



2013-06-13

# Student Understanding of Limit and Continuity at a Point: A Look into Four Potentially Problematic Conceptions

Miriam Lynne Amatangelo  
*Brigham Young University - Provo*

Follow this and additional works at: <https://scholarsarchive.byu.edu/etd>

 Part of the [Science and Mathematics Education Commons](#)

---

## BYU ScholarsArchive Citation

Amatangelo, Miriam Lynne, "Student Understanding of Limit and Continuity at a Point: A Look into Four Potentially Problematic Conceptions" (2013). *All Theses and Dissertations*. 3639.  
<https://scholarsarchive.byu.edu/etd/3639>

This Thesis is brought to you for free and open access by BYU ScholarsArchive. It has been accepted for inclusion in All Theses and Dissertations by an authorized administrator of BYU ScholarsArchive. For more information, please contact [scholarsarchive@byu.edu](mailto:scholarsarchive@byu.edu), [ellen\\_amatangelo@byu.edu](mailto:ellen_amatangelo@byu.edu).

Student Understanding of Limit and Continuity at a Point:  
A Look into Four Potentially Problematic Conceptions

Miriam Lynne Amatangelo

A thesis submitted to the faculty of  
Brigham Young University  
in partial fulfillment of the requirements for the degree of  
Master of Arts

Steve Williams, Chair  
Daniel Siebert  
Dawn Teuscher

Department of Mathematics Education  
Brigham Young University

June 2013

Copyright © 2013 Miriam Lynne Amatangelo

All Rights Reserved

## ABSTRACT

### Student Understanding of Limit and Continuity at a Point: A Look into Four Potentially Problematic Conceptions

Miriam Lynne Amatangelo  
Department of Mathematics Education, BYU  
Master of Arts

Mathematics students and teachers are familiar with the difficulty of learning and teaching concepts of continuity and limits. Research has expanded our knowledge of how students think about these concepts, including different conceptions and metaphors students use to reason about continuity and limits at a point. From the literature I have identified four potentially problematic conceptions (PPCs) students may use when reasoning about limit and continuity at a point. Questionnaires were administered to 861 BYU students in various mathematics courses to determine how prevalent and persistent the PPCs are among the students in each course. Interviews were conducted with nine first semester calculus to get an idea of how students reason about continuity and limit at a point and how that influences whether they use the PPCs. Students showed evidence of holding the four PPCs with a decrease in these conceptions typically after they took a course in analysis. Participants also did not understand the Formal definition of a Limit until they took a course in Analysis. Students were able to reason appropriately using many different conceptions of continuity. Considering limit conceptions, students using a Dynamic conception of Limit tended to be better able to reason about continuity and limit at a point. Students who did not use a Dynamic conception of limit tended to use the PPCs in general and incorrectly more often.

Keywords: Calculus, Limit, Continuity, Conceptions, Misconceptions, Concept Image

## ACKNOWLEDGEMENTS

I would like to thank all those who supported me as I worked on my research and prepared this thesis. I thank my committee chair, Dr. Steve Williams for allowing me to study what I was most interested in and helping me make this research my own. I appreciate the help and feedback from my other committee members as well, Dr. Daniel Siebert and Dr. Dawn Teuscher. I am particularly grateful for the advice from my committee that helped my research answer the questions I was interested in and made the work appropriate for a thesis. I thank my parents who supported me in pursuing a master's degree. I also thank my husband, Joseph, and son, Louis for their patience and support as I researched and worked to complete my degree.

## TABLE OF CONTENTS

List of Tables .....	vii
List of Figures .....	viii
Chapter 1: Rationale .....	1
Learning with Understanding .....	1
Calculus Learning and Misconceptions .....	2
Chapter 2: Theoretical Framework and Literature Review .....	5
Constructivism, Connections, and Conceptions .....	5
Misconceptions, Potentially Problematic Conceptions, and Concept Images .....	6
Summary of Conceptions Framework .....	10
The PPCs of Focus: Continuity and Limit at a Point.....	11
General Student Conceptions.....	13
Practical and Theoretical Conceptions.....	14
Dynamic Conceptions.....	15
Other General Conceptions.....	17
Formal Definitions .....	20
Research Questions .....	22
Chapter 3: Methodology .....	24
Participants and Setting.....	24
Data and Data Collection .....	25

Questionnaire .....	25
Interviews.....	29
Differences in Questionnaire and Interview Data.....	33
Management and Analysis.....	33
Form of the Results.....	40
Summary.....	41
Chapter 4: Results.....	42
Pervasiveness of the PPCs .....	42
Continuity and Limit Conceptions.....	48
Practical Conceptions.....	49
Dynamic Conceptions.....	50
Other Conceptions Identified in Previous Research.....	53
Formal Conceptions.....	57
New Conceptions from the Interviews .....	60
Conceptions from Research Left Unobserved .....	65
Preferred Conceptions on the Questionnaire .....	65
Summary of Conceptions.....	68
Potentially Problematic Conceptions.....	71
The Defined Means Continuous PPC.....	72
The Limit Equals Function Value PPC.....	75

The Limit Means Continuous PPC .....	79
Correct in General use of the Limit and Continuity PPCs.....	81
The Limit Means Discontinuous PPC.....	81
Comments on Differences in Questionnaire and Interview Results .....	84
Chapter 5: Summary and Conclusions.....	86
Summary of Research Questions .....	86
Limitations .....	88
Implications.....	90
Future Research .....	92
References.....	93
Appendix A.....	95
Questionnaire .....	95
Interview .....	96
Appendix B.....	101
Extensive List of Codes and their Explanations .....	101

## LIST OF TABLES

Table 1: Sample Size for Each Course .....	26
Table 2: Continuity and Limit Conceptions on Questionnaire .....	27
Table 3: PPCs and Questionnaire Statements.....	28
Table 4: Preliminary Codes Used for Analysis of Interviews .....	36
Table 5: Specific PPC Codes .....	40
Table 6: Students Conceptions on Questionnaire .....	49
Table 7: Student Conceptions in Interviews .....	49
Table 8: Preferred Limit Conceptions on the Questionnaire .....	66
Table 9: Preferred Continuity Conceptions on the Questionnaire .....	67
Table 10: Continuity Conception Counts from Each Interviewed Student .....	69
Table 11: Limit Conception Counts from Each Interviewed Student.....	70
Table 12: Conception Correlations with the Defined Means Continuous PPC in Interviews .....	73
Table 13: Conception Correlations with the Limit Equals Function Value PPC in Interviews ...	76
Table 14: Conception Correlations with the Limit Means Continuous PPC in Interviews .....	80
Table 15: Conception Correlations with the Limit Mean Discontinuous PPC in Interviews .....	82



## LIST OF FIGURES

Figure 1: Formal definitions of limit and continuity at a point.....	21
Figure 2: Interview questions 1-3. ....	30
Figure 3: Graphs from the interviews. ....	31
Figure 4: Average PPC scores in each course. ....	43
Figure 5: Evidence of PPCs. ....	46
Figure 6: Formal limit understanding .....	59

## Chapter 1: Rationale

### Learning with Understanding

Learning with understanding is an important goal for mathematics education, and it is generally accepted that this type of learning occurs when pieces of knowledge are mentally connected together (Hiebert & Carpenter, 1992). The number and strength of the connections students make determine the degree of a student's understanding; the better the connections, the better the understanding. As learners encounter new external situations or experiences these interact with their existing knowledge (von Glasersfeld, 1983), thus allowing such connections to be made. New ideas can be connected into the thinker's existing mental schemes, or experiencing something new may result in preexisting knowledge being connected in new ways (Siegler, 1986). Connections can be created, dissolved, changed, strengthened, or weakened depending on how the thinker reacts to the new information, such as if the new ideas resonate with or contradict his existing knowledge. It is also possible for the learner to fail to make connections, resulting in disconnected or compartmentalized knowledge, which is more likely to result in flawed reasoning (Hiebert & Carpenter, 1992). Knowledge is thus seen as a connected network that is constructed over time as old knowledge interacts with new information.

During the process when knowledge is constructed, what the thinker constructs may not always be *correct*, by which I mean, what is accepted by the community. This concept of correctness borrows a social constructivist view that the truth of any particular piece of knowledge resides within a community of practice, in the case of this thesis, the community of mathematicians. This may happen when inappropriate connections are made, such as connecting multiplication to making numbers larger based on limited examples taken from whole number arithmetic. It may also happen if students fail to make connections, such as failing to connect

piecewise functions into an existing schema for function, resulting in the conception that functions are always defined by only one equation, which could in turn result in the claim that a piecewise relation is not a function. Such misconceptions, i.e., conceptions of a particular situation which contradict or are considered incorrect by the mathematical community, can arise from students' existing knowledge as the thinker tries to incorporate new information into his existing mental schemes or as he fails to do so (Smith III, diSessa, & Roschelle, 1994). Thus prior knowledge becomes a critical factor in learning. For this reason, it is generally accepted that understanding student knowledge is a critical factor in teaching (Ball, Thames, & Phelps, 2008). It seems, then, that effective teaching should consider the students' existing knowledge, including their misconceptions, in instruction so the teacher can better facilitate the connections students make as they learn. In fact, much research has been done under the generally accepted notion that it is important for instruction to address student misconceptions (Bezuidenhout, 2001; Resnick et al., 1989; Smith III et al., 1994; Tall, 1990); however, Smith III et al. (1994) suggested that research needs to move beyond simply identifying the misconceptions students have and they propose a focus on why misconceptions exist and how they are overcome. This research will move beyond identifying the misconceptions to consider how prevalent they are and how students actually reason about particular misconceptions.

### **Calculus Learning and Misconceptions**

Calculus is an important mathematical course for many college students, including students in majors beyond mathematics and science (Ferrini-Mundy & Graham, 1991; Robert & Speer, 2001). Moreover, it is a topic for which many different connections, understandings, and misconceptions can be constructed (Bezuidenhout, 2001; Duru, Köklü, & Jakubowski, 2010; Eade, 2003; Grabiner, 1983; Jordaan, 2009; Monaghan, 1991; Oehrtman, 2009; Tall, 1990; Tall

& Vinner, 1981; Williams, 1991, 2001). Calculus misconceptions are particularly important for instructors to be aware of and are common throughout the entire curriculum, including misconceptions about functions, continuity, sequences, limits, derivatives, and integrals (Bezuidenhout, 2001; Tall, 1990). Two of these topics, continuity and limit, are the focus of this study and are particularly important for students' understanding of calculus. Continuity and limit are fundamental pieces of calculus understanding as the other topics listed above are developed from continuity and limit (Ervynck, 1981; Williams, 1991). The conceptions students use to make sense of continuity and limit will be brought to bear as they make sense of each of the other calculus concepts (Oehrtman, 2009; Tall & Vinner, 1981).

Continuity and limit, have received a great deal of attention in research. There are various documented ways students think about these topics, from practical to theoretical and from intuitive to formal (Núñez, Edwards, & Filipe Matos, 1999; Oehrtman, 2009; Williams, 1991). Although there are a number of misconceptions that have been identified and studied in this research, there is one which has been identified but has not been studied beyond its identification. This misconception relates to the idea of continuity and limit at a point. Although I speak of this as one misconception, it may be more accurate to think of it as a group of related misconceptions students hold about what happens at the point at which the limit is being evaluated or what is happening at a point where a function is continuous or discontinuous.

Students hold various misconceptions about what is happening at a point with respect to the limit or continuity of the function. Some students believe that if the limit exists, the function must be continuous. Others believe the limit and the function to be the same thing (Bezuidenhout, 2001; Duru et al., 2010; Jordaan, 2009; Przenioslo, 2004). Both of these beliefs are false since there are discontinuous functions whose limit exists at a point, but is not equal to

the function value. Bezuidenhout (2001) conjectured that such misunderstandings may come from the students focusing on strategies to evaluate a limit, such as substitution, rather than actually understanding what a limit is. Some students also believe that if the function is defined at a point, it is continuous there too (Duru et al., 2010). Another misconception some students hold is almost the opposite belief, that if the limit exists the function must be discontinuous, which is a misconception when the considered function is continuous. This ties into, and possibly is a result of, student thinking that there is no purpose in taking a limit if the function is continuous because we already know what the value is, a misconception Williams (1991) and Juter (2006) discussed. Although various papers have identified such misconceptions about points of continuity or limit points and some have conjectured about where these misconceptions come from, few researchers have studied students beyond the first or second semester of calculus, and no research has focused on the types of reasoning that allow for particular misconceptions. This is the problem the present research will address.

## **Chapter 2: Theoretical Framework and Literature Review**

This section will outline the theoretical framework used as I conducted this study as well as a simultaneous analysis of the existing literature and how my research contributes to it. I describe the constructivist perspective under which this research is conducted in conjunction with a brief review of how other researchers have studied conceptions and misconceptions and I describe my own use of these ideas. I then introduce what others have studied with respect to continuity and limits, including student conceptions and formal definitions. These works are the building blocks for the framework of the present study, which is presented in combination with this literature review. I conclude with the presentation of the research questions.

### **Constructivism, Connections, and Conceptions**

This study was conducted through a constructivist lens. Constructivism defines learning in a way centered on the thinker's experiences, beliefs, and existing knowledge. When students learn, it is not a simple process where the knowledge is stored away like a box in a warehouse. Rather, what they experience (hear, observe, explore, etc.) is interpreted through what they have already experienced in the past (von Glasersfeld, 1983). The mental constructions of knowledge which already exist for the learner are added to or altered when they learn something new, and thus new knowledge is actively built by the learner. Knowledge, then, is considered as a connected web of information<sup>1</sup>. When students encounter a new experience, the experience may be interpreted as fitting with their current understanding and thus the experience may be added to their existing knowledge structure as evidence supporting their understanding. On the other hand, the experience may be interpreted as conflicting with prior knowledge, leaving the learner with a choice of how to deal with the new information. This clash between the existing knowledge and the perceived experience may form a new part of the learner's cognitive construction

---

<sup>1</sup> Such a network is often referred to as a *schema* (Piaget, 1928; Schraw, 2006).

(compartmentalized or separated from the conflicting knowledge) or their knowledge structure may be changed (Vinner & Dreyfus, 1989). These options mirror Piaget's ideas of assimilation and accommodation (Siegler, 1986).

Regardless of how an individual learner incorporates new information into their organized knowledge structure, it is clear that learning occurs as new information interacts with the thinker's existing knowledge. Moreover, different learners may interpret and deal quite differently with similar experiences, thus creating different understandings from experiences which seem the same to an outside observer. Thus a constructivist view provides a platform to explain the existence of multiple conceptions for a given mathematical concept. By conception I mean a knowledge structure that describes how one reasons about a given topic, and by concept I mean a mathematical topic as generally understood and agreed upon in the larger mathematical community. For example, for the concept of function, a student may hold the conception *for every input of a function, there is exactly one output*. Note that this conception is part of the definition of a function. One situation a student may use this conception in is determining if a particular relation is, in fact, a function.

### **Misconceptions, Potentially Problematic Conceptions, and Concept Images**

It is natural, then, for there to be many student conceptions of a given topic, and much research has been devoted to studying various conceptions in mathematics (Artigue, Batanero, & Kent, 2007; Confrey, 1990; Oehrtman, 2009; Resnick et al., 1989; Smith III et al., 1994; Tall & Vinner, 1981). These many conceptions are consequences of students interpreting mathematical concepts through their personal experiences and existing knowledge. We can describe several different levels of conceptions: misconceptions, alternative, or partial conceptions, correct conceptions as accepted by the mathematical community, etc. Olivier (1989) distinguishes

between slips, errors, and misconceptions. A slip is an incorrect answer as a result of a careless processing error, such as mistakenly claiming the product of 2 and 3 to be 5. Slips, then, are not conceptions, nor are they the result of a conception, because they are the result of a processing error rather than a result of reasoning. Slips are often made by both novices and experts and are typically simple and quick to overcome. Errors are incorrect answers which are the result of reasoning on the part of the thinker. The same errors are made each time a student is reasoning in a particular situation, such as claiming .237 is larger than .52 and .41 is larger than .8.

Misconceptions are the underlying conceptual structures, or conceptions, which cause errors. A misconception which the student claiming .237 is larger than .52 may be reasoning with is that *more digits makes a number bigger*, a misconception which may have come from prior knowledge about whole numbers. I will use the terms *slip* and *error* as Oliver did, and I will describe my use of the term *misconception* more specifically below. This framework aligns with a significant point made by Smith III et al. (1994) that misconceptions are more than just mistakes, they are deeper structure used to reason about (in this case) mathematics. Smith III et al. (1994) identified phrases other researchers have used to describe incorrect ideas like *naïve conceptions* or *alternative frameworks* and antonymic descriptions such as *expert*, but Smith ultimately used the familiar term *misconception*, a term I will use and defined below.

Smith III et al. (1994) and Nesher (1987) described a misconception as part of a complex knowledge system, and claimed that misconceptions are beneficial to overcome and reason through as part of the learning process. Smith III et al. (1994) focused on mental schemas and the role misconceptions play within them, while Nesher discussed similar ideas by talking about how to teach using student misconceptions. Both articles argued that it is not necessarily beneficial for students to simply be told their conceptions are incorrect because this will not convince the



students and it is possible that students may interpret what they are told differently than intended. It is important for students to bump up against a contradicting experience rather than simply being told their conception is incorrect, otherwise their misconceptions will likely go unchanged. The articles both took a constructivist stance, both describing how students will learn from their experiences, and thus how misconceptions can both arise and be overcome as the learner incorporates new information with existing knowledge.

I define a misconception a little differently than others. A misconception is a conception applied to a mathematical *situation* which is considered incorrect by the community and which results in an error. Thus, a misconception is the reasoning or conception used which produced the error. I use this definition because a student may hold a conception which is not considered correct by the mathematical community, but may or may not use that conception in a situation where it would be problematic. Thus I consider a student to actually hold a misconception if they apply the conception in a situation where it results in an error. I introduce another term that will help clarify my use of these terms and that will describe the focus of my study. A *potentially problematic conception* (PPC) is a conception that has been observed to be held by some students, and that can lead to either correct or incorrect conclusions depending on how and whether it is applied to a particular mathematical situation. Thus a PPC could be applied to some situations with no resulting error, while in other situations a student's conclusion would be incorrect if they applied the PPC.

PPCs can be reasoned with in three ways: 1) *generally*, without reference to a particular situation, 2) *harmlessly*, in a situation such that the reasoning is not appropriate, but the resulting conclusion is not an error, and 3) *incorrectly*, in a situation that makes their conclusions incorrect, thus making the PPC a misconception for the student. One example of a PPC is the

belief that *the limit value and function value are the same*. This is a PPC because it only sometimes leads to incorrect reasoning or conclusions and has been observed in student reasoning (Bezuidenhout, 2001; Jordaan, 2009; Przenioslo, 2004). A student reasoning *generally* may simply state this PPC or describe it when explaining what a limit is or how to determine a limit value in general, void of, i.e. without considering a particular situation. A student could *harmlessly* reason that the limit value and function value are always the same, and conclude that this must necessarily be the case with the function  $f(x) = x + 3$ , even though that reasoning is not complete. The student's conclusion is correct but their reasoning was nevertheless lacking. The PPC is correct when applied to continuous functions, but unless the student consciously determined this was the case, their reasoning was incorrect, although harmless because there was not error in their conclusion. On the other hand, the student may apply the PPC *incorrectly*, so that it is a misconception, in a situation involving a discontinuous function because use of this PPC in this situation will result in an error. The misconception is thus specific to the situation, that is, a discontinuous function. This study will focus on four particular PPCs involving limit and continuity. It is important to note that conceptions, misconceptions, and PPCs can all be part of a larger knowledge structure, such as described by Tall and Vinner (1981).

Tall and Vinner (1981) focused their work on student conceptions in general, rather than only incorrect conceptions. The authors referenced a student's *concept image*, being all of the connections a student has made with respect to a particular topic or word –the whole knowledge structure associated with a concept-- and a *concept definition*, which uses words to define a concept. The authors also distinguish between an individual's *personal concept definition* and the *formal concept definition*, which is accepted by the mathematical community. In the present study, I will also consider students' *concept images*. A student's concept image includes their

conceptions, personal concept definition for a topic and any other knowledge relating to the concept, such as a mental image or connection to another concept. The focus of this study is on different conceptions that students reason with. My study also considers models and metaphors described by other researchers, which I consider types of conceptions. It is important to note that a student's concept image may include multiple conceptions of a topic. The student may use different conceptions in different situations allowing for contradicting conceptions to go undetected by the student (Tall & Vinner, 1981). The *formal definition* of a concept is just as Tall and Vinner (1981) described it, the conceptual definition as accepted by the mathematical community. The formal definition may or may not be part of a student's concept image. Great variation in students' concept images can be expected, since each constructs their mental schemas based on their own experiences. This also suggests that there may be many PPCs and thus misconceptions of mathematical topics.

