



2011-07-29

# Efficient, Accurate, and Non-Gaussian Error Propagation Through Nonlinear, Closed-Form, Analytical System Models

Travis V. Anderson

*Brigham Young University - Provo*

Follow this and additional works at: <https://scholarsarchive.byu.edu/etd>



Part of the [Mechanical Engineering Commons](#)

---

## BYU ScholarsArchive Citation

Anderson, Travis V., "Efficient, Accurate, and Non-Gaussian Error Propagation Through Nonlinear, Closed-Form, Analytical System Models" (2011). *All Theses and Dissertations*. 2675.

<https://scholarsarchive.byu.edu/etd/2675>

This Thesis is brought to you for free and open access by BYU ScholarsArchive. It has been accepted for inclusion in All Theses and Dissertations by an authorized administrator of BYU ScholarsArchive. For more information, please contact [scholarsarchive@byu.edu](mailto:scholarsarchive@byu.edu), [ellen\\_amatangelo@byu.edu](mailto:ellen_amatangelo@byu.edu).

Efficient, Accurate, and Non-Gaussian Statistical Error Propagation  
Through Nonlinear, Closed-Form, Analytical System Models

Travis V. Anderson

A thesis submitted to the faculty of  
Brigham Young University  
in partial fulfillment of the requirements for the degree of  
Master of Science

Christopher A. Mattson, Chair  
David T. Fullwood  
Kenneth W. Chase

Department of Mechanical Engineering  
Brigham Young University  
Thesis Completed July 2011

Copyright © 2011 Travis V. Anderson  
All Rights Reserved



## ABSTRACT

### Efficient, Accurate, and Non-Gaussian Statistical Error Propagation Through Nonlinear, Closed-Form, Analytical System Models

Travis V. Anderson  
Department of Mechanical Engineering, BYU  
Master of Science

Uncertainty analysis is an important part of system design. The formula for error propagation through a system model that is most-often cited in literature is based on a first-order Taylor series. This formula makes several important assumptions and has several important limitations that are often ignored. This thesis explores these assumptions and addresses two of the major limitations. First, the results obtained from propagating error through nonlinear systems can be wrong by one or more orders of magnitude, due to the linearization inherent in a first-order Taylor series. This thesis presents a method for overcoming that inaccuracy that is capable of achieving fourth-order accuracy without significant additional computational cost. Second, system designers using a Taylor series to propagate error typically only propagate a mean and variance and ignore all higher-order statistics. Consequently, a Gaussian output distribution must be assumed, which often does not reflect reality. This thesis presents a proof that nonlinear systems do not produce Gaussian output distributions, even when inputs are Gaussian. A second-order Taylor series is then used to propagate both skewness and kurtosis through a system model. This allows the system designer to obtain a fully-described non-Gaussian output distribution. The benefits of having a fully-described output distribution are demonstrated using the examples of both a flat rolling metalworking process and the propeller component of a solar-powered unmanned aerial vehicle.

Keywords: uncertainty analysis, statistical error propagation, system modeling, Taylor series expansion, variance propagation, skewness propagation, kurtosis propagation



## ACKNOWLEDGMENTS

I would like to thank Dr. Christopher Mattson for supporting me in this research. He has helped open my eyes to the world of research. He has pushed me to explore the field of engineering design and find solutions to some problems in statistical uncertainty analysis. He has been a great mentor to me.

I would like to thank Dr. David Fullwood for providing his insights and lending his mathematical expertise to furthering this work. Without his help, the equations upon which this research is based could not have been derived.

I would like to thank Dr. Kenneth Chase for his expertise in uncertainty analysis, particularly in the field of tolerance analysis. His previous work has been a great reference and source of ideas for this research.

I would like to thank Brad Larson for his work in compositional system models, for his advice, and for his patience and willingness in answering my incessant questions.

I would like to thank BYU's Department of Mechanical Engineering for its financial support of this research.

Lastly, I would like to thank my parents for teaching me the value of hard work, my Heavenly Father for inspiring me at every step in this work, and my wife, Lacey, for her never-ending patience, support and love.



## TABLE OF CONTENTS

<b>LIST OF TABLES</b> . . . . .	<b>vi</b>
<b>LIST OF FIGURES</b> . . . . .	<b>viii</b>
<b>NOMENCLATURE</b> . . . . .	<b>x</b>
<b>Chapter 1 Introduction</b> . . . . .	<b>1</b>
<b>Chapter 2 Literature Survey</b> . . . . .	<b>3</b>
2.1 Error Propagation via Taylor Series Expansion . . . . .	3
2.2 Non-Deterministic Analysis via Brute Force . . . . .	4
2.3 Deterministic Error by Model Composition . . . . .	5
2.4 Error Budgets . . . . .	5
2.5 Univariate Dimension Reduction . . . . .	6
2.6 Interval Analysis Methods . . . . .	6
2.7 Bayesian Inference . . . . .	6
2.8 Response Surface Methodologies . . . . .	6
2.9 Anti-Optimization Techniques . . . . .	7
<b>Chapter 3 Propagation of Variance</b> . . . . .	<b>9</b>
3.1 Fundamental Concepts . . . . .	9
3.1.1 Mathematical Expectation . . . . .	9
3.1.2 Central Moments . . . . .	10
3.2 Propagation of Variance Using First Order Taylor Series . . . . .	11
3.2.1 First-Order Formula Derivation . . . . .	11
3.2.2 First-Order Accuracy . . . . .	13
3.3 Propagation of Variance Using Higher-Order Taylor Series . . . . .	14
3.3.1 Second-Order Formula Derivation . . . . .	15
3.3.2 Second-Order Accuracy . . . . .	16
3.4 Taylor Series Error Propagation Assumptions and Limitations . . . . .	17
3.5 Higher-Order Accuracy With First- or Second-Order Cost . . . . .	18
3.5.1 Propagation of Variance Using Higher-Order Terms . . . . .	18
3.5.2 Reducing Truncation Error . . . . .	20
3.5.3 Correction Factors for Other Nonlinear Functions . . . . .	22
3.5.4 Model Composition . . . . .	24
3.6 Example: Compositional System Model of a Dual Propeller, Three Degree- of-Freedom Helicopter . . . . .	24
3.7 Example: Kinematic Model of a Mechanism that Simulates Flapping Flight Wing Motion . . . . .	28
3.8 Comments on Variance Propagation . . . . .	30
<b>Chapter 4 Higher-Order Statistics</b> . . . . .	<b>33</b>



4.1	Motivation for Propagating Higher-Order Statistics . . . . .	33
4.2	Propagation of Skewness . . . . .	35
4.2.1	Definition of Skewness . . . . .	35
4.2.2	Skewness Propagation Formula Derivation . . . . .	36
4.2.3	Skewness Propagation Assumptions and Limitations . . . . .	37
4.2.4	Proof: Nonlinear Functions Produce Non-Gaussian Output Distributions	38
4.3	Propagation of Kurtosis . . . . .	39
4.3.1	Definition of Kurtosis . . . . .	39
4.3.2	Definition of Excess Kurtosis . . . . .	39
4.3.3	Kurtosis Propagation Formula Derivation . . . . .	40
4.4	Example: Solar-Powered Unmanned Aerial Vehicle Propeller Thrust . . . . .	41
4.4.1	The Analytical Model of Thrust . . . . .	42
4.4.2	Statistical Component Model Output Prediction . . . . .	44
4.5	Example: Flat Rolling Metalworking Process . . . . .	45
4.5.1	The Analytical Model . . . . .	47
4.5.2	Statistical System Model Output Prediction . . . . .	49
4.5.3	Ramifications of Neglecting Higher-Order Statistics . . . . .	50
4.6	Comments on Propagation of Higher-Order Statistics . . . . .	51
<b>Chapter 5 Conclusion . . . . .</b>		<b>53</b>
<b>REFERENCES . . . . .</b>		<b>55</b>
<b>Appendix A Step-by-Step Solution to Dual-Propeller, Three Degree-of-Freedom Helicopter Problem . . . . .</b>		<b>59</b>
A.1	Variance Propagation Through Component 1 . . . . .	60
A.2	Variance Propagation Through Component 2 . . . . .	60
A.3	Variance Propagation Through Component 3 . . . . .	62
A.4	Variance Propagation Through System-Level Compositional Model . . . . .	63

## LIST OF TABLES

3.1	Partial Derivatives Required for Taylor Series Variance Propagation . . . . .	19
3.2	Truncation Error Correction Factors for Common Nonlinear Functions . . .	23
3.3	Design Parameters for Dual-Propeller, 3-DOF Helicopter . . . . .	26
3.4	Mean Values and Standard Deviations of the 16 Model Input Parameters . .	29
4.1	Comparison of Positive and Negative Skew . . . . .	35
4.2	Statistical Properties of the Angular Velocity Distribution of a Propeller . .	43
4.3	Central Moments of the Distribution of Angular Velocity, $\omega$ . . . . .	44
4.4	Friction Conditions in Mechanical Devices and Metalworking Processes . . .	47
4.5	Statistical Properties of the Coefficient of Friction Distribution . . . . .	48
4.6	Central Moments of the Distribution of the Coefficient of Friction, $\mu_f$ . . . . .	49
A.1	Design Parameters for Dual-Propeller, 3-DOF Helicopter . . . . .	59



