teachers gave their students a test prior to the first lesson and the same test was repeated in the week following the lesson sequence. In total 204 students completed both pre- and post- tests. The test consisted of ten items that reflected the content of the lessons. Each item showed a number line, similar in layout to a physical string number line, and students were required to insert one or more missing numbers, as shown in Figure 3.

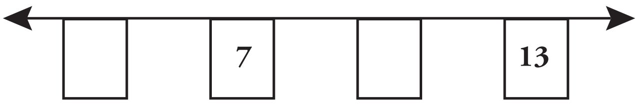

Figure 3. Sample test item.

Did the students' reasoning improve?

The results for the pre- and post- tests were compared using paired-sample t-tests. There was a statistically significant difference between the scores for the

Figure 2. Rescaling and placing numbers on the string number line.

Table 1. The lesson sequence and main ideas of each.

Lesson	Idea
1	Naming and locating the middle number between 0 and an even number
2	Manipulating the starting number, while keeping numbers even and maintaining the interval length
3	Determining the starting number or the end number having been given the midpoint
4	Determining the starting number or the end number when the middle number changes (re-scaling)
5	Determining the value of multiple numbers given the first two numbers (and thus the interval value)
6	Determining the value of two non-consecutive numbers (thereby determining the interval value)
7	Repeating Lesson 6 but with more missing numbers or larger numbers (teacher judgment used here)
8	Rescaling - interval distance is changed so numbers need to be relocated

Note: All numbers were whole numbers, no fractional thinking was involved.

pre- and post- tests for all three year levels. The teachers were also asked to annotate their instructional scripts to record their reflections during the lessons and to note the ways in which students responded to the questions and activities. The annotations on the scripts indicated a number of patterns in the thinking of the students. Early in the lesson sequence, the teachers noted that:

- some students used counting strategies to determine the value when one number was missing;
- once the interval value was established, additive strategies were sometimes used to determine missing consecutive numbers;
- many children wanted to draw blank number lines to help them solve the problem, used trial and error, or asked to measure the distances; and
- some students could give the correct answer but weren't able to articulate why.

When solving the problems:

- the students found the items much easier when the starting number was zero;
- if the starting number was not zero, students sometimes used erroneous logic (e.g., When solving

  some students calculated the difference between 2 and 8, i.e., 6 and halved it before realising that 3 is not the midpoint of 2 and 8);
- when the first number was missing, some students counted on and then counted back by the same amount; and
- pre-algebraic strategies were evident in some students' reasoning (e.g., "I found that 7+ □ = 10, which was 3 so then I said 7 – 3 = 4").

By the end of the lesson sequence, the teachers noted that the children were better able to use the known values to determine the interval value and solve for the missing values. They also noted a shift from additive and absolute thinking, to multiplicative and relative thinking. There was also a distinct increase in the children's use of the mathematical language.

Teachers' perceptions of the string number line lesson sequence

The teachers perceived the string number line lessons as valuable and their responses indicated a number of different benefits for both teaching and learning. It was useful for diagnostic assessment because it allowed teachers to easily determine which students were using additive/absolute thinking or multiplicative/relative thinking. The teachers felt that because the use of the string number line required physical manipulation, the children developed an understanding more quickly and effectively. They noted that the structured nature of the activity supported their teaching by modelling a developmental sequence and providing model questions and explanations. They felt that this approach prompted them to use different strategies, for example, varying the starting number rather than always starting at zero. Teachers liked the short repetitious nature of the activities, which were used daily, to support students who were struggling with the concepts. At the same time, the teachers noted that there were still opportunities for challenging more able students because some prompts elicited higher-order thinking.

The lesson sequence and script promoted the children's use of the mathematical language and they began to use such terms as line, interval, and interval value. Teachers described this as empowering for the students because they were better able to articulate their reasoning. Some teachers used the opportunity to develop word cards or word walls to support the development and use of mathematical language. The teachers also felt that the lessons and script provided them with a personal learning opportunity, describing how their confidence and knowledge of scaling and re-scaling were enhanced, and noting their increased fluency in the use of the mathematical terminology. They also noted that the lesson sequence and script acted as a model for scaffolding students' development of understanding the concept of relative thinking.