### **Summary of Conceptions Framework**

I will summarize how the terms conception, misconception, and PPC are used throughout this work. A *conception* is a cognitive schema which can be used in various situations. Thinkers assign truth values to the conceptions, essentially true or false, when it is applied in a particular situation. A conception becomes a misconception when a student uses it to reason about a mathematical situation and their conclusion is an error. A student can also harmlessly use a conception in a situation where its use does not result in an error, even if the conception alone is not enough to mathematically justify their conclusion. A student reasoning in that situation would not be considered to hold a misconception, as I have defined a misconception to specifically be when the resulting conclusion is an error. Thus, conceptions can be considered true or harmless in some situations and false in others. If a conception can be true or harmless,

meaning there is no resultant error, in some mathematical situations, but false resulting in an error in other situations, I call these conceptions *potentially problematic conceptions* (PPCs).

PPCs can be reasoned with in three ways: 1) generally, without situation, 2) harmlessly, where the reasoning is inappropriate or incomplete but the conclusion is not an error, and 3) incorrectly, making it a misconception, when the PPC is used and the resulting conclusion is an error.

### **The PPCs of Focus: Continuity and Limit at a Point**

My study considers a particular group of PPCs about continuity and limit at a point, and I begin by describing what researchers have found already about these potentially problematic conceptions. There are four conceptions I considered which are identified in research: 1) *if the function is defined at a point, it is continuous at that point*, 2) *the limit value and the function value are the same at a point*, 3) *if the limit exists at a point, the function is continuous at that point*, 4) *if you take the limit at a point, the function must be discontinuous at that point*. The first three conceptions are specifically identified as misconceptions for some students in various studies while the final one was identified in studies focused on a more general conception of limit under which such a PPC can appear.

The first PPC I will discuss is *if a function is defined at a point, then the function must be continuous at that point*, what I call the Defined Means Continuous PPC. This PPC becomes a misconception when students apply it to a defined function with a discontinuity. Bezuidenhout (2001) identified this as a misconception for some first year calculus students who participated in his study. Similarly, Duru et al. (2010) identified that 22% of pre-service teachers at various levels in their studies showed evidence of the misconception. It may be that this PPC comes from students assuming the inverse of the commonly taught statement ‘if a function is undefined at a

point, then it is discontinuous at that point' is also true. The reasoning which allows for and the sources of this PPC have not been studied, the misconception has simply been identified.

The second PPC is *the limit value and the function value are the same at a point*. The Limit Equals Function Value PPC becomes a misconception when students use it to reason about discontinuous functions. This conception relates to the commonly taught method for evaluating a limit of a continuous function – simply plug in the value  $x$  is approaching to the function, or more simply, the substitution method (Bezuidenhout, 2001). Bezuidenhout identified students with the idea that the limit and the function value at a given point are identical mathematical terms. Jordaan (2009) verified the existence of the conception through a study very similar to Bezuidenhout's study and also with first-year calculus students. Similarly, Prezenioslo (2004), in his study about conceptions in general, found university students who held conceptions about a function being defined at the point at which the limit is being evaluated or these values being the same. This conception was conjectured by Bezuidenhout to have come from the use of a substitution method when evaluating a limit.

The third PPC is *if the limit exists at a point, the function is continuous at that point*. The Limit Means Continuous PPC becomes a misconception when students apply this conception to functions with removable discontinuities. This is a more common conception than the first two. Bezuidenhout (2001) found that 37% of the first year calculus students in his study held this conception and Prezenioslo (2004) identified this conception as well. Bezuidenhout conjectured that this PPC may come from students believing the converse of 'if a function is continuous at a point, then the limit exists at that point.' Although this is a common PPC for students, there is another PPC that reaches the opposite conclusion.

The final PPC is *if you take the limit at a point, the function must be discontinuous at that point*. The Limit Means Discontinuous PPC becomes a misconception when students apply it to continuous functions. This conception has not been the focus of any study, yet it was identified by Williams (1991) when students were reasoning that a limit was unapproachable. A few students in this study reasoned that a limit cannot be reached by a function, so taking the limit of a continuous function does not make sense. One student went as far as to conclude that if the limit is considered, the function must be discontinuous or there would be no point in even thinking about a limit. Juter (2006) also identified this conception relating to students claiming a limit is unreachable. This PPC has not been studied other than the identification through viewing more general conceptions of limit.

Each of these PPCs has been identified to some extent in the research, and for a few of them conjectures have been made about what previous knowledge is contributing to the development of the conceptions. It seems students' conceptions of continuity and limit are likely to influence the existence of these PPCs and misconceptions. This is an assumption the present study makes. There are certainly other factors that may contribute to the students reasoning with these four PPCs, but I focus primarily on the impact of the conceptions they hold of continuity and limit. I will continue by describing the literature on conceptions of continuity and limits students may hold, which will provide the framework for how PPCs are evaluated.

### **General Student Conceptions**

Much of the existing literature on continuity and limits includes a description of how the researcher observes and interprets student thinking about these topics. The result is a range of conceptions described in different ways, but sometimes referring to the same type of cognitive structure. I have organized these many conceptions, both correct and incorrect, giving a name to

each of those I refer to throughout this work. I call these the *general conceptions* of continuity and limit. I describe each conception and its supporting literature in detail. Conceptions of limits and conceptions of continuity will be interspersed, although there is much more research on the former. I also define the formal definitions of continuity and limit at a point, because students may use these definitions in their reasoning. Since conceptions are often interrelated, I begin with two conceptions seeming to relate to many others.

**Practical and theoretical conceptions.** Student conceptions of limit and continuity can be considered on a continuum between practical conceptions and theoretical ones. A purely practical conception is one related entirely to a procedure used to solve a problem and may be used, for example, when a student is asked to simply determine the existence and value of a particular limit. A theoretical conception is one relating to the meaning of a limit and may be used, for example, when a student is asked what a limit represents or why a limit is used in a particular situation. Przenioslo (2004) suggested that practical conceptions are those most significant to students because they often returned to using these conceptions in his study, even if the practical conceptions contradicted other conceptions in the student's concept image. Recognizing the difference between the practical and theoretical conceptions one holds is something Williams (2001) identified as fundamental, suggesting that students should be able to distinguish how they think about limits from how they actually evaluate limits. How students think about limits (theoretical conceptions) and how students evaluate limits (practical conceptions) are undoubtedly broad categories in which other conceptions may also be considered, but the conceptions described next are completely describable within these categories.

Przenioslo (2004) identified *algorithm conceptions* as those centered on schematically applying memorized algorithms for particular types of questions. These are practical conceptions because the student uses this type of conception to perform a procedure and determine a solution. One such conception may lead students to evaluate the limit of a piece-wise function as  $x$  approaches  $a$  by evaluating the two formulas of the function around  $a$  and seeing if they “match up.” If they match up, the function value there is also the limit, if they do not match up, the limit does not exist. Another conception Przenioslo (2004) identified will simply be referred to as the *substitution* conception. This conception is a practical conception where the student simply substitutes the value  $x$  is approaching into the function and claims the output to be equal to the limit. Supportive of the existence of this conception, Bezuidenhout (2001) found that some students claim that  $\lim_{x \rightarrow 2} f(x)$  and  $f(2)$  are identical, one of the PPCs I will study. These algorithm conceptions can be described as exclusively practical because students use them when they are only concerned with performing some operation or procedure. Other conceptions, however, can be described using the categories of practical or theoretical in ways different from those I have described so far. Thus, the practical and theoretical descriptors will also be used to describe other conceptions in the following sections.

**Dynamic conceptions.** The most prevalent student conception found in the literature for continuity and limits is a dynamic conception, or one relating to motion imagery or language. Núñez et al. (1999) described the type of reasoning one uses when describing continuity from a dynamic perspective. Students may say that a function is continuous if the line produced by that function can be drawn without picking up the pencil or that the line moves on the graph without jumping over any holes. The individual may realize that the line is static, but they imagine the line as a product of movement, which enables the dynamic conception. Tall and Vinner (1981)



identified this same dynamic conception with continuity, but this type of conception extends to limits as well.

When reasoning about limits, students often use language such as “approaches,” “tends to,” or “moves in on.” This is no surprise since students are taught to read  $\lim_{x \rightarrow a} f(x)$  as “the limit of function  $f$  as  $x$  **approaches**  $a$ ”. This common way of describing limits was identified in several studies (Eade, 2003; Oehrtman, 2009; Przenioslo, 2004; Tall & Vinner, 1981; Williams, 1991, 2001) and labeled as a few different conceptions. Williams (1991) broke the dynamic conception into two more specific pieces, dynamic-practical and dynamic-theoretical. The dynamic-practical conception leads students to evaluate points moving closer and closer to the limit point and see what these values approach. The dynamic-theoretical conception describes how the function (perhaps an image of the graph) moves as the  $x$ -values move toward the limit point. When given a list of possible limit conceptions, a majority of students chose the dynamic-theoretical conception to be true. Similar to these conceptions, Przenioslo (2004) identified 34% of his participants having a concept image where the key conception was “graph approaching” for which points of a graph were approaching some value. In the same study 16% were labeled as having a concept image where the key conception was “values approaching” which is the same as the dynamic-practical conception (Williams, 1991). I refer to the dynamic-theoretical conception as simply the Dynamic conception. I refer to the dynamic-practical conception as such. Regardless of the type of dynamic conception a student holds in their concept image, studies have found that students tend to turn back to the dynamic conception, even when more precise or efficient conceptions are accessible to the student (Eade, 2003; Ervynck, 1981; Williams, 1991).

Oehrtman (2009) also recognized the common use of motion language in his study of introductory calculus students, but he suggests that the dynamic conception is not as significant as it may appear. When students used language such as “approaches” the researcher probed further to identify what the student was actually meaning. Most students admitted that they were not, in fact, imagining anything moving at all, and students did not use these ideas of motion when solving problems. Thus Oehrtman suggested that the dynamic conception is not a way that students necessarily *think* about limits, but it is more likely a way in which students simply *talk* about limits. He also suggested, however, that his conclusion is not necessarily contradictory to the prevalence of a dynamic conception in other studies. Oehrtman stated that some of the more prevalent metaphors identified in his study could be reasoned with from a motion or dynamic perspective.

**Other general conceptions.** Conceptions other than practical, theoretical, and dynamic have been identified as well, but their occurrence in the literature is not as common as the conceptions described above. I will describe the other conceptions in an order representative of their prevalence in the literature.

Oehrtman (2009) identified a *proximity* metaphor for limits and this conception was recognized, though labeled differently, by other researchers as well (Bezuidenhout, 2001; Przenioslo, 2004; Williams, 2001). The proximity conception relates to the “closeness” of inputs and outputs as Bezuidenhout (2001) and Williams (2001) referred to it. Przenioslo (2004) called this a *neighborhood* conception because it often results in students talking about a neighborhood around  $a$ . One thinking with a proximity model may reason that when you plug in  $x$ -values close to  $a$ , the  $f(x)$  values will be close to  $L$  when  $L$  is the value of the limit. This idea can be seen with practical conceptions, such as plugging in points *close* to  $a$  and seeing what the outputs get *close*

to as well as theoretical conceptions like the inequalities of the formal definitions (defined below). Oehrtman (2009) also described how students reasoned with this metaphor for continuity as well, thinking that a function would be continuous as long as inputs close to each other provided outputs which were also close to each other.

The *approximation* conception was a student model for limit identified by both Williams (1991) and Oehrtman (2009). This conception claims that the limit serves as an approximation for something, such as an approximation for a function value or an approximation of the slope of a tangent line. Williams identified this conception with half of the participants choosing a statement representing this conception to be true and the other half choosing false. From the same sample of participants, only 4% chose this as the best description of a limit. In contrast, Oehrtman identified this as the most commonly observed metaphor in his study, meaning when students answered the questions posed, they used language that conveyed this metaphor. The differences in these results may come from the differences in focus. Williams focused on limits of a function, while Oehrtman was broader, including topics such as the definition of a derivative or infinite series.

Williams (1991) identified a conception that limits are *unreachable* to be one chosen as correct by a majority of students. This conception holds that a limit cannot be reached or that the function value can never equal the limit value. This conception of limit is appropriate for many functions, but there are an infinite number of functions for which the limit can be reached, for example, continuous functions. When students were questioned about such situations, one responded that if the function is defined at a point, there is really no reason to even look at the limit, also concluding that if you are considering a limit at a point, the function is discontinuous at that point, one of the PPCs I am studying. Juter (2006) found similar results with participants.

A similar conception is that limits act as a bound. This conception holds that a limit is a value the function cannot pass. This *bounding* conception of limit may differ from the *unreachable* conception, because the latter may consider limits from both sides of an x-value, whereas the former is typically considered from one side.

Duru et al. (2010) and Tall and Vinner (1981) identified a conception of continuity I will call the *lack of discontinuities* conception. A student reasoning with this conception typically looks for any discontinuities when determining if a function is continuous or not. Students typically look for holes, jumps, breaks, or asymptotes. The lack of any such discontinuity is what the student uses to reason that the function is continuous.

Oehrtman (2009) described a *collapse* metaphor. The collapse metaphor, although difficult to define, is easier to describe through example. A student reasoning through the collapse metaphor may describe the fundamental theorem of calculus by describing a Riemann Sum as follows: When the widths of the rectangles approach zero, right when it is zero and there is no width, then it is just the sum of all the heights, or the area under the curve. This description makes no sense physically, because a height with no width cannot have an area (or be a rectangle, for that matter!). This metaphor, which also describes limit situations with words like “collapse,” “disappears,” or “magic,” is common and strong, resulting in statements like “I know that makes no sense, but I get it that way.”

Núñez et al. (1999) presented a metaphor for which students may think about continuity through their understanding of a line. There are two ways one could reason about a line. A line could be its own entity, where points are simply locations on that line. In contrast, a line could be a collection of points, which are not locations on the line, but they actually form and define the line. This second provides a metaphor which allows students to reason with a *gapless* conception

of continuity, which holds that between any two points which are part of the curve there is no space or gap. This is a little different than the idea that there are no jumps, which was described as a dynamic conception. The gapless conception considers static points, while the dynamic conception considers the movement of points (or a pencil) along the line.

Tall and Vinner (1981) identified a few other conceptions about continuity. When given a few graphs and equations of functions, students were asked to explain if each was continuous and to justify their response. Various conceptions were identified. For example, some students felt that in order for a function to be continuous, it must be defined by one equation (meaning it is not piece-wise). Similarly, students would claim a function continuous simply because it *was* defined by one equation, even if the argument was a rational expression. Students also held in their concept image of continuity, the importance that the curve itself is in one piece, not have any sudden or sharp curves, and not grow infinitely at any point. I have also observed students using such visual cues when determining the continuity of a function and will call using reasoning such as a graph being in one piece, being smooth, or straight to determine continuity the *visual* conception of continuity.

**Formal definitions.** Finally, I identify the formal definitions for function continuity and the limit of a function as they are used throughout this research. When I describe a formal definition for either of these topics, I refer to the precise or rigorous definitions, both of which use quantifiers, inequalities, and the symbols  $\epsilon$  and  $\delta$ . Most students do not intuitively reason with the formal definitions, but because students are presented with these conceptions it is possible for these definitions to influence the development of PPCs. Formal continuity can be described in two particular ways: continuity at a point and continuity on an interval. Formal definitions for limits can be considered as  $x$  approaches a particular point or as  $x$  approaches

infinity. The focus of this study is on continuity and limit at a point, so only these formal definitions are provided. I also define what I call the *semi-formal* definition for continuity at a point, which relates continuity to the limit value. Still a formally accepted definition, this is slightly more intuitive to students than the first two I define. The formal definitions as accepted by the mathematical community and as they will be referred to throughout the paper are in Figure 1.

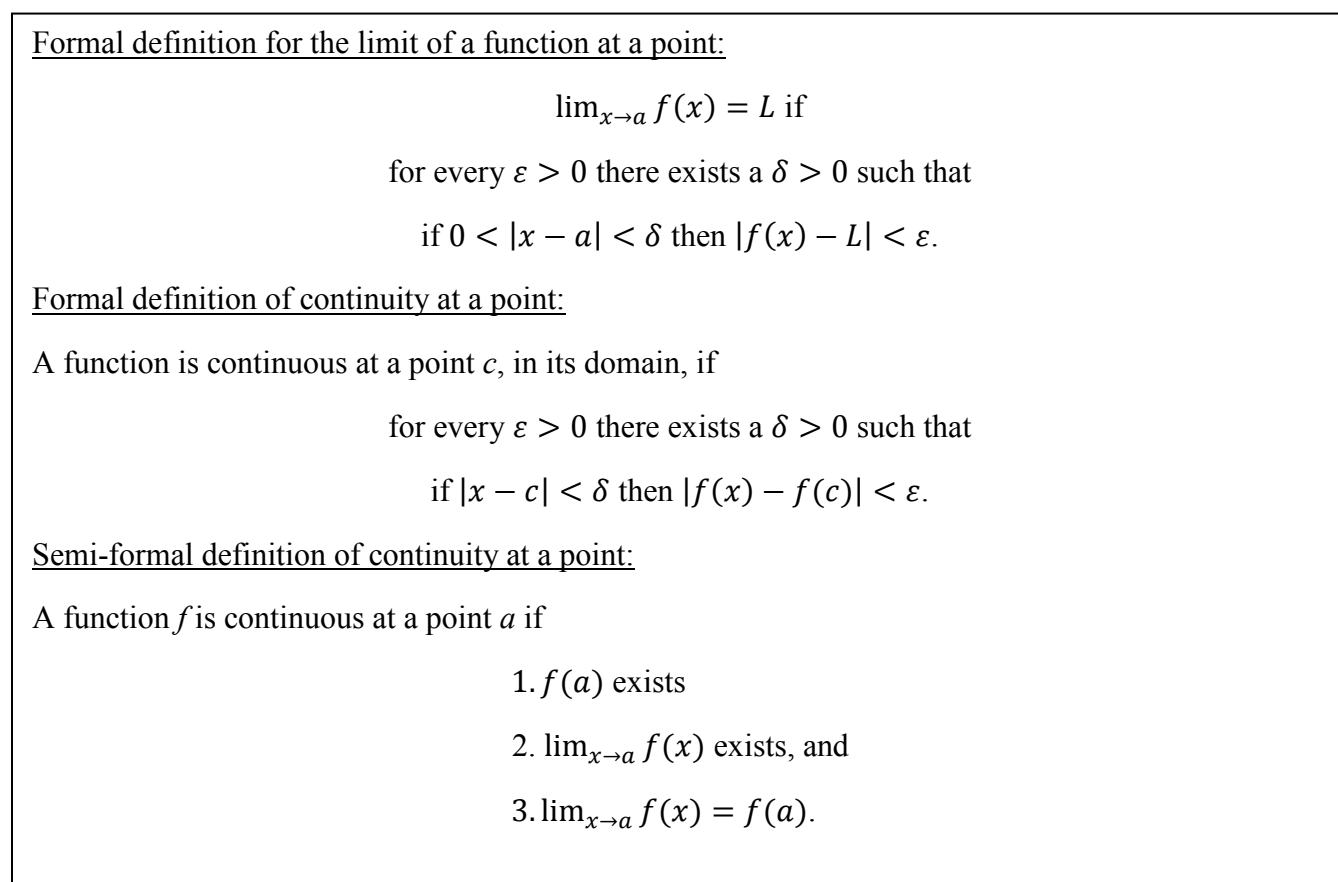


Figure 1. Formal definitions of limit and continuity at a point.

Thus we see there are many conceptions students may use to reason about continuity and limits. It seems that student conceptions about continuity and limit would influence how they reason about the PPCs about continuity and limit at a point. I used the described general conceptions for continuity and limit as I studied the PPCs students hold about continuity and limit at a point and which of those general conceptions they use to reason about the PPCs. For

example, some students could reason that you evaluate a limit by substituting the value  $x$  is approaching into the function. Students reasoning with this algorithmic conception of limit may claim that the limit value and function value were the same, i.e., they may hold the Limit Equals Function Value PPC. Students could also use general conceptions to reason that a PPC does not apply in a particular situation. For example, students using a semi-formal conception of continuity may not hold the Limit Means Continuous PPC because they know a limit could exist without the function being continuous since limit existence is only one of three conditions in the semi-formal conception of continuity.

### **Research Questions**

This study focuses on student conceptions of continuity and limits, specifically looking at the PPCs students have about continuity or limit at a point. It is informed by existing research because students used many of the conceptions listed above as they reason about limit and continuity at a point. It will differ from much of the existing research because I studied the nature of PPCs about continuity and limit at a point, which many have only recognized and identified in earlier studies. I went beyond the more simple identification of the potential misconceptions and I studied the general conception of continuity and limit students use to reason about the four PPCs (Nesher, 1987; Smith III et al., 1994). The sample size of this research also differs from existing literature because most other studies only look at calculus students in their first semester or year, with the exception of Duru (2010) who looked at pre-service teachers and Przenioslo (2004) who looked at Calculus students at various points after taking calculus, but did not compare their reasoning on these specific four PPCs. This research focuses on students at various levels up into introductory analysis classes, thus extending the reach well beyond what the existing literature addresses. The present research attempts to answer the following questions:

How pervasive are potentially problematic conceptions (PPCs) about continuity and limit at a point? What conceptions of continuity and limit do students utilize when reasoning about continuity and limit at a point?



### **Chapter 3: Methodology**

In this chapter I describe the research methodology I used as well as why it was designed the way it is. I begin by describing the participants and setting of the study. Following this background introduction, I discuss my use of questionnaire and interview data and how these data were collected. My plan for organizing and analyzing the data are described before concluding with my expectations for the form or categories of the results.