## LIST OF FIGURES

3.1	Relative Error in Output Variance Using a First-Order Taylor Series Expansion for the Function $y = 1000 \sin(x)$ . . . . .	14
3.2	Relative Error in Estimate of Variance Propagation Using a Second-Order Taylor Series Expansion for the Function $y = 1000 \sin(x)$ . . . . .	17
3.3	Relative Error in Variance Propagation Using Taylor Series Approximations	19
3.4	Relative Computational Cost of Taylor Series Approximations, as Determined by MATLAB Execution Time . . . . .	20
3.5	Relationships Between the Second-Order Bias and both Input Standard Deviation $\sigma_x$ and Input Variance $\sigma_x^2$ for a <i>sin</i> Function . . . . .	21
3.6	Relative Error in Estimations of Variance Propagation . . . . .	22
3.7	Mean Relative Error in Variance Estimations . . . . .	22
3.8	Dual-Propeller, Three Degree-of-Freedom Helicopter Used in the Design of Unmanned Aerial Vehicles . . . . .	25
3.9	Relative Error in Estimations of Variance Propagation . . . . .	27
3.10	Mean Relative Errors that Indicate the Compositional Model with a Correction Factor Predicts Output Variance with About Half the Error as the First-Order Approximation . . . . .	27
3.11	Mechanism Used by the BYU Flapping Flight Research Team to Simulate Three-Degree-of-Freedom Motion of a Flapping Wing . . . . .	28
3.12	Computational Time to Predict Output Distributions Using Various Error Propagation Methods . . . . .	29
3.13	Relative Error in Predictions of Output Variance Obtained from Various Orders of a Taylor Series . . . . .	30
4.1	Predicted Gaussian Output Distribution Obtained from Propagating Mean and Variance Only (a) Compared with Actual System Output (b) . . . . .	34
4.2	Predicted Non-Gaussian Output Distribution Obtained from Propagating Mean, Variance, Skewness, and Kurtosis (a) Compared with Actual System Output (b) . . . . .	34
4.3	Examples of Negative (Left) and Positive (Right) Skewness . . . . .	36
4.4	Excess Kurtosis of Various Common Statistical Distributions . . . . .	40
4.5	Power and Propulsion System of a Solar-Powered Unmanned Aerial Vehicle (UAV) . . . . .	42
4.6	Compositional System Model of a Solar-Powered Unmanned Aerial Vehicle (UAV) . . . . .	43
4.7	Distribution of the Angular Velocities of a Propeller in a Solar-Powered UAV	44
4.8	Predicted Gaussian Output Distribution of Thrust Obtained by Propagating a Mean and Variance Only (a) Compared with Actual Thrust Measurements (b) . . . . .	45
4.9	Predicted Non-Gaussian Output Distribtuion of Thrust Obtained by Propagating a Mean, Variance, Skewness, and Kurtosis (a) Compared with Actual Thrust Measurements (b) . . . . .	46

4.10 Flat Rolling Manufacturing Process Whereby Plates or Sheets of Metal are Made—Material is Drawn Between Two Rollers, which Reduces the Material’s Thickness . . . . .	46
4.11 Distribution of the Coefficient of Friction in a Flat Rolling Metalworking Process	48
4.12 Predicted Gaussian Output Distribution Obtained from Propagating a Mean and Variance Only (a) Compared with Actual System Output (b) . . . . .	50
4.13 Predicted Non-Gaussian Output Distribution Obtained from Propagating a Mean, Variance, Skewness, and Kurtosis (a) Compared with Actual System Output (b) . . . . .	50

## NOMENCLATURE

$N$	Number of data points in a distribution
$n$	Number of inputs to a system model
$x$	An input distribution to a system model
$x_i$	The $i$ -th system input distribution
$x_j$	The $j$ -th element of a system input distribution
$\bar{x}$	The mean value of input $x$
$\sigma_x$	Standard deviation in the input distribution $x$
$\sigma_x^2$	Variance in the input distribution $x$
$\sigma_{x_i x_j}^2$	Covariance in between input distributions $x_i$ and $X_j$
$\gamma_1$	Skewness in a distribution
$\gamma_2$	Kurtosis in a distribution
$\beta_2$	Excess kurtosis in a distribution
$\kappa_i$	The $i$ -th cumulant of a distribution
$\mu_k$	The $k$ -th central moment of a distribution
$\mu_{k,i}$	The $k$ -th central moment of the $i$ -th input distribution
$\partial_1$	First partial derivative of the system model, evaluated at the mean
$\partial_2$	Second partial derivative of the system model, evaluated at the mean
$y$	System output
$\bar{y}$	Mean system output value
$\sigma_y$	Standard deviation in output distribution $y$
$\sigma_y^2$	Variance in output distribution $y$
$\epsilon$	Relative error in a model's prediction compared with actual output
$e$	Correction factor
$C_k$	The $k$ -th component of a compositional system model
$E[ ]$	indicates the expectation operator performed on [ ]

### **Example: Dual Propeller, Three Degree-of-Freedom Helicopter**

$\theta$	Helicopter pitch angle (rad)
$\ddot{\theta}$	Angular acceleration in pitch (rad/s <sup>2</sup> )
$m_1$	Mass of helicopter motors, propellers, and circuitry (kg)
$m_2$	Mass of helicopter counterbalance weight (kg)
$L_k$	Length of helicopter segment $k$ (m)
$J_y$	Mass moment of inertia about the pitch axis of rotation (kg-m <sup>2</sup> )
$u_L$	Motor command given to left motor
$u_R$	Motor command given to right motor
$k_m$	Motor calibration constant

### **Example: Kinematic Model of Flapping Flight Wing Motion**

$t$	Simulation time (s)
$\omega$	Flapping frequency (Hz)
$\phi$	Positional angle (deg)
$\theta$	Elevation angle (deg)
$\alpha$	Feathering or attack angle (deg)

$A_{xx}$  Fourier series coefficients for *cos* terms  
 $B_{xx}$  Fourier series coefficients for *sin* terms

**Example: Solar-Powered Unmanned Aerial Vehicle Propeller Thrust**

$T$  Thrust generated by the propeller (N)  
 $C_t$  Unitless calibration coefficient of thrust  
 $\rho$  Density of air (kg/m<sup>3</sup>)  
 $\omega$  Propeller angular velocity (rad/s)  
 $D$  Propeller Diameter (m)

**Example: Flat Rolling Metalworking Process**

$\Delta H$  The change in material thickness with each pass (m)  
 $\Delta H_{max}$  The maximum change in material thickness attainable in a single pass (m)  
 $\mu_f$  Coefficient of friction  
 $R$  Roller radius (m)

## CHAPTER 1. INTRODUCTION

In the system design process, designers frequently experience significant uncertainty in predicting whether a proposed design will meet the design objectives. Design decisions often cannot be validated until a physical prototype is built and tested. If the proposed design proves to be faulty, it can lead to a very costly design iterations. The inability of the system designer to verify design decisions early in the design process and to determine if a proposed design will accomplish design objectives is a limiting obstacle in system design [1].

Consequently, system behavioral modeling is an important part of system design. If a system behavioral model could be obtained and its accuracy quantified, it would enable the designer to verify design decisions early in the design process. This would greatly reduce the risk of a creating a failed system design.

Determining system model accuracy is difficult, especially when the system is still theoretical and actual system behavior is not known [2]. In many situations, a statistical distribution of possible outputs is more meaningful than a simple max/min error bound [3]. If the system input distributions are known, such an output distribution can be obtained for closed-form differentiable models using a Taylor series expansion to propagate input distributions through a system model. This is typically accomplished with the first-order Taylor series approximation, but there are two major limitations of using this method.

First, the first-order Taylor series takes a derivative-based weighted sum of independent input variances to estimate the variance in the output distribution. Since all higher-order terms in the series are neglected, this predicted output variance may be wrong by one or more orders of magnitude for nonlinear functions [4]. While higher-order terms clearly improve the accuracy of the approximation, they also require greater computational cost. This thesis shows that the truncation error in lower-order approximations can be predicted



and accounted for. Consequently, fourth-order accuracy in error propagation can be obtained without significant additional computational cost.

Second, system designers typically use the Taylor series to *only* propagate a mean and variance. All higher-order statistics are neglected. As all non-Gaussian distributions cannot be fully described with a mean and variance alone, the output distribution is typically assumed to be Gaussian. This often is an erroneous and costly assumption. This thesis proves that nonlinear functions do not produce Gaussian output distributions, even when inputs are Gaussian. A second-order Taylor series expansion can be used to propagate higher-order statistical properties through a system model without incurring significant additional computational cost.

This thesis explores error propagation through nonlinear system models using a Taylor series expansion. Chapter 2 provides an overview of nine common methods of uncertainty analysis. Chapter 3 shows how high accuracy in variance propagation can be obtained with low-order computational cost. Chapter 4 presents a method for propagating the higher-order statistical properties skewness and kurtosis through a closed-form system model. Chapter 5 concludes this thesis and suggests areas for future research.

## CHAPTER 2. LITERATURE SURVEY

Many methods are currently in use or being researched that quantify the accuracy of a system model by propagating error through the system. These methods include the following:

- Error propagation via Taylor series expansion
- Non-deterministic analysis via brute force (Monte Carlo, Latin hypercube, etc.)
- Deterministic model composition
- Error budgets
- Univariate dimension reduction
- Interval analysis
- Bayesian inference
- Response surface methodologies
- Anti-optimizations (sub-optimizations)

While this thesis focuses on error propagation via Taylor series expansion, the other uncertainly analysis techniques listed above are also briefly described in the remainder of this chapter.

### 2.1 Error Propagation via Taylor Series Expansion

A Taylor series can be used to propagate variation in system inputs through a system model to produce an estimate of the variation in system outputs. This derivatives-based method can be very simple and fast, and is the method most-often cited in literature [5, 6].

Nevertheless, it may produce results that are inaccurate by one or more orders of magnitude [4], particularly for nonlinear systems. This inaccuracy is a result of linearizing the system model, as the higher-order derivatives are truncated from the Taylor series. This linearization also means accuracy decreases as input variances increase.

Typically, this method is used only to propagate variance. Data is typically assumed to be independent and Gaussian, as variable interactions and higher-order statistics are ignored.

## 2.2 Non-Deterministic Analysis via Brute Force

Uncertainty analysis methods are usually classified as either deterministic or non-deterministic, and non-deterministic analysis methods are further grouped into two main categories: 1) reliability-based design methods [7, 8, 9], and 2) robust-design-based methods [10, 11, 12, 13, 14, 15].

A deterministic model always produces the same output values from the same input values, but a non-deterministic model does not. Consequently, a non-deterministic model, executed repeatedly with the same input values, can produce a statistical output distribution instead of a single nominal value.