Conclusion

This study showed that the use of a structured lesson sequence utilising a string number line and script to target concepts and mathematical language was effective for developing children's understanding of scale and their ability to use relative thinking. It also provided some insight into the ways in which children reason when thinking about scale and the number line. An added benefit of the structured lessons and script was the improvement in the use of mathematical language by both teachers and students, which is significant because the use of mathematical language is a key aspect of developing conceptual understanding. Certainly, the teachers felt that the consistent use of language with their students supported its development and provided students with a means by which to articulate their thinking.

While this small-scale study only involved eight teachers, it provided an opportunity to investigate the effectiveness of a sequence of lessons that targeted the specific needs of those teachers and their students.

The findings suggest that the string number line is an effective tool when accompanied by sequenced questions and activities that target children's development of scale and its underpinning concepts. The string number line is simple, quick to set up, and, perhaps most importantly, it is both a physical and visual representation that allows children to position and reposition numbers while explaining their thinking. Further research is ongoing and will focus on investigating similar lesson sequences for developing fractional understanding.

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Appendix 1

Extract from the lesson sequence: The first two lessons

Setting up the number line

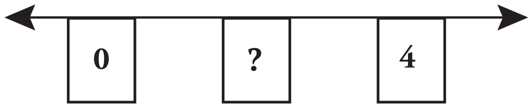
The best resources for the string number line are a thick string/cord and pegs that fit the string snugly. This allows the number cards to be held firmly and prevents them from spinning around on the string. The best approach is to secure the ends of the string so that students are not required to hold the ends, thereby allowing all students to participate in the activity.

Day 1

Naming and locating the middle number between 0 and an even number. All examples keep the numbers whole and avoid fractional answers at this stage.

Example 1

Ask students to locate the position of half way and then name that position. For these exercises the interval distance stays the same but the interval value may vary.



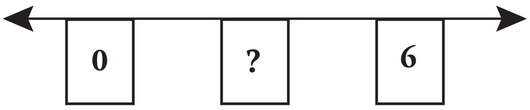
Q: How was the midpoint determined?
Q: How was the number value determined?

Note: children who say "halve the end number (4)" are only correct when the starting number is 0.

Even at this early stage, there is an opportunity to identify whether children are using additive or multiplicative thinking. This is also an opportunity for teachers to use these terms with the children. For example, a child who determines the interval value by counting on is using additive thinking and trial and error (e.g., 0, 1, 2, no; 0, 2, 4, yes) but one who knows that the interval value is found by dividing (4 – 0) by 2 is using multiplicative thinking.

Example 2

Repeat as for Example 1 but use 0 and 6.
Do not change the position of the number cards.



Q: What is the middle number now?
How do you know?

Q: Why did the middle number change?

Q: Why did the end number increase by 2 but the middle number increased by only 1 from the previous example?

Q: Did the interval length change?

Q: Did the interval value change? What is it now?

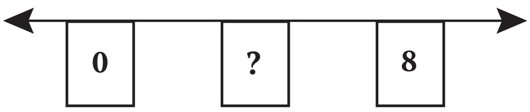
Q: Why did the interval value change but the interval length did not?

Day 2

The purpose today is to manipulate the starting number. We are keeping numbers even and whole and the interval length does not change.

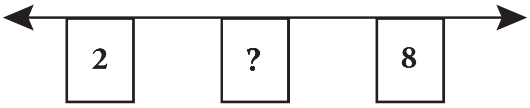
Example 1

Start with the number line below and repeat questions from Day 1 examples for how to determine the middle number.



Example 2

Change the starting number to 2.
Do not change the positions of the number cards.



Q: What is the middle number now?
How do you know?

Q: Why did the middle number change?

Q: Why did the starting number increase by 2 but the middle number increase by 1?

Q: Did the interval distance change?

Q: What numeric value does the interval now represent?

Q: Why did the number that the interval represents change but the length of the interval did not?

Repeat the exercise with the starting number at 4 and 6. Repeat any questions you feel are needed to consolidate concepts.