#### **Participants and Setting**

This study was carried out at Brigham Young University, a large private university in the mountain west. BYU is recognized by the MAA as unusually successful with Calculus instruction (Bressoud, 2012). Participants consisted of university students taking various courses: first, second, and third semester calculus students, students taking either of two math courses for engineers, and first semester analysis students. The particular sections (within each course) I studied were a convenience sample: specifically, sections with instructors who were willing to let me into their classroom and have their students participate in this study. I chose to study how pervasive the PPCs about continuity and limits at a point are in various classes from beginning calculus through introductory analysis to gain some insight into how pervasive and widespread these PPCs were. A pilot interview with a multivariable (third semester) calculus student showed that students in that course may have great success (pass with an A or A-) and still hold PPCs and misconceptions. Through personal conversation with various graduate students, I found that even students graduating with a degree in mathematics education, requiring a number of upper-division math courses, could still have such PPCs and misconceptions. Students from all courses were invited to complete a questionnaire.

In addition to the questionnaire, first-semester calculus students were also interviewed. The goals of the interviews were to see how different students reason about continuity and limit at a point, what PPCs they hold, and which conceptions of continuity and limit they employ as they reason. I needed to interview enough students to have a variation in student reasoning, the conceptions they used, and the PPCs they employed while still keeping the study and work involved appropriate for a master's thesis. The interviewed students were chosen based on their answers to the questionnaire. Selecting students based on their responses allowed for the richest data possible from the interviews. I began by only choosing students who seem to have one or more of the PPCs. The first three interviewees only showed evidence of one or two PPCs and their understandings seemed very similar to each other. I then chose students with evidence of three or four of the PPCs and I also chose students with varying responses about their preferred way to think about continuity and limits. This helped me see how students use different conceptions of continuity and limit as they reason about continuity and limit at a point. Recommendation from the instructors was also used to ensure the productivity of the interviews. For example, a student seemed to have promising questionnaire responses, showing evidence of all four PPCs, but the teaching assistant familiar with this student identified him as one whose answers on the questionnaire didn't seem to match his understanding, suggesting language deficiencies may have gotten in the way. Ten students agreed to be interviewed and one dropped out. After these nine students were interviewed, I had observed many different conceptions of continuity and limit being used and evidence of all four PPCs used.

### **Data and Data Collection**

**Questionnaire.** Data for this study were collected in the form of a questionnaire and personal interviews. Eight hundred sixty one students submitted a completed questionnaire. The

sample size for each participating course is as follows (see Table 1): 392 first semester calculus students, 275 second semester calculus students, 104 first semester math for engineering students, 32 second semester math for engineering students, 37 third semester calculus students, and 21 theory of analysis students.

Table 1  
*Sample Size for Each Course*

Course	Sample Size
Calculus 1	392
Calculus 2	275
Math for Engineering 1	104
Math for Engineering 2	32
Multivariable Calculus	37
Theory of Analysis	21

The questionnaire had four purposes: (1) to determine the pervasiveness of PPCs about continuity and limit at a point, (2) to determine what conceptions students held, (3) to determine correlations between PPCs and conceptions, and (4) to choose interviewees who hold such PPCs. The entire questionnaire is available for reference in Appendix A. The questionnaire included questions about major and the particular mathematics course the student was taking. Asking these questions allowed me to organize the results both by course and by major. The most important questions in the questionnaire were those identifying if the students had any of the PPCs about continuity or limit at a point discussed in the previous chapter. The mathematics questions had been adapted from questionnaires and interviews conducted by Bezuidenhout (2001) and Williams (1991, 2001). The mathematics questions began with four questions which indicated how students thought they reasoned about continuity and limit.

Students were given four descriptions of continuity and seven descriptions of limit listed in Table 2. Each relates to a particular conception or definition. It is important to note that the fourth continuity conception was typed incorrectly on the questionnaire, as reflected in Table 2,

and will not be discussed further because it does not reflect student understanding. This conception was only added for students taking Theory of Analysis, for which the text referred to continuity in terms of converging sequences (Abbott, 2001). Students who had not taken Theory of Analysis were not expected to have ever seen such a definition or have the abilities to reason about it. The first two questions showed the continuity conceptions with the first asking students to mark each as true or false and the second asking students to choose the conception which best represented their understanding of continuity. The third and fourth questions asked the same of the limit conceptions. These questions helped indicate what level the students were reasoning on and the type of conceptions they held for continuity and limit.

Table 2  
*Continuity and Limit Conceptions on Questionnaire*

Statement Number	Description	Conception/ Definition
<b>Continuity</b>		
1	A graph is continuous if you can draw it without picking up your pencil.	Dynamic
2	A graph is continuous if there are no holes or jumps in the line.	Lack of Discontinuities
3	A graph is continuous if it is continuous at every point on that line. To be continuous at a point the function values has to be equal to the limit of the function at that point.	Semi-Formal definition
4	A graph is continuous when there exists a convergent sequence in the domain produces a convergent sequence in the range.	Sequence convergence
<b>Limits</b>		
1	A limit is a type of math problem you solve, like determining the value of $\sin \frac{5\pi}{6}$ .	Algorithm
2	A limit is a point or number the function gets close to but never reaches.	Unreachable
3	A limit describes how a function moves as $x$ moves toward a certain point.	Dynamic (Theoretical)
4	A limit is a number or point past which the function cannot go.	Boundary
5	A limit is a number that the $y$ -values of a function can be made arbitrarily close to.	Proximity
6	A limit is used to approximate a value you cannot determine.	Approximation
7	A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.	Dynamic-Practical

The fifth and sixth questions gave students that  $\lim_{x \rightarrow 2} f(x) = 3$  or  $g(5) = 7$ , respectively, and asked which of a given set of statements must also be true (see Table 3). Students were able to select as many choices as they believed to be true. Each PPC was represented in one or more of the options provided. The statements and the PPC students may be using if they mark it as true are presented in Table 3. It is important to note that all of these mathematical situations presented on the questionnaire were general, without a specific example. Thus students who showed evidence of using a PPC on the questionnaire showed general use of the PPC, and we cannot make claims about how the student would use the PPC if a specific mathematical example were given. In other words, we cannot know from the questionnaire how the students would reason (correctly, incorrectly, harmlessly, etc.) when given a particular mathematical situation.

Table 3  
*PPCs and Questionnaire Statements*

<b>PPC student may hold</b>	<b>Statement student marked as true</b>
Defined Means Continuous	Given $g(5) = 7$ , $g$ must be continuous.
Limit Equals Function Value	Given $\lim_{x \rightarrow 2} f(x) = 3$ , $f(2) = 3$ .
Limit Equals Function Value	Given $g(5) = 7$ , $\lim_{x \rightarrow 5} g(x) = 7$ .
Limit Means Continuous	Given $\lim_{x \rightarrow 2} f(x) = 3$ , $f$ must be continuous.
Limit Means Discontinuous	Given $\lim_{x \rightarrow 2} f(x) = 3$ , $f$ must be discontinuous.

While analyzing all six mathematics questions, I was able to consider conceptions for continuity and limit that tend to mean students may have a PPC, or those conceptions that mean students are not likely to have the PPCs I am studying. The questionnaire helped answer the research questions, specifically about how pervasive the PPCs are and which conceptions of continuity and limit students are using. To get further insight into which conceptions students were using while they reasoned about continuity and limit at a point, individual interviews were conducted 1-4 weeks after the questionnaires were collected.

**Interviews.** Interviews were conducted in order to gather richer data about which conceptions of continuity and limit students utilized when reasoning about continuity and limit at a point and to potentially provide examples of patterns identified in the questionnaire. Nine first-semester calculus students were chosen to participate in a one-hour long personal interview. The interviews were conducted close to the end of the course, after the students have been introduced to and tested on continuity and limits. Paper and pen were available to the student throughout the interview and questions with a visual component were printed on separate pages so the student could write on it if they wanted. This procedure was standard for all interviews. All interviews were audio recorded and the interviewee's written work was collected for the researcher's reference, transcription, and coding. The examples in the interviews were chosen to provide students examples where the PPCs may be used incorrectly, to determine if students who seem to hold the PPCs in general also use them as misconceptions. The examples were also intended to give students the opportunity to push the boundaries of applicability of the PPCs – for example, to realize the incomplete reasoning behind the PPCs. This gave some indication of how deeply the PPCs were held and how pervasive their use was. The interview questions can be found in Appendix A.

The goal of the interview was to get an idea of the types of conceptions of continuity and limit students used as they reasoned about continuity and limits at a point. The purpose of the first three questions was to gather evidence as to whether the student was reasoning with one of the problematic conceptions about continuity or limit at a point (See Figure 2). The most important data from these questions came as the interviewer inquired why the students answered the way they did. When the interview question (see Appendix A) includes the prompt, "Why?" the interviewer asked more specific questions to help determine which conceptions of continuity

or limit students were using as they reasoned about these situations. The questions varied from student to student, but the following were typical: How were you thinking about the function being continuous/discontinuous as you answered this question? How did what you said show that the limit exists? Can you think of a counterexample? Could you give me an example of what you mean by “continuous” (or any other term for which the student’s meaning is not clear to the interviewer)? How would you describe a limit/continuity to someone who had not learned about it yet? Would you say a limit is (or continuity is) \_\_\_\_\_ (describing a conception of limit or continuity the interviewer thinks the student may be using)? These questions attempted to provide evidence of students reasoning with PPCs as well as evidence of which conceptions of continuity and limit they used while reasoning about continuity and limit at a point.

1. Is it possible to draw a function such that the limit as  $x$  approaches 3 exists but,
  - a. The function is discontinuous at  $x = 3$ ? Why?
  - b. The function is continuous at  $x = 3$ ? Why?
  - c. The function is undefined at  $x = 3$ ? Why?
  - d. The function is defined at  $x = 3$ ? Why?
2. Is it possible to draw a discontinuous function where the limit exists at every point in the domain? Why?
3. Is it possible to draw a continuous function where the limit does not exist? Why?

*Figure 2.* Interview questions 1-3.

The student was then given 5 graphs and was asked to identify the graphs that represent a function which is continuous at the point  $x=4$ , the functions can you take the limit of as  $x$  approaches 4, and the graph(s) for which the limit exists as  $x$  approaches 4. The graphs and the misconceptions that may be identified as students reasoned about them are presented in Figure 3. They are misconceptions rather than only PPCs because the graphs provide the situation that would make the PPC incorrect. Larger versions of the graphs can be found in Appendix A.

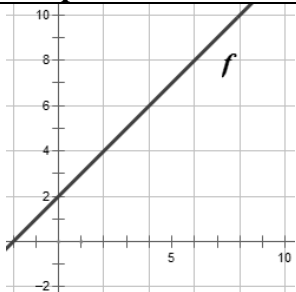
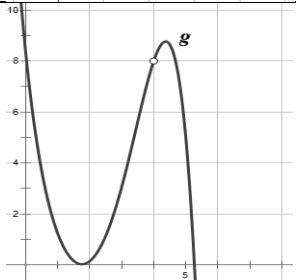
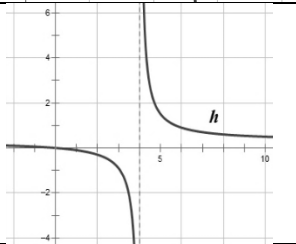
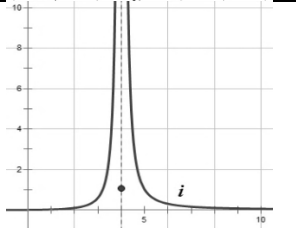
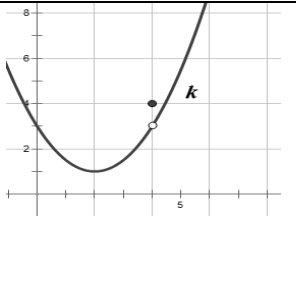
Graph	Misconception
	<p>The student may hold the conception that limits only refer to functions with discontinuities, the Limit Means Discontinuous PPC, if they do not say you can take the limit or that the limit exists for this function.</p>
	<p>If the student does not say that the limit exists here, they may believe that limit means continuous (or discontinuous means the limit does not exist), the Limit Means Continuous PPC.</p>
	<p>If the student does not say that the limit exists here, they may believe that limit means continuous (or discontinuous means the limit does not exist), the Limit Means Continuous PPC. This is simply a different type of discontinuity than the previous graph.</p>
	<p>The student might believe that when a function is defined it is continuous at that point if they say this is continuous, the Defined Means Continuous PPC. This is also a place where the substitution conception can come in if students say the limit exists and it is the value of the function, the Limit Equals Function Value PPC.</p>
	<p>The student might believe that when a function is defined it is continuous at that point if they say this is continuous, the Defined Means Continuous PPC. This is also a place where the substitution conception can come in if students say the limit exists and it is the value of the function, the Limit Equals Function Value PPC. The student could also say the limit does not exist here because the function is discontinuous, the Limit Means Continuous PPC.</p>

Figure 3. Graphs from the interviews. This figure shows each graph students were given during the interview along with the misconception which may be identified while students reason.

The student was then asked the same questions that were in the questionnaire about which statements are true given that  $\lim_{x \rightarrow 2} f(x) = 3$  and given  $g(5) = 7$ . I asked these questions again because I expected that some students may realize the potential problem of the



PPC by explicitly thinking about limits and continuity at the same time, or simply because they were thinking differently than when they initially completed the questionnaire. After the student answered the questions again, the researcher compared their answers to their answers on the original questionnaire to see if *any* of the PPCs that arose during the questionnaire were not being reasoned with anymore (or if any new ones developed). The researcher explored each PPC in greater depth by asking the “Why?” questions described above. The interviewer again attempted to identify the conceptions about continuity and limit the students are using when they reason about these situations. This included a discussion of why some answers changed for the student, paying close attention to what general conceptions the student was using to describe the corrected conception. After the first few interviews I wondered if the order of the questions would influence how students reasoned. For example, students referenced examples provided earlier during the interview in their reasoning about general situations such as those on the questionnaire. It seemed like students’ reasoning could be different if they were given the opportunity to reason generally, by reviewing the questionnaire questions, before being asked about specific examples. So for six interviews the questions were asked in a different order. This change in question order helped the general reasoning brought on by the questionnaire questions to be answered entirely by the student rather than borrowing the examples presented by the interviewer. Noticing most of the questions related to a graphical approach to the functions, the last three interviewees were also asked about a piecewise function and the limit and continuity at two points therein (See interview questions in Appendix A). At the conclusion of the interview, the interviewer took some time with the student to try and help the student learn more about the PPCs and misconceptions they still held if the students wanted such help.

**Differences in questionnaire and interview data.** As described above, the two instruments had different purposes. It was important to use both instruments because although they both were used to evaluate how students thought about the continuity and limit conceptions and the four PPCs, the claims we can make from each instrument are a bit different. On the questionnaire, students marked true or false each of a predetermined list of conceptions. When they chose a conception to best represent their understanding, they were also choosing from this predetermined list. Their answers to the mathematical questions where they may have used the PPCs were about general mathematical situations rather than specific mathematical examples. The nature of the questionnaire allowed for many students from various courses to participate. On the other hand, the interviews allowed the students to speak freely about how they reason about continuity and limit, thus providing richer data than the student simply marking true or false about something provided by the researcher. Students in the interviews also responded to various specific mathematical examples, allowing them to show evidence of using the PPCs as misconceptions or express which conceptions of continuity or limit they used as they reasoned. Thus from the questionnaire, many students participated providing general information about which conceptions are considered true, false, or preferred and if they used the PPCs generally. The interviews gathered richer information about fewer students' reasoning about continuity, limits, and the PPCs in particular mathematical situations.

### **Management and Analysis**

I began analysis after the questionnaires were collected. The questionnaire had four purposes: to determine the pervasiveness of PPCs about continuity and limit at a point, to determine what conceptions students held, to determine correlations between PPCs and conceptions, and to choose interviewees who hold such PPCs.

Questionnaires were analyzed by looking for recurring patterns in student responses. The first analysis was to look at the percentages of students in each course which have PPCs about continuity and limit at a point. This provided an answer to the first research question regarding the pervasiveness of the PPCs. The second analysis was to determine the most common conceptions students chose, beginning to answer the second research question which the interviews would develop further. The third analysis done with the questionnaires was to see if there are patterns or correlations between the PPCs students hold and the conceptions they have of continuity and limit. A Chi-squared analysis was done comparing the continuity and limit conceptions and the evidence that a student held a PPC. This was done to look for relationships between particular conceptions and the PPCs. Of course, the questionnaires were not enough to provide a complete answer to my questions, so the interviews were included in the data as well.

Those portions of each interview were transcribed for which the discussion related to the student's conception of continuity and limit or their reasoning about continuity and limit at a point. The entire interview focused on these types of discussion, meaning the only parts of interviews which were not transcribed were those in which the student lost focus or strayed from answering the question at hand, such as discussing a broken pencil. The data from the interviews were then coded, organized, and analyzed. The unit of analysis was a student's answer to a question or follow-up question, provided that it included a *reason* for their response. I called such a unit an *excerpt*. For example, when asked the value of the limit, if a student answered "4" it was not an excerpt until the student then answered a follow-up question and described *why* the limit was 4. When that response was added to the original answer, it formed a full excerpt to be analyzed. When describing their answers to the questionnaire, some students addressed each part of the questions separately and some addressed a few at once. Those that did it separately have

different excerpts, but those where their answers ran together are one excerpt. I chose this unit of analysis because research shows that students may use more than one conception of continuity or limit at once. A smaller unit such as a sentence may break up two conceptions that the student was reasoning with at the same time, but said in different sentences during his response. Both needed to be included in the same excerpt so patterns could be considered about the conceptions students often used together. A larger unit would gather too much into one excerpt. Students sometimes used one conception with specific examples and a different conception in another context. For example, a student may think of limit dynamically with a removable discontinuity, but as a tangent line when the function is continuous. Before coding began, reviewing the existing literature enabled me to predict a few patterns I might see and identify some preliminary codes (see Table 4). These codes were simply the conceptions of continuity and limit identified and described from the literature discussed in the previous chapter as well as each of the four PPCs.

Table 4  
*Preliminary Codes Used for Analysis of Interviews*

<b>Conception</b>	<b>Description/Example</b>	<b>Page</b>
Formal	Statement or reference to the formal definition see Figure 1 in Chapter 2.	18
Algorithm	Use of an algorithm such as L'Hospital's Rule in evaluating a limit	13
Substitution	Evaluating a limit by plugging the value $x$ is approaching into the function.	13
Dynamic	Use of language such as "approaching" or "tends to" for a limit or references drawing a function without lifting a pencil for continuity.	13
Bounding	Describing a limit as something the function cannot cross or pass.	16
Proximity	Use of language such as "when $x$ is close to $a$ , $f(x)$ is close to $L$ ." Also the idea of a neighborhood around $a$ or $L$ .	15
Approximation	Describing the point of a limit as to get an estimate of what the function value would be.	16
Unreachable	Describing a limit as a value for which the function cannot be defined or that the graph cannot touch.	16
Lack of Discontinuities	Student says a function is continuous because there are no holes, gaps breaks, jumps, or asymptotes.	21
Gapless	Describing continuity as when the points are so close together that there is no space between them.	17
Visual	Describing a line as continuous because it is "smooth," "in one piece," or "straight."	23
Defined Means Continuous PPC	Claiming a function to be continuous because it is defined at every point	9-11
Limit Equals Function Value PPC	Claiming the function value and the limit value are the same in a case where the student does not know if the function is continuous	9-11
Limit Means Continuous PPC	Assuming that the limit existing means the function is continuous	9-11
Limit Means Discontinuous PPC	Describing how limits do not matter or do not apply to continuous functions	9-11

I did not expect that these codes would be the only ones necessary for my analysis, so as the data were transcribed and coded, new codes emerged and were used to organize the data. I used the steps of constant comparison described by Glaser and Strauss (1967) with reference to grounded theory, with the exception that I was not generating a theory. I used the first two steps they identify: comparing incidents and integrating categories and their properties. From the

preliminary codes I developed, categories were the conceptions students used to think about continuity or limits. When new categories emerged, I used them, comparing the characteristics to other conceptions to see if those could be combined. This allowed me to develop new codes as they emerged while keeping track of the relationships between PPCs about continuity and limit at a point. I will now briefly describe the new codes formed during the coding process.

Two types of new codes were created, conception codes and PPC codes. New conception codes were created for conceptions of continuity and limit that did not seem to fall into the codes designed from previous research. An example of a conception code was the Point Doesn't Matter code. Students would discuss the limit and describe everything around the limit making a claim that the particular point we are approaching does not matter. This is similar to the Unreachable code, except some students were aware that the point could be reached, they noted that what happened at the point simply did not influence the existence or value of the limit. This new code was often used along with another code such as Dynamic Limit which would also describe their reasoning. The Point Doesn't Matter code was important because some students put great importance on what happened at the point, sometimes resulting in a misconception from using the Limit Equals Function Value PPC, the limit value and function value are the same, incorrectly. Other conception codes were created when similar circumstances arose during the coding process. An extensive list of codes used and their descriptions can be found in Appendix B. The other type of codes created during the coding process was PPC codes. These were codes involving the pieces of the PPCs I am studying. For example, the Defined Means Continuous PPC is the belief that if a function is defined at a point, it is continuous at that point. The two pieces of this PPC relate to the function's Definition at a Point and Continuity at a Point. Each

PPC was broken down into such pieces so a refined coding could occur such that any opportunity where students may have used a PPC could be evaluated.