However, due to the complexities of non-deterministic modeling, it is more common to represent uncertainty with probabilistic methods and propagate these uncertainties through a deterministic model [16]. This approach can also result in a statistical output distribution by executing the model repeatedly. This brute-force approach to non-deterministic uncertainty analysis commonly uses Monte Carlo, quasi Monte Carlo [17, 18], Monte Carlo hybrid [19], Latin hypercube, Latin supercube [20], or some other sampling technique.

This technique does not need to assume a Gaussian output distribution. Consequently, an estimate of the fully-described output distribution can be obtained. However, this approach comes at great computational cost, which only grows exponentially with an increase in the number of statistically distributed inputs. Furthermore, the entire simulation must be executed again each time the model or any input value changes. This can be prohibitive in an iterative design process.

### 2.3 Deterministic Error by Model Composition

Current research is being done [21] to deterministically propagate error through a compositional system model to obtain max/min error bounds. In order to do this, the system model error must be included in the model itself. Augmenting the compositional system model with component error models allows the component error interactions and propagations to be evaluated together with the model. System model accuracy can be determined by comparing the results of the regular system model with those from the error-augmented system model [2]. Monte Carlo simulations or optimization routines can be executed to find the max/min error bounds of the system model.

With this approach to error propagation, the accuracy of even complex systems and interactions can be obtained. Errors do not need to be independent. Component models do not have to be mathematical or closed-form functions, but rather can include dynamic models, nonlinear models, non-differentiable models, lookup-tables, software programs, digital logic, CAD models, finite-element models, and any other model capable of mapping inputs to outputs.

Deterministic error analysis requires known component error models and relationships. In some situations, the max/min error bounds obtained from this method are so large that they are not helpful. In many practical applications, a statistical error distribution is more useful than a max/min error envelop [3].

### 2.4 Error Budgets

The method of error budgets involves propagating the error of each component through the system separately, and resolving each component's error to the contribution it makes on the total system error [22, 23]. This is done by perturbing one error source at a time and observing the effect this has on the total system error. Consequently, this method requires either that component errors be independent or that a separate model showing component error interactions be developed, which typically is not done [24]. If the error sources are not actually independent, this method will not necessarily describe the full range of possible model error.

## 2.5 Univariate Dimension Reduction

Univariate dimension reduction methods transform data from a high-dimensional space to a lower-dimensional space. In some situations, data analysis may even be more accurate in the reduced space than in the original space [25].

## 2.6 Interval Analysis Methods

Interval analysis methods bound rounding and measurement errors in mathematical computation. Arithmetic can then be performed using intervals instead of a single nominal value [26]. These techniques can be used to propagate error envelopes, or intervals, through a system model. These methods, however, are typically limited to basic arithmetic operations.

Currently there are many software languages, libraries, compilers, and data types that implement interval arithmetic. These include XSC, Profil/BIAS, Boost, Gaol, Frink, and a MATLAB extension named Intlab. There is also a working group currently developing an IEEE Interval Standard (P1788).

## 2.7 Bayesian Inference

Bayesian inference is a method of statistical inference whereby the probability that a hypothesis is true is inferred based on both observed evidence and the prior probability that the hypothesis was true [27]. It combines common-sense knowledge with observational evidence in an attempt to eliminate needless complexity in a model by declaring only meaningful relationships [28] and disregarding the influences of all other variables on system outputs.

## 2.8 Response Surface Methodologies

Response surface methodology is commonly used in design of experiments. It is a modeling technique whereby an  $n$ -dimensional response surface showing the relationship between  $n$ -input variables is created, typically from using empirical data [29, 30]. A few common experimental design setups include full factorial, partial factorial, central composite, Plackett-Burman, Box-Behnken, and others.

## 2.9 Anti-Optimization Techniques

Anti-optimization techniques allow the designer to find the worst-case scenario for a given problem. This results in a two-level optimization problem, where the uncertainty is anti-optimized on the lower level and the overall design is optimized on a higher level [31].



## CHAPTER 3. PROPAGATION OF VARIANCE

This chapter demonstrates how a Taylor series is used to propagate variance by estimating a distribution's central moments. The mathematical expectation operator and central moments are first discussed. The derivations of the first- and second-order Taylor series error propagation formulas are then presented along with an in-depth discussion of their accuracies. The assumptions and limitations of using a Taylor series to propagate error through a system model are then summarized.

This chapter then demonstrates how fourth-order accuracy can be obtained in error propagation with only first- or second-order computational cost. This is accomplished by applying a correction factor to the lower-order estimate, which accounts for the higher-order truncation error in the Taylor series. These correction factors were determined empirically for several common nonlinear engineering functions.

Lastly, this chapter demonstrates the effectiveness of this method using the model of a dual-propeller, three degree-of-freedom helicopter, and the kinematic system model of a flapping wing.

### 3.1 Fundamental Concepts

Before continuing, it is essential to understand two statistical concepts that are fundamental to this research. These are the mathematical expectation operator and the statistical central moment property, both of which are described in this section.

#### 3.1.1 Mathematical Expectation

The mathematical expectation operator,  $E[ \ ]$  denotes the calculation of a weighted average of all possible values, as shown in Eq. (3.1). The weights correspond to the probability of a particular value. When all possible values have an equal probability, the expected



value is equal to the arithmetic mean, which is also equal to the limit of the sample mean as the sample size increases to infinity.

$$E[X] = x_1p_1 + x_2p_2 + \dots + x_np_n \quad (3.1)$$

where  $X$  is some population,  $x_i$  are the possible values,  $p_i$  are their respective probabilities, and all  $p_i$  add to 1. Some common expectation properties are presented below:

$$\begin{aligned} E[a] &= a \\ E[X + a] &= E[X] + a \\ E[X + Y] &= E[X] + E[Y] \\ E[a \cdot X] &= a \cdot E[X] \\ E[aX + b] &= a \cdot E[X] + b \\ E[aX + bY] &= a \cdot E[X] + b \cdot E[Y] \end{aligned}$$

where  $X$  and  $Y$  are statistical distributions and  $a$  and  $b$  are constants.

### 3.1.2 Central Moments

A central moment is a statistical property commonly used in statistical analysis [32, 6, 33]. The  $k$ -th central moment of a distribution is defined in Eq. (3.2).

$$\begin{aligned} \mu_k &= E[(x - \bar{x})^k] \\ &= \frac{1}{N} \sum_{j=1}^N (x_j - \bar{x})^k \end{aligned} \quad (3.2)$$

where  $x$  represents some distribution of  $N$  values,  $\bar{x}$  represents the input mean, and  $E$  is the mathematical expectation operator. The central moments of a population can easily be estimated using any appropriate population-sampling technique.

The zeroth central moment is always equal to one, the first central moment is always equal to zero [34], and the second central moment is equivalent to the variance. The third and fourth moments are used in the calculation of higher-order statistics, such as skewness and kurtosis. Propagation of these higher-order statistics is the subject of Chapter 4.

## 3.2 Propagation of Variance Using First Order Taylor Series

Generally, a distribution of input values propagates through a given system to produce a distribution of output values. While many sources of error might be present, including modeling error (unmodeled behavior, emergent behavior) and measurement error, this research focuses on the output distribution caused by variation in system model inputs only.

The analytical formula most-often cited in literature that estimates this variance propagation is based on a first-order Taylor series expansion. It makes several important assumptions that are limiting in many practical situations. These assumptions and limitations cannot be fully understood (much less overcome) without understanding the derivation of this first-order variance propagation formula.

The author of this thesis has found the complete derivation of this formula to be absent in textbooks and archival journal literature. Consequently, the derivation is presented in this section in order to more fully explain the assumptions, limitations, and accuracy of this error propagation technique. The assumptions and limitations mentioned are discussed in Section 3.4.

### 3.2.1 First-Order Formula Derivation

Let  $y$  be some function of  $n$  inputs  $x_i$ . The first-order Taylor series approximation expanded about the input means  $\bar{x}_i$  is shown in Eq. (3.3) [35].

$$y \approx f(\bar{x}_1, \dots, \bar{x}_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} (x_i - \bar{x}_i) \quad (3.3)$$

where the partial derivatives are evaluated at the mean  $x_i = \bar{x}_i$ . An approximation of the output mean  $\bar{y}$  is given in Eq. (3.4).

$$\begin{aligned}
\bar{y} &= E[y] \\
&\approx E \left[ f(\bar{x}_1, \dots, \bar{x}_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} (x_i - \bar{x}_i) \right] \\
&\approx f(\bar{x}_1, \dots, \bar{x}_n) + \sum_{i=1}^n \frac{\partial f}{\partial x_i} \mu_{1,i} \\
&\approx f(\bar{x}_1, \dots, \bar{x}_n)
\end{aligned} \tag{3.4}$$

where  $E$  is the expectation operator and  $\mu_{k,i}$  is the  $k$ -th central moment for the  $i$ -th input, as given previously in Eq. (3.2). (Recall that the first central moment is equal to zero.) Subtracting Eq. (3.3) from Eq. (3.4) produces Eq. (3.5).

$$y - \bar{y} \approx \sum_{i=1}^n \frac{\partial f}{\partial x_i} (x_i - \bar{x}_i) \tag{3.5}$$

Squaring and taking the expectation of Eq. (3.5) produces Eq. (3.6).

$$\begin{aligned}
E[(y - \bar{y})^2] &\approx \sum_{i=1}^n \left[ \left( \frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 + 2 \sum_{j=i+1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \sigma_{x_i x_j}^2 \right] \\
&= \sigma_y^2
\end{aligned} \tag{3.6}$$

where  $\sigma_y^2$  and  $\sigma_x^2$  are the variances in  $y$  and  $x$ , respectively. Recall that variance  $\sigma^2$  is the second central moment, which is defined in Eq. (3.7).

$$\begin{aligned}
\sigma_x^2 &= \mu_2 \\
&= E[(x - \bar{x})^2] \\
&= \frac{1}{N} \sum_{j=1}^N (x_j - \bar{x})^2
\end{aligned} \tag{3.7}$$

The second term in Eq. (3.6) is the covariance term, where  $\sigma_{x_i x_j}^2$  is the covariance between inputs  $x_i$  and  $x_j$ . Covariance is defined in Eq. (3.8).

$$\sigma_{x_i x_j}^2 = E [(x_i - \bar{x}_i)(x_j - \bar{x}_j)] \quad (3.8)$$

When inputs are independent, the covariance term is equal to zero, and Eq. (3.6) reduces to Eq. (3.9).

$$\sigma_y^2 \approx \sum_{i=1}^n \left( \frac{\partial f}{\partial x_i} \right)^2 \sigma_{x_i}^2 \quad (3.9)$$

This simplifying assumption of independence is typically made, both in literature and in practice. Consequently, Eq. (3.9) is the formula typically given for statistical error propagation through an analytical system model [36, 37, 38, 4].

