2007-05-23

On Connections Between Univalent Harmonic Functions, Symmetry Groups, and Minimal Surfaces

Stephen M. Taylor
Brigham Young University - Provo

ON CONNECTIONS BETWEEN UNIVALENT HARMONIC FUNCTIONS, SYMMETRY GROUPS, AND MINIMAL SURFACES

by

Stephen Taylor

A thesis submitted to the faculty of

Brigham Young University

in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics

Brigham Young University

August 2007

BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Stephen Taylor

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	Michael Dorff, Chair
Date	Jim Cannon
Date	David Cardon
Date	Tiencheng Ouyang
Date	Jean-Francois Van Huele

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the dissertation of Stephen Taylor in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

Date

Michael Dorff
Chair, Graduate Committee

Accepted for the Department

Tyler Jarvis, Chair
Department of Mathematics

Accepted for the College

Earl M. Woolley, Dean
College of Mathematics and Physical Sciences

ABSTRACT

ON CONNECTIONS BETWEEN UNIVALENT HARMONIC FUNCTIONS, SYMMETRY GROUPS, AND MINIMAL SURFACES

Stephen Taylor

Department of Mathematics

Master of Science

We survey standard topics in elementary differential geometry and complex analysis to build up the necessary theory for studying applications of univalent harmonic function theory to minimal surfaces. We then proceed to consider convex combination harmonic mappings of the form $f = sf_1 + (1 - s)f_2$ and give conditions on when f lifts to a one-parameter family of minimal surfaces via the Weierstrauss-Enneper representation formula. Finally, we demand two minimal surfaces M and M' be locally isometric, formulate a system of partial differential equations modeling this constraint, and calculate their symmetry group. The group elements generate transformations that when applied to a prescribed harmonic mapping, lift to locally isometric minimal surfaces with varying graphs embedded in $\mathbb{R}^3$.

ACKNOWLEDGMENTS

First, I would like to thank my advisor and coauthor Michael Dorff for motivating and helping me address many of the problems in this thesis. I wish to thank David Cardon, Tiencheng Ouyang, Jean-Francois Van Huele, and Aaron Peterson for critically reviewing the text. Many thanks to Jim Cannon for teaching a course on differential geometry, organizing a seminar on the Ricci Flow, and providing advice for this thesis.

Finally, I would like to thank my family and the tithe payers of the Church of Jesus Christ of Latter Day Saints for making Brigham Young University possible.

This work was supported by a BYU graduate physics research grant, a BYU mathematics teaching assistantship and research assistantship, and NSF grant PHY-0502218.

NOTATION

The following are common notations in the text:

$\mathbb{C}$ - the complex numbers

C^0- the space of continuous functions

C^n - the space of n -times differentiable functions

C^∞ - the space of infinitely differentiable functions

iff - if and only if (logical equivalence)

M - n-dimensional C^∞ manifold

$\mathbb{R}$ - the real numbers

$v_a v^a$ - indicates implicit Einstein summation convention over a

$\otimes$ - tensor product

S^n - the unit n sphere

$T_p(M)$ - the tangent space at a point p in a manifold M

$T_P^*(M)$ - the cotangent space at a point p in a manifold M

A NOTE ON GRAPHICS

Graphics in this thesis were produced in two ways.

All anti-aliased graphics were produced with the Persistence of Vision Raytracer (POV-Ray). A version of Ingo Janssen's param.inc file was used for mesh generation. Mike Williams helped alter this file to produce coordinate curve grids on surface plots.

All other plots were produced with a combination of the ParametricPlot3D command in Mathematica 5.0 and the Smooth3D grid refinement package written by Allan Hayes and Hartmut Wolf.

Some graphics were altered in Adobe PhotoShop CS2 to eliminate subtle rendering errors near singular points of plots.

Any source code for the graphics in this thesis will be made available upon request.

Contents

List of Figures

Preface

Geometric analysis techniques have been fruitful in resolving many diverse problems in geometry, topology, and physics. We mainly study geometric analysis problems related to minimal surfaces embedded in Euclidean three space. Specifically, we are concerned with variations of the problem of deforming a prescribed minimal surface into another minimal surface.

In Chapter 1, we survey standard topics in the differential geometry of surfaces with an emphasis on minimal surfaces. In Chapter 2, we develop topics in planar univalent harmonic function theory to address minimal surface questions.

Chapter 3 is a version of a paper called *A Minimal Surface Convex Combination Theorem* submitted to *Computational Methods and Function Theory*. We study the following question: Consider two minimal surfaces M, M' constructed from harmonic functions f, f' via the Weierstrass-Enneper representation formula. When does the harmonic mapping $f = sf + (1-s)f'$ lift to a minimal surface for all s in the closed unit interval? The theorems in chapter 3 resolve this issue. It would be beneficial to final general necessary and sufficient conditions on f and f' for their convex combination to be univalent. This would simplify our proofs.

Chapter 4 is a version of a paper called *Locally Isometric Families of Minimal Surfaces*. This paper was motivated by the construction of associated surfaces from two prescribed minimal surfaces. We fix a minimal surface M, and construct a method for generating multiple one parameter families of minimal surfaces locally isometric to M via a Lie point symmetry analysis. We find non-trivial symmetries and work out an example for the half-catenoid. We would like to generalize our methods to minimal surfaces embedded in more general manifolds but require a more general Weierstrass-Enneper representation to do this.

Chapter 1

Differential Geometry

In this chapter we provide a summary of elementary differential geometric tools that will be pertinent to the subsequent chapters. We define the standard elements of manifold theory and stress that ideas pertaining to the differential geometry of surfaces immersed in $\mathbb{R}^3$ follow as special cases. We proceed to give definitions and examples that will be relevant to forthcoming material.

1.1 Basic Manifold Theory

The manifold is the fundamental object of differential geometry. Before we can define geometric quantities (i.e. tangent vectors, curvature, etc.) we must first provide a precise definition of the manifold we wish to define them upon. Heuristically, a manifold is a locally Euclidean set. Stated in other words, if we restrict to a sufficiently small region of an n-dimensional manifold, it would appear locally as $\mathbb{R}^n$.

Definition 1. A n-dimensional, C^r **manifold** M is a set M with **charts** (U_α, ϕ_α) where $U_\alpha \subset M$ and $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ are injective such that

i) $M = \bigcup_\alpha U_\alpha$

ii) if $U_\alpha \cap U_\beta$ is non-empty, then the map

$$\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$$

is a C^r map of an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$ for all α, β. If $\phi_\beta \circ \phi_\alpha^{-1} \in C^r$ for all $r \in \mathbb{N}$, then we say M is a C^∞ manifold (see figure 1.1).

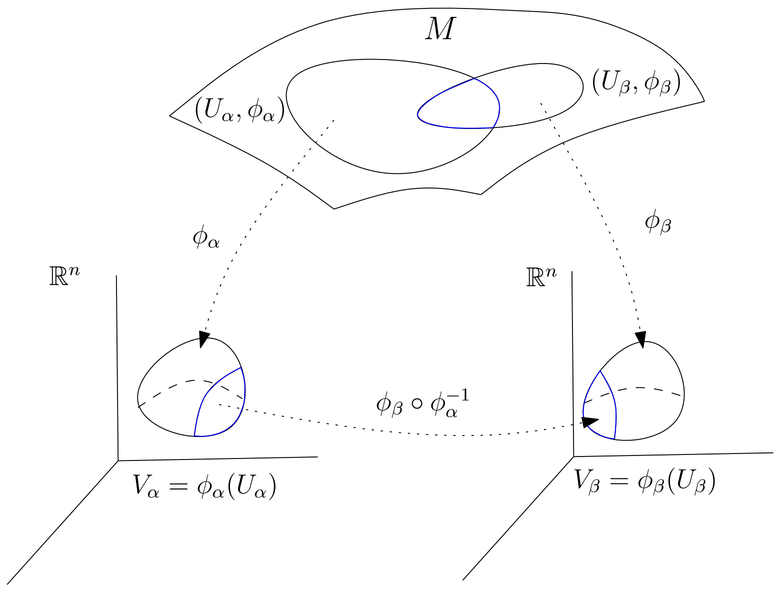

Figure 1.1 Manifold Definition.

The following examples of manifolds will provide intuition for this definition:

Example 1. (Euclidean Space) $\mathbb{R}^n$ is a manifold covered by $U_1 = \mathbb{R}^n$ with coordinate function $\phi_1 = \text{id}$.

Most manifolds cannot be covered by one chart, as the following shows:

Example 2. (S^2) Define six hemispheres

$$U_i^\pm = \{(x^1, x^2, x^3) \in S^2 | \pm x^i > 0\}$$

where $i = 1, 2, 3$. Note the $U_i^{\pm}$ cover S^2. Define coordinate maps by their projections onto the unit disc, i.e. $\phi_1^{\pm} : U_1^{\pm} \to \mathbb{D}$, given by $\phi_1^{\pm}(x^1, x^2, x^3) = (x^2, x^3)$. The overlap functions $\phi_i^{\pm} \circ (\phi_j^{\pm})^{-1}$ are C^∞ [7], from which we conclude S^2 is a C^∞ manifold.

We will always work on C^∞ manifolds. We now give several definitions in order to restrict the class of manifolds we will consider in this work.

Definition 2. A manifold is said to be **orientable** if there is a collection of charts $\{U_\alpha, \phi_\alpha\}$ such that for every non-empty intersection $U_\alpha \cap U_\beta$, the Jacobian $|\partial x^i / \partial x'^j|$ is positive, where the x^i and x'^i label the local coordinates of U_α and U_β respectively.

The Möbius strip and Klein bottle are two examples of non-orientable manifolds.

Definition 3. A topological space M is said to be **Hausdorff** if it satisfies the following separation axiom: For $\{p, q\} \in M$ where $p \neq q$, there exist open sets $U \subset M$ and $V \subset M$, such that $p \in U$, $q \in V$ and $U \cap V = \emptyset$. Note if one defines elements of the topology of a manifold to be the U_α, then the manifold is a topological space.

Definition 4. A collection of charts $\{U_\alpha, \phi_\alpha\}$ of a manifold M is **locally finite** if for every point $p \in M$ there is an open neighborhood which intersects only a finite number of the U_α. M is said to be **paracompact** if for every collection $\{U_\alpha, \phi_\alpha\}$ there exists a locally finite atlas $\{V_\beta, \phi_\beta\}$ where each V_β are contained in some U_α, where we define an atlas to be the collection of all coordinate patches on a manifold.

All manifolds in this work will be paracompact, connected, C^∞, Hausdorff manifolds unless otherwise stated. Now that we have developed the idea of a manifold, we proceed to define geometric quantities.

Definition 5. Let M be a manifold. A differentiable function $c : (-\epsilon, \epsilon) \subset \mathbb{R} \to M$ is called a **curve** in M. By reparametrization if necessary, set $c(0) = p \in M$ and let $C^1(M)$ be the space of differentiable functions on M. The tangent vector to the curve c at $t = 0$ is a function $c'(0) : C^1(M)|_p \to \mathbb{R}$ defined by

$$c'(0)f = \left.\frac{d(f \circ c)}{dt}\right|_{t=0} \qquad f \in C^1(M)|_p.$$

Definition 6. For a manifold M and $p \in M$, a **tangent vector** at p is the tangent vector at $t = 0$ of some curve $c : (-\epsilon, \epsilon) \to M$ with $c(0) = p$. The union of all tangent vectors to M at p defines the **tangent space** T_pM. The **tangent bundle** of M is defined by

$$TM = \bigcup_p T_pM \qquad p \in M.$$

Definition 7. A **one-form** ω is a linear functional on T_pM. The set of all one forms at p constitutes a vector space called the **cotangent space** which we denote T_p^*.

Definition 8. A **tensor** T of type (r, s) at $p \in M$ is a mapping

$$T : \bigotimes_r T_p \otimes \bigotimes_s T_p^* \to \mathbb{R}$$

which is linear in each argument.

Definition 9. A Riemannian **metric** is a positive definite symmetric $(0, 2)$ tensor.

Riemannian metrics are the fundamental calculational tool in differential geometry.

1.2 Differential Geometry of Surfaces

This thesis will mainly be concerned with differential geometry on 2-manifolds imbedded in $\mathbb{R}^3$ which we call **regular surfaces**.

Definition 10. If there is a map $\mathbf{x} : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ whose image is a regular surface S, we call $\mathbf{x}$ a **parametrization** of S. We will take $\mathbf{x} \in C^\infty$ unless otherwise stated.

Definition 11. Given a parametrization $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$, we define a map $N : U \to S^2$ by

$$N(u, v) = \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}$$

which we call the **Gauss Map** where subscripts indicate partial differentiation and $\wedge$ is the wedge product.

Definition 12. Given a parametrization $\mathbf{x}(u, v) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ of a surface M, let $S(p) : T_pM \to T_pM$ act on $v \in T_pM$ represented by $v = v^a \partial_a$ be given by $S(p)v = -v^a \partial_a N$. We call S the **shape operator** or **Weingarten map**.

Theorem 13. *S is a Hermitian operator on T_pS. Stated another way $\langle Sv, w \rangle = \langle v, Sw \rangle$, where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^3$ for $v, w \in T_pM$.*

Proof:

By definition of the Gauss map, $\langle N, \partial_a \mathbf{x} \rangle = 0$. Differentiating, we have

$$\langle \partial_a N, \partial_b \mathbf{x} \rangle + \langle N, \partial_a \partial_b \mathbf{x} \rangle = 0.$$

Thus by the product rule we have $\langle \partial_b N, \partial_a \mathbf{x} \rangle = \langle N_a, \mathbf{x}_b \rangle$ from which it follows

$$\langle Sv, w \rangle = -\langle \partial_b N v^b, \partial_b \mathbf{x}\, w^b \rangle = -\langle \partial_b N, \partial_a \mathbf{x} \rangle v^b w^a$$

$$= -\langle \partial_b \mathbf{x}, \partial_a N \rangle v^b w^a = -\langle \partial_a \mathbf{x} v^a, \partial_b N w^b \rangle = \langle v, Sw \rangle.$$

□

The most important tensors in differential geometry are defined in terms of the shape operator.

Definition 14. We define three symmetric bilinear two-forms on $v, w \in T_pS$ by

$$I(v,w) = \langle v, w\rangle \qquad II(v,w) = \langle Sv, w\rangle \qquad III(v,w) = \langle Sv, Sw\rangle$$

which are called the **first**, **second**, and **third** fundamental forms.