I will now more thoroughly describe the refined coding used to analyze the students' use of the PPCs or their reasoning in situations where a PPC could be used. To continue the example with the Defined Means Continuous PPC, when an excerpt was coded with both the Definition at a Point and Continuity at a Point codes, the excerpt was then coded with respect to how the student used or did not use the Defined Means Continuous PPC. There were five PPC Specific codes to describe the student's use of the PPC: General, Incorrect, Harmless, Overcome, and PPC Not Used. General reasoning with a PPC would occur when a student made a statement similar to the conception itself without considering a specific example. This type of reasoning does not necessarily lead to an error because there is no example they are reasoning about. For example, students asked what they know about a function if they know the function value at a point may conclude that they know the limit value. Depending on the situation, the limit value may equal the function value or it may not. Without the situation we cannot determine if the student's reasoning lead to an error, so we cannot determine if it is a misconception. The student may reason in general that the limit value and function value are the same, but given an example where this reasoning would lead to an error, the student may realize it is incorrect. Thus, without a specific example we can only claim that the student holds the PPC as a general belief. Incorrect reasoning is when the student considers a specific example, say a function with a defined jump discontinuity, and the student reasons that since they know the function value, the limit value must be the same (the Limit Equals Function Value PPC), an error in this situation. Their reasoning lead to an error meaning the PPC in this case is a misconception and such a case would have been coded Incorrect. A student can also use a PPC with a specific example such that their

conclusion is not an error. This is considered Harmless use of a PPC. For example, if given a continuous function a student claims they know the limit value because they know the function value, they have used the Limit Equals Function Value PPC and their conclusion is not an error, but reasoning that the limit value and function value are the same is not complete reasoning. In this case, the reasoning would be complete if the student also indicated that it was the continuity of the function that made their reasoning appropriate for this function. Without using the important characteristic of continuity, their reasoning is incomplete although harmless because their conclusion is not an error. When a student has Overcome a PPC, this means they used the PPC in general or incorrectly, as a misconception, at some point during the interview, but they have realized their incomplete or incorrect reasoning. For example, the student who reasoned the value of the function at the jump was the limit value at that point may indicate later that they remembered the limit must be the same from both sides, then realizing their previous reasoning and conclusion were incorrect. Finally, a student's reasoning may be PPC Not Used if they are in a situation when the PPC may have been utilized incorrectly, but the student reasons appropriately with some other reasoning or conception. For example, when reasoning about a defined jump discontinuity a student may simply reason that the limit does not exist because the limits on the left and right are not equal. Thus, when coded PPC Not Used, the student has not used the PPC in an instance where if they had used it their conclusion and reasoning would have been incorrect. These codes are PPC specific because they would be coded along with the Defined Means Continuous, Limit Equals Function Value PPC, Limit Means Continuous PPC, or Limit Means Discontinuous PPC. Table 5 shows a summarized meaning of each of these codes. Once this final refined coding was completed, I compared the conceptions of continuity



and limit students were using in each excerpt with the way they were reasoning with the PPC to look for any patterns that emerged. This completed the coding and analysis from the interviews.

Table 5  
*Specific PPC Codes*

<b>PPC specific codes</b>	<b>Meaning</b>
General	Student used the PPC to reason about a general situation of limit or continuity at a point. No particular example was being reasoned about, so it cannot be determined if their reasoning was correct or incorrect.
Incorrect	Student used the PPC to reason in a situation where the PPC was incorrect.
Harmless	Student used the PPC to reason in a situation where the PPC alone was an incorrect or incomplete way to reason but the resulting conclusion was not an error.
Overcome	Student reasoned appropriately in a situation where the PPC may have been used incorrectly. The student previously used the PPC as a misconception or used it generally in the same or a recent excerpt.
PPC Not Used	Student reasoned appropriately in a situation where the PPC may have been used incorrectly. The student had not reasoned with the PPC as a misconception in this instance where it would have been a misconception to use the PPC.

### **Form of the Results**

This study provides insight into how pervasive PPCs about continuity and limit at a point are and the types of general conceptions about continuity and limit students use when reasoning about continuity and limit at a point. The results are in the form of percentages of students in each course which hold such PPCs, correlations between PPCs and conceptions of continuity and limit, and descriptions of the conceptions of continuity and limit students utilize as they reason about continuity and limit at a point. Although the sample was from only one university, the results provide insight into the research questions and extend the existing literature on student conceptions of continuity and limit.

## **Summary**

Through this study I attempted to answer the following questions: How pervasive are PPCs about continuity and limit at a point? What conceptions of continuity and limit do students utilize when reasoning about continuity and limit at a point? Data were collected through a questionnaire completed by 861 students in various university mathematics courses and through personal interviews with nine first semester calculus students. Questionnaire data were organized to determine the pervasiveness of PPCs about continuity and limit at a point, to determine what conceptions students held, and to determine correlations between PPCs and conceptions. Interview data were organized to identify and describe which conceptions students employ when reasoning about continuity and limit at a point and the PPCs and to provide examples of patterns identified in the questionnaire.

## Chapter 4: Results

In this chapter I describe the results of the study. I begin by addressing the first research question: How pervasive are potentially problematic conceptions about continuity and limit at a point? I then address the second research question: What conceptions of continuity and limit do students utilize when reasoning about continuity and limit at a point? Additional patterns were revealed through data analysis and I describe such results for each of the four PPCs with the Limit Means Continuous PPC and the Limit Means Discontinuous PPC combined because of their related nature.

### Pervasiveness of the PPCs

The pervasiveness of each PPC was evaluated primarily through the questionnaire. Questions on the questionnaire allowed for students to show evidence of using each PPC in general. For the Limit Equals Function Value PPC there were two such questions and for each of the other PPCs there was one such question. Each student was given a PPC Score from 0-5 where one point was assigned for each question for which they showed evidence of holding a PPC. It is important to consider the nature of the questionnaire while interpreting these results. Evidence of the PPCs was collected through true or false questions. If students were not familiar with components of answering true or false questions, such as thinking of counterexamples, students may have marked true for an item, thus providing evidence of holding the PPC in general, even if the student would not have used the PPC incorrectly making it a misconception. Thus, as we consider the questionnaire results it is always evidence of holding the PPC *in general* that is the matter of discussion. The questionnaire only detects students' general reasoning, and thus cannot be used to determine if the student would continue to use the PPC in a

situation that would cause it to be a misconception. I will describe the general results using the PPC scores and then the results from each specific PPC.

The average PPC score per course slightly dropped when students took Math for Engineering 1, but did not significantly drop until students took Theory of Analysis. The average PPC scores for each course are displayed in Figure 4. It can be seen that the average score for each of the first five courses (First, second and third semesters of Calculus, and first and second semesters of Math for Engineering) is within 0.2 points of their mean of 1.4. It is not until students take Theory of Analysis that the average PPC score drops below 0.5 out of 5 points.

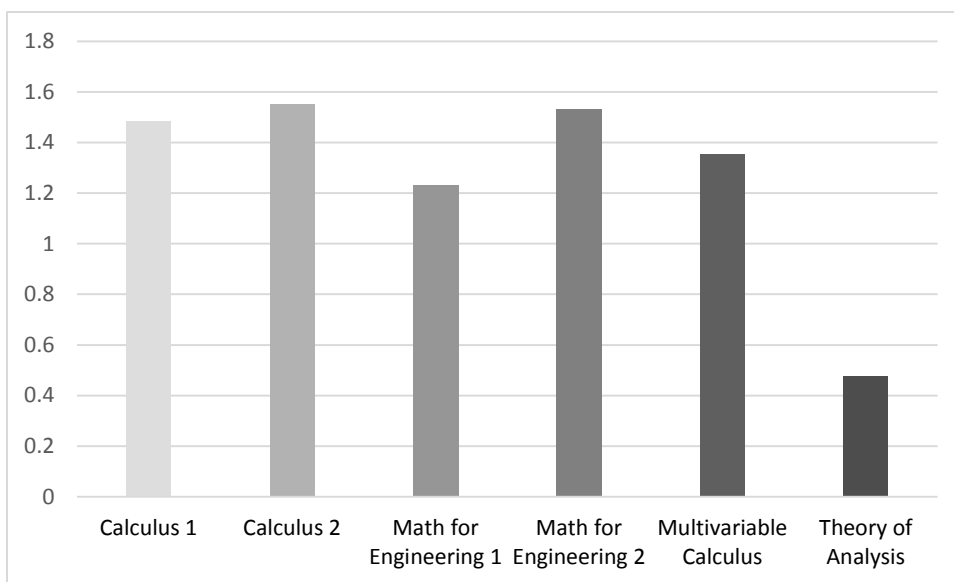


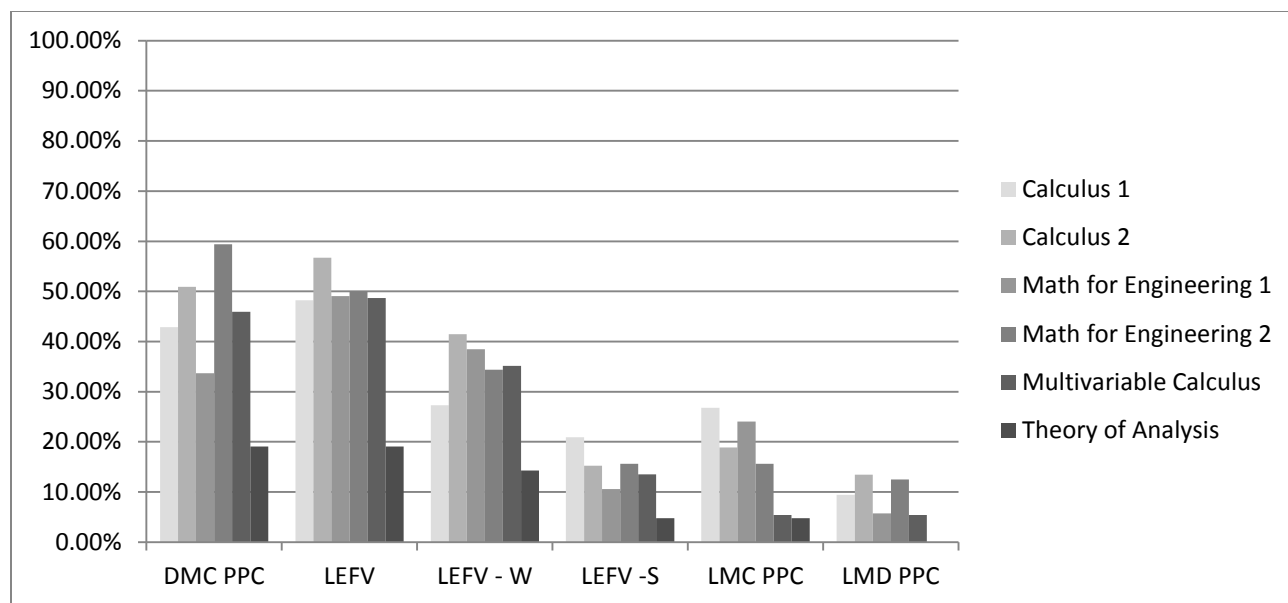
Figure 4. Average PPC scores in each course.

A similar pattern emerges when we consider the percentage of students in each course who completed the questionnaire who showed evidence of one or more of the PPCs (a PPC score of one or higher). In the first five courses at least 68.27% (Math for Engineering 1) of the students showed evidence of having at least one of the PPCs. Similarly, eight of the nine interviewed first semester calculus students used a PPC generally or incorrectly at least once.

This drops significantly once students take a course in analysis, so that only 28.57% of the analysis students showed evidence of having at least one of the PPCs. We can see that general belief in the PPCs is pervasive through the mathematics courses studied until students take a course in analysis.

When considering the percentages of students showing evidence of the Defined Means Continuous PPC and the Limit Means Discontinuous PPC across the courses, a similar pattern was detected (see Figure 5). Evidence of the Defined Means Continuous PPC was shown when students answered that a function must be continuous given  $g(5) = 7$ . Evidence of the Limit Means Discontinuous PPC was given when students answered that a function must be discontinuous when given  $\lim_{x \rightarrow 2} f(x) = 3$ . The trend shows the percentage of students increasing from first to second semester calculus, dropping for first semester math for engineering, rising again in second semester math for engineering, and then falling in the last two courses to the lowest percentage in the analysis course. For both PPCs the courses with the highest percentages of students showing evidence of holding them are Calculus 2 and Math for Engineering 2. The drop in percentage for Math for Engineering students may be because limits and continuity are in the course objectives for first semester math for engineering but are not for second semester, causing students to forget about continuity and limit and possibly causing the second semester students to show greater evidence of holding these PPCs. Both are lowest in Theory of Analysis, as would be expected after considering the drop in average PPC score once students take this course. Although the patterns across the courses are the same, it is important to note that the percentage of students holding the Defined Means Continuous PPC is much higher than that of the Limit Means Discontinuous PPC. Perhaps this is because continuity, the concept in the Defined Means Continuous PPC, is not studied as thoroughly as limits, one of the concepts

from the Limit Means Discontinuous PPC. Other studies have identified the Defined Means Continuous PPC as a common conception students hold (Bezuidenhout, 2001; Duru et al., 2010). Two studies identified the Limit Means Discontinuous PPC in students' reasoning but those were only examples of particular students, not the identification of general acceptance of the PPC as common for students (Juter, 2006; Williams, 1991). Thus a smaller percentage of students holding the Limit Means Discontinuous PPC than the Defined Means Continuous PPC seems reasonable. The average percentage for the Defined Means Continuous PPC across the courses was 50.36% while the average percentage for the Limit Means Discontinuous PPC across the courses was only 9.31%. The Limit Means Discontinuous PPC has the lowest percentage of students in all 6 courses, with no students showing evidence of the Limit Means Discontinuous PPC in the analysis course. These percentages for the Defined Means Continuous PPC are higher than those identified in previous research (Bezuidenhout, 2001; Duru et al., 2010), although these studies were focusing more on the misconceptions. Misconceptions are the PPCs used inappropriately, so it is reasonable that their percentages would be lower. Although the percentages for the Limit Means Discontinuous PPC are low, it provides evidence that some students do reason like the examples from Williams (1991) and Juter (2006) described previously in the literature review.



*Figure 5.* Evidence of PPCs. This figure shows the percentage of students in each course showing evidence of each PPC. Each PPC is represented and the Limit Equals Function Value PPC is also broken down into weak evidence, LEFV-W, such that students only showed evidence in one of the two instances on the questionnaire and strong evidence, LEFV-S, such that students showed evidence in both instances on the questionnaire.

The Limit Equals Function Value PPC (LEFV PPC) has similar patterns across the courses as did the average PPC scores, with higher percentages in the first five courses and a drop finally when students take a course in analysis. There were two instances on the questionnaire through which students could show evidence of having the Limit Equals Function Value PPC. Thus students were categorized as showing strong evidence of the Limit Equals Function Value PPC (LEFV-S) if they did so with both problems and weak evidence of the Limit Equals Function Value PPC (LEFV-W) if in only one instance they showed evidence of the Limit Equals Function Value PPC. I only considered the percentages of students with either weak or strong evidence of the Limit Equals Function Value PPC in this discussion, although results for weak and strong conceptions are available in Figure 5 as well. Students displayed evidence of the Limit Equals Function Value PPC by answering  $\lim_{x \rightarrow 5} g(x) = 7$  given  $g(5) = 7$  or by answering  $f(2) = 3$  given  $\lim_{x \rightarrow 2} f(x) = 3$ , thus reasoning that the limit value and

function value at a point are the same. Nearly 50% of students in the first 5 courses, with a jump up to 56.73% in second semester calculus, showed evidence of the Limit Equals Function Value PPC in at least one instance. Similar to the average PPC score and other PPCs this did not drop until Theory of Analysis when only 19.05% showed evidence of the Limit Equals Function Value PPC. Again these results are consistent with previous research which also identified this PPC in general and as a misconception (Bezuidenhout, 2001; Jordaan, 2009; Prezenioslo, 2004).

An interesting observation was made with students who showed weak evidence of the Limit Equals Function Value PPC. To show evidence students either marked that given a function value they knew the limit value would be the same or given the limit value the function value would be the same. Of the students displaying weak evidence, meaning they only answered in one of the above ways, 82.99% did so given the function value, and the remaining 17.01% did so given the limit value. Perhaps then, students are less certain about what it means for a function to be defined at a point and what information that provides about the limit than they are about what limit existence guarantees about the function value.

Evidence of holding the Limit Means Continuous PPC (LMC PPC) is highest (26.29%) in first semester Calculus then decreases through the remainder of the courses except for a jump with students in Math for Engineering 1. The Limit Means Continuous PPC was evidenced when students answered that the function must be continuous given  $\lim_{x \rightarrow 2} f(x) = 3$ . It is interesting to note that only for the Limit Means Continuous PPC does the percentage of students go up for those taking Math for Engineering 1. Given the course objectives for all the courses except Math for Engineering 2 have limits and continuity included to some extent, it seems that simply having the topics in the course does not necessarily dictate whether students are more or less likely to show evidence of these PPCs. Another way in which the Limit Means Continuous PPC was



different from the others is that the drop in percentage (5.41%) occurs in Multivariable Calculus, rather than when the students take the analysis course. These results confirm the existence of the Limit Means Continuous PPC in first semester Calculus students found by Bezuidenhout (2001), although 37% of his participants showed evidence using a similar questionnaire question compared to 26.29% of first semester Calculus students in the present study.

### **Continuity and Limit Conceptions**

Students used various conceptions of continuity and limit on the questionnaire and during the interviews. Conceptions used by students in the present study are capitalized throughout the remainder of this work. The capitalization of a conception indicates that I am using that term for the conception to represent how I specifically have defined it. Table 6 presents student conceptions (beyond the PPCs) identified on the questionnaire from all participants. It includes the percentage of students who marked each conception as true and the percentage of students who chose each conception as the one which best represents their understanding, their preferred conception. Table 7 presents student conceptions identified during the interviews. It includes the percentage of excerpts in which each conception was used out of all the excerpts in which the student used a conception of either limit or continuity. For example, in 37.68% of excerpts where students described or reasoned about continuity students used a Dynamic conception of continuity. It also includes the number of students who used each conception at least once out of the nine interviewees. I follow the order used in my theoretical framework to discuss each of the conceptions of continuity and limit observed in the data. I first describe the observed conceptions which have previously been identified in other research, then the observed conceptions other researchers have not described, followed by a discussion of the previously identified conceptions

which did not appear in this study. I conclude this section with an analysis of how the preferred student conceptions from the questionnaire were similar or different across the different courses.

Table 6  
*Students Conceptions on Questionnaire*

Questionnaire Results	% True	% Preferred
Continuity Conceptions	-	-
Dynamic	87.19	27.59
Lack of Discontinuities	84.27	32.48
Semi-Formal	88.69	38.18
Limit Conceptions	-	-
Algorithm	54.76	2.33
Unreachable	67.33	27.54
Dynamic	91.93	42.36
Bounding	28.59	2.8
Proximity	81.91	12.02
Approximation	74.94	6.65
Dynamic-Practical	64.96	7.12

Table 7  
*Student Conceptions in Interviews*

Interview Results	% of Excerpts	# of Students	Interview Results	% of Excerpts	# of Students
<b>Continuity</b>	-	-	<b>Limit</b>	-	-
Dynamic	37.68	8	Dynamic	31.84	8
Faulty	17.39	3	Dynamic-Practical	2.79	3
Lack of Discontinuities	55.07	9	Bounding	2.23	1
Semi-Formal	4.35	2	Proximity	5.05	5
Semi-Semi-Formal	4.35	1	Approximation	0.00	0
Gapless	2.90	1	Unreachable	2.79	3
Predictable	4.35	1	Faulty	20.67	7
Visual	21.74	7	$x$ Approaches	2.79	3
			Tangent Line	5.03	3
-	-	-	Asymptote	3.35	3
-	-	-	Limit is a Rift	6.15	2
-	-	-	Formal	0.00	0
-	-	-	Substitution	4.47	4
-	-	-	Left/Right	21.23	8
-	-	-	Graphical	5.59	4
-	-	-	Zooming	1.68	2
-	-	-	Point Doesn't Matter	3.35	4

**Practical conceptions.** There were two entirely practical conceptions identified in previous research also identified in my study: the Algorithm conception of limit (Przenioslo,

2004), and the Substitution conception of limit (Bezuidenhout, 2001; Przenioslo, 2004). An Algorithm conception of limit is a practical conception used simply to evaluate a limit. The questionnaire presented this conception of limit as a belief that a limit is a math problem intended to be solved. Of the students who completed the questionnaire, 54.76% marked this conception as true while only 2.33% chose it as the conception which best represented their understanding of limit. A Substitution conception of limit is used when students reason that limits are evaluated by plugging the value  $x$  is approaching into the function and claiming the resulting output as the limit value. The following description of limit by Student D is representative of the types of Substitution conceptions observed during the interviews.