### 3.2.2 First-Order Accuracy

Clearly, Eq. (3.9) is an approximation only and can be wrong by one or more orders of magnitude. This is especially evident when dealing with nonlinear functions [4].

For example, let  $y$  be modeled by the function  $y = 1000 \sin(x)$ . An estimation of the output variance  $\sigma_y^2$  obtained from Eq. (3.9) is given in Eq. (3.10).

$$\sigma_{y,1st}^2 \approx 10^6 \cos^2(\bar{x}) \sigma_x^2 \quad (3.10)$$

Throughout this entire thesis, “actual” output variance was determined using a Monte Carlo simulation. While convergence in statistical properties was obtained for every model in this thesis after 1 million executions, each system’s “actual” output is the result of 10 million executions.

The relative error  $\epsilon$  of Eq. (3.10) can be calculated using Eq. (3.11) and the result is plotted as a function of input mean  $\bar{x}$  in Figure 3.1.

$$\epsilon = \frac{|\sigma_{y, predicted}^2 - \sigma_{y, Monte Carlo}^2|}{\sigma_{y, Monte Carlo}^2} \quad (3.11)$$

As illustrated in Figure 3.1, the first-order approximation of variance propagation is fairly accurate for most values of  $\bar{x}$ . However, for certain input values, the approximation can be wrong by one or more orders of magnitude, as indicated by the 100% jump in relative

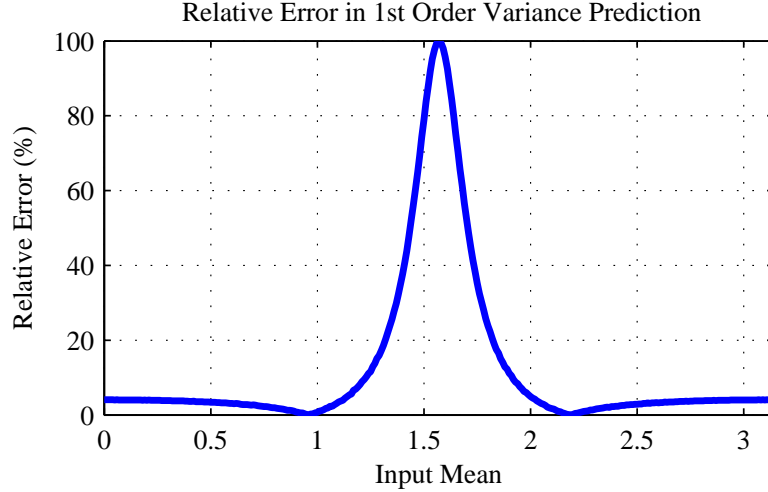


Figure 3.1: Relative Error in Output Variance Using a First-Order Taylor Series Expansion for the Function  $y = 1000 \sin(x)$

error at  $\bar{x} = \frac{\pi}{2}$ . This spike in relative error occurs because the *sin* function is nonlinear and the higher-order terms in the Taylor series used to derive Eq. (3.9) were neglected.

### 3.3 Propagation of Variance Using Higher-Order Taylor Series

As shown in the preceding section, Eq. (3.9) is based on a first-order Taylor series. For nonlinear and higher-order polynomial functions, Taylor series truncation error becomes significant and Eq. (3.9) can become extremely inaccurate (i.e., wrong by one or more orders of magnitude [4]).

As expected, the accuracy of this estimate of variance propagation through a system can be improved by including higher-order terms in the Taylor series. In situations where increased accuracy is required, a second-order Taylor series is sometimes used to propagate statistical error. Similar to the first-order approximation, some simplifying assumptions are commonly made in the derivation of the second-order approximation, which are presented in Section 3.4.

This section presents the derivation of the second-order error propagation formula and then discusses the accuracy of this formula.

### 3.3.1 Second-Order Formula Derivation

For the sake of brevity, the second-order derivation is presented in this section for a monovariate function,  $y = f(x)$ . Extending this derivation to multivariate functions is trivial, as it follows the same derivation steps.

The second-order Taylor series taken about the input mean  $\bar{x}$  is given in Eq. (3.12), where the partial derivatives are again evaluated at the mean,  $x = \bar{x}$ .

$$y \approx f(\bar{x}) + \frac{\partial f}{\partial x}(x - \bar{x}) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x - \bar{x})^2 \quad (3.12)$$

The second-order approximation of the output mean  $\bar{y}$  is given in Eq. (3.13).

$$\begin{aligned} \bar{y} &= E[y] \\ &\approx E \left[ f(\bar{x}) + \frac{\partial f}{\partial x}(x - \bar{x}) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x - \bar{x})^2 \right] \\ &\approx f(\bar{x}) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \mu_2 \end{aligned} \quad (3.13)$$

Subtracting Eq. (3.13) from Eq. (3.12) gives Eq. (3.14).

$$y - \bar{y} \approx \frac{\partial f}{\partial x}(x - \bar{x}) + \frac{1}{2} \frac{\partial^2 f}{\partial x^2}(x - \bar{x})^2 - \frac{1}{2} \frac{\partial^2 f}{\partial x^2} \mu_2 \quad (3.14)$$

Squaring and taking the expectation of Eq. (3.14) produces Eq. (3.15).

$$\begin{aligned} E[(y - \bar{y})^2] &\approx \mu_2 \left( \frac{\partial f}{\partial x} \right)^2 + \mu_3 \frac{\partial f}{\partial x} \frac{\partial^2 f}{\partial x^2} + \frac{1}{4} (\mu_4 - \mu_2^2) \left( \frac{\partial^2 f}{\partial x^2} \right)^2 \\ &\approx \sigma_y^2 \end{aligned} \quad (3.15)$$

If  $x$  is Gaussian, all odd moments ( $\mu_k$  where  $k$  is odd) are zero and Eq. (3.15) reduces to Eq. (3.16).

$$\begin{aligned}
\sigma_y^2 &\approx \left(\frac{\partial f}{\partial x}\right)^2 \mu_2 + \frac{1}{4} \left(\frac{\partial^2 f}{\partial x^2}\right)^2 (\mu_4 - \mu_2^2) \\
&\approx \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \frac{1}{4} \left(\frac{\partial^2 f}{\partial x^2}\right)^2 (\mu_4 - \sigma_x^4)
\end{aligned} \tag{3.16}$$

Furthermore, if  $x$  is Gaussian, the substitution  $\mu_4 \approx 3\sigma^4$  can be made [39], which eliminates the need to know the input's higher-order moments. This substitution is made in Eq. (3.17).

$$\sigma_y^2 \approx \left(\frac{\partial f}{\partial x}\right)^2 \sigma_x^2 + \frac{1}{2} \left(\frac{\partial^2 f}{\partial x^2}\right)^2 \sigma_x^4 \tag{3.17}$$

If  $y$  is a function of multiple independent inputs, the generalized form of Eq. (3.17) is given in Eq. (3.18).

$$\sigma_y^2 \approx \sum_{i=1}^n \left(\frac{\partial y}{\partial x_i}\right)^2 \sigma_{x_i}^2 + \frac{1}{2} \sum_{j=1}^n \sum_{i=1}^n \left(\frac{\partial^2 y}{\partial x_i \partial x_j}\right)^2 \sigma_{x_i}^2 \sigma_{x_j}^2 \tag{3.18}$$

Equation (3.18) is the second-order formula most often cited in literature [40, 36] for analytical statistical error propagation. Note that the covariance terms in Eq. (3.18) have been neglected.

### 3.3.2 Second-Order Accuracy

Continuing with the same function  $y = 1000 \sin(x)$ , the relative errors obtained from the second-order approximations in Eq. (3.15) (full approximation) and Eq. (3.16) (Gaussian) are plotted as a function of input mean  $\bar{x}$  in Figure 3.2.

Figure 3.2 illustrates the noise introduced by assuming  $x$  is Gaussian when in reality the distribution is never perfect. Nevertheless, in both cases the second-order approximation successfully filters the large spikes in relative error present in the first-order approximation. However, the second-order approximation still overestimates the actual variance propagation, which could degrade system performance and lead to failure or infeasibility [41]. This bias (about 4%, in this case) is a result of truncating the higher-order terms in the Taylor series.

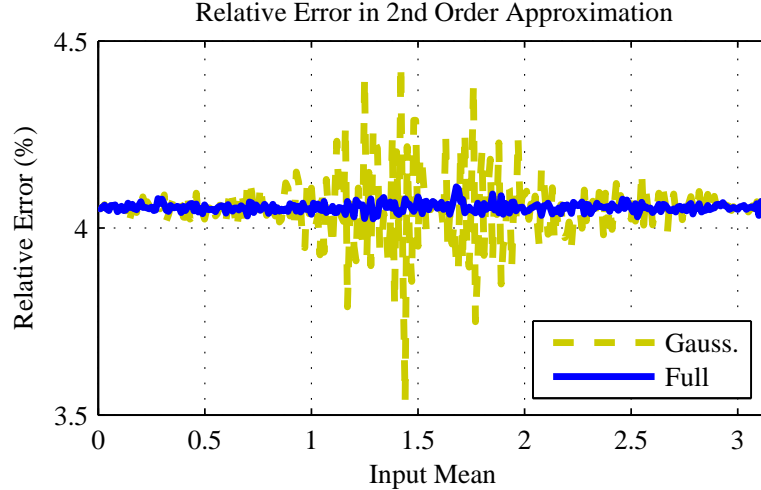


Figure 3.2: Relative Error in Estimate of Variance Propagation Using a Second-Order Taylor Series Expansion for the Function  $y = 1000 \sin(x)$

### 3.4 Taylor Series Error Propagation Assumptions and Limitations

Though common in engineering literature and academia, Eqs. (3.9) and (3.18) have many significant limitations. Often, designers use these equations without knowing or understanding these limitations. The following list summarizes these assumptions and limitations:

1. Only variance is propagated and higher-order statistics are neglected. Consequently, the output distribution is generally assumed to be a Gaussian distribution, which is not the case in many practical applications.
2. The system model  $y$  must be representable as a closed-form, differentiable, mathematical equation.
3. Taking the Taylor series expansion about a single point ( $\bar{x}$ ) causes the approximation to be of local validity only [36, 19]. Consequently, the accuracy of the approximation generally decreases with an increase in the input variance  $\sigma_x^2$ .
4. The approximation is generally more accurate for linear models.
5. All inputs  $x_i$  are assumed be Gaussian. Consequently, only means and variances are used to fully describe the input distributions. When inputs are not Gaussian, the non-Gaussian terms (i.e., odd moments) cannot be neglected.