1.2.1 The Metric

We will refer to the first fundamental form as the **metric tensor** and note that it is defined identically as the preceding definition of metric. Moreover, it defines an inner product on the tangent space of a surface. We will denote its components with respect to the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ by g_{ab}, and occasionally adopt the convention $g_{11} = E, g_{12} = F$, and $g_{22} = G$. Explicitly, we may calculate metric components via the following:

Given an m-dimensional submanifold M of an n-dimensional manifold N with metric g_N and $\phi : M \to N$, the components of the **induced metric** g_M are given by

$$g_{M_{ab}}(x) = g_{N_{cd}}(\phi(x))\frac{\partial \phi^c}{\partial x^a}\frac{\partial \phi^d}{\partial x^b}.$$

Example 3. Consider the metric on S^2 in $(\mathbb{R}^3, \delta_{ab})$. If (θ, ϕ) are polar coordinates on S^2, then we may define its embedding by

$$\phi : (\theta, \phi) \to (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

from which we compute

$$ds^2 \equiv g_{ab}\, dx^a dx^b = \delta_{cd}\frac{\partial f^c}{\partial x^a}\frac{\partial f^d}{\partial x^b}dx^a dx^b = d\theta^2 + \sin^2\theta d\phi^2.$$

We call ds^2 the **line element** for the sphere. Note that this provides a general method for constructing line elements of manifolds embedded in Euclidean space.

We have the following corollary from the definition of the induced metric:

Corollary 15. *The induced metric tensor for a C^∞ manifold embedded into Euclidean space is given by $g = J^T J$ where J is the Jacobian of the embedding.*

We now construct the line element for the sphere using this corollary:

Example 4. S^2 is embedded into $\mathbb{R}^3$ by

$$(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

where $0 < \theta \leq \pi$ and $-\pi < \phi \leq \pi$ and has Jacobian

$$J = \begin{pmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & \cos\phi\sin\theta \\ -\sin\theta & 0 \end{pmatrix}.$$

Thus after simplification we have

$$g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix}$$

which when written as a line element gives the above result

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2.$$

1.2.2 Second Fundamental Form

To develop intuition for the second fundamental form, we require a few more definitions.

Definition 16. Let α be a curve in a surface S passing through $p \in S$, and let k be its curvature at p. Define an angle θ by $\cos\theta = \langle n, N\rangle$ where n is the unit normal to

α, N is the unit normal to S, and $\langle \cdot, \cdot \rangle$ is the Euclidean inner product. Then $k_n \equiv k \cos\theta$ is called the **normal curvature** of α at p.

Now let α be a parametrization for a curve in S parameterized by arc-length, with $\alpha(0) = p$. Let $N(s)$ be the Gauss map restricted to α. Then $\langle N(s), \alpha'(s) \rangle = 0$ and therefore $\langle N, \alpha'' \rangle = -\langle N', \alpha' \rangle$, from which we compute

$$II_p(\alpha'(0)) = -\langle dN_p(\alpha'(0)), \alpha'(0) \rangle = -\langle N'(0), \alpha'(0) \rangle$$
$$= \langle N(0), \alpha''(0) \rangle = \langle N, kn \rangle(p) = k_n(p).$$

So II_p acting on a unit vector $v \in T_pS$ gives the normal curvature of a curve passing through p that is tangent to v.

Definition 17. The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the **principal curvatures** at p.

Let ξ_1 and ξ_2 be elements of T_pM such that $k_1 = II(\xi_1, \xi_1)$ and $k_2 = II(\xi_2, \xi_2)$. Then by definition of the second fundamental form

$$S\xi_1 = k_1\xi_1 \qquad S\xi_2 = k_2\xi_2.$$

Thus the principle curvatures may be thought of as the eigenvalues of the shape operator with corresponding eigenvectors ξ_1, ξ_2.

Definition 18. The **mean curvature** H and the **Gaussian curvature** K are defined by

$$H = \frac{1}{2}(k_1 + k_2) \qquad K = k_1k_2.$$

Since k_1 and k_2 are the eigenvalues of S, they are given by the equation

$$0 = \det(S - \lambda I) = (\lambda - k_1)(\lambda - k_2) = \lambda^2 - 2H\lambda + K$$

which by the Cayley-Hamilton theorem gives the equation

$$S \cdot S - 2H \cdot S + K \cdot I = 0$$

where I is the identity matrix. We may reexpressed this equation as

$$K \cdot I - 2H \cdot II + III = 0$$

which relates the three fundamental forms.

Theorem 19. *Let S be a surface and $p \in S$. Then the mean curvature of S at p is given by*

$$H = \frac{1}{\pi} \int_0^{\pi} k_n.$$

Proof: Let e_1 and e_2 be the eigenvectors of the shape operator. Then since $\{e_1, e_2\}$ are an orthonormal basis for T_pS [7, p. 214-216], for a unit vector $v \in T_pS$ we write

$$v = e_1 \cos\theta + e_2 \sin\theta$$

Thus we have

$$k_n = II_p v = \langle Sv, v\rangle = \langle S(e_1 \cos\theta + e_2 \sin\theta), e_1 \cos\theta + e_2 \sin\theta\rangle$$

$$= \langle e_1 k_1 \cos\theta + e_2 k_2 \sin\theta, e_1 \cos\theta + e_2 \sin\theta\rangle = k_1 \cos^2\theta + k_2 \sin^2\theta.$$

Integrating we find

$$\frac{1}{\pi}\int_0^\pi k_n = \frac{1}{\pi}\int_0^\pi (k_1\cos^2\theta + k_2\sin^2\theta) = \frac{k_1+k_2}{2} = H.$$

□

Note that the above gives a geometrical interpretation of the mean curvature as the average of all normal curvatures at a point.

1.2.3 Coordinate Formulas

We will often use coordinate formulas in this work. Given a parameterization $\mathbf{x}$, the coefficients of the three fundamental forms are given by

$$g_{ab} = \langle \partial_a \mathbf{x}, \partial_b \mathbf{x}\rangle$$

$$b_{ab} = -\langle \partial_a N, \partial_b \mathbf{x}\rangle$$

$$c_{ab} = \langle \partial_a N, \partial_b N\rangle.$$

We now write the shape operator eigenvalue problem in the form

$$\langle Sv, w\rangle = k\langle v, w\rangle$$

which in component notation becomes

$$b_{ab}v^a w^b = k g_{ab} v^a w^b.$$

Since w is arbitrary, we find $b_{ab}v^a = k g_{ab}v^a$, which in tensor notation is

$$bv = kgv \quad \rightarrow \quad g^{-1}bv = kv.$$

Thus the principal curvatures are given by

$$0 = \det(b - kg) = (\det g)k^2 - (b_{11}g_{22} + b_{22}g_{11} - b_{12}g_{21} - b_{21}g_{12})k + \det b$$
$$= (\det g)(k - k_1)(k - k_2) = (\det g)[k^2 - (k_1 + k_2)k + k_1k_2]$$

from which we make the identifications

$$K = k_1k_2 = \frac{\det b}{\det g} = \det(g^{-1}b)$$
$$2H = k_1 + k_2 = \mathrm{tr}(g^{-1}b).$$

These are explicit formulae for the curvature scalars in terms of the first and second fundamental forms. We now provide an example of calculating the above quantities for Enneper's Surface

Example 5. Enneper's Surface is given by the parametrization

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

We compute

$$g_{11} = g_{22} = (1 + u^2 + v^2)^2 \qquad g_{12} = 0$$
$$b_{11} = -b_{22} = 2 \qquad b_{12} = 0.$$

Since $b_{12} = 0$, we see from the mean curvature formula that the principal curvatures are given by

$$k_1 = \frac{b_{11}}{g_{11}} = \frac{2}{(1 + u^2 + v^2)^2} \qquad k_2 = \frac{b_{22}}{g_{22}} = -k_1$$

Thus we find

$$K = -\frac{4}{(1 + u^2 + v^2)^4} \qquad H = 0.$$

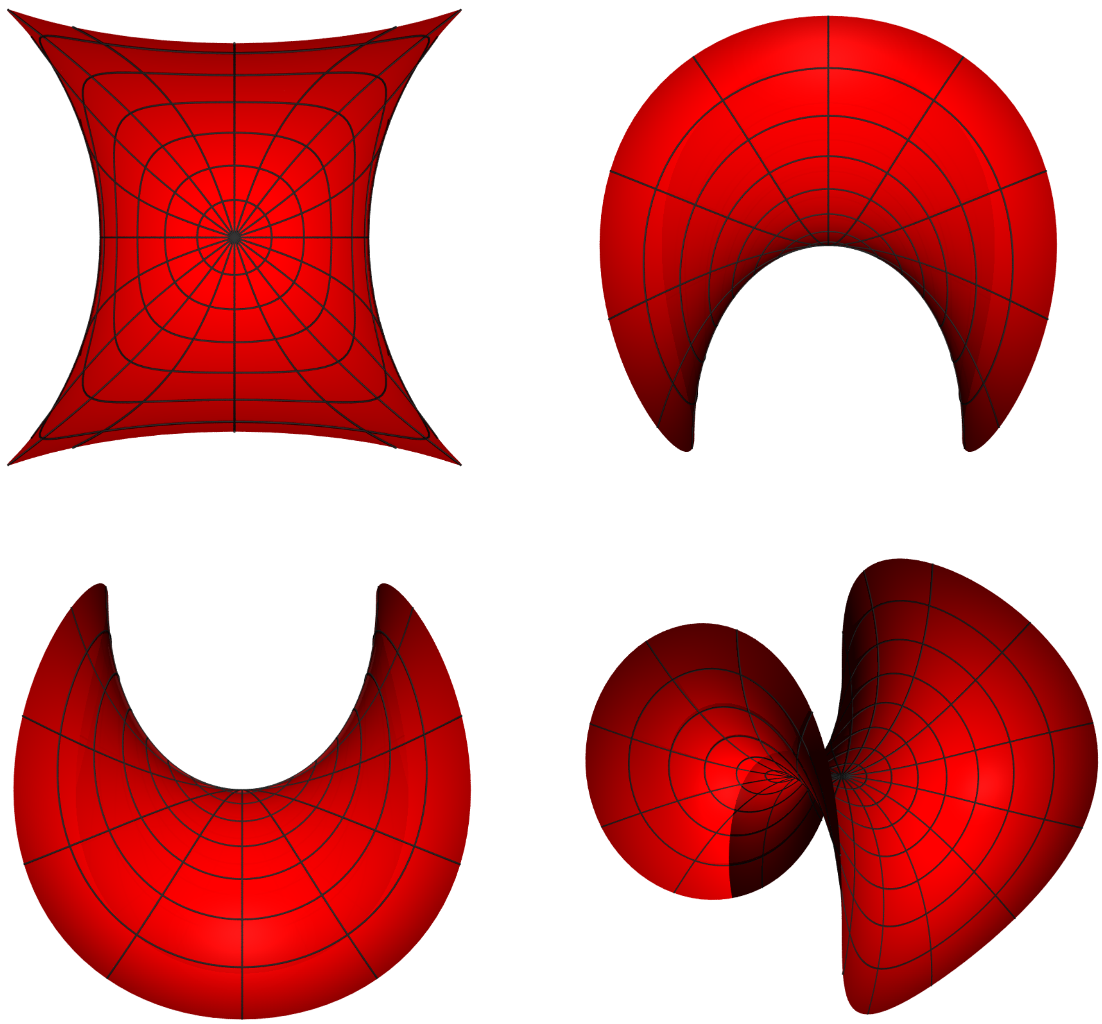

Figure 1.2 Four views of Enneper's Surface.

1.3 Minimal Surfaces

Definition 20. A **minimal surface** S is a two-manifold with $H = 0$ for every $p \in S$.

Suppose a surface S can be represented by a graph given by $\phi(x, y) : U \subset \mathbb{R}^2 \to \mathbb{R}^3$. Then the surface area of S is given by the functional

$$A[x, y, \phi, \phi_x, \phi_y] = \int_S = \int_U \sqrt{1 + \phi_x^2 + \phi_y^2}.$$

Demanding the first variation of A vanishes is equivalent to requiring the Lagrangian $L = \sqrt{1 + \phi_x^2 + \phi_y^2}$ to satisfy the Euler-Lagrange equation

$$0 = \frac{\partial L}{\partial \phi} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \phi_x} - \frac{\partial}{\partial y}\frac{\partial L}{\partial \phi_y}$$

.

Preforming the differentiation and simplifying we find

$$\phi_{xx}(1 + \phi_y^2) - 2\phi_x\phi_y\phi_{xy} + \phi_{yy}(1 + \phi_x^2) = 0$$

which is called the **minimal surface equation**. Note that this is a non-linear partial differential equation that is not easily solvable without restrictive ansatz. It is straightforward to show that if a surface is given by $z = z(x, y)$, then the metric and second fundamental form coefficients are given by

$$g_{11} = 1 + z_x^2 \qquad g_{12} = z_x z_y \qquad g_{22} = 1 + z_y^2$$

$$e = b_{11} = \frac{z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}} \qquad f = b_{12} = \frac{z_{xy}}{\sqrt{1 + z_x^2 + z_y^2}} \qquad g = b_{22} = \frac{z_{yy}}{\sqrt{1 + z_x^2 + z_y^2}}$$

.

Applying our coordinate formulae we find

$$H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)} = \frac{(1 + z_x^2)z_y y - 2z_{xy}z_x z_y + (1 + z_y^2)z_{xx}}{\sqrt{1 + z_x^2 + z_y^2}}$$

which reduces to the previous definition of minimal surface.

The above shows minimal surfaces are in a sense a generalization of geodesics. The geodesic equation is given by demanding the first variation of the arc-length functional for a curve vanishes, and the minimal surface equation is given by requiring the first variation of the area functional for a surface is zero.

We now show a relation between minimal surfaces and harmonic functions, but first require two lemmas.

Definition 21. A parameterized surface $\mathbf{x}(u, v)$ is **isothermal** if $g_{11} = g_{22}$ and $g_{12} = 0$.

Lemma 22. *Let* $\mathbf{x}(u, v)$ *be a parameterized isothermal surface. Then*

$$\Delta \mathbf{x} = \mathbf{x}_{uu} + \mathbf{x}_{vv} = 2\lambda^2 HN$$

.

Proof:

Since $\mathbf{x}$ is isothermal, $g_{11} = g_{22} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle = \langle \mathbf{x}_v, \mathbf{x}_v \rangle$, and $g_{12} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle = 0$. Differentiating with respect to u, we find

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle = \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle = -\langle \mathbf{x}_u, \mathbf{x}_{vv} \rangle$$

which gives the equation

$$\langle \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{x}_u \rangle = 0.$$

Similarly, differentiating with respect to v yields

$$\langle \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{x}_v \rangle = 0.$$

From these two equations, we conclude $\mathbf{x}$ is parallel to the Gauss map N. Since the parametrization is isothermal, the formula for mean curvature takes the form

$$H = \frac{1}{2}\frac{b_{11} + b_{22}}{E^2}.$$

Thus we find

$$\langle N, \mathbf{x}_{uu} + \mathbf{x}_{vv}\rangle = b_{11} + b_{22} = 2E^2H$$

which may be written

$$\Delta\mathbf{x} = \mathbf{x}_{uu} + \mathbf{x}_{vv} = 2E^2NH.$$

□

This next lemma is a corollary to a theorem of Gauss which was simplified by Chern that states every surface has an isothermal parametrization.