D: I would explain that as you see the limit and as it approaches a certain number,  $x$ , say  $x$  equals, as  $x$  approaches 2, you set  $x$  equal to 2 in your function and you solve and that's how you do a limit.

During the interviews, 4.47% of the excerpts about limit showed students reasoning with this conception and it was used by four of the nine students.

**Dynamic conceptions.** A Dynamic conception of continuity was identified by Núñez et al. (1999) and Tall and Vinner (1981). Students using this conception tend to describe continuity using motion language such as describing a continuous function as one you can draw without lifting your pencil or describing how the function “keeps going” or “flows.” This conception was one of the continuity conceptions presented to students on the questionnaire. It was a common conception in that 87.19% marked this conception as true and 27.59% of students chose it as the conception best representing their understanding. The following descriptions from the interviews are typical of students reasoning with a Dynamic conception of continuity. Student D was describing why the Dynamic conception is true, Student A was explaining why a function she drew and a provided piece-wise function were continuous, and Student B was explaining what it means for a function to be continuous.

D: Well, a graph is continuous if you can draw it without picking up your pencil which will mean it flows, it continues.

A: ... I could take my pencil and draw all the way through and there's nothing that would stop it.

A: ... so that means that it's, keeps going!

B: Like everyone would say you don't pick up your pencil, like you can just keep on drawing ...

During the interviews, 37.68% of the excerpts about continuity included students using a Dynamic conception of continuity. Eight of the nine students used this conception of continuity at least once.

One type of dynamic limit conception identified in previous research describes the theoretical motion of a function, a conception I call the Dynamic conception of limit (Eade, 2003; Erynck, 1981; Oehrtman, 2009; Przenioslo, 2004; Tall & Vinner, 1981; Williams, 1991, 2001). A Dynamic conception of limit is used as students talk about a limit describing the movement of a function. On the questionnaire the Dynamic conception was marked as true by 91.93% of the students and 42.36% of the students chose it as the conception best representing their understanding of limit. This makes the Dynamic conception of limit the most common from the questionnaire. During the interview, students were considered to be reasoning with a Dynamic conception of limit when they described the limit as how the function was moving or what the output values were approaching. If students simply used the word "approaching" this did not mean their reasoning was coded as Dynamic. Simply reading a limit statement usually involves a student using the word "approaching" so it was necessary for the students to indicate that they were reasoning about what the *output* of the function was approaching. The following excerpts from student A show examples where she reasoned with a Dynamic conception of limit because she is reasoning about what the output of the function is approaching. First she is simply describing what a limit is. In the second excerpt, she is describing why a point defined

somewhere else when there is a removable discontinuity does not influence the existence of the limit.

A: The limit is like as a function approaches that value, it's its y value, like that y value of x, is like how I would try to describe it.

A: Well just, because the value exists at a different point than what it's approaching at 4 [what the inputs are approaching], that like doesn't affect the limit. It's still approaching the same value [what the outputs are approaching] from the left and the right.

Students used a Dynamic conception of limit in 31.84% of excerpts about limit. Eight of the nine students reasoned with a dynamic conception at least once. The Dynamic conception of limit was also the most common limit conception used in the interviews.

Oehrtman (2009) identified the common use of dynamic language in his interview participants, but conjectured that this dynamic language represents more how students *talk* about limits than how they *think* about limits. In the present study, when students used what seemed to be a Dynamic conception to the interviewer they were asked to clarify what they meant. Most students simply repeated what they said or added hand motions indicating a curve where two pieces (traced by the student's hands) met at the point of interest. They did seem to be reasoning with motion. Student J thought aloud about what he meant by "moving" and although uncertain at first, seemed to conclude that it is both how he talks and thinks about limits.

J: I don't know. I don't think something's moving. Um, I think it's just a figure of speech people use to explain it (M: Uh huh) or teach a concept or make it, you know, 'cause there's, yeah I don't know many better words we use to explain it.... That's why I think 3 [the Dynamic conception on the questionnaire] is probably the best way to explain it because, you know, it moves to this point. You know you can think about it like, they're both like, the graph is moving or what not. Whether or not it's actually moving or time or anything like that, you know, I just think it's the way I like to think about it.

The interview data from this study confirm Oehrtman's suggestion that students may use dynamic language to *talk* about limits, but also confirms what others have found, that students do *think* dynamically about limit situations as well (Eade, 2003; Eryvynck, 1981; Przenioslo, 2004;

Tall & Vinner, 1981; Williams, 1991, 2001). The Dynamic conception, in fact, was the most commonly used conception of limit both on the questionnaire and during the interviews, which is also congruent with the results of the previous studies already cited.

Another previously identified dynamic conception of limit is a practical conception which I call a Dynamic-Practical conception of limit (Przenioslo, 2004; Williams, 1991, 2001). The Dynamic-Practical conception involves plugging in values closer and closer to the limit point and observing what value the outputs are approaching. This conception was marked as true by 64.96% of the students and chosen as the preferred conception by 7.12%. The following statement from student N shows her first giving a Dynamic conception of limit and then she clarified her thinking with a Dynamic-Practical conception as she explained what a limit is.

N: I'd say like in a function as  $x$  approaches a certain number the limit would be what  $y$  value you get as you get closer and closer to that particular  $x$  value and that it'd be, it's what  $y$  value you're getting as you're coming from the left so if you're going 2.8, 2.9 to 3 or if you're going 3.2 3.1 to 3 it'd be whatever  $y$  value you get going from both sides and if they're the same then that would be your limit and if they are two different things then, you'd the limit would not exist at that point.

During interviews 2.79% of the excerpts about limit were coded with the Dynamic-Practical conception and three of the nine students used it at least once in their interviews.

**Other conceptions identified in previous research.** The Proximity or Neighborhood conceptions of limit were identified by various researchers (Bezuidenhout, 2001; Oehrtman, 2009; Przenioslo, 2004; Williams, 2001). These conceptions were originally coded differently with the Proximity conception used when students referenced the closeness of inputs or outputs to the limit of a function. The Neighborhood conception was when this closeness was specifically described with the use of some interval around what  $x$  was approaching or the limit value. Both relate to the idea of closeness with a limit, and because only one student in one excerpt used an interval to describe this closeness, I decided to combine these codes into one

conception as I did in the framework. The combined code simply took on the name of the more general conception, the Proximity conception. The proximity conception was represented on the questionnaire and was marked as true by 81.91% of the students. It was chosen as a preferred conception by 12.02% of the students. The following excerpt from student N as she explained what she means by “approaching” during the interview is representative of how students reasoned with a Proximity conception of limit.

N: Like, it gets closer as like  $x$  gets closer and closer to three it's gonna get closer closer to whatever value the function approaches when you put in three.

During the interviews 5.05% of the excerpts about limit used a Proximity conception of limit and five of the nine students used a Proximity conception at least once.

The Approximation conception of limit was identified by Oehrtman (2009) and Williams (1991). The Approximation conception was represented on the questionnaire and was marked as true by 74.94% of students. It was chosen as a preferred conception by 6.65% of the students. This conception was not observed during the interviews. This may be because none of the questions related to the formal definition of the limit, an instance when students in Oehrtman's study used this conception or simply because the sample size of the interview was only nine students.

Williams (1991) and Juter (2006) identified the Unreachable conception of limit in their work and Williams (1991) also identified the similar Bounding conception of limit. An Unreachable conception describes the limit as something the function cannot *reach* while a Bounding conception describes the limit as something the function cannot *go beyond*. On the questionnaire, the Unreachable conception was marked as true by 67.33% of the students and chosen as a preferred conception by 27.54%. This made the Unreachable conception of limit the second most common preferred limit conception. A typical description of the Unreachable

conception is represented by the following statement from Student B as he described what a limit is.

B: So, I would describe the limit as something, like, that it's unattainable.

The common use of the Unreachable conception on the questionnaire was not identified in the interviews. Only 2.79% of the interview excerpts about limits showed the use of the Unreachable limit conception and three of the nine students used it at least once. This may be explained by the use of a new code which will be described below, the Point Doesn't Matter limit conception. The Bounding conception was marked as true by 28.59% of students who completed the questionnaire and was chosen as a preferred conception by 2.80% of the students. The following statements from Student S as he reasoned about what a limit is and what can be understood knowing a function is defined at a point show a couple ways one could reason with a Bounding limit conception.

S: It's the, basically I think of it like a boundary (M: OK) that a graph has to function within.

S: Like if it's, if that's the maximum value, or the maximum on a graph then, like, you could have a limit there.

The Bounding limit conception was used in 2.23% of excerpts about limit and was only used by one student, Student S.

Duru et al. (2010) and Tall and Vinner (1981) identified the Lack of Discontinuities conception of continuity. On the questionnaire, 84.27% of students marked this conception as true and 32.48% marked it as the conception that best represented their understanding of continuity. The commonality of this conception was also identified in the interviews. Students were considered using this conception when they reasoned that a function was continuous because there were no discontinuities such as holes, jumps, or asymptotes. This conception was coded in excerpts where students used it to reason a function was either continuous or



discontinuous. The following excerpts show examples of students reasoning with a Lack of Discontinuities conception of continuity. Student K uses it to claim a function continuous and Student A uses it to justify why a few different functions are discontinuous.

K: 'Cause there's no holes and there's not jump discontinuities and no asymptotes

A: These two have infinite discontinuities. This one has a removable discontinuity and that one has a hole.

The Lack of Discontinuities conception of continuity was the most common continuity conception used during the interviews. Of the excerpts about continuity, 55.07% showed use of the Lack of Discontinuities conception. All nine students used this conception during the interview.

A much less common conception of continuity was the Gapless conception (Núñez et al., 1999). This conception was not included on the questionnaire because Núñez et al. (1999) described this conception theoretically, without evidence of students actually thinking this way. A Gapless conception of continuity is used when students reason that a line is the compilation of points such that there is no gap between one point and the next. No students seemed to reason very clearly with this conception of a line. One student used the Gapless conception in two excerpts (2.90% of continuity excerpts). The portions of the excerpts where he uses the Gapless conception are below and both were from within larger excerpts where he describes other conceptions of continuity as well. As he explained why a function was continuous and whether it is possible to draw a continuous function where the limit does not exist, he was able to reason about continuity with many different conceptions, never using the same conception more than twice.

J: ... the line is completely connected.

J: Well, continuous function requires that, you know, every number as  $y$ , as  $x$  gets positive or more negative, every number is directly connected to the number directly positive and negative to it.

Tall and Vinner (1981) identified a conception I labeled the Visual conception of continuity. A student reasons with a Visual conception of continuity when they use visual cues such as the graph being in one piece, the curve being smooth, or the curve being straight as justification as to why a graph is continuous. The colloquial nature of this conception (what does “smooth” mean, for example) made it difficult to describe in one statement and this conception was not included as a statement on the questionnaire. In the interviews, students reasoned with a Visual conception of continuity in various ways. Below are a few examples of excerpts coded with the Visual continuity conception.

D: (explaining what continuous means) So I'd have to say it just, the function continues and has smooth, and just runs... it looks like it's continuous when you look at the function

J: (explaining why a function is continuous) Um, I would say  $f$  is continuous on 4 'cause it's a straight line.

Tall and Vinner (1981) identified students reasoning that a graph with a sharp corner is not continuous. This was also considered a Visual conception of continuity. Student D made this argument, that a function with a sharp corner is not continuous, while he reasoned about if a limit can exist where a function is discontinuous.

D: Um... the only thing I know of having a discontinuous graph (mumbles). If there is like a point. So if it's, say it's continuous and it goes up and then there's a point at three, then it's discontinuous.

M: OK, like a sharp (D: right) corner.

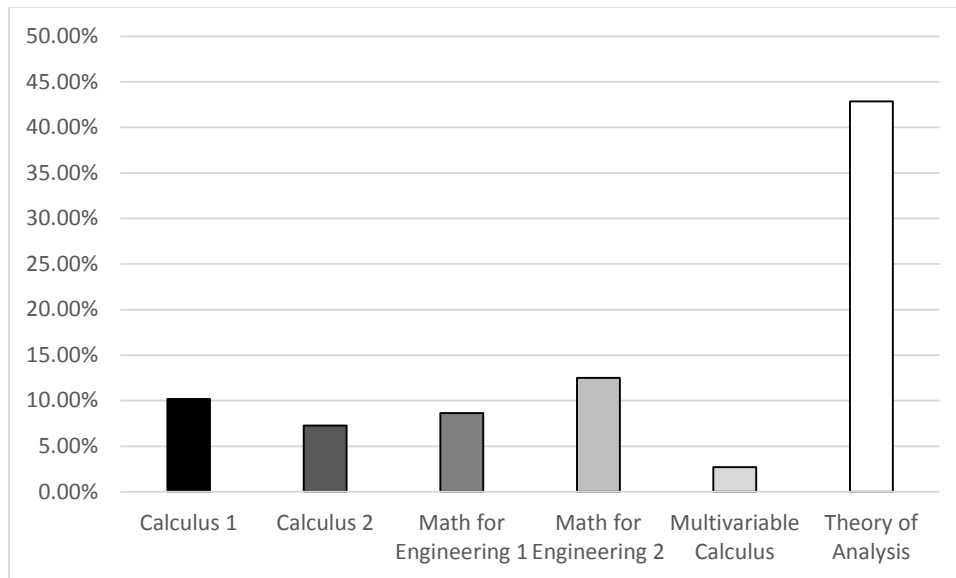
D: Mhmm

From the interviews, 21.74% of excerpts about continuity showed students reasoning with a Visual continuity conception. Seven of the nine students used this conception at least once. This makes the Visual conception of continuity one of the most common continuity conceptions during the interviews.

**Formal conceptions.** Some students who completed the questionnaire showed understanding of the Formal definition of Limit, but no interviewees showed such an

understanding. The interviewed students did not know how to reason about the questions relating to the Formal definition of limit and the questionnaire indicates that this may be the case with students until they take a course in analysis. There was no Formal limit statement on the questionnaire, but two questions were designed to determine if students could reason with ideas similar to the Formal definition of limit (see Figure 1). Both gave  $\lim_{x \rightarrow 2} f(x) = 3$  and asked students to mark which of a set of statements must be true. The first statement was a correct use of the Formal limit idea, but in a slightly different form:

For every positive integer  $n$ , there is a real number  $\delta > 0$  such that if  $0 < |x - 2| < \delta$ , then  $|f(x) - 3| < \frac{1}{n}$ . The second was similar to the Formal definition, but the hypothesis and conclusion were switched. The intention was that only students who could appropriately reason using a Formal conception of the limit would correctly mark the first as true and the second as false. In comparing the percentage of students in each course with such an understanding of the Formal definition of limit we see that few students could correctly answer both of these items until they took a course in analysis (see Figure 6).



*Figure 6.* Formal limit understanding. This figure shows the percentage of students from the questionnaire who answered both formal limit questions correctly.

This is reasonable because students are only briefly introduced to the Formal definition of limit in Calculus 1 at Brigham Young University, and the other courses do not focus on it either. It is not until Theory of Analysis that students are required to reason and develop proofs using the Formal definition of limit. Student in the interview did not show much evidence of understanding or confidence in the Formal definition of limit. The following excerpt shows student K not sure how to reason through the answer choices from the questionnaire. Other students were less equipped and less confident than she was.

K: Um, (reads delta epsilon option) Ok, one sec, I got to think of this for a second (mumbles as rereads). OK, so this is just the whole delta epsilon proving limit, right? (M: OK) That's what I'm guessing, um, which I didn't really get that stuff that well (M: That's OK). So, um, I just had to like, I get that delta, or yeah delta's on the x axis, the amount above or below will create an epsilon, er, I think I'm getting this backwards now, oh my gosh, (M: That's OK) But, either way, (Reads next delta epsilon option) See, I wouldn't know which one's true between e and f honestly, because I didn't get that that well, the epsilon delta.

Considering it takes students in the participating courses until an analysis course to understand the Formal definition of limit, this may indicate that students do not need to have the Formal

definition integrated into their concept image for limit unless they are advancing in a major requiring an analysis course.

The Semi-Formal conception of continuity was used by both students who completed the questionnaire and students who participated in the interviews. The Semi-Formal conception of continuity was a statement on the questionnaire and was marked as correct by 88.69% of the students and 38.18% chose it as their preferred conception of continuity. It was the most common continuity conception on the questionnaire. This was not the case, however, during the interviews. The Semi-Formal conception was used in 4.35% of the interview excerpts about continuity. Only two students used the Semi-Formal conception during the interview and both developed it or remembered it as they reasoned about various situations. Student K developed the Semi-Formal conception over time as she looked at various examples. The excerpts during which she had not yet fully developed this conception were coded as Semi-Semi-Formal and comprised another 4.35% of continuity excerpts. She realized the limit value and function value must both exist, but then finally determined through another example that those values must be equal to each other. The Semi-Formal conception was not her most common conception, which was Lack of Discontinuities. Student J also developed the Semi-Formal conception, but only through one example. He did not use it again after he developed it. He did not have a common conception, but held several conceptions in his concept image, using different ones with different situations.

**New conceptions from the interviews.** Some students reasoned about continuity and limit in faulty ways which had not previously been identified in the literature. With continuity, the faulty conceptions were not easy to describe and were characterized by the student discussing why a function is or is not continuous erroneously without showing evidence of an identifiable

conception. For example, student S claimed the function  $k$  was continuous when, in fact, it has a removable discontinuity and is defined somewhere else. No identifiable reasoning is described to determine what conception of continuity he might be using to reach this conclusion.

S: Just like at  $k$  if you approach uh, you know, both points on the graph, uh, (mumbles)... It does look like a jump discontinuity on  $k$ . Um, simply because it's at the same  $x$  and it gets a different value. Um, I still think it's continuous because it doesn't have a part of the graph with it. Um, but it would be undefined at that point.

M: OK, so by 'a part of the graph with it', do you mean that extra point at  $(4, 4)$  (S: Yeah) doesn't have a line attached to it?

S: Right. Yeah. So if you had like, the rest of the graph came from this point (M: Oh, OK) Then um, OK, kay, then you could have like on the defined point on  $k$  ... (draws on paper) ... So if you had something like that, that's an obvious jump discontinuity and therefore the graph would be discontinuous (M:OK) at um, um, at that point. And (mumble) a hole. But the, the limit at that point automatically wouldn't exist either.

M: OK

S: But uh... just a side note, this is (M: Sure) a, this is, I think this is a continuous function with a limit that does not exist.

M: The way it was originally drawn,  $k$ ?

S: Yeah.

Students also used Faulty conceptions of limit, but some of the conceptions were describable. Four faulty conceptions of limit gave rise to new codes as well. The four new codes describing conceptions that lead to faulty conclusions were  $x$  Approaches, Tangent Line, Limit is a Rift, and Asymptote. The  $x$  Approaches conception was used when a student reasoned that the limit value was equal to whatever  $x$  was approaching. Student J used this when reasoning about the limit value at an infinite discontinuity. He shortly after concluded the limit should be infinity.

J: It approaches three, but the limit, yeah the limit is three because it's approaching three, but like... Wait, limit as  $x$  approaches three if it's an infinity one, the limit...oh, it's been too long since I've done this. ... Um, limit approaches three with an infinity one, the limit, is three.

Three students used the  $x$  Approaches conception of limit in 2.79% of limit excerpts. The Tangent line conception of limit is used when students reason that the limit is a tangent line, likely simply confusing limit with derivative. The Limit is a Rift conception involves a student reasoning the limit is some sort of "rift" in the function such as an asymptote, a jump, or a point

defined outside what one would expect. The word "rift" comes from student B, the most persistent and predominant user of this conception. In the following excerpts student B describes what happens when there is a limit.

B: Yeah you're either jumping or there's some kind of rift where you're like going to infinity or something like that.

B: If the limit exists then at that point, then yes, I believe, at least, I think, if there's a limit then it's discontinuous, just 'cause the limit is causing some kind of rift in the, in the graph.

The Limit is a Rift conception was used in 6.15% of limit excerpts, all the excerpts except one were from student B. Student S reasoned with this conception once as well. The Asymptote conception of limit is the belief that a limit is an asymptote and is a special case of the Limit is a Rift conception. From the interviews, 3.35% of limit excerpts showed students using the Asymptote conception of limit. Three of the nine students used this conception at least once.

The Left/Right limit conception was not explicitly identified in previous research and was not including in the preliminary coding, but emerged through the coding process. Students were considered to be using the Left/Right conception when they reasoned about or described the importance of considering the limit from the left and right of the point of interest. Although this conception was not a focus of any study, Williams (2001) described the reasoning of a student who seemed to be using this conception. Gerry, the student in Williams' study, talked about the limit "sandwiching" in on the limit point from both sides (p. 356). In the present study, students sometimes used the Left/Right conception alone, while in other situations students used the Left/Right conception along with another conception of limit. The difference can be seen in the following two excerpts.

T: Because we can find the limit again from the, from the right and from the left and that's what matters to us.

A: "It's still approaching the same value from the left and the right."

Student T uses the Left/Right conception alone as she describes why the limit exists and student A uses it along with a Dynamic conception. The Left/Right conception was used in 21.23% of excerpts about limits and eight of the nine interviewees used this conception at least once. It was the second most common limit conception from the interviews.