6. The input means and standard deviations must be known.
7. All inputs  $x_i$  are assumed to be independent. When inputs are not independent, the covariance terms cannot be neglected [37, 42, 4].

The first assumption listed above is addressed in detail in Chapter 4.

### 3.5 Higher-Order Accuracy With First- or Second-Order Cost

While an improvement over the first-order formula, the second-order equation given by Eq. (3.18) is far from perfect. This section shows that accuracy can be improved by adding higher- and higher-order terms. However, the resulting computational cost quickly becomes prohibitive. This is a common problem in engineering analysis and design [43].

This section then presents a solution to that problem. This is accomplished by predicting the higher-order truncation error and applying a resultant correction factor to the lower-order approximation. This can yield fourth-order accuracy in the estimation of error propagation with first- or second-order computational cost. These correction factors were empirically determined for trigonometric, logarithmic, and exponential functions.

#### 3.5.1 Propagation of Variance Using Higher-Order Terms

As expected, adding higher-order terms reduces the second-order bias. This is shown in Figure 3.3. With an infinite number of terms, the Taylor series approximation eventually converges to zero error.

However, computational cost grows exponentially as higher-order terms are included. This growth in computational cost is accelerated at an exponential rate with an increase in the number of system inputs, as shown in Table 3.1. Furthermore, higher-order terms also require the calculation of higher-order moments and covariance terms for the system inputs. This exponential growth in cost causes higher-order terms to quickly become prohibitively expensive for complex systems.

For the purposes of this research, the computational costs of propagating error were determined using the MATLAB execution time. Figure 3.4 shows the relative computational

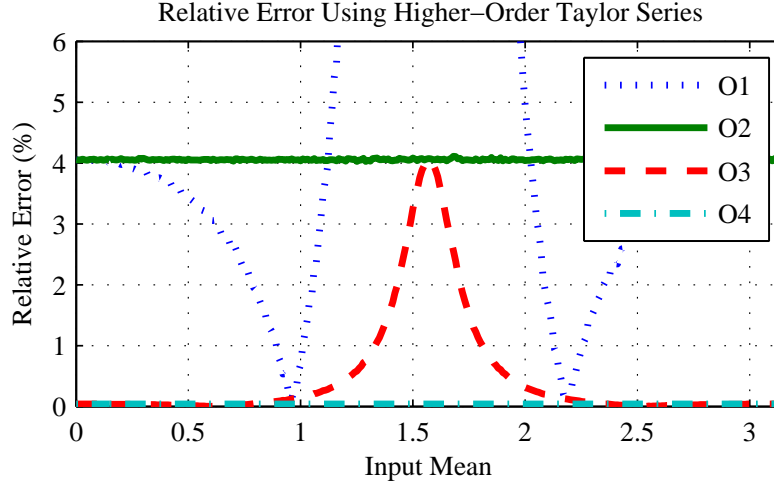


Figure 3.3: Relative Error in Variance Propagation Using Taylor Series Approximations

Table 3.1: Partial Derivatives Required for Taylor Series Variance Propagation

Inputs	O1	O2	O3	O4	O5	O6	O7
2	2	3	6	10	15	21	28
4	4	10	20	35	56	84	120
6	6	21	56	126	252	462	792
8	8	36	120	330	792	1716	3138
10	10	55	220	715	2002	5005	11146

cost associated with including higher-order terms in the calculation of variance propagation. (The second-order approximation with a correction factor seen at the end of Figure 3.4 is presented and discussed in Section 3.5.2.)

Furthermore, computational cost grows exponentially with an increase in the number of system inputs and with an increase in model complexity. As this high cost can be prohibitive, higher-order terms are usually not considered. The author of this thesis has shown that higher-order truncation error can be predicted. Consequently, a correction factor can be applied to the lower-order formula, yielding higher-order accuracy without significantly increasing computational cost.

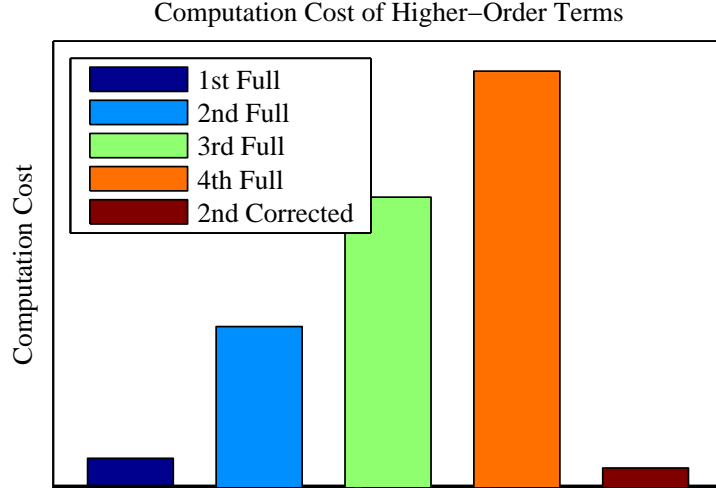


Figure 3.4: Relative Computational Cost of Taylor Series Approximations, as Determined by MATLAB Execution Time

### 3.5.2 Reducing Truncation Error

As Figure 3.2 illustrates, the truncation error in the second-order approximation for a *sin* function is essentially a constant bias for all  $\bar{x}$ . Furthermore, this bias has a linear relationship to the input variance,  $\sigma_x^2$ , as shown in Figure 3.5.

Consequently, the second-order truncation error can be easily be estimated empirically. A correction factor  $e$  corresponding to this truncation error can be calculated using Eq. (3.19).

$$e = \frac{1}{1 + 1.022\sigma_x^2} \quad (3.19)$$

This correction factor can then be applied to the second-order approximation of variance propagation, as shown in Eq. (3.20), which reduces the higher-order Taylor series truncation error.

$$\sigma_{y,CF}^2 \approx \sigma_y^2 e \quad (3.20)$$

By way of example, consider the same function  $y = 1000 \sin(x)$ . The corrected second-order approximation produces an estimate of error propagation with fourth-order accuracy, but with only second-order computational cost. Figure 3.4 (already displayed in

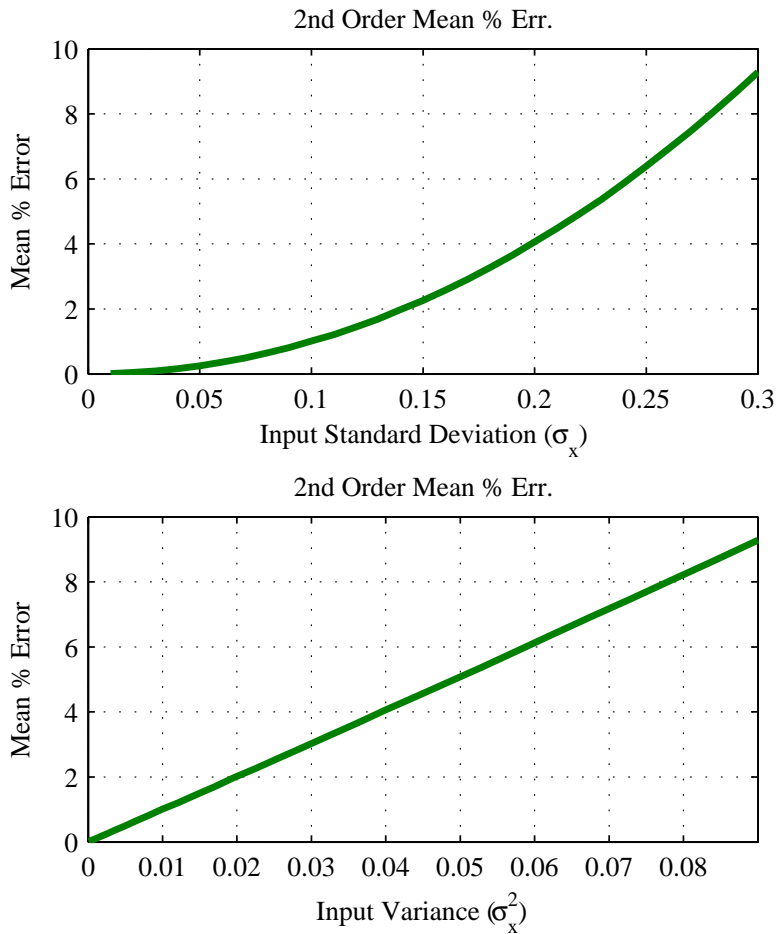


Figure 3.5: Relationships Between the Second-Order Bias and both Input Standard Deviation  $\sigma_x$  and Input Variance  $\sigma_x^2$  for a *sin* Function

Section 3.5.1) compares the computational costs of the uncorrected Taylor series approximations with the corrected second-order approximation.

Figure 3.6 compares the relative errors in variance estimations. Note that the fourth-order and second-order corrected approximations are almost identical in Figure 3.6.