Lemma 23. *[18, Vol 4, p. 265-267] Isothermal coordinates can be introduced around any minimal surface.*

Proof:

Assume S is the graph of a function $\phi : U \to \mathbb{R}$ for $U \subset \mathbb{R}^2$. Thus S is the image of a map $f(x, y) = (x, y, \phi(x, y))$. Define the quantities

$$p \equiv \phi_x \qquad q \equiv \phi_y \qquad r \equiv \phi_{xx} \qquad s \equiv \phi_{xy} \qquad t \equiv \phi_{yy}.$$

Then the minimal surface equation becomes

$$(1 + q^2)r - 2pqs + (1 + p^2)t = 0$$

.

Define $W = \sqrt{1 + p^2 + q^2}$ and note by the minimal surface equation we have the identities

$$\frac{\partial}{\partial x}\left(\frac{1+q^2}{W}\right)-\frac{\partial}{\partial y}\left(\frac{pq}{W}\right)=-\frac{p}{W^3}\left[(1+q^2)r-2pqs+(1+p^2)t\right]=0$$

$$\frac{\partial}{\partial x}\left(\frac{pq}{W}\right)-\frac{\partial}{\partial y}\left(\frac{1+p^2}{W}\right)=0.$$

Thus we can locally define functions α, β by the equations

$$\frac{\partial\alpha}{\partial x}=\frac{1+p^2}{W}\qquad\frac{\partial\beta}{\partial x}=\frac{pq}{W}=\frac{\partial\alpha}{\partial y}\qquad\frac{\partial\beta}{\partial y}=\frac{1+q^2}{W}.$$

Consider a transformation given by

$$T(x,y)=(x+\alpha(x,y),y+\beta(x,y))$$

with Jacobian given by

$$J(T)=\frac{1}{W}\begin{pmatrix}W+(1+p^2) & pq\\ pq & W+(1+q^2)\end{pmatrix}$$

and determinant

$$|J(T)|=2+\frac{2+p^2+q^2}{W}\geq 2.$$

Thus T is locally invertible, with inverse

$$J^{-1}(T)=C\begin{pmatrix}1+W+q^2 & -pq\\ -pq & 1+W+p^2\end{pmatrix}$$

for some constant C. We thus compute the composition

$$J(f\circ T^{-1})=C\begin{pmatrix}1 & 0\\ 0 & 1\\ p & q\end{pmatrix}\begin{pmatrix}1+W+q^2 & -pq\\ -pq & 1+W+p^2\end{pmatrix}=C\begin{pmatrix}1+W+q^2 & -pq\\ -pq & 1+W+p^2\\ p+pW & q+qW\end{pmatrix}$$

from which we note the two column vectors are orthogonal with norm $(1+p^2+q^2)(2W+2+p^2+q^2)$, from which we conclude that $f \circ T^{-1}$ is conformal (preserves angles), and its inverse is an isothermal coordinate system.

□

Theorem 24. *If* $\mathbf{x}$ *is an isothermal parametrization for a surface* S*, then* S *is minimal iff* $\Delta \mathbf{x} = 0$.

($\Rightarrow$) If S is a minimal surface, then by Lemma 22 we have

$$\Delta \mathbf{x} = 2E^2 NH = 0$$

($\Leftarrow$) If $0 = \triangle \mathbf{x} = 2E^2 NH$. Then either E, N or H vanishes. If E or N are zero, then $\mathbf{x}$ does not parameterize a surface. Thus $H = 0$ and S is minimal.

□

The following example demonstrates the usefulness of this theorem.

Example 6. The Catenoid parameterized by

$$\mathbf{x} = (c\cosh(v)\cos u, c\cosh(v)\sin u, cv) \qquad u \in (0, 2\pi),\ \ v \in (-\infty, \infty)$$

is a minimal surface.

Proof:

The metric coefficients of the catenoid are

$$E = G = c^2\cosh^2 v \qquad F = 0$$

and $\Delta \mathbf{x} = 0$. Thus the catenoid is a minimal surface.

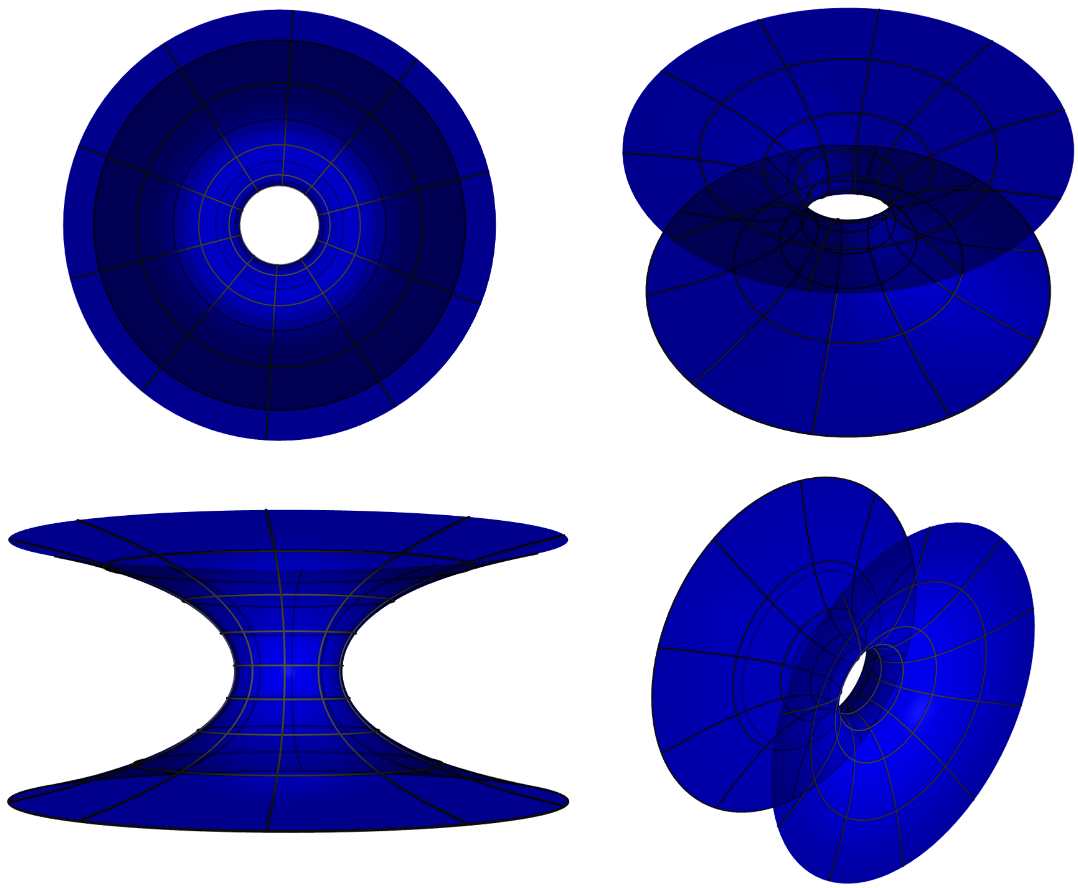

Figure 1.3 Four views of the Catenoid.

□

Definition 25. A **surface of revolution** is a surface with a parametrization of the form

$$\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, v) \qquad u \in (0, 2\pi) \qquad v \in (a, b).$$

Theorem 26. *The catenoid and the plane are the only minimal surfaces of revolution.*

Proof:

If $\mathbf{x}$ is a minimal surface, then it must have an isothermal parametrization. Computing the metric components of $\mathbf{x}$ we find $g_{11} = f^2$ and $g_{22} = (f')^2 + 1$. Demanding these are equal requires $f = 1$ or $f = c_1 \cosh(v) + c_2 \sinh(v)$. The first solution gives a plane. Computing the mean curvature with the form of the second solution substituted into $\mathbf{x}$, we find $c_1 = 1$ and $c_2 = 0$ are required for $\mathbf{x}$ to be minimal. The result gives the parametrization for the catenoid.

□

Definition 27. When two differentiable functions $f, g : U \subset \mathbb{R}^2 \to \mathbb{R}$ satisfy the equations

$$f_u = g_v \qquad f_v = -g_u$$

then f and g are called **harmonic conjugates**.

Definition 28. Let $\mathbf{x}$ and $\mathbf{y}$ be isothermal parameterizations of minimal surfaces with pairwise harmonic conjugate component functions. Then $\mathbf{x}$ and $\mathbf{y}$ are called **conjugate minimal surfaces**.

We now introduce another canonical minimal surface called the helicoid, given in isothermal coordinates by

$$\mathbf{x} = (\sinh u \cos v, \sinh u \sin v, v) \qquad u \in (0, 2\pi),\;\; v \in \mathbb{R}$$

which is graphed in Figure 1.4.

Theorem 29. *The helicoid and catenoid are conjugate surfaces*

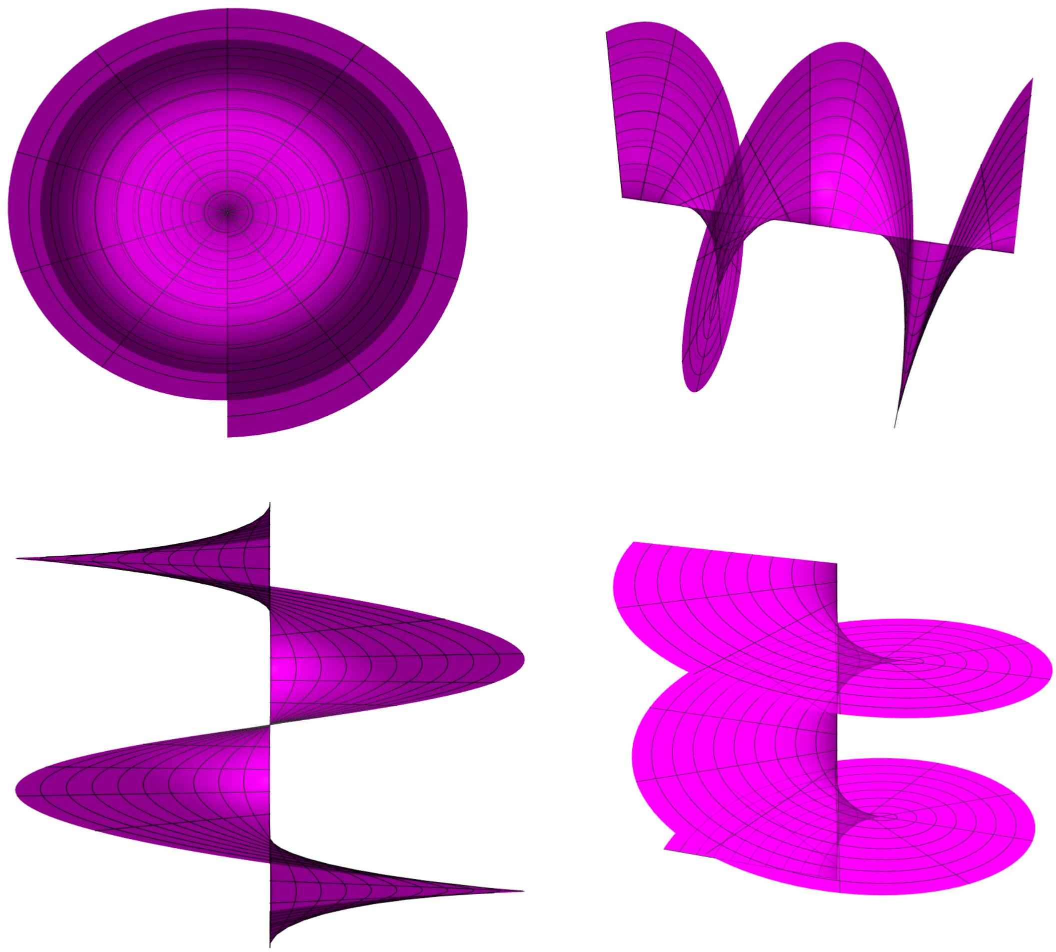

Figure 1.4 Four views of the Helicoid.

This proof is trivial by differentiating the isothermal parameterizations of the helicoid and catenoid. The extension of the following theorem to complex functions will be the main consideration of Chapter 3.

Theorem 30. *Given two conjugate minimal surfaces* $\mathbf{x}$ *and* $\mathbf{y}$*, the surface*

$$\mathbf{z}_t = (\cos t)\mathbf{x} + (\sin t)\mathbf{y} \qquad t \in [0, \pi/2]$$

is minimal for all t*, and* $\mathbf{x}$ *and* $\mathbf{y}$ *have identical induced metrics.*

This proof is an exercise in differentiation. We call the surfaces $\mathbf{z}_t$ the **associated**

surfaces of **x** and **y**. Figure 4.1 is a plot of some of the associated surfaces of the catenoid and helicoid.