The Graphical conception of limit was added during the coding process as well. The Graphical conception was used when a student described the limit as something used to determine how the graph of the function "looks," "acts," or "behaves." Williams (1991) identified a *generic metaphor for limit* to describe students' trust in the graphical representation of function in evaluating a limit. This is similar to the Graphical conception because the Graphical conception relates to students referencing the graph of a function. The following two excerpts are representative of students reasoning with a Graphical conception of limit because they reference the behavior of the graph and what is happening (how the graph is behaving) at the limit point. Both students were reasoning about what a limit is.

S: ... Um, you'll find the behavior of the graph specifically at that point. ...

P: Um, it's like being blind to a certain part on the graph and you're just observing it as it gets closer. So, you never know exactly what's going on at the point, but you know what hap, what happens as you get close to that. ... As you travel along the line from both sides. You just don't know what's happening, but you can infer from what you see from the sides.

The Graphical conception of limit was used in 5.59% of the limit excerpts and was used by four of the nine students. Related to the Graphical conception is another new conception students used to describe what a limit is, the Zooming conception of limit. A student using a Zooming conception of limit explained that the limit essentially zooms in on the graph so that one can determine how it behaves. The students using the Zooming conception were conveying that the limit zooms in on a graph and the purpose for taking the limit was to determine its behavior near



a point. All three excerpts coded with the Zooming conception were also coded with the Graphical conception.

Another new conception from the coding process is the Point Doesn't Matter conception of limit. This conception may begin to explain why the Unreachable conception was so common on the questionnaire and so rare in the interviews. Many students would reason about limit and make statements not quite reasoning with an Unreachable conception, such as the following excerpts from student S and student K as they each reasoned about what a limit is.

S: [The limit] will tell you the behavior of the graph at, almost at that point. But not exactly at that point (M: Uh huh). So, uh, basically it's telling you, you don't need to know the exact value of the point.

K: limits are um, as it's getting arbitrarily closer and closer but not necessarily hitting it.

These examples of student reasoning include the idea that the limit does not deal with the point it is approaching, but rather with what the function is doing nearby, with the conclusion that what is happening at the point does not influence the limit. Some students made statements about the point being unreachable, especially when describing asymptotic behavior, and those instances were coded as Unreachable, but other students like student S or student K made a point in saying what happened at that point, such as if it was reached or not reached, did not influence the existence or value of the limit. Students reasoning with a Point Doesn't Matter conception may have marked Unreachable as their preferred conception on the questionnaire as student S did, even though it may not have been entirely consistent with how they reasoned about limit. The Point Doesn't Matter limit conception was used in 3.35% of the limit excerpts and was used by four of the nine students.

Only one new conception of continuity was added during the coding process, the Predictable conception of continuity. This was used when students reasoned that a function is continuous because it behaves as one would expect it to behave. Student P was the only student

to use this conception and it was used in 4.35% of continuity excerpts. The following excerpt shows him reasoning that a function is discontinuous because it does something he would not expect.

P: ... Um. I would say it's probably actually not continuous, but it is defined at that point.

M: And it's not continuous because?

P: Because it jumps. And there's, where you would expect the point to be, there isn't a point. Instead it's removed from the actual function.

**Conceptions from research left unobserved.** One conception from the research described in the literature review was not identified in the present research, the *collapse* metaphor. This conception was identified by Oehrtman (2009) as he studied student reasoning about limit conceptions in contexts such as derivatives and integrals. The focus of this study did not include limit applications such as derivatives and integrals. It is reasonable, then, that this conception, which specifically related to derivative or integral application of limits, would not be observed in the present study.

**Preferred conceptions on the questionnaire.** A few patterns emerged about students' preferences of the different limit and continuity conceptions across the six courses studied. Table 8 shows the percentage of students in each course who chose each limit conception as their preferred conception. The percentage of students choosing the Algorithm, Boundary, and Dynamic-Practical conceptions remains somewhat constant through the courses. The Unreachable conception has a slow decrease as students progress through these mathematics courses. This may be because students in upper-division courses have seen more examples of functions or understand that the limit does not relate to if the function reaches a particular value or point. As described above, the interviews suggested that fewer students may really reason with the Unreachable conception than the questionnaire suggests. The decrease across the courses may reflect students simply being more aware of how they reason about limits so fewer chose the

Unreachable conception as their preferred conception. The Proximity conception, the conception from the questionnaire closest to the Formal definition of limit, was chosen by few (3% - 15%) students in the first five courses but jumps to 30% in the analysis course. This is reasonable considering the analysis course is the first of these courses at BYU in which students are expected to be able to reason with the formal definition. The Approximation conception is slightly higher in the engineering courses and no students chose it as their preferred conception in Multivariable Calculus or Theory of Analysis. Perhaps this is because the Approximation conception relates to an application of limits. Students may see some such approximation examples in first and second semester Calculus and perhaps they see more of these applications in the engineering courses. It may be the case that the students who continue to take the engineering courses do not see more of these examples, but this conception simply resonates with what they consider an important use of limits. This conception of limit also relates to the kinds of ideas found in the Formal definition (Oehrtman, 2009). Perhaps this Approximation conception is sufficient in that it allows students to reason somewhat formally without the formal mathematical language or notation.

Table 8  
*Preferred Limit Conceptions on the Questionnaire*

Limit Conception	Calculus 1	Calculus 2	Math for Engineering 1	Math for Engineering 2	Multivariable Calculus	Theory of Analysis
Algorithm	2.08%	1.87%	1.96%	3.13%	2.70%	0.00%
Unreachable	27.60%	32.58%	22.55%	21.88%	16.22%	5.00%
Dynamic	44.53%	35.96%	42.16%	56.25%	51.35%	55.00%
Boundary	2.34%	4.12%	0.98%	3.13%	2.70%	0.00%
Proximity	11.98%	10.86%	12.75%	3.13%	13.51%	30.00%
Approximation	6.51%	5.62%	9.80%	12.50%	0.00%	0.00%
Dynamic-Practical	4.95%	8.99%	9.80%	0.00%	13.51%	10.00%

Table 9 shows the percentage of students in each course who chose each continuity conception as their preferred conception. The Dynamic continuity conception was used by many (21% -32%) calculus students in all three semesters. Fewer (0% - 16%) of the students outside a calculus course chose this conception of continuity. Perhaps this is because engineering students are not asked to evaluate the continuity of a function or graph, so being able to draw a function without lifting ones pencil is not useful in those courses. The analysis book used in the BYU courses describes continuity using sequence convergence, so it is reasonable to think that students in this course do not use this Dynamic conception often. The Lack of discontinuities conception of continuity remained common (27% - 35%) in the first five courses with a drop (17%) in the analysis course. This again may have to do with the type of reasoning typical for this analysis course at BYU. The Semi-Formal conception of continuity tended to increase as students progressed through the course. This seems reasonable as it is the most thorough and complete conception on the questionnaire. It is highest with 77.78% of analysis students choosing the Semi-Formal conception as their preferred continuity conception.

Table 9  
*Preferred Continuity Conceptions on the Questionnaire*

Continuity Conceptions	Calculus 1	Calculus 2	Math for Engineering 1	Math for Engineering 2	Multivariable Calculus	Theory of Analysis
Dynamic	31.64%	29.37%	16.00%	9.38%	21.62%	0.00%
Lack of Discontinuities	32.44%	32.71%	30.00%	34.38%	27.03%	16.67%
Semi-Formal	33.78%	36.80%	49.00%	50.00%	45.95%	77.78%

On the questionnaire, some students marked “other” for their preferred conception of limit or continuity and described the conception that best represented their understanding. A few aligned with one of the conceptions provided. For example, some matched with a Dynamic limit conception, but were simply worded differently than the limit conception listed on the

questionnaire, *a limit describes how a function moves as  $x$  moves toward a certain point*. One student wrote in “what the function APPROACHES as  $x$  APPROACHES a value.” The only other written limit conception from the questionnaire that did not align with one of the others was a belief about limits many students may agree with: “The limit is the most confusing part of Calculus.” For continuity conceptions two students referenced the importance that the function be differentiable or have a continuous derivative. One student wrote in that a function is continuous if every  $x$  has a corresponding  $y$ -value. Two of the analysis students wrote in a conception using open sets. Very few students wrote in a conception and it seems the provided conceptions were sufficient for most students.

**Summary of conceptions.** The most common conceptions of continuity were Semi-Formal, Dynamic, and Lack of Discontinuities from the questionnaire and Lack of Discontinuities, Dynamic, and Visual from the interviews. The most common conceptions of limit were Dynamic and Unreachable from the questionnaire and Dynamic and Left/Right from the interviews. Table 10 presents the number of times each continuity conception was used (number of excerpts coded with the conception) by each interviewed student and Table 11 presents the number of times each limit conception was used by each interviewed student. There were many more conceptions for limit than continuity used in the interviews. This can also be seen by looking at the average number of conceptions for each interviewee. The average number of continuity conceptions was 3.78 conceptions per student while the average number of limit conception was 6.22 conceptions per student. The two highest continuity conceptions were used in 55.07% and 33.33% of the excerpts about continuity whereas the two highest limit conceptions were used in only 31.84% and 21.23% of the excerpts about limit. This shows that students hold and use more limit conceptions in their concept images of limit than they may for

continuity conceptions in their concept image of continuity. This may indicate why there has been a greater focus on limit conceptions than continuity conceptions in previous research, there is simply more to study. This may also result from limit being the more fundamental mathematical concept. Continuity depends on limit in the semi-formal definition and the formal definition of continuity using delta and epsilon seems to be a specialized form of the formal definition of limit.

Table 10  
*Continuity Conception Counts from Each Interviewed Student*

Student	T	S	P	N	K	J	D	B	A
Dynamic Continuity	2	3	3	2	0	2	6	5	4
Faulty	0	6	0	0	0	0	3	3	0
Semi-Formal	0	0	0	0	2	1	0	0	0
Semi-semi-Formal	0	0	0	0	3	0	0	0	0
Gapless	0	0	0	0	0	2	0	0	0
Lack of Discontinuities	3	2	7	2	8	2	3	5	6
On the Line	0	0	1	2	0	0	1	0	0
Predictable	0	0	3	0	0	0	0	0	0
Visual	0	1	3	0	0	2	1	2	1
Total Number of Conceptions Used	2	4	5	3	3	5	5	4	3

Table 11  
*Limit Conception Counts from Each Interviewed Student*

Student	T	S	P	N	K	J	D	B	A
Dynamic Limit	3	4	7	13	13	7	0	2	8
Dynamic-Practical	1	2	0	2	0	0	0	0	0
Bounding	0	4	0	0	0	0	0	0	0
Faulty	1	10	2	0	2	4	6	13	0
Asymptote	0	2	0	0	0	0	1	3	0
Limit is a Rift	0	1	0	0	0	0	0	10	0
Tangent	1	6	2	0	0	0	0	0	0
$x$ Approaches	0	2	0	0	0	2	0	1	0
Graphical	0	4	2	1	0	3	0	0	0
Left/Right	5	3	2	8	6	6	0	1	7
Point Doesn't Matter	1	3	0	0	1	0	0	0	2
Proximity	1	0	3	2	2	1	0	0	0
Substitution	0	0	1	1	0	2	4	0	0
Unreachable	0	2	1	0	0	0	0	2	0
Zooming	0	1	0	0	0	2	0	0	0
Total Number of Conceptions Used	6	12	7	6	5	8	3	6	3

Students in the interviews showed a few patterns in their reasoning about continuity and limit (See Table 10 and Table 11). Some students held many conceptions of a concept, such as student S using 12 different limit conceptions during his interview. Other students held few conceptions of a concept, such as student T only using two conceptions of continuity during her interview. Some students tended to have one or two main conceptions of a concept which they used most often, such as student P who tended to use the Lack of Discontinuities conception of continuity most often even though he was able to reason with other continuity conceptions as well. On the other hand students could also reason with many conceptions without favoring a particular one, such as student J who used five different continuity conceptions but never used any one of them more than twice during the interview. There was no evidence that the number of conceptions students held for one or both of these concepts or the frequency of use of the conceptions corresponded to their use of the PPCs either generally or incorrectly.

## Potentially Problematic Conceptions

Analysis of correlations between conceptions of continuity and limit and the four PPCs were done with both interview transcripts and questionnaire data. During interview analysis, each instance where a student may have utilized a PPC was coded specifically to reflect how they used the PPC with one of five codes (see Table 5). Two of the codes (General and Incorrect) related to a general (thus incomplete) or inappropriate way of reasoning with the PPC and two of the codes (Overcome and PPC Not Used) reflected an appropriate reasoning with or without utilizing the PPC. The other code, Harmless, represents a reasoning where students seemed to be using only the PPC to reason, which left their reasoning incorrect or incomplete while their conclusion was not an error. I consider this different than appropriate or inappropriate reasoning because the student may have identified additional information about the situation which makes their reasoning complete, but they may not have considered all necessary information and their reasoning could still have been incomplete. For example, reasoning with the Defined Means Continuous PPC is harmless if the student is considering a continuous function. Their reasoning is appropriate if they reasoned that the function is defined *and the limit value equals this defined function value*, so the function is continuous. This reasoning is correct. Without the addition of this piece of information about the limit value, reasoning that the function is continuous only because the function is defined is incomplete and incorrect reasoning. Excerpts coded Harmless are coded this way because the student simply did not show identifiable evidence that their reasoning was complete or incomplete. If students showed evidence of adding appropriate reasoning to a PPC to come to a correct conclusion, this is noted and considered appropriate reasoning. For each excerpt the continuity or limit conceptions used in the same excerpt were noted. Some excerpts showed no evidence of the use of a particular continuity or limit



conception. A Chi-Squared analysis was done to determine correlations between conceptions and evidence of holding a PPC from questionnaire data. For each PPC, I will describe the conceptions used when reasoning about a PPC in incompletely or incorrectly (General or Incorrect), harmlessly, and appropriately (Overcome or PPC Not Used). I will also describe a few correlations found through the statistical analysis of the questionnaire data.

**The Defined Means Continuous PPC.** If a function is defined at a point, then it is continuous at that point. Table 12 presents the number of excerpts in which students used each continuity conception for each possible use of the Defined Means Continuous PPC (General, Incorrect, Harmless, Overcome, PPC Not Used). Only the conceptions used at least once are included in the table. Under each code identifying a possible use of a PPC is the number of excerpts coded with this code in parentheses. During the interviews there were six excerpts from four students reasoning about the Defined Means Continuous PPC in general and one excerpt with a student reasoning incorrectly. These excerpts were typically when students were explaining why they thought a function was continuous when given  $g(5) = 7$  or when describing what it means for a function to be continuous. The most common continuity conception used was the Defined Means Continuous PPC. Indeed, one student, and perhaps all four, used it as a ‘rule of continuity.’ One student also used a Lack of Discontinuities conception in the same excerpt while making a claim using the Defined Means Continuous PPC. Student T used the Defined Means Continuous PPC to reason about graph  $i$ , a graph with an infinite discontinuity but with a point defined where the vertical asymptote was. She claimed that since the point was defined at the point “that solves the problem” of the function being undefined there, so it must be continuous. In the very next moment she laughed and said she knew that function  $i$  was not continuous, even though she just reasoned that it was because it was defined

everywhere. It seems student T suspended some sort of reasoning, perhaps the Lack of Discontinuities conception she used in the excerpt before her Defined Means Continuous claim, or a Dynamic conception which she used immediately after. The reasoning was suspended as she used the Defined Means Continuous PPC, but when she stepped back from dealing with if the function was defined, she laughed at her claim, stating she knew the function was not continuous. Other similar instances occurred where students made a claim while seemingly suspending some reasoning and then changing their mind once the suspended reasoning was enacted again. In these instances, students typically used the PPC generally or incorrectly and then realized their error when the suspended reasoning was reenacted. This is reasonable considering knowledge organized into a concept image such that certain conceptions can be enacted during particular situations and may go unused in others.

Table 12  
*Conception Correlations with the Defined Means Continuous PPC in Interviews*

	General (6)	Incorrect (1)	Harmless (1)	Overcome (5)	PPC Not Used (5)
Defined Means Continuous PPC	4	1	1	-	-
Counterexample	-	-	-	3	-
<b>Continuity Conceptions</b>					
Lack of Discontinuities	1	-	-	2	3
On the Line	-	-	-	1	2
Semi-Formal	-	-	-	1	1
Gapless	-	-	-	-	1
Predictable	-	-	-	1	-

One excerpt from student J used the Defined Means Continuous PPC harmlessly, claiming a function continuous because it was defined when it was, in fact continuous. He did not show evidence of using any other conception during the excerpt. Five excerpts from four students showed how they overcame either a misconception use of the Defined Means Continuous PPC or a general use of the PPC from the same or a recent excerpt. Students used

functions  $k$  or  $i$  as counterexamples or came up with their own counterexample when reasoning about what was known given  $g(5) = 7$ . Three students referenced specific counterexamples, two used Lack of Discontinuities, and Semi-Formal, Predictable, and On the Line conceptions were each used once. Five excerpts from four students showed students reasoning appropriately in situations when the Defined Means Continuous PPC could have been used incorrectly. Most of these were about function  $k$  or a similar situation with a removable discontinuity where the function was defined elsewhere at the  $x$ -value where the hole occurred. It is observed that in the two ways students can reason appropriately in Defined Means Continuous situations (Overcome and PPC Not Used), many different conceptions of continuity could be used (see Table 12). Lack of Discontinuities remained the most common, but students could use one or more of various conceptions to reason appropriately.

Various students made statements about the Defined Means Continuous PPC which may help clarify why so many students showed evidence of the PPC on the questionnaire and in the interviews. Four of the five students who reasoned generally or incorrectly about the Defined Means Continuous PPC made statements indicating that a function being continuous when it is defined seems to make sense. When describing the reasoning she used making a general claim of the Defined Means Continuous PPC right after she realized her error, student N explained, “I was thinking there’s a function  $g$  and the value 5 gives an actual value – it would make sense for the function to be continuous.” Others explained similarly saying defined “automatically made me think” continuous or “I feel like” defined would mean continuous. It seems likely that the Defined Means Continuous PPC seems reasonable to students until they consider or think of a counterexample like those who overcame it in the interviews. Perhaps this PPC seems reasonable to students because they are used to working with continuous functions, for which use of this

PPC would be harmless and cause no error. Three of the four students who overcame their general or misuse of the Defined Means Continuous PPC did so with a counterexample. When students initially consider this PPC, perhaps it seems reasonable because it is true with most of the functions they see, even if the student is capable of determining the error it causes with some counterexamples. Perhaps, then, it is important for students to encounter many examples of functions so they are equipped to consider counterexamples to this and other potentially problematic conceptions.

There were no correlations found between specific continuity conceptions and the Defined Means Continuous PPC. The interviews showed students using many conceptions of continuity when appropriately reasoning about Defined Means Continuous situations. Four of the six general Defined Means Continuous excerpts showed students using the Defined Means Continuous PPC as a rule of continuity. No correlations were found from the questionnaire data.

**The Limit Equals Function Value PPC.** The limit value and function value are the same at a point. Table 13 presents the number of excerpts in which students used each limit conception for each possible use of the Limit Equals Function Value PPC (General, Incorrect, Harmless, Overcome, PPC Not Used). During interviews there were nine excerpts from five students reasoning with the Limit Equals Function Value PPC in general. All the excerpts were about the final two questionnaire questions, given  $g(5) = 7$  or  $\lim_{x \rightarrow 2} f(x) = 3$ , or the students was explaining what a limit is. The conceptions of limit students were using made the Limit Equals Function Value PPC seem reasonable for them. Five of the nine excerpts used the Limit Equals Function Value PPC as a rule for limits and three used a Substitution conception, supporting the conjecture made by Bezuidenhout (2001) that the substitution method for evaluating limits may lead to such a misconception. A few students used other means of

reasoning about limits to justify their use of the Limit Equals Function Value PPC. For example, student B reasoned that because it said the limit was *equal* to 3, rather than having “the little squiggly equal sign” (approximate), he knew that the limit value and function value were the same. Student K argued that the function value was equal to the limit value because the function could get arbitrarily close to the limit value, and comparing her reasoning to why  $\overline{.9} = 1$ , there is no number between the two, she argued that the function must actually equal the limit value at that point. Both of these students seemed to be attempting to reason with something they were told, such as what an equal sign means or an unconventional interpretation of what it means to be arbitrarily close. It seems then that most of the students who used the Limit Equals Function Value PPC generally did so as a direct result of the conception of limit they were using, five excerpts where students used the Limit Equals Function Value PPC as a limit conception and three where the Substitution conception was employed.

Table 13  
*Conception Correlations with the Limit Equals Function Value PPC in Interviews*

	General (9)	Incorrect (6)	Harmless (0)	Overcome (6)	PPC Not Used (8)
Limit Equals Function Value PPC	5	6	-	-	-
<b>Limit Conceptions</b>					
Substitution	3	-	-	-	-
Dynamic Limit	1	-	-	5	7
Left/Right	1	-	-	1	3
Proximity	1	-	-	2	-
Limit is a Rift	-	1	-	-	-
Point Doesn't Matter	-	-	-	-	1
<b>Continuity Conceptions</b>					
Lack of Discontinuities	-	-	-	-	2
Semi-Formal	-	-	-	-	1

Four students used the Limit Equals Function Value PPC incorrectly in six excerpts. These excerpts were about functions  $k$  or  $i$  or a jump discontinuity the student drew. All six excerpts involved the student using the Limit Equals Function Value PPC as a rule about limits.