Figure 3.7 compares the mean relative error averaged over the range  $0 \leq \bar{x} \leq \pi$ , showing that fourth-order accuracy can be obtained by adding a correction factor to the second-order formula for error propagation.

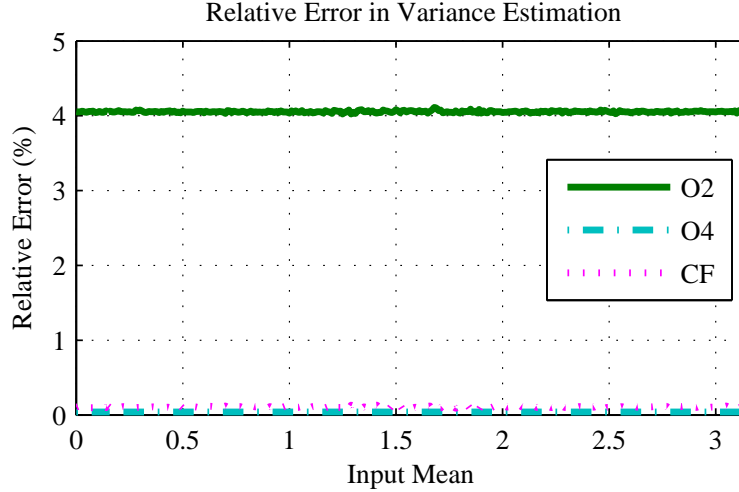


Figure 3.6: Relative Error in Estimations of Variance Propagation

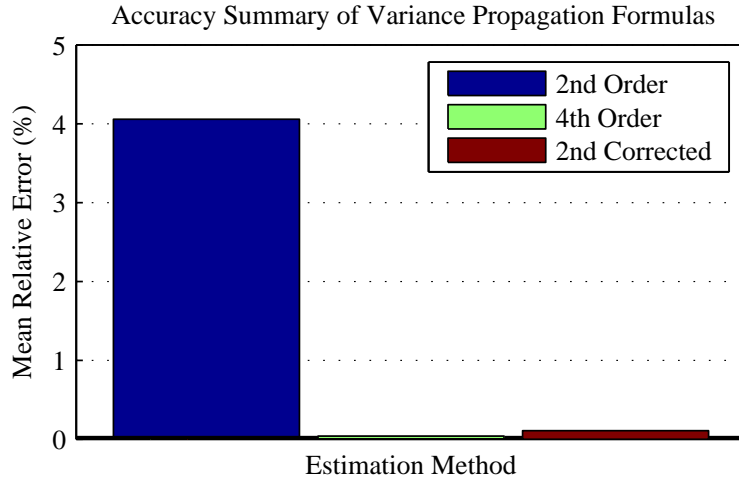


Figure 3.7: Mean Relative Error in Variance Estimations

### 3.5.3 Correction Factors for Other Nonlinear Functions

Table 3.2 gives the correction factors for some common nonlinear functions. Note that the cyclical nature of trigonometric functions and derivatives require a second-order approximation before the higher-order truncation error can easily be determined, but the exponential and logarithmic functions only require a first-order calculation. The correction factors for  $y = \ln(x)$ ,  $y = \exp(x)$ , and  $y = b^x$  are given in Eqs. (3.21), (3.22), and (3.23), respectively.

Correction factor for  $y = \ln(x)$ , where  $r = \ln\left(\frac{\bar{x}}{\sigma_x}\right)$ :

Table 3.2: Truncation Error Correction Factors for Common Nonlinear Functions

Func.	Ord.	Correction Factor ( $e$ )
$y = \sin(x)$	$2^{nd}$	Eq. (3.19)
$y = \cos(x)$	$2^{nd}$	Eq. (3.19)
$y = \ln(x)$	$1^{st}$	Eq. (3.21)
$y = \exp(x)$	$1^{st}$	Eq. (3.22)
$y = b^x$	$1^{st}$	Eq. (3.23)

$$e = \begin{cases} \exp(-1.9772r + 0.9128) + 1 & \text{if } r \geq 0 \\ -3.88r^3 - 4.9835r^2 - 1.5704r + 1.3302 & \text{if } r < 0 \end{cases} \quad (3.21)$$

Correction factor for  $y = e^x$ , where  $e$  is constrained to a maximum value of 2 and a minimum value of 1:

$$e = \max(1, \min[2, 0.3375\sigma_x^2 + 0.4937\sigma_x + 0.959]) \quad (3.22)$$

Correction factor for  $y = b^x$ , where  $e$  is constrained to a maximum value of 2 and a minimum value of 1:

$$e = \begin{cases} \max(1, \min[2, \mathbf{X} \cdot (\mathbf{Z}_1 \cdot \mathbf{B})^T]) + 1 & \text{if } b < 1 \\ \max(1, \min[2, \mathbf{X} \cdot (\mathbf{Z}_2 \cdot \mathbf{B})^T]) + 1 & \text{if } b > 1 \end{cases} \quad (3.23)$$

where

$$\mathbf{X} = \begin{bmatrix} 1 & \sigma_x & \sigma_x^2 & \sigma_x^3 & \sigma_x^4 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 & b & b^2 & b^3 \end{bmatrix}^T$$

$$\mathbf{Z}_1 = \begin{bmatrix} 0.224 & 0.012 & -2.570e3 & 0 & 0 \\ -0.992 & 0.076 & 1.544e4 & 0 & 0 \\ -0.037 & 7.803 & -2.061e4 & 7.135 & -38.093 \\ 0 & 0 & 0 & 23.556 & 61.532 \end{bmatrix}$$

$$\mathbf{Z}_2 = \frac{1}{1000} \begin{bmatrix} -8.612 & 0.018 & -897.1 & 0.006 & 0.005 \\ 1.182 & 0.084 & 986.2 & -0.002 & -0.008 \\ 0.101 & -6.891 & 54.93 & -240.9 & 26.52 \\ -0.002 & 0.089 & -0.276 & -1.743 & 11.06 \end{bmatrix}$$

### 3.5.4 Model Composition

It should be noted that the correction factors given in Table 3.2 are only pertinent to a particular function. If a system model contains this function along with other operators, the system should be decomposed into components (with  $\sin(x)$  being a single component, for example). The error should then be propagated through each component individually. The variances in each component's output can then be propagated through the rest of the system model. This process of model decomposition is demonstrated in Section 3.6 with the example of a dual-propeller, three degree-of-freedom helicopter.

## 3.6 Example: Compositional System Model of a Dual Propeller, Three Degree-of-Freedom Helicopter

Consider the dual-propeller, three degree-of-freedom helicopter shown in Figure 3.8. This helicopter is a useful tool for the design of unmanned aerial vehicles (UAVs). Among other benefits, it allows the system designer to 1) test pitch, roll, and yaw control systems without endangering actual aircraft, 2) observe the effects of changing the payload and component weights and positions, and 3) calibrate the conversion factor (given as  $k_m$  below) that converts a motor's throttle command into units of thrust. A UAV's position and orientation are the inputs to its control system. An efficient prediction of the error in the UAV's position allows the designer to easily create a robust control system capable of accomplishing mission objectives.

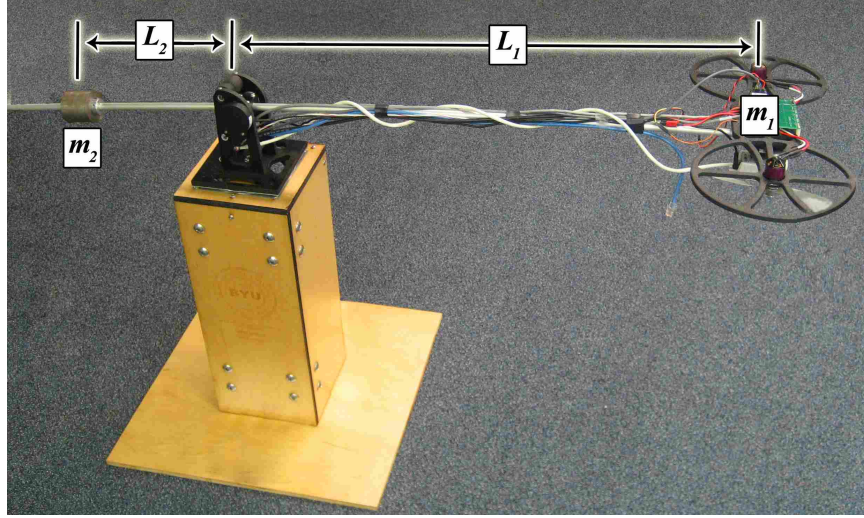


Figure 3.8: Dual-Propeller, Three Degree-of-Freedom Helicopter Used in the Design of Unmanned Aerial Vehicles

The system model for the longitudinal (pitch) acceleration as a function of the system's design parameters and inputs is given in Eq. (3.24).

$$\ddot{\theta} = \frac{(m_2 L_2 - m_1 L_1)g}{m_1 L_1^2 + m_2 L_2^2 + J_y} \cos \theta + \frac{L_1 k_m (u_L + u_R)}{m_1 L_1^2 + m_2 L_2^2 + J_y} \quad (3.24)$$

where the masses  $m_i$  and lengths  $L_i$  are indicated in Figure 3.8,  $J_y$  is the mass moment of inertia about the pitch axis of rotation,  $g$  is the acceleration of gravity,  $u_L$  and  $u_R$  are the throttle commands (0-100) given to the left and right motors, respectively, and  $k_m$  is a motor calibration constant that converts throttle commands to units of thrust.

For this demonstration, the assumed mean and standard deviation of the dependent and independent system design parameters are given in Tab. 3.3. Three system inputs are also required:  $u_L$  and  $u_R$ , which were each fixed at 45, and the pitch angle  $\theta$ . The mean value of  $\theta$  was varied from  $-45^\circ$  to  $45^\circ$ , with a standard deviation of  $1^\circ$ .