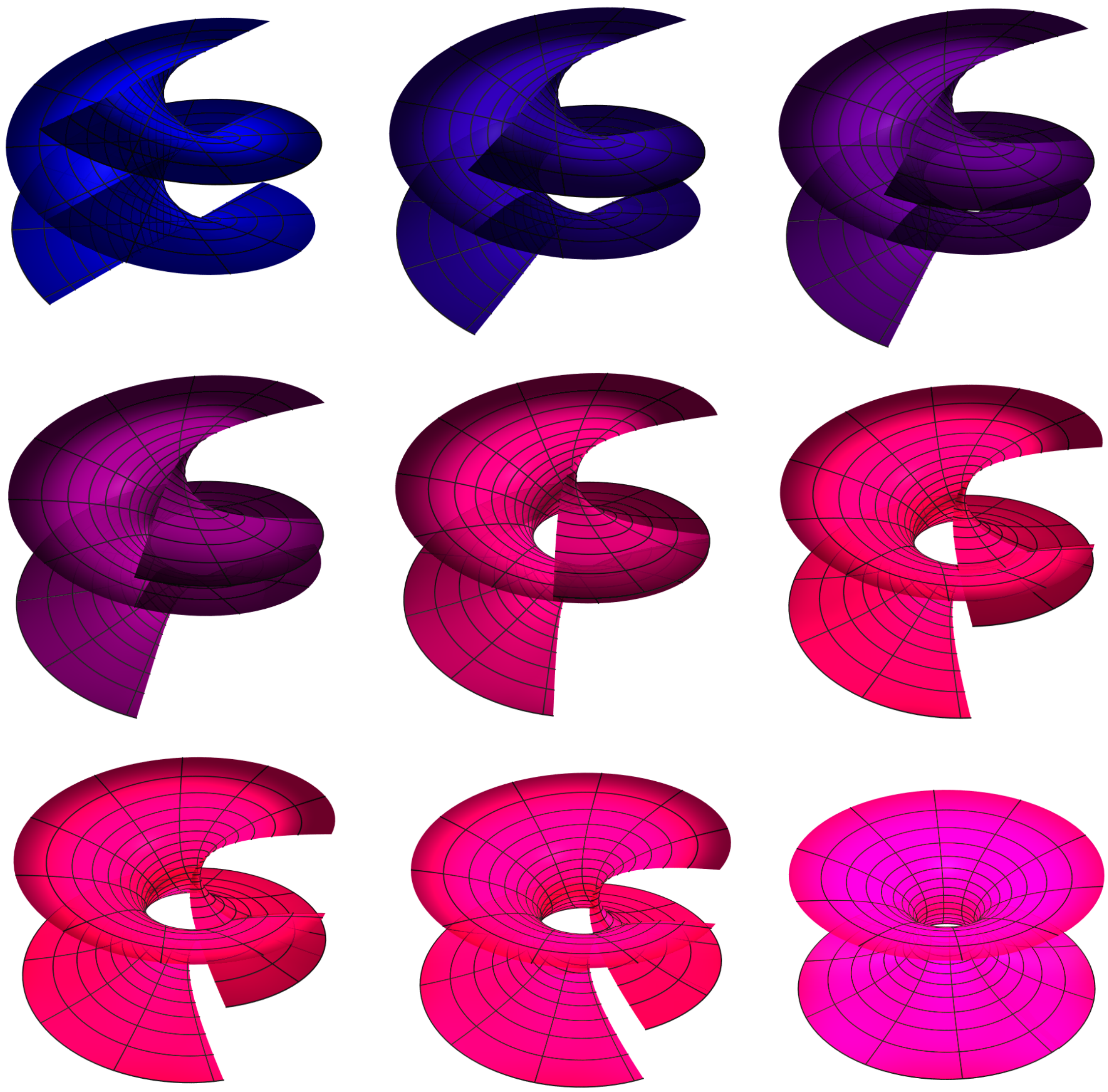

Figure 1.5 Helicoid to Catenoid Transformation.

Chapter 2

Complex Geometry

We present background material in complex analysis and harmonic function theory necessary to study complex parameterizations of minimal surfaces.

2.1 Complex Analysis

We define the set of complex numbers $\mathbb{C} = \{a + ib | a, b \in \mathbb{R}\}$. Addition and multiplication of $z, w \in \mathbb{C}$, where $z = z_1 + iz_2 = (z_1, z_2)$ and $w = (w_1, w_2)$ are given by $z + w = (z_1 + w_1, z_2 + w_2)$, and $z \cdot w = (z_1w_1 - z_2w_2, z_1w_2 + z_2w_1)$ which immediately yields $i^2 = -1$. We may represent $z \in \mathbb{C}$ in the form $z = re^{i\theta}$ where $0 \leq r < \infty$, and $0 \leq \theta < 2\pi$, which we will often use to simplify calculations. For $z \in \mathbb{C}$ written as $z = a + ib$, we define its real and imaginary parts to be $\text{Re}\{z\} = a$ and $\text{Im}\{z\} = b$, respectively.

Definition 31. The **modulus** function $|\cdot| : \mathbb{C} \to \mathbb{R}$ is defined for $z \in \mathbb{C}$ by

$$|z| = |z_1 + iz_2| = \sqrt{z_1^2 + z_2^2}.$$

Note this generalizes the absolute value function.

Definition 32. The function $f(x)$ is said to **limit** to L as $x \to a$ written:

$$\lim_{z \to a} f(z) = A$$

iff for every $\epsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - A| < \epsilon$ for all x such that $|x - a| < \delta$ and $x \neq a$.

Definition 33. A function $f : \mathbb{C} \to \mathbb{C}$ is **continuous** at $a \in \mathbb{C}$ iff

$$\lim_{z \to a} f(z) = f(a).$$

For $D \subset \mathbb{C}$, we say f is continuous on D if the above holds for all $a \in D$.

Definition 34. The derivative of $f : \mathbb{C} \to \mathbb{C}$ at $z \in \mathbb{C}$ is defined to be

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

for any $h \in \mathbb{C}$.

Definition 35. For $D \subset \mathbb{C}$ and $f : D \to \mathbb{C}$, we say that f is **holomorphic** in D if its derivative exists for all $a \in D$. We say f is **entire** if it is holomorphic in $\mathbb{C}$.

Definition 36. A function $f : D \to \mathbb{C}$ has a **pole** of order m at $z = a$ if m is the smallest positive integer such that $f(z)(z-a)^m$ has a removable singularity at $z = a$ (The limit is defined).

Definition 37. If G is open and f is a function defined and holomorphic in G except for poles, then f is a **meromorphic function** on G.

Theorem 38. *Any holomorphic function has a Taylor series expansion.*

The limit in the definition of the derivative must remain the same independent of the path in which h approaches zero on the complex plane. Thus for $f = u + iv$ and $h \to 0$ along the real axis we find

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

and for $h \to 0$ along the imaginary axis ($h = ik$) we have

$$f'(z) = \lim_{h \to 0} \frac{f(z + ik) - f(z)}{ik} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

from which we see that f must satisfy the partial differential equation

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}.$$

Equating real and imaginary parts we find the **Cauchy-Riemann** equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Note any holomorphic function must satisfy the Cauchy-Riemann equations, but the converse is not generally true.

Theorem 39. *f satisfies the Cauchy-Riemann equations and has continuous first order derivatives iff f is holomorphic.*

Definition 40. A real function which satisfies Laplace's equation $\Delta u = \nabla \cdot (\nabla u) = 0$ is said to be **harmonic**.

Definition 41. Let u and v be real valued functions. u and v are harmonic conjugates if they are harmonic and satisfy the Cauchy-Riemann equations.

Theorem 42. *If $u : D \to \mathbb{R}$ is a real valued harmonic function, then u has a harmonic conjugate.*

2.2 Harmonic Analysis

It is immediate from the Cauchy-Riemann equations that every holomorphic function is harmonic. However the converse of this statement is false as can be seen by the counterexample $f(z) = \overline{z}$.

Definition 43. A function $f : D \to \mathbb{C}$ is **univalent** (one-to-one), if for any distinct $z_1, z_2 \in D$, $f(z_1) \neq f(z_2)$.

Definition 44. A complex-valued harmonic function is a **harmonic mapping** of a domain $D \subset \mathbb{C}$ if it is univalent (injective) in D.

We will mainly be concerned with univalent harmonic mappings.

Theorem 45. *$f = u + iv$ is a harmonic function, iff there exist holomorphic g, h such that $f = h + \overline{g}$.*

Proof: ($\Rightarrow$) Since u and v are real harmonic by assumption, we know they have harmonic conjugates u_c, v_c. Thus we compute

$$f = u + iv = \frac{u + iu_c + \overline{u + iu_c}}{2} + i\frac{v_c + iv - \overline{v_c + iv}}{2i}$$
$$= \frac{(u + v_c) + i(u_c + v)}{2} + \frac{\overline{(u - v_c) + i(u_c - v)}}{2} \equiv h + \overline{g}.$$

($\Leftarrow$) By hypothesis we have $h = h_1 + ih_2$, $g = g_1 + ig_2$, and $f = h + \overline{g}$. Thus $\text{Re}(f) = h_1 + g_1$ and $\text{Im}(f) = h_2 - g_2$. Since h and g are holomorphic, then h_1, h_2, g_1, and g_2 are harmonic, which by linearity of the Laplacian operator, implies $h_1 + g_1$ and $h_1 - g_2$ are harmonic.

□

We will call such a form, for a harmonic function f, the canonical decomposition of f.

Definition 46. Let $f : D \to \mathbb{C}$. The **Jacobian** of $f = u + iv$ is defined by

$$J_f = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

Theorem 47. *[15] For a harmonic function $f : D \to \mathbb{C}$, $J_f \neq 0$ iff f is locally univalent at z.*

Definition 48. A locally univalent harmonic mapping is **sense-preserving** at $z \in \mathbb{C}$ if $J_f(z) > 0$ or **sense-reversing** if $J_f(z) < 0$.

If f is sense preserving then its conjugate $\overline{f}$ is sense-reversing and vice-versa.

2.3 Weierstrass-Enneper Representations

The following theorem found in [11, p. 168] provides the link between harmonic univalent functions and minimal surfaces:

Theorem 49. *(Weierstrass-Enneper Representation). Every regular minimal surface has locally an isothermal parametric representation of the form*

$$\left(\operatorname{Re}\left\{\int^z p(1+q^2)dw\right\}, \operatorname{Im}\left\{\int^z p(1-q^2)dw\right\}, 2\operatorname{Im}\left\{\int^z pq dw\right\}\right). \tag{2.1}$$

in some domain $D \subset \mathbb{C}$, where p is holomorphic and q is meromorphic in D, with p vanishing only at the poles (if any) of q and having a zero of precise order $2m$ wherever q has a pole of order m. Conversely, each such pair of functions p and q holomorphic and meromorphic, respectively, in a simply connected domain D generate through the formulas (2.1) an isothermal parametric representation of a regular minimal surface.

We will use (2.1) in the following form:

Corollary 50. *For a harmonic function $f = h + \overline{g}$, define the holomorphic functions h and g by $h = \int^z p d\zeta$ and $g = -\int^z pq^2 d\zeta$. Then the minimal surface representation* (2.1) *becomes*

$$\left(\mathrm{Re}\{h+g\}, \mathrm{Re}\{h-g\}, 2\,\mathrm{Im}\left\{\int_0^z \sqrt{h'g'}d\zeta\right\}\right). \tag{2.2}$$

In general converting between the Weierstrass-Enneper representation of a surface and its classical coordinate parametrization is a non-trivial task. We demonstrate a process in the following for the case of the catenoid.

Example 7. (Converting between coordinate and Weierstrass-Enneper representations of a minimal surface) Consider a coordinate parametrization of the catenoid

$$x = c\cosh u \cos v$$

$$y = -c\cosh u \sin v$$

$$z = cu.$$

The Weierstrass Enneper representation must be of the form

$$x = c + \mathrm{Re}\int_1^p (1-p^2)f(p)dp$$
$$y = \mathrm{Re}\int_1^p i(1+p^2)f(p)dp$$
$$z = \mathrm{Re}\int_1^p 2pf(p)dp.$$

Setting $w = u + iv$, make the change of variables $\omega = e^{-w}$ and define $r = |\omega|$, $\theta = \mathrm{arg}\omega$ $(\omega = re^{i\theta})$. Noting $\log\omega = \log r + i\theta = -u - iv$, the coordinate representation becomes

$$x = \frac{c}{2}\left(\frac{1}{r} + r\right)\cos\theta = \text{Re}\left\{\frac{\alpha}{2}\left(\frac{1}{\omega} + \omega\right)\right\}$$
$$y = \frac{c}{2}\left(\frac{1}{r}\right)\sin\theta = \text{Re}\left\{\frac{ic}{2}\left(\frac{1}{\omega} - \omega\right)\right\}$$
$$z = -c\log r = -\text{Re}\left\{c\log\omega\right\}.$$

Setting $f(p) = -c/2\omega^2$ gives the equivalent Weierstrass-Enneper representation.

We will often consider only the Weierstrass-Enneper representation of a minimal surface.

Example 8. Scherk's doubly periodic surface is defined by the Weierstrass-Enneper representation with $p = (1 - z^4)^{-1}$ and $q = iz$. Preforming the integration, we find it has the parametrization

$$\mathbf{x} = \text{Re}\left(\frac{i}{2}\log\left(\frac{z+i}{z-i}\right), -\frac{i}{2}\log\left(\frac{1+z}{1-z}\right), \frac{1}{2}\log\left(\frac{1+z^2}{1-z^2}\right)\right).$$

This surface is plotted in Figure (2.1) and can be translated about the plane like a checkerboard to form a new minimal surface (Figure 2.2).

Example 9. Its conjugate surface called Scherks singly periodic surface is given by taking p the same and $q = z$. The resulting surface is given by

$$\mathbf{x} = \text{Re}\left(\frac{1}{2}\log\left(\frac{z+i}{z-i}\right), -\frac{1}{2}\log\left(\frac{1+z}{1-z}\right), -\frac{i}{2}\log\left(\frac{1+z^2}{1-z^2}\right)\right).$$

This surface is plotted in Figure (2.3) and can be stacked upon itself to form the Scherk tower (Figure 2.4).

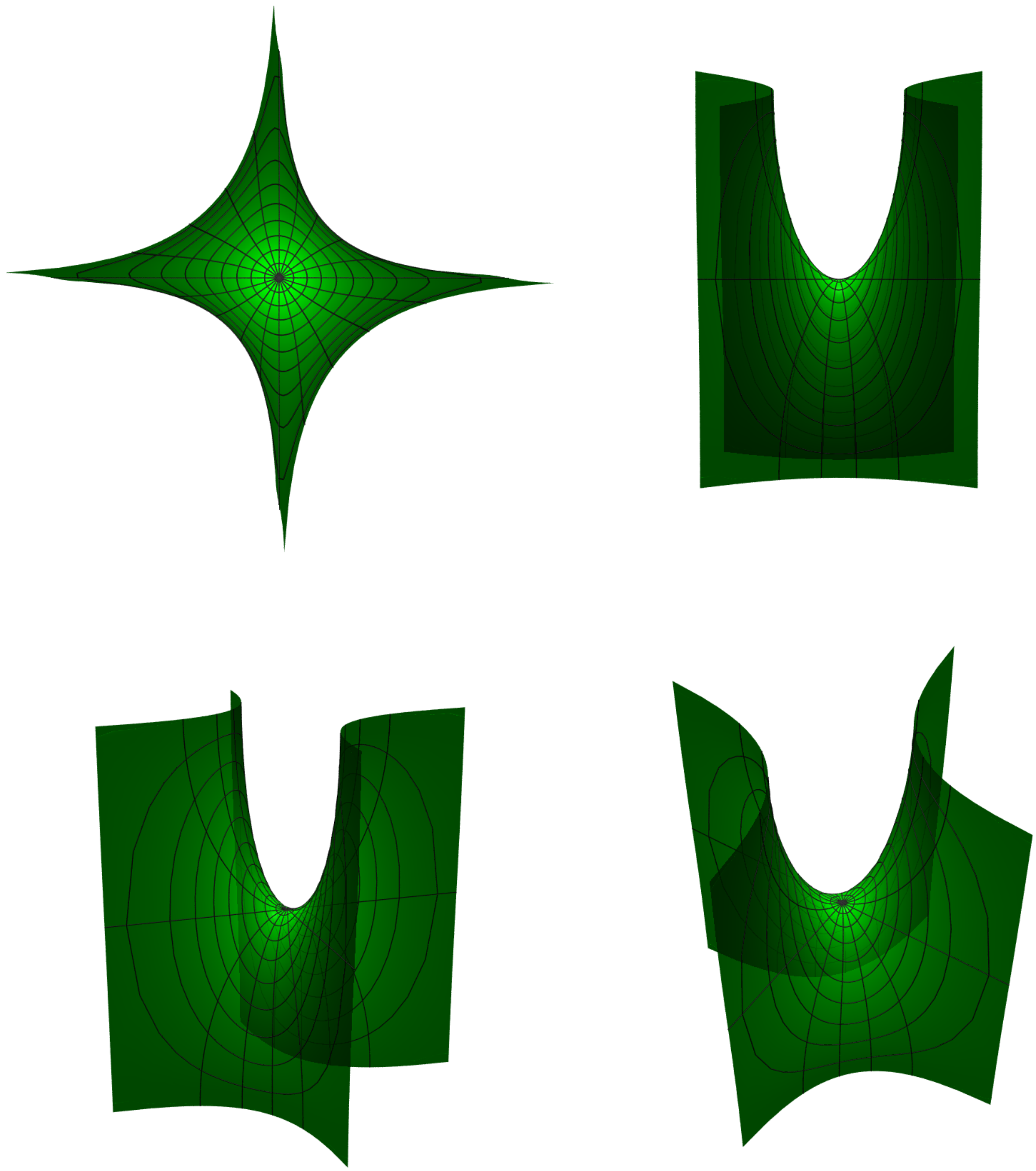

Figure 2.1 Scherk's Doubly Periodic Surface.