One of them, student B, also included the student using the Limit is a Rift faulty conception of limit. It is evident that all four of these students reasoned incorrectly using the Limit Equals Function Value PPC because they were reasoning with the PPC as their main conception of limit in these excerpts.

Students reasoned appropriately in the Limit Equals Function Value PPC situations in 15 excerpts. In one excerpt a student reasoned with the Limit Equals Function Value PPC correctly by simply stating that the function value and limit value were the same on a continuous function. This was considered a correct use of the PPC rather than a harmless one because the student reasoned that the limit value and function value were the same *because the function was continuous*. This addition in the students reasoning made their thinking appropriate. Students did not comment often about continuous functions, perhaps because the functions seemed commonplace and not worth commenting on. Four students in eight excerpts reasoned appropriately in situations where the Limit Equals Function Value PPC could have been used incorrectly as a misconception. This PPC Not Used reasoning was done with many different examples of continuous functions. It seems that this type of appropriate reasoning is often simultaneous with a Dynamic reasoning about limits as seven of the eight excerpts showed use of this limit conception. Three students in six excerpts overcame their use of the Limit Equals Function Value PPC. In five of the excerpts, students overcoming their misconception, an incorrect use of the Limit Equals Function Value PPC, did so while reasoning about functions  $k$  or  $i$  or a jump discontinuity. Student N reasoned generally about the Limit Equals Function Value PPC when given  $\lim_{x \rightarrow 2} f(x) = 3$ , but realized her error in using the PPC in the same excerpt. A similar pattern arises with this type of appropriate reasoning in that five of the six

excerpts where students overcame the Limit Equals Function Value PPC showed use of a Dynamic conception of limit.

Students reasoning appropriately in Limit Equals Function Value situations used a Dynamic conception of limit more often than students reasoning generally or incorrectly about the Limit Equals Function Value PPC in the interviews. This pattern was not identified from the questionnaire data, perhaps because students were only given one statement to represent the dynamic conception of limit and may have interpreted it differently than how they actually reasoned. Also the questionnaire only gave information on if students used the PPC generally, thus it would not have been expected that the questionnaire would identify this type of difference in students reasoning appropriately or inappropriately with the Limit Equals Function Value PPC. In Table 13 we see that twelve of fourteen of the excerpts where students reasoned appropriately (Overcome or PPC Not Used) used a Dynamic conception compared to one of fifteen excerpts where students reasoned Incorrectly or Generally. It seems that students reasoning inappropriately or generally about the Limit Equals Function Value PPC did not reason with a Dynamic conception of continuity or had momentarily suspended using that part of their concept image. In contrast, students often used a Dynamic conception when reasoning appropriately. The two students who never overcame their misconception or general use of the Limit Equals Function Value PPC rarely used a Dynamic conception. Student D never used a Dynamic conception for limit and student B only used a Dynamic conception twice during his entire interview, both excerpts about an infinite limit. Perhaps an ability to reason with the more intuitive Dynamic conception is important to be able to reason about continuity and limit at a point.

From the questionnaire, two interesting patterns were observed about the Limit Equals Function Value PPC. It was discussed above that of the students showing weak evidence of the Limit Equals Function Value PPC, meaning they only showed evidence in one of the possible instances on the questionnaire, more used a general Limit Equals Function Value conception when given the function value than when given the limit value. There was no significant difference in the preferred limit conceptions of those showing evidence of the Limit Equals Function Value PPC given the function value and those showing evidence given the limit value. However, a chi-squared analysis showed some significant differences in preferred limit conceptions of students who showed weak evidence of the Limit Equals Function Value PPC and students who showed strong evidence of the Limit Equals Function Value PPC. These differences related to three limit conceptions, Unreachable, Proximity, and Dynamic-Practical. Students who showed strong evidence of the Limit Equals Function Value PPC chose the Unreachable conception more, the Proximity conception less, and the Dynamic-Practical conception less than students who only showed weak evidence of the Limit Equals Function Value PPC. It has already been discussed that the Unreachable conception may bring students difficulties when reasoning about limit and continuity at a point, but it is interesting that when considering a situation where the Limit Equals Function Value PPC may have been employed, students who used the general PPC more consistently did not choose the Proximity or Dynamic-Practical conceptions as much as students who only used the PPC in one of two instances.

**The Limit Means Continuous PPC.** If the limit exists at a point, the function is continuous at that point. Table 14 presents the limit and continuity conceptions students used while reasoning about the Limit Means Continuous PPC. During the interviews there were five excerpts from two students reasoning in general about the Limit Means Continuous PPC. Of the



five excerpts where students reasoned generally about the Limit Means Continuous PPC, four were while discussing the questionnaire and one was when discussing limit existence in general. In two of the excerpts, both from student S, he uses the contrapositive of the Limit Means Continuous PPC: If a function is not continuous, the limit does not exist. There was only one excerpt from a student reasoning incorrectly about the Limit Means Continuous PPC. This incorrect use occurred when student S continued to use the contrapositive of the Limit Means Continuous PPC as a rule for limits with a function where the limit did exist but it was discontinuous.

Table 14  
*Conception Correlations with the Limit Means Continuous PPC in Interviews*

	General (5)	Incorrect (1)	Harmless (0)	Overcome (1)	PPC Not Used (4)	Correct in General (4)
Limit Means Continuous PPC	2	1	-	-	-	-
Counterexample	-	-	-	1	-	-
<b>Limit Conceptions</b>						
Dynamic Limit	2	-	-	-	3	4
Left/Right	-	-	-	-	2	3
Point Doesn't Matter	-	-	-	-	1	1
Graphical	1	-	-	-	-	-
<b>Continuity Conceptions</b>						
Lack of Discontinuities	1	-	-	1	2	1
Semi-Formal	-	-	-	-	1	-
Predictable	1	-	-	1	-	-

No students reasoned harmlessly with the Limit Means Continuous PPC. One student, student P, overcame his general use of the Limit Means Continuous PPC from earlier in the interview. He did this as he saw the counterexample of function  $i$ , again pointing to the possible benefit of exposing students to many examples of functions so they are equipped to reason about these and other PPCs. Four excerpts from three students showed appropriate reasoning about a situation where the Limit Means Continuous PPC could have been applied incorrectly. The pattern arises again that students reasoning appropriately (Overcome or PPC Not Used) used the

Dynamic conception of limit more often in these excerpts than those using the PPC in General or Incorrectly. No correlations were identified about the Limit Means Continuous PPC from the questionnaire.

**Correct in General use of the limit and continuity PPCs.** The Limit Means Continuous PPC and the Limit Means Discontinuous PPC have opposite conclusions given that the limit exists. There were general situations where students explained that knowing the limit exists does not provide information about if the function is continuous or discontinuous. These excerpts were coded and will be described hereafter as Limit Means (Dis)continuous Correct in General and their results are included in Table 14. Four students in four excerpts reasoned correctly in general about a situation where the Limit Means Continuous PPC or the Limit Means Discontinuous PPC could have been used incorrectly. All four used a Dynamic conception of limit, again showing a common use of the Dynamic conception of limit when students are reasoning appropriately about limit and continuity at a point.

**The Limit Means Discontinuous PPC.** If we take the limit at a point, the function must be discontinuous at that point. Table 15 presents the limit and continuity conceptions students used while reasoning about the Limit Means Discontinuous PPC. During the interviews there were two excerpts where the Limit Means Discontinuous PPC was used in general and three excerpts where the PPC was used incorrectly. All of these excerpts were from student B, who never overcame the Limit Means Discontinuous PPC in general or as a misconception. Student B's main conception for limit was Limit is a Rift. He used this conception throughout his interview. He also used a Dynamic conception of continuity and a Visual conception of continuity in the general and incorrect excerpts. Student B also used the contrapositive of the Limit Means Discontinuous PPC, if a function is continuous, then you cannot take the limit,

while reasoning generally with the Limit Means Discontinuous PPC. A statement by student B, as he reasoned about if the Semi-Formal continuity conception was true, along with his strong conception of Limit is a Rift describe his tendency toward the Limit Means Discontinuous PPC.

B: When I think of limit, I just don't think of continuous just 'cause they seem to be opposites.

Student B reasoned this way throughout the entire interview and never overcame his general or misuse of the Limit Means Discontinuous PPC.

Table 15

*Conception Correlations with the Limit Means Discontinuous PPC in Interviews*

	General (2)	Incorrect (3)	Harmless (0)	Overcome (0)	PPC Not Used (3)
Limit Means Discontinuous PPC	1	-	-	-	-
<b>Limit Conceptions</b>					
Dynamic Limit	-	-	-	-	2
Left/Right	-	-	-	-	2
Limit is a Rift	1	2	-	-	-
<b>Continuity Conceptions</b>					
Lack of Discontinuities	-	-	-	-	2
Dynamic Continuity	1	-	-	-	-
Visual	-	1	-	-	-

Two students reasoned appropriately about Limit Means Discontinuous situations when the conception could have been used incorrectly in three excerpts. Both students were explaining situations with continuous functions where the limit existed. The Dynamic conception of limit is used more often when students reasoned appropriately about the Limit Means Discontinuous PPC than when the PPC is reasoned with in General or Incorrectly. Thus we see that throughout the use of all three PPCs that involve the limit at a point (Limit Equals Function Value PPC, Limit Means Continuous PPC, and Limit Means Discontinuous PPC), the Dynamic conception was often used when students reasoned appropriately and not often used when students reasoned in general or incorrectly with the PPCs.

Two correlations were identified involving the Limit Means Discontinuous PPC from the questionnaire. A Chi-Squared analysis was done comparing the seven limit conceptions from the questionnaire with the PPCs involving limit at a point. A significant correlation was found between the Unreachable conception of limit and the Limit Means Discontinuous PPC, in that students who chose the Unreachable conception as the one best describing their understanding of limit were more likely to show evidence of holding the Limit Means Discontinuous PPC. An example of this can be seen from student B during his interview. He explained that the Unreachable limit conception made sense to him and reasoned with it twice in his interview. His main conception of limit was Limit is a Rift which is in line with the idea that a limit cannot be reached. Student B was the only student who reasoned with the Limit Means Discontinuous PPC in general and incorrectly and he never overcame his misconception. The following excerpt from the interview is representative of the type of reasoning student B used throughout the interview. It shows his use of the Unreachable conception by explaining the limit is approaching 2, not equaling or reaching it, as well as the Limit Means Discontinuous PPC in general.

M: ... why are you thinking that if  $\lim_{x \rightarrow 2} f(x) = 3$ , that  $f$  would be discontinuous at  $x = 2$ ?

B: Mostly it's the part right at the beginning when I read it like "as  $x$  approaches 2" I'm thinking because it's *approaching* 2 it's not *equaling* 2, so it means it's not defined, so it means it's discontinuous at  $x = 2$ .

The Chi-Squared analysis also showed a significant correlation between the Dynamic conception of limit and the Limit Means Discontinuous PPC, in that students who chose the Dynamic conception to best represent their understanding of limit were less likely to show evidence of the Limit Means Discontinuous PPC. The discussion above describes the support for this correlation from the interviews as students reasoning appropriately about the Limit Means Discontinuous PPC used the Dynamic conception of limit more often than those reasoning generally or incorrectly about the Limit Means Discontinuous PPC. In fact, the interviews showed this type

of correlation with all three PPCs involving limit at a point (Limit Equals Function Value PPC, Limit Means Continuous PPC, and Limit Means Discontinuous PPC).

### **Comments on Differences in Questionnaire and Interview Results**

The differences in results from the questionnaire and interviews may be related to how the students participated with each instrument. Sometimes the preferred conceptions students marked on the questionnaire did not seem to be their most commonly used conception during the interviews, such as with Student S and the Unreachable conception as described above.

Similarly, students may have shown evidence of using the PPCs generally on the questionnaire, but using the PPC neither generally nor incorrectly during the interviews. For example, Student A showed evidence of the Limit Equals Function Value PPC and the Defined Means Continuous PPC on the questionnaire. She never, however, used any of the PPCs generally or as misconceptions during the interview. Thus she showed two instances where what seemed to be general uses of PPCs on the questionnaire were not identified during the interviews. From the nine students interviewed, there were 20 instances where they used the PPCs generally on the questionnaire. Ten of those instances, including the two from student A, were not identified during the interview, meaning the students never used the PPC generally or incorrectly during the interview even though they showed evidence of using the PPC on the questionnaire. When questioned about her seemingly different understanding on the questionnaire and during the interview, Student A commented that she was just answering the questionnaire by “circling things that looked like they could possibly be true” as she tried to finish the questionnaire quickly at the end of class. When she answered the questionnaire questions again during the interviews she used statements such as “not necessarily true” and provided counterexamples which made this the case. Considering counterexamples seems to be a different activity than

simply considering if the statements “could possibly be true.” This example may point to a difference in how students participated in the questionnaire and interviews. Perhaps students read the statements in the questionnaire and reasoned that they were generally true statements, being able to think of examples in which they were true. While participating in the interview and considering both general mathematical situations and specific examples students seemed to consider the truth of such statements while reasoning about counterexamples, determining if these statements were, in fact, true. This consideration of students marking statements true or false because the statements seem to be generally true may support why there was a drop in PPC use once students take a course in analysis, a course where such general consideration of mathematical statements is not appropriate in mathematical proof. Students in the first five courses may have shown evidence of using these PPCs even if they would not have shown evidence during an interview when their reasoning was more example driven. On the other hand, the analysis students were used to reasoning by considering counterexamples and did so even on the questionnaire. The implications of this difference between questionnaire and interview results for teachers and researchers will be taken up in the following section.

## **Chapter 5: Summary and Conclusions**

In this chapter I summarize the results of my research. I begin by describing how my research questions were answered. I then describe the limitations of this research, the implications it has for both researchers and teachers, and the directions such work can take in future research.

### **Summary of Research Questions**

The first research question the present work addressed was How pervasive are potentially problematic conceptions about continuity and limit at a point? This question was addressed by distributing a questionnaire to 861 students in various mathematics courses at BYU with at least one true or false question corresponding to each PPC on the questionnaire. Responding that the statement must be true was indication that the student held that PPC in general, although it was not determined on the questionnaire if the student would use the PPC as a misconception in a situation where using it would have been incorrect. The average PPC score (out of 5 points) for each course remained somewhat consistent around 1.4 per student until students took Theory of Analysis when it dropped to 0.48. A similar pattern occurred when considering each PPC individually, although there was more variation across the first five courses. The general pattern was that students showed evidence of holding PPCs in general until they took an analysis course when the percentage of students showing evidence finally decreased. One exception was the Limit Means Continuous PPC in which case the decrease occurred when students took Multivariable Calculus and the low percentage remained in Theory of Analysis. Thus it seems the general belief of these PPCs are rather pervasive as students continue on in mathematics perhaps until they reach upper division courses such as Theory of Analysis. The wide range of courses studied extends existing research beyond the study of first semester calculus.

The second research question was What conceptions of continuity and limit do students utilize when reasoning about continuity and limit at a point? I first describe how this work addressed this question with respect to continuity. Students were given the questionnaire with three conceptions of continuity provided. They were asked to mark each as true or false and to choose one conception which best represented their understanding of continuity. From the questionnaire, all three conceptions, Dynamic continuity, Lack of Discontinuities, and Semi-Formal continuity, were marked true by a majority (84% - 89%) of students and about 1/3 of the students chose each as their preferred conception for continuity. Dynamic continuity and Lack of Discontinuities were conceptions students also used often in the interview, although only two students used the Semi-Formal conception and they used it in only a couple instances in the interview. The Visual continuity conception was also common during the interviews. Also from the interview it was made evident that students can reason appropriately about continuity at a point using various conceptions of continuity. The number of continuity conceptions students utilized in the interview were fewer than the number of limit conceptions utilized. This more detailed study of student conceptions of continuity contributes to continuity research what Williams (1991, 2001) and Oehrtman (2009) contributed to limit research, filling the previous void.

The second research question was addressed with respect to limits as well. A majority (54% - 92%) of students marked six of the seven questionnaire conceptions of limit as true: Algorithm, Unreachable, Dynamic, Dynamic-Practical, Proximity, and Approximation. The most commonly chosen preferred conceptions on the questionnaire were Dynamic and Unreachable. The Dynamic conception was also one of the most common in the interviews, but the Unreachable conception was seldom used. In addition to the Dynamic conception, the Left/Right



conception of limit was also a commonly used limit conception during the interviews. The interviews showed a relationship between students' use of the PPCs and their ability to reason with the Dynamic conception of limit. Students reasoning in general or incorrectly with the Limit Equals Function Value PPC, Limit Means Continuous PPC, or Limit Means Discontinuous PPC (all relating to limits), tended to use reasoning other than the Dynamic conception of limit. On the other hand, students reasoning appropriately about these limit PPCs tended to use the Dynamic conception of limit most often in these situations. This result was supported by the questionnaire from data with students in all courses with respect to the Dynamic conception of limit and the Limit Means Discontinuous PPC. The interviews and questionnaire also showed a relationship between the Unreachable conception of limit and the Limit Means Discontinuous PPC. Students who chose the Unreachable conception of limit as their preferred conception were more likely to use the Limit Means Discontinuous PPC in general. Similarly, the one student who used the Limit Means Discontinuous PPC in general and incorrectly in the interviews was one of the few who reasoned with the Unreachable conception of limit. These results compared conceptions from existing research with the three limit PPCs, beginning to determine what these previously researched conceptions of limit could say about students understanding about limit and continuity at a point.

### **Limitations**

There were a few limitations to this study with respect to the sample size, the questionnaire, and the interviews:

- The participants for this study were all from one university. A more general sample may have produced different results simply because the courses at different universities may focus on or include more or less about limit and continuity.

- Interview participants were chosen from first semester calculus students only. If other students were included, perhaps different conceptions would have been used more often, although the questionnaire did not indicate large differences in conception preferences between first semester calculus students and the other participants.
- Interviewees were also chosen if they showed evidence of holding at least one of the PPCs in general on the questionnaire because the PPCs were the focus of this study. Perhaps different results would have been found if students showing no evidence of the PPCs were also interviewed.
- It was described in the previous chapter how student conceptions from the questionnaire were not necessarily congruent with the conceptions used in the interviews. For example, the Unreachable conception was one of the two commonly preferred conceptions on the questionnaire but it was rarely used during the interviews. A limitation of the questionnaire may be that students were not able to distinguish which of a set of conceptions they actually reason with most often.
- The questionnaire also had a limited number of conceptions to choose from. Even though students could write in their own preferred conception, few used this option. It was seen that one limit conception, the Left/Right conception, was not included on the questionnaire but was one of the most commonly used conceptions in the interviews.
- The questionnaire also could only evaluate if students showed evidence of the PPCs in general, not if they used them as misconceptions.
- A limitation of the interview was that most of the questions were either general or about graphical examples. When this was observed an algebraic function was added to the interview, but only two of the nine interviewees were asked this question. An interview

with more varied function representations may lead students to reason differently about limit and continuity at a point.

## **Implications**

This study has implications for both researchers and teachers. A few conceptions came out of this research that were not identified in previous research such as the Left/Right limit conception or the Point Doesn't Matter limit conception. Those researching limit and continuity may benefit from including these and the other conceptions in their work so that students have adequate conception choices to choose from on a questionnaire, for example. Those teaching calculus or planning curriculum for calculus courses may also benefit from the results of this study. Similarly, considering the use of counterexamples in overcoming the PPCs, teachers may find it helpful to expose students to many examples of functions to address the situations in which the PPCs become misconceptions for students. This may also help students engage with the mathematics in the more application- and example-based ways students reasoned during the interviews, considering the mathematical truth of statements by reasoning about counterexamples.

Two implications relate to the importance of particular limit conceptions. First, for students to reason appropriately about limit and continuity at a point, it was important for them to be able to reason with the Dynamic conception of limit, a conception researchers have considered a more intuitive conception of limit. Second, the Formal conception of limit did not seem necessary for students to reason appropriately about continuity and limit at a point. It was not studied whether or not a Formal conception of limit would have also been more beneficial for students as they reason in such situations, but it did not seem necessary given that none of the interviewees could reason about the Formal definition but many could still reason appropriately

about limit and continuity at a point. Similarly, students did not tend to understand the Formal definition until they took a course in analysis. Although this may be because the Formal definition was not covered or required for the previous courses and not because students were unable to understand it, it brings to question the importance of incorporating the Formal definition into students' concept image unless students will continue in a major requiring a course in analysis. On the other hand, perhaps understanding the formal definition helped the analysis students reason more appropriately and achieve a lower PPC score on the questionnaire. Researchers, teachers, and curriculum developers should consider these results as they plan studies, lessons, and curricula.