The first-order approximation in Eq. (3.9) can be used to propagate error through this system (see Figures 3.9-3.10). However, the nonlinearity of the  $\cos$  function causes significant error in the results. Consequently, the model should be decomposed into its components, as shown in Eq. (3.25). Error can then be propagated through each component. Once the variance in each component is obtained, each component can then be treated as a variable



Table 3.3: Design Parameters for Dual-Propeller, 3-DOF Helicopter

Design Parameter	Mean	Std. Dev.
$m_1$	0.891 (kg)	$10^{-4}$ (kg)
$m_2$	1.000 (kg)	$10^{-4}$ (kg)
$L_1$	0.850 (m)	$10^{-4}$ (m)
$L_2$	0.3048 (m)	$10^{-4}$ (m)
$J_y$	0.0014 ( $kg \cdot m^2$ )	$10^{-5}$ ( $kg \cdot m^2$ )
$k_m$	0.0546	$10^{-5}$

and its variance propagated through the system-level compositional model. A full step-by-step solution is presented in Appendix A.

$$\begin{aligned}
 \ddot{\theta} &= C_1 C_2 + C_3 \\
 C_1 &= \frac{(m_2 L_2 - m_1 L_1)g}{m_1 L_1^2 + m_2 L_2^2 + J_y} \\
 C_2 &= \cos \theta \\
 C_3 &= \frac{L_1 k_m (u_L + u_R)}{m_1 L_1^2 + m_2 L_2^2 + J_y}
 \end{aligned} \tag{3.25}$$

The first-order Taylor series approximation given in Eq. (3.9) is used to propagate error from the input parameters through components  $C_1$  and  $C_3$ . The second-order approximation with a correction factor is used to propagate error through the nonlinear trigonometric component  $C_2$ . Figure 3.9 compares the results of this compositional model with a correction factor applied with the results of the first-order approximation from Eq. (3.9) applied to the full system model in Eq. (3.24). These results are shown for  $-45^\circ \leq \bar{\theta} \leq 45^\circ$ .

Figure 3.10 shows the relative error for each of these methods, averaged across the entire input domain  $-45^\circ \leq \bar{\theta} \leq 45^\circ$ .

The computational cost required to determine the output variance using the compositional system model was approximately three times less than the cost using the first-order approximation. Thus, twice the accuracy was obtained with one-third of the computational

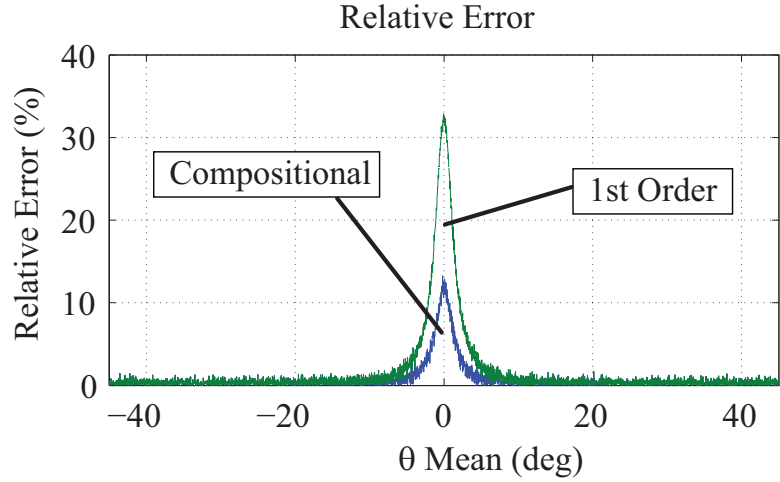


Figure 3.9: Relative Error in Estimations of Variance Propagation

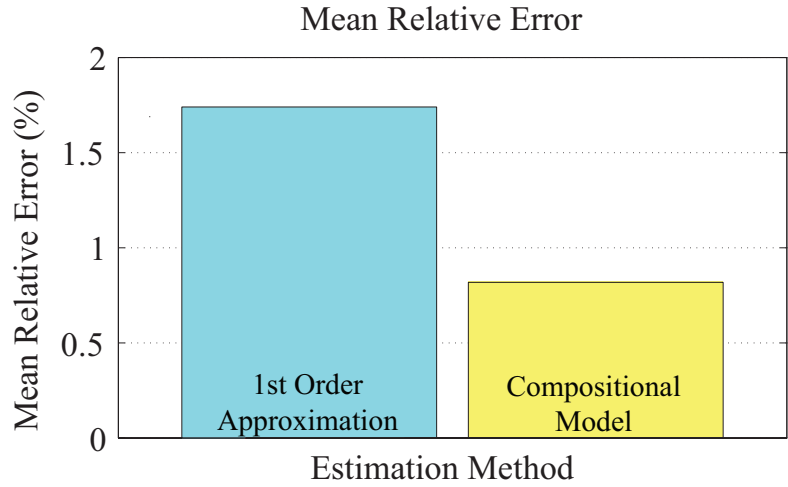


Figure 3.10: Mean Relative Errors that Indicate the Compositional Model with a Correction Factor Predicts Output Variance with About Half the Error as the First-Order Approximation

cost. This efficiency in error propagation allows the system designer to more easily create a robust system capable of accomplishing mission objectives.

### 3.7 Example: Kinematic Model of a Mechanism that Simulates Flapping Flight Wing Motion

Consider the flapping flight wing mechanism shown in Figure 3.11. The kinematic model used in the design and optimization of this mechanism is the Fourier series in Eq. (3.26) [44, 45].

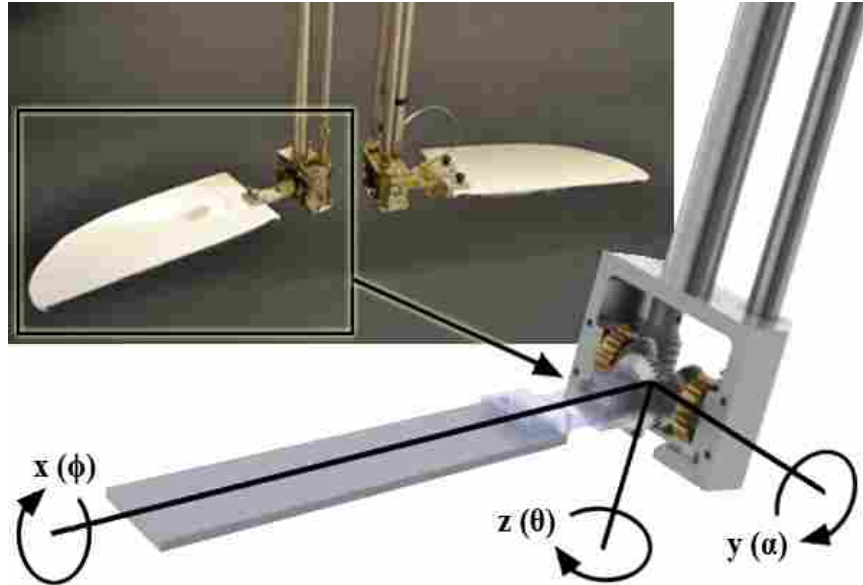


Figure 3.11: Mechanism Used by the BYU Flapping Flight Research Team to Simulate Three-Degree-of-Freedom Motion of a Flapping Wing

$$\begin{bmatrix} \phi(t) \\ \theta(t) \\ \alpha(t) \end{bmatrix} = \sum_{n=0}^2 \begin{bmatrix} A_{\phi n} \\ A_{\theta n} \\ A_{\alpha n} \end{bmatrix} \cos(n\omega t) + \begin{bmatrix} B_{\phi n} \\ B_{\theta n} \\ B_{\alpha n} \end{bmatrix} \sin(n\omega t) \quad (3.26)$$

where  $\phi$  is the positional angle (deg),  $\theta$  is the elevation angle (deg),  $\alpha$  is the feathering or attack angle (deg), the  $A$ s and  $B$ s are Fourier series coefficients (deg),  $\omega$  is the flapping frequency (Hz), and  $t$  is time (s). This three-output system model has 16 Gaussian inputs (time does not vary), which are statistically described in Table 3.4 [45].

Various orders of a Taylor series were used to estimate the variance in the three output wing angles based on the variance in these system inputs. The system model was then decomposed (as demonstrated in the previous example) and the trigonometric correction

Table 3.4: Mean Values and Standard Deviations of the 16 Model Input Parameters

	Mean	StdDev		Mean	StdDev
$A_{\phi 0}$	-20	0.1	-	-	-
$A_{\phi 1}$	-4	1.5	$B_{\phi 1}$	44	1.5
$A_{\phi 2}$	8	3.0	$B_{\phi 2}$	33	8.0
$A_{\theta 0}$	0	0.1	-	-	-
$A_{\theta 1}$	43	0.75	$B_{\theta 1}$	0	0.1
$A_{\theta 2}$	17	0.5	$B_{\theta 2}$	0	0.1
$A_{\alpha 0}$	12	4.0	-	-	-
$A_{\alpha 1}$	0	0.1	$B_{\alpha 1}$	50	0.1
$A_{\alpha 2}$	-2	0.75	$B_{\alpha 2}$	0	0.1
$\omega$	0.298	0.1			

factors given in Table 3.2 were then applied. Figure 3.12 shows the computational cost of error propagation (in minutes) using these different methods. These costs are summed over the time interval  $0s \leq t \leq 5s$  at time steps of 0.001s for all three output angles. The cost of calculating input moments and input covariances from 10k input samples is also included. Computational cost was determined using MATLAB execution time.

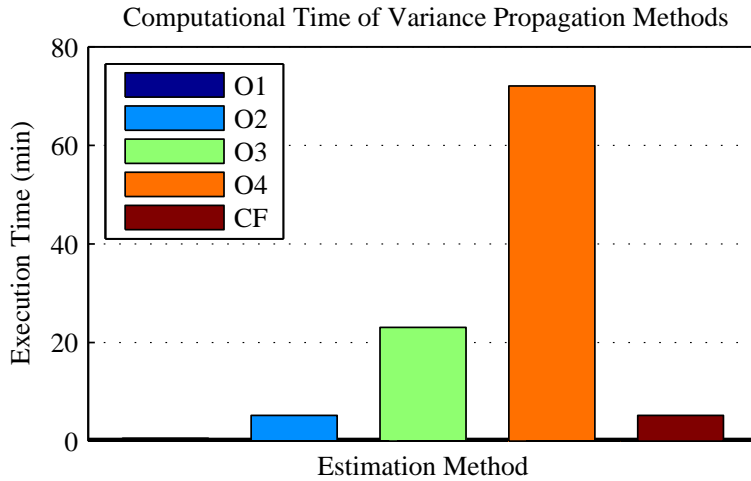


Figure 3.12: Computational Time to Predict Output Distributions Using Various Error Propagation Methods

The fourth-order prediction took approximately 70 minutes to execute, where the corrected second-order approximation executed in approximately 4 minutes—a significant reduction in computational cost.