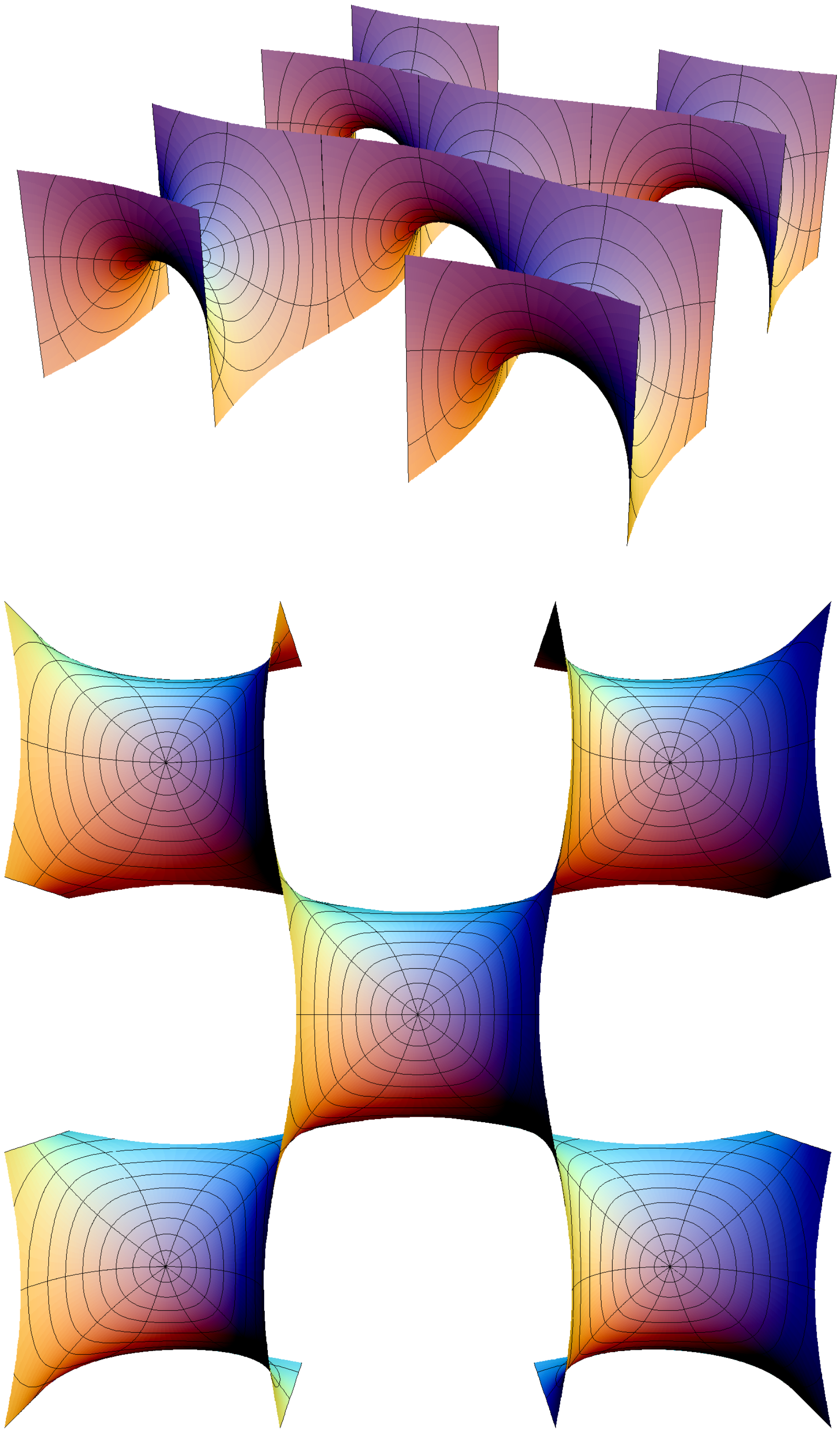

Figure 2.2 Double Scherk Checker Table.

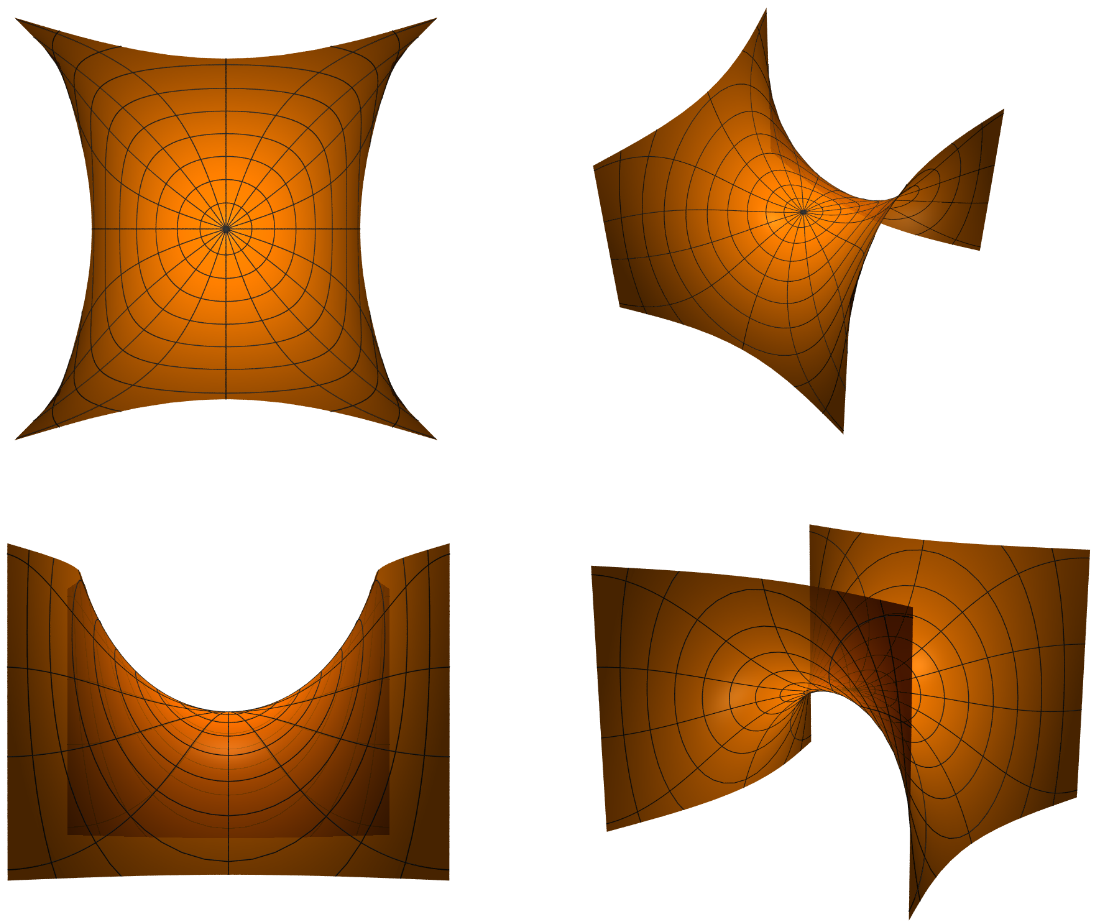

Figure 2.3 Scherk's Singly Periodic Surface.

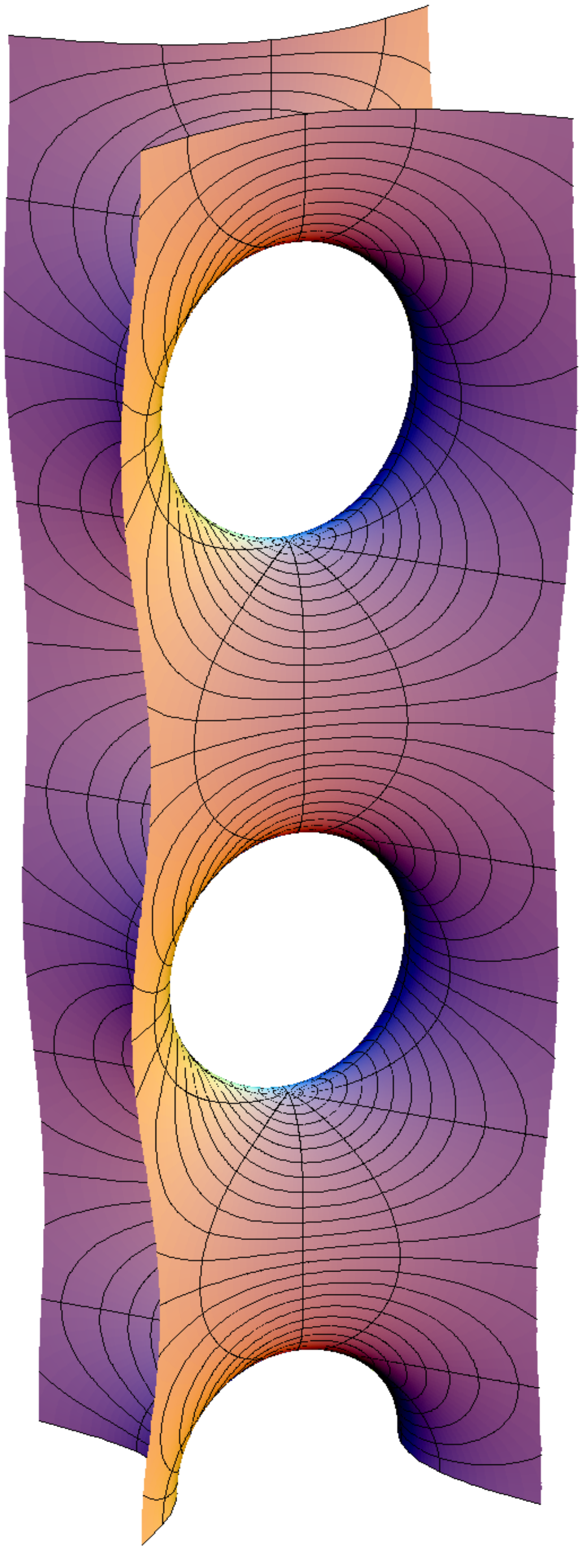

Figure 2.4 Singly Periodic Scherk Tower .

Chapter 3

Minimal Linear Combinations

Given two univalent harmonic mappings f_1 and f_2 on the unit disk $\mathbb{D}$, which lift to minimal surfaces via the Weierstrass-Enneper representation theorem, we give conditions for linear combinations of the form $f_3 = \alpha_1(t)f_1 + \alpha_2(t)f_2$ to lift to a minimal surface for $t \in [0, 1]$. We then take $\alpha_1 = t$ and $\alpha_2 = (1 - t)$ in three examples involving well-known minimal surfaces.

3.1 Harmonic Linear Combinations

The main consideration of this work is the study of harmonic mappings of the form $f_3 = tf_1 + (1 - t)f_2$, where $t \in [0, 1]$ and f_1, f_2 are both harmonic mappings. We will provide conditions for f_3 to lift to a minimal surface via (2.1), and demonstrate several examples which further the work of [9] and relate seemingly disconnected minimal surfaces. Let $f_1 = h_1 + \bar{g}_1$ and $f_2 = h_2 + \bar{g}_2$ be two univalent harmonic mappings on $\mathbb{D}$, which lift to minimal surfaces, with dilatations $\omega_1^2 = g_1'/h_1'$ and $\omega_2^2 = g_2'/h_2'$ respectively, where q_1, q_2 are holomorphic. Construct a third harmonic mapping

$$\begin{aligned} f_3 =& t f_1(z) + (1-t) f_2 \\ =& \left[t h_1(z) + (1-t) h_2(z) \right] + \left[t\overline{g_1(z)} + (1-t)\overline{g_2(z)} \right] \\ =& h_3 + \overline{g_3} \end{aligned}$$

and define its dilatation to be $\omega_3 = g_3'/h_3'$. We require the following for the subsequent lemma:

Lemma 51. *Let h_1, h_2 be holomorphic mappings on $\mathbb{D}$ and $t \in [0,1]$ The following are sufficient conditions for the real valued function $h = |th_1 + (1-t)h_2|$ to be positive:*
i) $\mathrm{Re}\{h_1\} > 0$ and $\mathrm{Re}\{h_2\} > 0$.
ii) Let $z \in \mathbb{D}$ and consider the complex numbers $w_1 = h_1(z)$, $w_2 = h_2(z)$ on complex projective two space $\mathbb{P}(\mathbb{C}, 2)$. If $w_1 \neq w_2$ for all $z \in \mathbb{D}$, then the condition is satisfied.
iii) $h_1(z)/|h_1(z)| - h_2(z)/|h_2(z)| \neq 0$ for all $z \in \mathbb{D}$.

The proof of the lemma is straightforward. We note *iii*) is also necessary, and now investigate the univalence of the linear combination.

Lemma 52. *Assuming one condition in Lemma 51 holds, if $\omega_1 = \omega_2$, then ω_3 is a perfect square of a holomorphic function and hence f_3 is locally univalent.*

Proof: Suppose that $\omega_1 = \omega_2$. Then we have

$$\omega_3 = \frac{t h_1' \omega_1 + (1-t) h_2' \omega_1}{t h_1' + (1-t) h_2'} = \omega_1,$$

which shows ω_3 is a perfect square of a holomorphic function. Since f_1 is univalent and a condition in Lemma 51 holds we find $|\omega_3| = |\omega_1^2| > 0$ which by Lewy's theorem [15] implies f_3 is locally univalent.