The difference between the results from the questionnaire and the interviews also has implications for teachers and researchers. If students tended to reason more generally during the questionnaire, considering conceptions or statements and marking them as true if they seemed to sometimes be true or "generally" be true, then questionnaires may only be useful in collecting a particular type of information about student understanding. Questionnaires may simply collect data whether students think a statement is "generally" true. Of course, for students who are familiar with considering the truth value of a statement by considering if it is always true (such as students who have taken an analysis course), the questionnaire may provide more information about their mathematical reasoning. Researchers must consider the activity in which students are engaging while they complete a questionnaire if they are to use the results to make claims about student understanding. Teachers should similarly consider how students answer questions on assessments if they plan on using the student responses to make evaluations of student understanding.

## **Future Research**

There are various ways in which this research can be extended including continued research about the PPCs, extended research about continuity and limit conceptions, and research into the importance of the Formal definition of limit. One could take this research further by administering a questionnaire designed to determine if students are using the PPCs as misconceptions rather than if they are just using them in general. It would also be interesting to know if students' belief or use of the PPCs would go back up after they take an analysis course or if their understanding is more permanent. This study also indicated that students overcame general use and misuse of the PPCs by using the dynamic conception and by encountering or thinking of counterexamples. Future research could study if a lesson or unit designed to develop the Dynamic conception of limit while encountering various function examples would help students appropriately reason about the PPC situations. Research could also look further into student understanding of limit and continuity by questioning what each conception means about students' understanding. Such research could ask what aspects of calculus are easier or more difficult for students reasoning with a particular conception of limit or continuity. Finally, research could ask the question, How important is it that calculus students understand the formal definition of limit? Similarly, it would be interesting to know if students' correct use of the formal conception of limit as observed in this study would decrease after they complete the analysis course or take subsequent courses. A study of students' understanding of the formal definition of limit and their use of the PPCs may also provide insight into the importance of such an understanding for calculus students. Thus this study provides a few new directions for research about these PPCs, limit and continuity conceptions, and the Formal definition of the limit.

## References

- Abbott, S. (2001). *Understanding Analysis*. New York: Springer-Verlag.
- Artigue, M., Batanero, C., & Kent, P. (2007). Mathematics thinking and learning at post-secondary level. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 1011-1049). Charlotte, NC: Information Age Publishing.
- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*, 389-407.
- Bezuidenhout, J. (2001). Limits and continuity: some conceptions of first-year students. *International Journal of Mathematical Education in Science and Technology*, *32*(4), 487-500. doi: 10.1080/00207390010022590
- Bressoud, D. M. (2012, 5 March 2012). [Letter to Tyler Jarvis, BYU].
- Confrey, J. (1990). A review of the research on student conceptions in mathematics, science, and programming. *Review of Research in Education*, *16*, 3-56.
- Duru, A., Köklü, Ö., & Jakubowski, E. (2010). Pre-service mathematics teachers' conceptions about the relationship between continuity and differentiability of a function. *Scientific Research and Essays*, *5*, 1519-1529.
- Eade, F. (2003). Secondary trainee teachers' understanding of convergence and continuity. *International Journal of Mathematical Education in Science and Technology*, *34*, 371-384.
- Ervynck, G. (1981). *Conceptual difficulties for first year university students in the acquisition of the notion of limit of a function*. Paper presented at the Fifth Conference of the International Group for the Psychology of Mathematics Education.
- Ferrini-Mundy, J., & Graham, K. G. (1991). An overview of the calculus curriculum reform effort: Issues for learning, teaching, and curriculum development. *The American Mathematical Monthly*, *98*, 627-635.
- Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory: Strategies for qualitative research*. Chicago: Aldine.
- Grabiner, J. V. (1983). Who gave you the epsilon? Cauchy and the origins of rigorous calculus. *The American Mathematical Monthly*, *90*, 185-194.
- Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 65-97). New York: Macmillan.
- Jordaan, T. (2009). *Misconceptions of the limit concept in a mathematics course for engineering students*. Master of Education, University of South Africa.
- Juter, K. (2006). Limits of functions as they developed through time and as students learn them today. *Mathematical Thinking and Learning*, *8*, 407-431.
- Monaghan, J. (1991). Problems with the language of limits. *For the Learning of Mathematics*, *11*(3), 20-24.
- Nesher, P. (1987). Towards an instructional theory: the role of student's misconceptions. *For the Learning of Mathematics*, *7*(3), 33-40.
- Núñez, R., Edwards, L., & Filipe Matos, J. (1999). Embodied cognition as grounding for situatedness and context in mathematics education. *Educational Studies in Mathematics*, *39*, 45-65. doi: 10.1023/a:1003759711966
- Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education*, *40*, 396-426.

- Olivier, A. (1989). Handling pupils' misconceptions. *Pythagoras*, 21, 10-19.
- Piaget, J. (1928). *The child's conception of the world*. London: Routledge and Kegan Paul Ltd.
- Przenioslo, M. (2004). Images of the limit of function formed in the course of mathematical studies at the university. *Educational Studies in Mathematics*, 55, 103-132. doi: 10.1023/b:educ.0000017667.70982.05
- Resnick, L. B., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual Bases of Arithmetic Errors: The Case of Decimal Fractions. *Journal for Research in Mathematics Education*, 20, 8-27.
- Robert, A., & Speer, N. (2001). Research on the teaching and learning of calculus/elementary analysis. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 283-299). Netherlands: Kluwer Academic Publishers.
- Schraw, G. (2006). Knowledge: Structures and processes. In P. A. Alexander & P. H. Winne (Eds.), *Handbook of educational psychology* (Vol. 2, pp. 245-260). New Jersey: Lawrence Erlbaum Associates, Inc.
- Siegler, R. S. (1986). *Children's thinking*. United States: Prentice-Hall.
- Smith III, J. P., diSessa, A. A., & Roschelle, J. (1994). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences*, 3, 115-163.
- Tall, D. (1990). Inconsistencies in the learning of calculus and analysis. *Focus on Learning Problems in Mathematics*, 12(3), 49-63.
- Tall, D., & Vinner, S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics*, 12, 151-169.
- Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education*, 20, 356-366.
- von Glasersfeld, E. (1983). Learning as a constructive activity. In J. C. Bergeron & N. Herscovics (Eds.), *Proceedings of the Fifth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 41-69). Montreal, Canada: Université de Montreal.
- Williams, S. R. (1991). Models of limit held by college calculus students. *Journal for Research in Mathematics Education*, 22, 219-236.
- Williams, S. R. (2001). Predications of the limit concept: An application of repertory grids. *Journal for Research in Mathematics Education*, 32, 341-367.

## Appendix A

### Questionnaire

#### Demographic Questions

1. Which math class are you taking this Fall 2012? (circle one)
  - a. Math 113: Calculus 2
  - b. Math 302: Math for Engineering 1
  - c. Math 303: Math for Engineering 2
  - d. Math 314: Calculus of Several Variables
  - e. Math 241: Theory of Analysis 1
2. What is your major? \_\_\_\_\_
3. Are you 18 years of age or older? (circle one)  
  
Yes                  No

#### Mathematics Questions

1. For each of the statements below mark if it is true (T) or false (F). (Circle T or F)  
  
T     F 1. A graph is continuous if you can draw it without picking up your pencil.  
T     F 2. A graph is continuous if there are no holes or jumps in the line.  
T     F 3. A graph is continuous if it is continuous at every point on that line.  
  
                To be continuous at a point the function value has to be equal to the limit of the  
                function at that point.  
T     F 4. A graph is continuous when there exists a convergent sequence in the domain  
                produces a convergent sequence in the range.
2. Of the four statements above, which describes continuity most closely to how **you**  
**understand** continuity of a function? (Circle one)  
1   2   3        4        Other (please describe below)
3. For each of the statements below mark if it is true (T) or false (F). (Circle T or F)



- T F 1. A limit is a type of math problem you solve, like determining the value of  $\sin \frac{5\pi}{6}$ .
- T F 2. A limit is a point or number the function gets close to but never reaches.
- T F 3. A limit describes how a function moves as  $x$  moves toward a certain point.
- T F 4. A limit is a number or point past which the function cannot go.
- T F 5. A limit is a number that the  $y$ -values of a function can be made arbitrarily close to.
- T F 6. A limit is used to approximate a value you cannot determine.
- T F 7. A limit is determined by plugging in numbers closer and closer to a given number until the limit is reached.

4. Of the seven statements above, which describes limit most closely to how **you** **understand** limit of a function? (Circle one)

1      2      3      4      5      6      7      Other (please describe below)

5. Which of the following must be true if  $f$  is a function for which  $\lim_{x \rightarrow 2} f(x) = 3$ ? Circle all that apply.
- $f$  is discontinuous at the point  $x = 2$ .
  - $f$  is continuous at the point  $x = 2$ .
  - $f(x)$  is defined at  $x = 2$
  - $f(2) = 3$
  - For every positive integer  $n$ , there is a real number  $\delta > 0$  such that if  $0 < |x - 2| < \delta$ , then  $|f(x) - 3| < \frac{1}{n}$ .
  - For every real number  $\epsilon > 0$ , there is a real number  $\delta > 0$  such that if  $|f(x) - 3| < \epsilon$ , then  $0 < |x - 2| < \delta$ .
  - None of the above statements are true
6. Which of the following must be true if for a function  $g$ ,  $g(5) = 7$ ?
- $\lim_{x \rightarrow 5} g(x) = 7$
  - $g$  is continuous at  $x=5$ .
  - $g$  is discontinuous at  $x=5$ .
  - None of the above statements are true.

## Interview

Goal: Identify which general conceptions of continuity and limit students are using to reason about continuity and limit at a point.

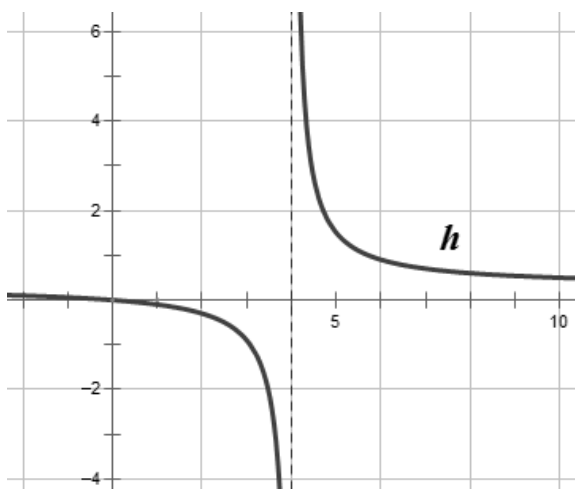
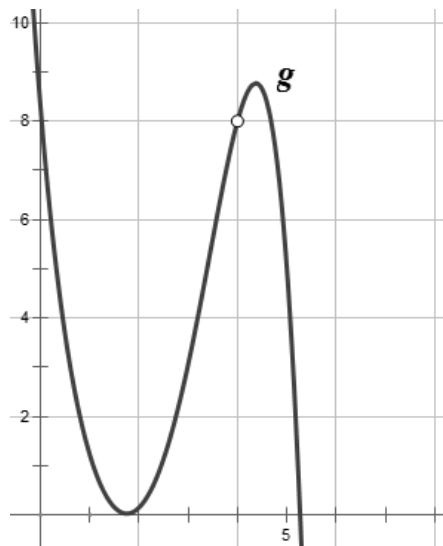
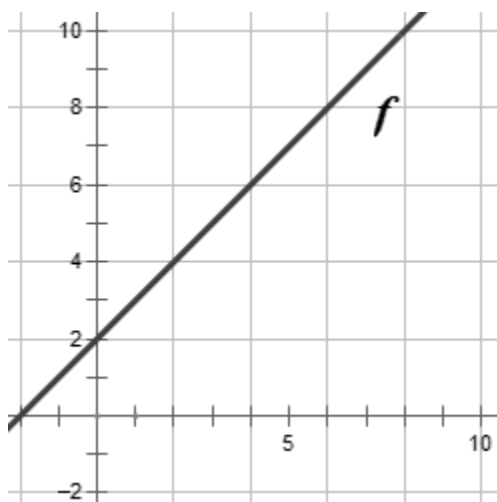
When asking “Why?” in the following interview questions, the interviewer should use questions like:

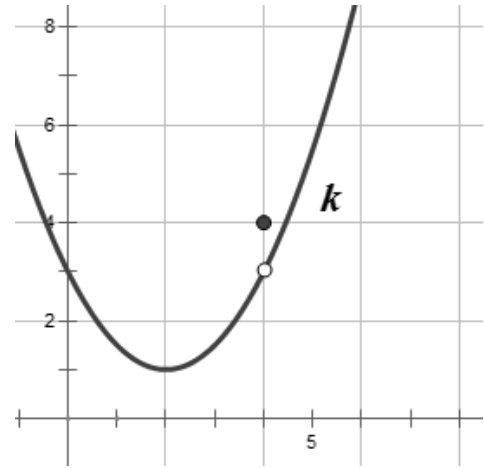
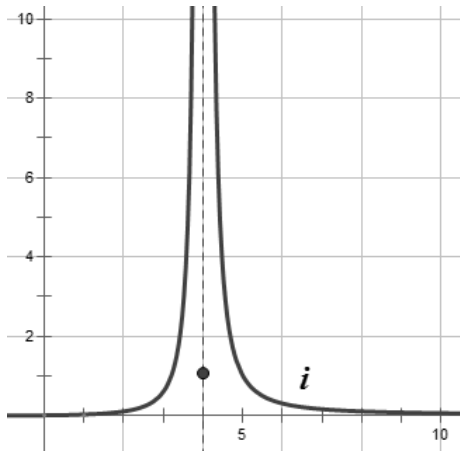
- How were you thinking about the function being continuous/discontinuous as you answered this question?
- How did what you said show that the limit exists?
- Can you think of a counterexample?
- Could you give me an example of what you mean by \_\_\_\_\_  
(continuous/discontinuous/defined/undefined/approaching or any other term you are not certain of what they mean)?
- Would you say a limit is (or continuity means) \_\_\_\_\_ (describing a conception of limit or continuity the interviewer thinks the student may be using)?

4. Is it possible to draw a function such that the limit as  $x$  approaches 3 exists but,
  - a. The function is discontinuous at  $x = 3$ ? Why?
  - b. The function is continuous at  $x = 3$ ? Why?
  - c. The function is undefined at  $x = 3$ ? Why?
  - d. The function is defined at  $x = 3$ ? Why?
5. Is it possible to draw a discontinuous function where the limit exists at every point in the domain? Why?
6. Is it possible to draw a continuous function where the limit does not exist? Why?
7. For each of the graphs below, determine
  - a. Which graph(s) represents a function which is continuous at the point  $x=4$ ?

- b. For which function(s) can you take the limit as  $x$  approaches 4?
- c. For which graph(s) does the limit exist as  $x$  approaches 4?

For each response the interviewer will inquire why the student responded the way they did.





8. Which of the following must be true if  $f$  is a function for which  $\lim_{x \rightarrow 2} f(x) = 3$ ? Circle all that apply.
- $f$  is discontinuous at the point  $x = 2$ .
  - $f$  is continuous at the point  $x = 2$ .
  - $f(x)$  is defined at  $x = 2$
  - $f(2) = 3$
  - For every positive integer  $n$ , there is a real number  $\delta > 0$  such that if  $0 < |x - 2| < \delta$ , then  $|f(x) - 3| < \frac{1}{n}$ .
  - For every real number  $\epsilon > 0$ , there is a real number  $\delta > 0$  such that if  $|f(x) - 3| < \epsilon$ , then  $0 < |x - 2| < \delta$ .
  - None of the above statements are true
9. Which of the following must be true if  $g(5) = 7$ ?
- $\lim_{x \rightarrow 5} g(x) = 7$
  - $g$  is continuous at  $x=5$ .
  - $g$  is discontinuous at  $x=5$ .
  - None of the above statements are true.
10. Comparing to the answers the student gave to this question on the questionnaire, determine which answers are different.
- Discuss why they answered each one the way they did, using the types of “Why?” questions described above.

- b. Discuss why the answers changed with the student. If it changed from incorrect to correct, try to pinpoint what conception of limit or continuity the student is using to correctly describe their solution.

Additional Question: What can you tell me about the limit and continuity of this function at  $x=-1$  and  $x=1$ ?

$$j(x) = \begin{cases} -x - 1 & \text{if } x < -1 \\ x^2 + 1 & \text{if } -1 \leq x < 1 \\ 2x & \text{if } x > 1 \end{cases}$$

## Appendix B

### Extensive List of Codes and their Explanations

Limit: Student is talking about limits but no specific conception of limit is identifiable

- **Dynamic Limit:** Student describes limits using terms like "approaching" or "tends to" or describes how a function is moving.
- **Dynamic Practical Limit:** The  $x$  values approach  $a$  as successively closer inputs result in outputs approaching a certain  $y$  value.
- **Bounding Limit:** Student describes a limit as a bound which the function cannot pass.
- **Proximity Limit:** Student describes limit using language such as "when  $x$  is close to  $a$ ,  $f(x)$  is close to  $L$ ."
- **Approximation Limit:** Describing the point of a limit as to get an estimate of what the function value would be.
- **Unreachable Limit:** Describing a limit as a value for which the function cannot be defined or that the graph cannot touch.
- **Faulty Conception Limit:** Student is using a definition for limit that does not relate to any accepted way to think about limits.
  - o  **$x$  approaches:** Limit value is the value  $x$  is approaching.
  - o **Tangent line:** The limit is the tangent line to the curve.
  - o **Asymptote is Limit:** Student describes a limit as an asymptote. Such as the limit is  $x=4$ , the  $x$  value at which the asymptote occurs with reference to a line.
  - o **Limit is a rift:** The limit is some sort of "rift" in the function such as an asymptote, a jump, or a point defined outside what one would expect. "Rift" language from Student B.
- **Formal Limit:** Student uses one of the formal ways to describe limit, such as with delta and epsilon.
- **Substitution Limit:** Student describes evaluating a limit by substituting  $x=a$ .
- **Left/Right Limit:** Student is describing evaluating the limit from one side, the other, or both.
- **Graphical Limit:** Student describes the limit as something used to determine how the function "looks" "acts" "behaves" etc.
- **Zooming Limit:** Student describes limit by referencing zooming in on a graph.
- **Neighborhood Limit (combined with Proximity after coding):** Student describes limit by talking about intervals around  $a$  or  $L$ .
- **Point Doesn't Matter Limit:** Student describes the limit as having to do with everything around a point, but what happens at the point itself doesn't matter. Might say it gets close to but doesn't necessarily touch. If they say it doesn't touch it would be unreachable.

Continuity: Student is talking about continuity but no specific conception can be identified.

- **Dynamic Continuity:** Student describes continuity as the ability to draw line without lifting pen from paper. Description includes on some sense of motion.

- **Faulty Conception Continuity** : Student is using a definition for continuity that does not relate to any accepted way of thinking about continuity.
- **Lack of discontinuities Continuity**: Student describes continuity as the lack of discontinuities. This includes saying there are no jumps, holes, asymptotes, etc. Also if student says discontinuous because there is a discontinuity like those listed.
- **Semi-Formal Continuity**: Student describes continuity as when limit value and function value are the same. All three parts are included: Limit exists, function exists, values are equal.
- **Semi-Semi-Formal Continuity**: Talking about pieces of formal continuity but not all three appropriately.
- **Gapless Continuity**: Student describes a line as a collection of points such that there are no gaps such that there are no points.
- **Predictable Continuity**: Student describes a function as continuous because the function does what you expect it to do.
- **Visual Continuity**: Student describes a function as continuous because it looks "smooth" or is "straight" or similar. Also if they say it is "one line." Student describes a function as continuous when all the points are on the line or describes it as discontinuous because there is a point not on the line.

**PPC Codes** – The following codes are the pieces of the PPCs or the topics students may be discussing when an opportunity to use a PPC is evident.

- **Definition at a point**: Student is discussing whether a point is defined or not.
- **Continuity at a point**: Student is discussing if a function is continuous at a point or not.
- **Limit value at a point**: Student is discussing the limit value at a point.
- **Function value at a point**: Student is discussing the value of the function at a point.
- **Limit existence at a point**: Student is discussing whether a limit exists or not at a point.

**Defined Means Continuous PPC**: If the function is defined at a point, it is continuous at that point.

**Limit Equals Function Value PPC**: The limit value and the function value are the same at a point.

**Limit Means Continuous PPC**: If the limit exists at a point, the function is continuous at that point.

**Limit Means Discontinuous PPC**: If you take the limit at a point, the function must be discontinuous at that point.

**PPC Specific Codes** – The following codes were used along with a specific PPC code to describe how the student reasoned with the PPC.

- **General:** Student used the PPC to reason about a general situation of limit or continuity at a point. No particular example was being reasoned about, so it cannot be determined if their reasoning was correct or incorrect.
- **Incorrect:** Student used the PPC to reason in a situation where the PPC was incorrect.
- **Correct:** Student used the PPC to reason in a situation where the PPC was correct.
- **Overcome:** Student reasoned appropriately in a situation where the PPC may have been used incorrectly. The student previously used the PPC as a misconception or used it generally in the same or a recent excerpt.
- **PPC Not Used:** Student reasoned appropriately in a situation where the PPC may have been used incorrectly. The student had not reasoned with the PPC as a misconception.

**Lack of examples:** Student describes their thinking or explains why they chose an answer by saying that they failed to think of the proper example, they jumped to conclusions too soon, or something similar. Also code if it seems as though the student has done this.