We now seek to study conditions under which f_3 is globally univalent and thus lifts to a minimal surface. To do this, we need a few definitions and theorems.

Definition 53. A domain $D \subset \mathbb{C}$ is said to be convex in the $e^{i\beta}$ direction if for all $a \in \mathbb{C}$ the set

$$D \cap \{a + te^{i\beta} : t \in \mathbb{R}\}$$

is either connected or empty. Specifically, a domain is convex in the direction of the imaginary axis if all lines parallel to the imaginary axis have a connected intersection with the domain.

Theorem 54 ([14], [17])**.** *Given a harmonic function $f = h + \overline{g}$, let $\phi = h - g$. ϕ is convex in the $e^{i\beta}$ direction if*

$$\mathrm{Re}\{\phi'(1 + ze^{i(\alpha+\beta)})(1 + ze^{-i(\alpha-\beta)})\} > 0$$

for some $\alpha \in \mathbb{R}$ and for all $z \in \mathbb{D}$

The following theorem will allow us to prove global univalence of a class of harmonic mappings.

Theorem 55 (Clunie and Sheil-Small, [4])**.** *A harmonic function $f = h + \overline{g}$ locally univalent in U is a univalent mapping of U onto a domain convex in the $e^{i\beta}$ direction if and only if $\phi = h - e^{i2\beta}g$ is a conformal univalent mapping of U onto a domain convex in the $e^{i\beta}$ direction.*

The following theorem allows us to determine if a function maps onto a domain convex in the direction of the imaginary axis:

Theorem 56 (Hengartner and Schober, [13])**.** *Suppose f is holomorphic and non-constant in $\mathbb{D}$. Then*

$$\mathrm{Re}\{(1 - z^2)f'(z)\} \geq 0, z \in \mathbb{D}$$

if and only if f is univalent in $\mathbb{D}$, f is convex in the imaginary direction, and there exists points z_n', z_n'' converging to $z = 1$, $z = -1$, respectively, such that

$$\lim_{n\to\infty} \mathrm{Re}\{f(z_n')\} = \sup_{|z|<1} \mathrm{Re}\{f(z)\}$$
$$\lim_{n\to\infty} \mathrm{Re}\{f(z_n'')\} = \inf_{|z|<1} \mathrm{Re}\{f(z)\}. \tag{3.1}$$

Note that the the normalization in (3.1) can be thought of in some sense as if $f(1)$ and $f(-1)$ are the right and left extremes in the image domain in the extended complex plane. Using the above results, we derive the following two theorems.

Theorem 57. *Let $f_1 = h_1 + \overline{g_1}$, $f_2 = h_2 + \overline{g_2}$ be harmonic mappings convex in the imaginary direction. Suppose $\omega_1 = \omega_2$ and $\phi_i = h_i - g_i$ is univalent, convex in the imaginary direction, and satisfies the normalization given in (3.1) for $i = 1, 2$. Then $f_3 = tf_1 + (1-t)f_2$ is convex in the imaginary direction* $(0 \le t \le 1)$.

Proof: We want to show that $\phi_3 = t\phi_1 + (1-t)\phi_2$ is convex in the imaginary direction. Then by Theorem 55, f_3 is convex in the imaginary direction. By the hypotheses, Theorem 56 applies to ϕ_1, ϕ_2. That is,

$$\mathrm{Re}\{(1-z^2)\phi_i'(z)\} \ge 0, \forall i = 1, 2.$$

Consider

$$\begin{aligned}\mathrm{Re}\{(1-z^2)\phi_3'(z)\} =& \mathrm{Re}\{(1-z^2)(t\phi_1'(z) + (1-t)\phi_2'(z))\} \\ =& t\,\mathrm{Re}\{(1-z^2)\phi_1'(z)\} + (1-t)\mathrm{Re}\{(1-z^2)\phi_2'(z)\} \ge 0.\end{aligned}$$

Hence, by applying Theorem 56 again, ϕ_3 is convex in the imaginary direction.

We need not only restrict to surfaces convex in the imaginary direction. The following gives a condition for a function to be convex in an arbitrary direction:

Theorem 58. *For a harmonic function $f = h + \overline{g}$, define $h - g = \phi = \phi_R + i\phi_I$. Then ϕ is convex in the $e^{i\beta}$ direction if*

$$[\cos\alpha + \cos(\beta+\gamma)]\,[\phi_R' \cos(\beta+\gamma) - \phi_I' \sin(\beta+\gamma)] > 0 \tag{3.2}$$

for some $\alpha \in \mathbb{R}$ and for all $z = re^{i\gamma} \in \mathbb{D}$.

Proof: This theorem follows by applying Theorem 54 to ϕ to get

$$\begin{aligned}
&\mathrm{Re}\{(\phi_R' + i\phi_I')(1 + re^{i(\alpha+\beta+\gamma)})(1 + re^{i(\gamma-\alpha+\beta)})\}\\
&\qquad = \phi_R' + 2r\cos\alpha(\phi_R'\cos\theta - \phi_I'\sin\theta) + r^2(\phi_R'\cos 2\theta - \phi_I'\sin 2\theta)\\
&\qquad = 2(\cos\alpha + \cos(\beta+\gamma))(\phi_R'\cos(\beta+\gamma) - \phi_I'\sin(\beta+\gamma)) > 0,
\end{aligned}$$

where $\theta = \beta + \gamma$.

3.2 Examples

We now proceed to give interesting examples resulting from Theorems 57 and 58.

Example 10 (Enneper's to Scherk's singly-periodic)**.**

Consider the harmonic maps

$$\begin{aligned}
f_E =& z + \frac{1}{3}\overline{z}^3\\
f_S =& \left[\frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) + \frac{i}{4}\ln\left(\frac{i-z}{i+z}\right)\right]\\
&\qquad + \overline{\left[\frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{i-z}{i+z}\right)\right]}.
\end{aligned}$$

It is straight forward to show that their dilatations are $\omega = z^2$ and both harmonic maps satisfy the hypotheses of Theorem 57. Hence

$$f_t = (1-t)f_E + tf_S$$

is globally univalent on $z \in \mathbb{D}$ and for every $t \in [0,1]$. By Corollary (2.1), f_t lifts to a family of minimal surfaces. Note that f_0 lifts to Enneper's surface parametrized by:

$$X_0 = \left(\mathrm{Re}\left\{z + \frac{1}{3}z^3\right\}, \mathrm{Im}\left\{z - \frac{1}{3}z^3\right\}, \mathrm{Im}\left\{z^2\right\}\right)$$

and f_1 lifts to Scherk's singly-periodic surface parametrized by

$$X_1 = \left(\mathrm{Re}\left\{\frac{1}{2}\ln\left(\frac{1+z}{1-z}\right)\right\}, \mathrm{Im}\left\{\frac{i}{2}\ln\left(\frac{i-z}{i+z}\right)\right\}, \mathrm{Im}\left\{\frac{1}{2}\ln\left(\frac{1+z^2}{1-z^2}\right)\right\}\right).$$

So for $t \in [0,1]$ we get a continuous family of minimal surfaces transforming from Enneper's to Scherk's singly-periodic surface. In Figure 3.1, we have shown six equal increments in this transformation.

Example 11 (Scherk's doubly-periodic to catenoid)**.**

Consider the harmonic maps $f_D = h_D + \overline{g}_D$, where

$$\begin{aligned} h_D(z) =& \frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{1+iz}{1-iz}\right) \\ g_D(z) =& -\frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{1+iz}{1-iz}\right), \end{aligned}$$

and $f_C = h_C + \overline{g}_C$, where

$$\begin{aligned} h_C(z) &= \frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) + \frac{1}{2}\frac{z}{1-z^2} \\ g_C(z) &= \frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{1}{2}\frac{z}{1-z^2}. \end{aligned}$$

and define

$$f_t = (1-t)f_D + tf_C.$$

While Theorem 57 does imply f_t is globally univalent on $z \in \mathbb{D}$ and $\forall t \in [0,1]$, we will instead prove this by using Theorem 58.

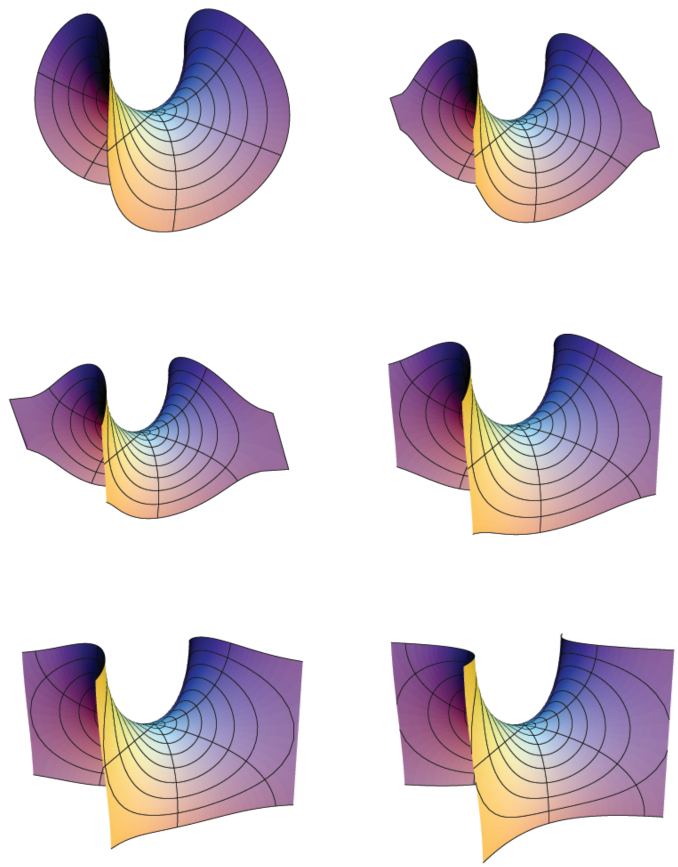

Figure 3.1 Enneper's to Scherk's singly-periodic transformation for $t = i/5$ for $i = 0, \ldots, 5$.

Note that the ϕ' associated with f is given by

$$\phi' = \frac{1-(1-2t)z^2}{(z^2-1)^2}.$$

Letting $z \to e^{i\gamma} \in \partial\mathbb{D}$, we compute

$$\mathrm{Re}\{\phi'\} = \frac{1}{2}(1 - t\csc^2\gamma), \qquad \mathrm{Im}\{\phi'\} = \frac{1}{2}\cot\gamma.$$

Then

$$\begin{aligned} S &= (\cos\alpha + \cos(\beta+\gamma))\,(\phi'_R\cos(\beta+\gamma) - \phi'_I\sin(\beta+\gamma)) \\ &= \frac{(1-t)}{2}\csc(\gamma)\,(\sin(\gamma) - \cos\alpha) \end{aligned}$$

Choosing $\beta = \pi/2$ and $\alpha = \pi/2$ gives $S > 0$. Thus, we conclude ϕ is convex in the imaginary direction, and by Theorem 55 and Corollary (2.1) we conclude that f lifts to a minimal surface for all t. Note that f_0 lifts to Scherk's doubly-periodic surface and f_1 lifts to a catenoid. So for $t \in [0, 1]$ we get a continuous family of minimal surfaces transforming from Scherk's doubly-periodic surface to a catenoid. In Figure 3.2, we have shown four equal increments in this transformation.

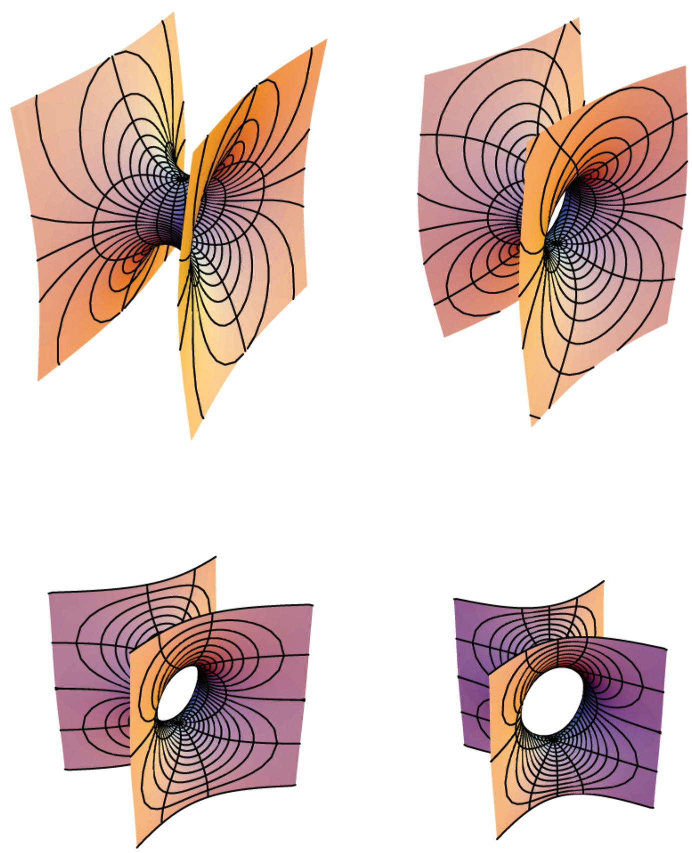

Figure 3.2 Catenoid to Scherk's doubly-periodic transformation for $t = i/3$ for $i = 0, \ldots, 3$.

3.3 Linear combinations that are not convex in one direction

In the previous section, we considered functions that are convex in the same direction and proved two theorems that show necessary conditions for the linear combinations of these functions to also be convex in that direction. Next, we consider the case in which the functions are not necessarily convex in some direction and look at linear combinations of such functions. While we do not prove a general theorem in this case. Instead we present a technique that can be used to prove that the linear combination of two specific functions is univalent and hence lifts to non-intersecting minimal surfaces.

Example 12 (The 4-noid to 4-Enneper)**.** The harmonic function that lifts to the 4-noid surface can be written as $f_{4N} = h_{4N} + \overline{g}_{4N}$, where

$$\begin{aligned} h_{4N}(z) &= \frac{1}{4}\frac{z}{1-z^4} + \frac{3}{16}\ln\left(\frac{1+z}{1-z}\right) - \frac{3i}{16}\ln\left(\frac{1+iz}{1-iz}\right) \\ g_{4N}(z) &= -\frac{1}{4}\frac{z^3}{1-z^4} + \frac{3}{16}\ln\left(\frac{1+z}{1-z}\right) + \frac{3i}{16}\ln\left(\frac{1+iz}{1-iz}\right), \end{aligned}$$

and the harmonic function that lifts to the 4-Enneper's surface can be written as $f_{4E} = h_{4E} + \overline{g}_{4E}$, where

$$\begin{aligned} h_{4E}(z) &= z \\ g_{4E}(z) &= -\frac{1}{7}z^7. \end{aligned}$$

Each of these functions is univalent on the ball $B(0, 0.95)$, the disk centered at the origin of radius 0.95, but not on the full unit disk. Neither of these surfaces is convex in any direction, and so the theorems from the previous section do not apply. However, we will show that the linear combination

$$f_t = (1-t)f_{4N} + tf_{4E} \tag{3.3}$$

is univalent for all $t \in [0, 1]$. To do so, we first need the following theorem.

Theorem 59 (argument principle, [11])**.** *Let f be a harmonic function in a Jordan domain D with boundary C. Suppose f is continuous in $\overline{D}$ and $f(z) \neq 0$ on C. Suppose f has no singular zeros in D and let N by the sum of the orders of the zeros of f in D. Then $\Delta_C arg f(z) = 2\pi N$.*

Using the argument principle and an approach presented in [8], we will prove that the specific linear combination given in (3.3) is univalent.

Lemma 60. *Let t be fixed such that $0 \leq t < 1$. Then f_t is univalent in $B = B(0, 0.95)$.*

Proof: Let $R = 0.95$ and define a closed contour Γ to be the union of $\Gamma_1 = \{r : 0 \leq r \leq R\}$, $\Gamma_2 = \{re^{i\pi/4} : 0 < r < R\}$, and $\Gamma_3 = \{Re^{i\pi\theta/4} : 0 \leq \theta \leq 1$. Let $\Omega = \mathrm{Int}(\Gamma)$. We will prove this claim in three steps. First, we will show that f is univalent in Ω and that $0 \leq Arg(f(\Omega)) \leq \frac{\pi}{4}$. Second, we verify that f is univalent in the sector $\Omega \cup \Omega'$, where Ω' is the reflection of Ω across the real axis, and $\frac{-\pi}{4} \leq Arg(f(\Omega \cup \Omega')) \leq \frac{\pi}{4}$. Finally, we will verify that f is univalent in B.

Step One: The argument principle for harmonic functions is valid if f is continuous on $\overline{D}$, $f(z) \neq 0$ on ∂D, and f has no singular zeros in D, where D is a Jordan domain. Note z_0 is a singular point if f is neither sense-preserving nor sense-reversing at z_0. We will show that for arbitrary $M > 0$, we may choose $r_0 < R$ so that each value in the region bounded by $|w| < M$ and $0 < Arg(w) < \frac{\pi}{4}$ is assumed exactly once in the sector bounded by $|z| < R$ and $0 < Arg(z) < \frac{\pi}{4}$, while no value in the region bounded by $|w| < M$ and $\frac{\pi}{4} < Arg(w) < 2\pi$ is assumed in this sector.

Observe that $f_1'(\Gamma_1) > 0$ Thus, on Γ_1, f_1 is an increasing function of r with trivial argument. Also, a direct calculations shows $\mathrm{Re}\{f_t(\Gamma_2)\} = \mathrm{Im}\{f_t(\Gamma_2)\}$, from which

we conclude $\text{Arg}(f_2(\Gamma_2)) = \pi/4$. We also note $f_t(\Gamma_3)$ differentiable curve which does not intersect itself. This can be seen from the fact that $\text{Re}\{f_t(\Gamma_3)\}$, $\text{Im}\{f_t(\Gamma_3)\}$, and $\text{Arg}\{f_t(\Gamma_3)\}$ are monotonic for $t \in [0, .75]$, $t \in [.8, 1]$, and $t \in [.75, .8]$ respectively. To show the argument bound on elements of Ω, it suffices to bound $\text{Arg}(f_t(\Gamma_3))$. A straight forward computation shows $\text{Arg}(f_t(\Gamma_3))|_{\theta=0} = 0$, and $\text{Arg}(f_t(\Gamma_3))|_{\theta=1} = \pi/4$. To show that are the minimum and maximum values the argument assumes, we need only consider $f_0(\Gamma_3)$ and $f_1(\Gamma_3)$ since $f_0(\Gamma_3) \leq f_t(\Gamma_3) \leq f_1(\Gamma_3)$ pointwise. For $t = 1$, $f_1(\Gamma_3) = e^{-7i\pi\theta/4}[Re^{2i\pi\theta} - a]$ where $a = 0.0997625$. Thus we have
$0 < \text{Im}\{f_1(\Gamma_3)\} \leq \text{Re}\{f_1(\Gamma_3)\}$, which implies the argument bound. The $t = 0$ case is analogous but yields a slightly more complicated calculation.

Step Two: Since f_t is univalent in Ω, we can use reflection across the real axis to establish that f_t is univalent in the sector Ω'. In particular, suppose $z_1, z_2 \in \Omega'$ with $f_t(z_1) = f_t(z_2)$. Then by symmetry $\overline{f_t(\overline{z_1})} = f_t(z_1) = f_t(z_2) = \overline{f_t(\overline{z_2})}$. Hence,
$f_t(\overline{z_1}) = f_t(\overline{z_2})$, or $\overline{z_1} = \overline{z_2}$. Arguing in the same manner as in Step One, we can show that $0 \geq Arg(f_t(\Omega')) \geq \frac{-\pi}{4}$. Therefore, f_t is univalent in $\Omega \cup \Omega'$ and its image is in the wedge between the angles $\frac{-\pi}{4}$ and $\frac{\pi}{4}$.

Step Three: First, it is true that $e^{i\pi j/2} f_t(ze^{-i\pi j/2}) = f_t(z)$, for all $z \in \mathbb{D}$ where
$j = 0, 1, ..., 4$. To see this note that
Now, using this fact that $e^{i\pi j/2} f_t(ze^{-i\pi j/2}) = f_t(z)$, we see that if z is any point in $\mathbb{D}$, it can be rotated so that it is in the sector Ω', in which f is univalent, and then rotated back by multiplying by the constant $e^{i\pi j/2}$ and hence preserving univalency.

In Figure 3.3 we have plotted six images of this transformation.

3.3.1 Further Avenues of Research

In the previous example, we showed that the linear combination of two specific harmonic mappings is univalent even though neither of the two harmonic mappings is convex in any direction. Are there some general conditions that will guarantee univalence of the linear combination of harmonic mappings that are not convex in any direction?

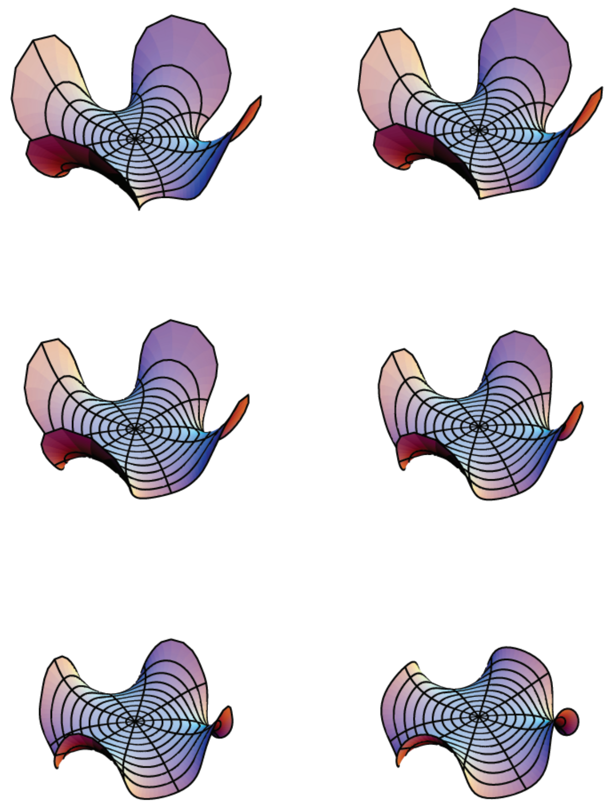

Figure 3.3 4-noid to 4-Enneper

Chapter 4

Isometric Families of Minimal Surfaces

We consider a surface M immersed in $\mathbb{R}^3$ with induced conformal metric $g = \psi\delta_2$ where δ_2 is the two dimensional Euclidean metric. We then construct a system of partial differential equations that constrain M to lift to a minimal surface via the Weierstrauss-Enneper representation demanding the metric is of the above form. It is concluded that the associated surfaces connecting the prescribed minimal surface and its conjugate surface satisfy the system. Moreover, we find a non-trivial symmetry of the PDE which generates a one parameter family of surfaces isometric to a specified minimal surface. We demonstrate an instance of the analysis for the half catenoid.

4.1 The Isometric Condition

Let $\mathbf{x}(u, v)$ be a parametrization for a surface M immersed in $\mathbb{R}^3$. Set $z = u + iv$ and define $\phi = \partial\mathbf{x}/\partial z$. Let E, F, and G be the coefficients of the metric induced in

$\mathbb{R}^3$ by $\mathbf{x}(u,v)$. We then have the relations

$$\phi^2 = \frac{1}{4}(E - G - 2iF) \tag{4.1}$$

$$\overline{\phi}^2 = \frac{1}{4}(E - G + 2iF) \tag{4.2}$$

$$|\phi|^2 = \frac{1}{4}(E + G) \tag{4.3}$$

where ϕ^2 is notation for $\phi \cdot \phi$. Inverting this system we find

$$E = \overline{\phi}^2 + \phi^2 + 2|\phi|^2$$

$$F = i(\phi^2 - \overline{\phi}^2)$$

$$G = -\overline{\phi}^2 - \phi^2 + 2|\phi|^2.$$

Since the Weierstrauss-Enneper representation theorem requires that M has an isothermal parametrization, we require $E = G$ and $F = 0$, which implies

$$\phi^2 = 0 \qquad \overline{\phi}^2 = 0 \qquad E = 2|\phi|^2.$$

The first two equations are identically satisfied. Expanding the constraint on E and using the identity

$$|\phi|^2 = \frac{1}{4}|p|^2\left((1+q^2)(1+\overline{q}^2) + (1-q^2)(1-\overline{q}^2) + 4q\overline{q}\right),$$

we find $E = |h'|^2 + |g'|^2$. Defining $\mathrm{Re}\{h\} = {}_1h$, $\mathrm{Im}\{h\} = {}_2h$, $\mathrm{Re}\{g\} = {}_1g$, and $\mathrm{Im}\{g\} = {}_2g$, we have the Cauchy Riemann and isometric conditions

$${}_1h_u - {}_2h_v = 0 \qquad {}_1h_v + {}_2h_u = 0 \tag{4.4}$$

$${}_1g_u - {}_2g_v = 0 \qquad {}_1g_v + {}_2g_u = 0 \tag{4.5}$$

$$ {}_1h_u^2 + {}_2h_u^2 + {}_1g_u^2 + {}_2g_u^2 - 2\psi = 0. \tag{4.6}$$

We now proceed to calculate the symmetry group of (4.4)-(4.6). For an introduction to symmetry methods see [3], [16], and [19].

4.2 Summary of Symmetry Analysis Techniques

Preforming a symmetry analysis on a system of nonlinear equations is widely regarded as the best way to find exact solutions of the system. We will now summarize the Lie method in [16] to outline our methods in the subsequent section. Consider a system of partial differential equations given by

$$\Delta_\nu(x, u^{(n)}) = 0 \quad \nu = 1, \ldots, l$$

with $x = (x^1, \ldots, x^p)$ the set of independent variables and $u = (u^1, \ldots, u^q)$ the set of dependent variables where $1 \ldots q$ run over the set of all partial derivatives of u up to order n. For $u = f(x)$, with $f : \mathbb{R}^p \to \mathbb{R}^q$ with components f^i, $i = 1 \ldots q$, we define the n-th prolongation of f to be

$$\mathrm{pr}^{(n)} f : \mathbb{R}^p \to U^{(n)}$$

given by $u^{(n)} = \mathrm{pr}^{(n)} f$, $u^a_J = \partial_J f^a$ where J is a multi-index running over the space of all possible derivatives. For example if we consider $u = f(x, y)$, we can compute

$$\mathrm{pr}^{(2)} f = (x, y) = (u; u_x, u_y; u_{xx}, u_{xy}, u_{yy}).$$

The space $\mathbb{R}^p \times U^{(n)}$ is called the n-th order jet space of $\mathbb{R} \times U$. The fundamental idea behind the method of symmetry analysis is to view Δ_ν as a map from the n-th order jet space into $\mathbb{R}^l$, and assuming derivative terms occur as polynomials in the system, we can identify Δ_ν with a subvariety in the jet space given by

$$\mathcal{L}_\Delta = \{(x, u^{(n)}) | \nabla(x, u^{(n)}) = 0\}.$$

Now let $M \subset \mathbb{R}^p \times U$ be open. A symmetry group of Δ_ν is a local group of transformations G acting on M such that when $u = f(x)$ solves Δ_ν, then $u = g \cdot f(x)$ solves Δ_ν for all $g \in G$ where defined.

Let X be a vector field on M, and assume X infinitesimally generates the symmetry ground G of Δ_ν. Then by projecting X into M via the exponential map, we may construct a local one-parameter group $exp(\epsilon X)$. We may then define the prolongation of X as

$$\mathrm{pr}^{(n)}X = \frac{d}{d\epsilon}\mathrm{pr}^{(n)}[exp(\epsilon X)](x, u^{(n)})\bigg|_{\epsilon=0}.$$

We also define the Jacobi matrix of Δ_ν to be

$$J_{\Delta_\nu}(x, u^{(n)}) = \left(\frac{\partial \Delta_\nu}{\partial x^i}, \frac{\partial \Delta_\nu}{\partial u^a_J}\right)$$

and say Δ_ν is maximal if the rank of $J_{\Delta_\nu} a = l$. The following theorem constrains the form of coefficients of the n-th prolongation of an infinitesimal generator for the symmetry group.

Theorem 61. *Let*

$$X = \xi^i(x, u)\frac{\partial}{\partial x^i} + \phi_a \frac{\partial}{\partial u^a}.$$

Then X has prolongation

$$\mathrm{pr}^{(n)}X = X + \phi^J_a(x, u^{(n)})\frac{\partial}{\partial u^a_J}$$

where

$$\phi_a^J(x, u^{(n)}) = D_J\left(\phi_a - \xi^a u_i^a\right) + \xi^i \partial_i u_J^a$$

and subscripts on u indicate partial derivatives.

The following may be called the fundamental theorem of the Lie method:

Theorem 62. *Let Δ_ν be a system of differential equations of maximal rank. If G is a local group of transformations acting on M and*

$$\mathrm{pr}^{(n)} X[\Delta_\nu(x, u^{(n)})] = 0$$

whenever $\Delta_\nu = 0$, for every infinitesimal generator X of G, then G is a symmetry group of Δ_ν.

We use these two theorems to calculate the coefficients of the infinitesimal generator. We then exponentiate the infinitesimal generator to obtain the symmetry group of the system. Finally, we apply these symmetries to known and usually simple solutions of the system to obtain new and hopefully more interesting solutions.

4.3 Symmetry Analysis

The infinitesimal generator of the above system is given by

$$\mathbf{v} = c^u\partial_u + c^v\partial_v + c^{1h}\partial_{1h} + c^{2h}\partial_{2h} + c^{1g}\partial_{1g} + c^{2g}\partial_{2g} + c^\psi\partial_\psi.$$

Since the system is first order, we need only consider the first prolongation

$$\mathrm{pr}^{(1)}\mathbf{v} = \mathbf{v} + {}_1h^u\partial_{1h_u} + {}_1h^v\partial_{1h_v} + {}_2h^u\partial_{2h_u} + {}_2h^v\partial_{2h_v}$$

$$+{}_1g^u\partial_{1g_u} + {}_1g^v\partial_{1g_v} + {}_2g^u\partial_{2g_u} + {}_2g^v\partial_{2g_v}$$

where the c^i are functions of $u, v, \psi, {}_1h, {}_2h, {}_1g$, and ${}_2g$. Applying the first prolongation to the PDE system and solving the resulting equations yields the following symmetry vectors:

$$\mathbf{v}_1 = \partial_u \quad \mathbf{v}_2 = \partial_v \quad \mathbf{v}_3 = \partial_{{}_1h} \quad \mathbf{v}_4 = \partial_{{}_2h} \quad \mathbf{v}_5 = \partial_{{}_1g} \quad \mathbf{v}_6 = \partial_{{}_2g}$$

$$\mathbf{v}_7 = -v\partial_u + u\partial_v \quad \mathbf{v}_8 = -{}_2h\partial_{{}_1h} + {}_1h\partial_{{}_2h} \quad \mathbf{v}_9 = -{}_2g\partial_{{}_1g} + {}_1g\partial_{{}_2g}$$

$$\mathbf{v}_{10} = u\partial_u + v\partial_v + {}_1h\partial_{{}_1h} + {}_2h\partial_{{}_2h} + {}_1g\partial_{{}_1g} + {}_2g\partial_{{}_2g}$$

Exponentiating these infinitesimal vector fields gives the solutions:

$$h \to h(z-s) \quad g \to g(z-s) \tag{4.7}$$

$$h \to h(z-is) \quad g \to g(z-is) \tag{4.8}$$

$$h \to h+s \quad g \to g \tag{4.9}$$

$$h \to h+is \quad g \to g \tag{4.10}$$

$$h \to h \quad g \to g+s \tag{4.11}$$

$$h \to h \quad g \to g+is \tag{4.12}$$

$$h \to h(e^{is}z) \quad g \to g(e^{is}z) \tag{4.13}$$

$$h \to e^{is}h \quad g \to g \tag{4.14}$$

$$h \to h \quad g \to e^{is}g \tag{4.15}$$

$$h \to e^{s}h(e^{-s}z) \quad g \to e^{s}g(e^{-s}z) \tag{4.16}$$

In [2] the minimal symmetry group for the real minimal surface equation

$$u_{xx}(1+u_y^2) + u_{yy}(1+u_x^2) - 2u_xu_yu_{xy} = 0$$

was calculated. Many of the translational symmetries and an $e^s f(e^{-s}x, e^{-s}y)$ symmetry were found. We note that the analogue of $\mathbf{v}^{10}$ in [2] is similar but different, since is constrains a Weierstrauss-Enneper representation of a surface and not a graph.

4.4 Symmetry Comments

Consider the transformation $h \to e^{i\theta}h$, $g \to e^{i\theta}g$ which preserves the metric $E = |h'|^2 + |g'|^2$. When $\theta = 0$, this is simply a minimal surface specified by defining ψ. When $\theta = \pi/2$ we get the conjugate surface. Thus all intermediate surfaces, called associated surfaces, are isometric. Since all minimal surfaces can be constructed from parts of a helicoid and catenoid [5], the following examples are of interest. First we draw attention to the catenoid, given by $\psi = \cosh(v)^2$. It's conjugate surface is the helicoid and associated surfaces between the two are plotted over $\mathbb{D}$ in Figure 4.1. Since all of the associated surfaces are isometric, geometrically they are equivalent. However, note topologically the catenoid is $S^1 \times \mathbb{R}$ where as the helicoid is $\mathbb{R}^2$.

We now turn our attention to the other symmetries found in the analysis for the half catenoid. We will see that the symmetries generate surfaces that are topologically distinct from the catenoid, but geometrically identical as in the above example. Let f be the harmonic mapping $f = h + \overline{g}$ where

$$h = \frac{1}{2}\left(\frac{1}{2}\log\left[\frac{1+z}{1-z}\right] + \frac{z}{1-z^2}\right)$$

and

$$g = \frac{1}{2}\left(\frac{1}{2}\log\left[\frac{1+z}{1-z}\right] - \frac{z}{1-z^2}\right)$$

which lifts to the catenoid. We make the transformation in equation (4.16) by letting $h \to e^s h(e^{-s}z)$ and $g \to e^s g(e^{-s}z)$. Figure 2 gives several plots of this

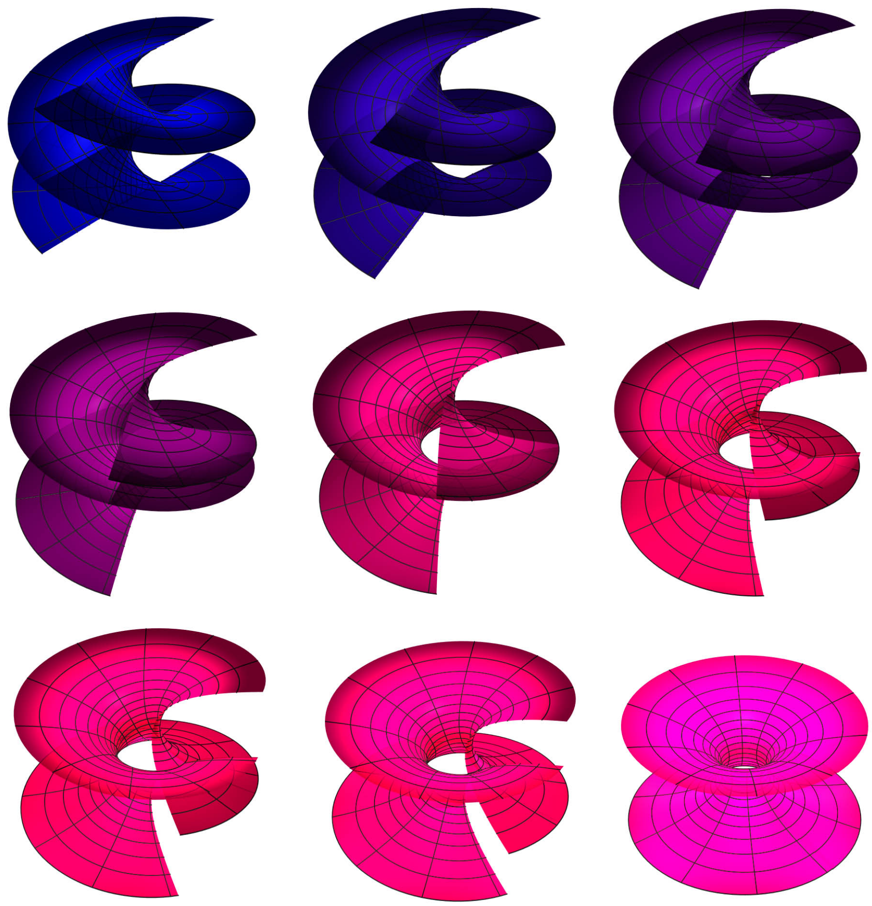

Figure 4.1 Helicoid to Catenoid Transformation.

transformation for various s values. The topology of the half catenoid is $\mathbb{R}^2$ for all s up to some value between $(0.3, 0.4)$ where it changes to a punctured cylinder. Note as $s \to \infty$ that the minimal surfaces eventually degenerate to a line, in a manner peculiarly similar to neckpinch singularities of the Ricci Flow. We note symmetry (4.8) is a scaled rotation, and can not comment on (4.7) in this example. The rest of the symmetries are translations.

When (4.16) is applied to the helicoid, we find that the number of rotations of the helicoid about its axis are scaled. Thus we have:

Theorem 63. *Let S be the helicoid over $\mathbb{D}$ parameterized isothermally by*

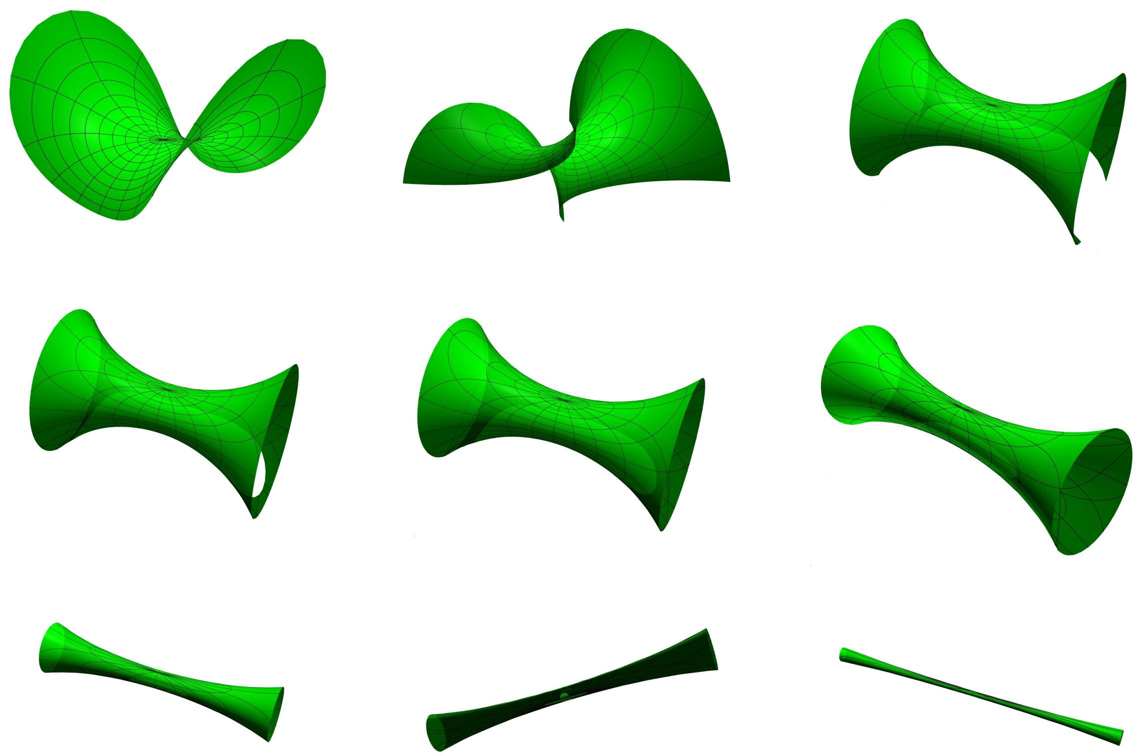

Figure 4.2 Symmetry (4.16) for $s = \{-1.2, -0.5, 0, 0.3, 0.4, 0.5, 1, 1.5, 3\}$.

$\mathbf{x} = (\sinh u \sin v, \sinh u \cos v, -v)$. *For helicoids* S_1 *given by* $u \in (0, 2\pi)$, $v \in (v_0, v_1)$, *and* S_2 *by* $u \in (0, 2\pi)$, $v \in (v_2, v_3)$ *where* $v_i \in \mathbb{R}$ *then* S_1 *and* S_2 *are locally isometric.*

It would be interesting to generalize the symmetry methods of this paper to higher dimensional Riemannian or Lorentzian manifolds. One would need a generalized Weierstrauss-Enneper which we are not aware exists. Moreover, we believe there are potential topological theorems coming from symmetry (4.16), which are connected to how the symmetry scales the domain of the graph under consideration. For instance, if one calculates the one parameter family of minimal surfaces given by symmetry (4.16) and a simply connected minimal surface, does the topology always change from the plane to $S^1 \times \mathbb{R}^1$ or some variant thereof?

Bibliography

[1] R. Berry, M. Dorff, and W. L. Petersen. Lie symetries of minimal surfaces, in preparation.

[2] N. Bîlă. Lie Groups Applications to Minimal Surfaces PDE. *DGDS* **1**: 1-9, 1999.

[3] B. J. Cantwell. *Introduction to Symmetry Analysis.* Cambridge University, 2002

[4] J. Clunie and T. Sheil-Small. Harmonic univalent functions. *Annales Academiæ Scientiarum Fennicæ* textbf9: 3-25, April 1985.

[5] T. H. Colding and William P. Minicozzi II. Shapes of embedded minimal surfaces. *PNAS* **103**: 11103-11105, July 2006.

[6] U. Dierkes, S. Hildebrandt, A. Kster, and O. Wohlrab, *Minimal surfaces I.* Grundlehren der Mathematischen Wissenschaften, 295, Springer-Verlag, 1992.

[7] M. Do Carmo. *Differential Geometry of Curves and Surfaces.* Pearson, 2004.

[8] M. Dorff. Some harmonic n-slit mappings. *Proceedings of the American Mathematical Society* **1**26: 569–576, 1998.

[9] M. Dorff and J. Szynal. Harmonic shears of elliptic integrals. *Rocky Mountain Journal of Mathematics*, **3**5: no. 2, 485-499, 2005.

[10] P. Duren, J. McDougall, and L. Schaubroeck. Harmonic mappings onto stars. *J. Math. Anal. Appl.* **307**: no. 1, 312-320, 2005.

[11] P. Duren. *Harmonic Mappings in the Plane.* Cambridge University Press, 2004.

[12] K. Driver and P. Duren. Harmonic shears of regular polygons by hypergeometric functions. *J. Math. Anal. App.* **2**39: 72-84, 1999.

[13] W. Hengartner and G. Schober. On schlicht mappings to domains convex in one direction. *Comment. Math. Helv.* **45**: 303-314, 1970.

[14] W. Koepf. Parallel accessible domains and domains that are convex in some direction. *Partial Differential Equations with Complex Analysis.* Pitman Res. Notes Math. Ser., 262, Longman Sci. Tech., Harlow, 93-105, 1992.

[15] H. Lewy. On the non-vanishing of the Jacobian in certain one-to-one mappings. *Bull. Amer. Math. Soc.* **4**2: 689-692, 1936.

[16] P. J. Olver. *Applications of Lie Groups to Differential Equations.* Springer-Verlag, second edition, 1993.

[17] W. C. Royster and M. Ziegler. Univalent functions convex in one direction. *Publ. Math. Debrecen.*, **23**: no. 3-4, 339-345, 1976.

[18] M. Spivak. *A Comprehensive Introduction to Differential Geometry.* Publish or Perish, 1999.

[19] H. Stephani. *Differential Equations: Their solution using symmetries.* Cambridge University Press, 1989.