The relative error in only one of these output angles,  $\phi$ , is shown in Figure 3.13, as the other two output angles have similar results. The root-mean-square of the relative error over the time interval shown ( $0s \leq t \leq 5s$ ) for the second-order approximation is 40.97%, third-order is 11.18%, fourth-order is 1.32%, and a second-order with a correction factor is 1.96%.

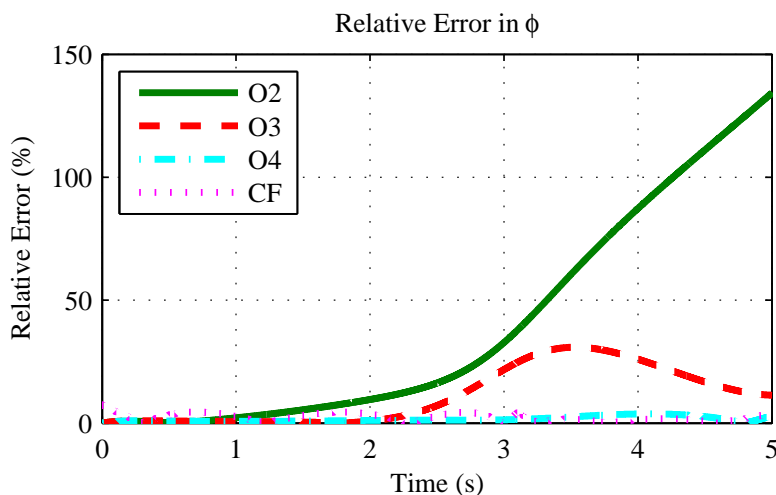


Figure 3.13: Relative Error in Predictions of Output Variance Obtained from Various Orders of a Taylor Series

Figures 3.12 and 3.13 illustrate that the correction factors presented in this paper can achieve near fourth-order accuracy in error propagation through this model with near second-order computational cost.

### 3.8 Comments on Variance Propagation

Using a first-order Taylor series to estimate error propagation through a closed-form model is common practice. Frequently, this approximation is used without a full appreciation of its limitations and the assumptions upon which it is based. This results in estimations that may be substantially inaccurate. Furthermore, the author has been unable to locate the full

derivation of this formula either in textbook or archival journal literature. Consequently, a useful contribution of the research presented in this chapter is the derivation and the discussion of the limitations and assumptions of the first- and second-order Taylor series approximations for variance propagation.

Additionally, the novel contribution of this chapter is the introduction of generic correction factors that account for some of the Taylor series truncation error for common nonlinear functions encountered in engineering. These correction factors are predictable, easy to calculate, and don't require significant computational cost. A system designer can use these correction factors to predict error propagation through trigonometric, logarithmic, and exponential functions with greater accuracy without greater computational cost. This enables the designer to better verify design decisions, which reduces the risk of developing a design that does not meet design objectives.

Future research may focus on the development of predictable correction factors for other nonlinear models, such as differential equations and state-space models. The author believes this can be accomplished using the same methods used in this chapter.



## CHAPTER 4. HIGHER-ORDER STATISTICS

For closed-form, analytical models, a statistical error distribution is usually obtained by propagating variance from system inputs to system outputs using a Taylor series. This has already been demonstrated in Chapter 3. As noted in Section 3.4, the variance propagation formulas given in Eqs. (3.9) and (3.18) assume all inputs are Gaussian. Since all higher-order statistics (e.g., skewness, kurtosis, etc.) are ignored, outputs are also typically assumed to be Gaussian. This assumption is often erroneous and does not accurately reflect reality, as proved later in Section 4.2.4.

In order to truly perform an accurate statistical error analysis, outputs cannot be assumed to be Gaussian. However, since all non-Gaussian distributions cannot be fully described by a mean and standard deviation alone, higher-order statistics (e.g., skewness and kurtosis) must also be used. Fortunately, this is relatively simple using the method and formulas presented in this chapter.

This chapter first demonstrates the need for propagating higher-order statistics. A second-order Taylor series is then used to show how skewness and kurtosis can both be propagated through a system model. Having a mean, variance, skewness, and kurtosis, the system designer can then fully describe the output distribution. The benefits of obtaining a fully-described output distribution are demonstrated using two examples: the propeller component of a solar-powered unmanned aerial vehicle, and a flat rolling metalworking process.

### 4.1 Motivation for Propagating Higher-Order Statistics

Consider the simple quadratic function,  $y = x^2$ . Assume the input  $x$  is a Gaussian distribution with a mean  $\bar{x}$  and a standard deviation  $\sigma_x$  both equal to 1. Equation (3.9) can be used to propagate this input distribution through the system model and predict the



Gaussian output distribution shown in Figure 4.1a. This predicted output is very different from the actual system output distribution, shown in Figure 4.1b.

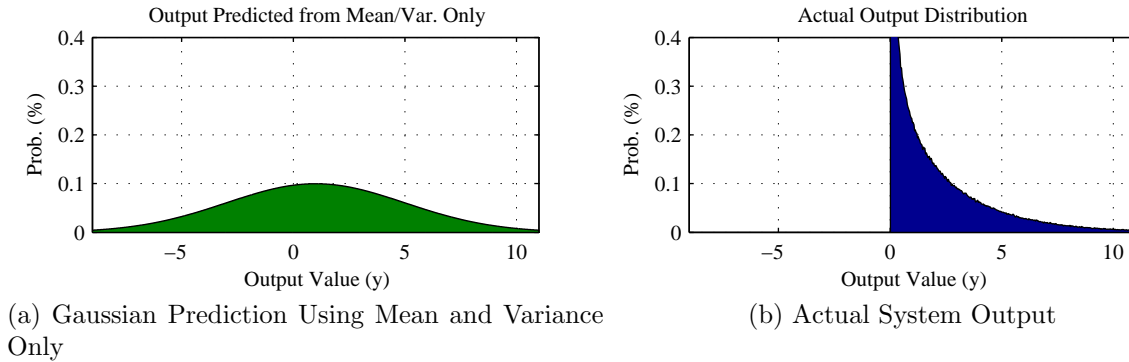


Figure 4.1: Predicted Gaussian Output Distribution Obtained from Propagating Mean and Variance Only (a) Compared with Actual System Output (b)

However, if skewness and kurtosis are also propagated through the system model, the predicted output distribution resembles actual system output much more closely [46]. This is illustrated in Figure 4.2.

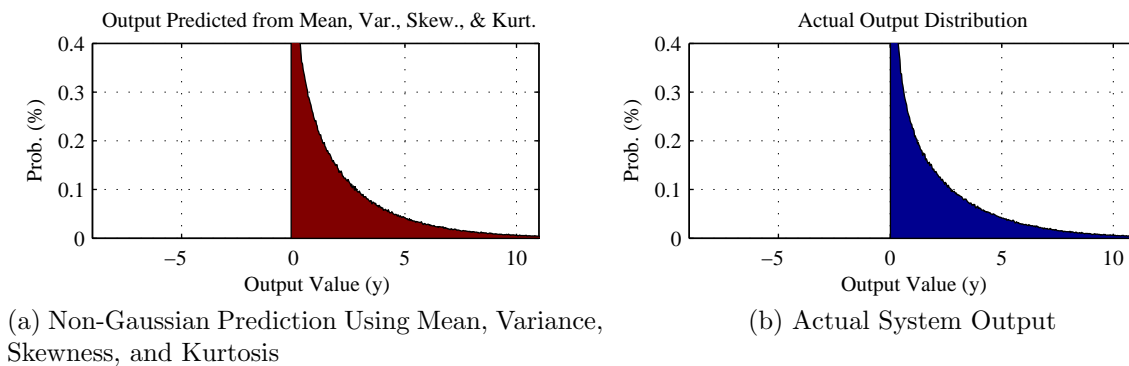


Figure 4.2: Predicted Non-Gaussian Output Distribution Obtained from Propagating Mean, Variance, Skewness, and Kurtosis (a) Compared with Actual System Output (b)

A proof that nonlinear systems produce non-Gaussian outputs even when inputs are Gaussian is presented later in Section 4.2.4.

## 4.2 Propagation of Skewness

As previously noted, non-Gaussian distributions cannot be fully described with a only mean and standard deviation. Consequently, higher-order statistics, such as skewness and kurtosis, must also be used to describe non-Gaussian distributions. This section considers the definition of skewness and derives a formula for propagating skewness through an analytical system model.

### 4.2.1 Definition of Skewness

The first-order statistic of a distribution is its mean, the second-order statistic is its standard deviation, and the third-order statistic is its skewness. Skewness is a measure of a distribution's asymmetry. Skewness (denoted  $\gamma_1$ ) is defined in Eq. (4.1).

$$\begin{aligned}\gamma_1 &= E \left[ \left( \frac{x - \bar{x}}{\sigma} \right)^3 \right] \\ &= \frac{\mu_3}{\sigma^3}\end{aligned}\tag{4.1}$$

where  $E$  is the expectation operator,  $\mu_3$  is the third central moment,  $\sigma$  is the standard deviation.

Table 4.1 and Figure 4.3 illustrate some characteristics and terminology of positively- and negatively-skewed distributions. A skewness of zero indicates a symmetric distribution.

Table 4.1: Comparison of Positive and Negative Skew

<b>Sign</b>	<b>Left/Right</b>	<b>Mean vs. Median [47]</b>
Negative	Left-skewed	Mean is typically (though not always) less than the median
Positive	Right-skewed	Mean is typically (though not always) greater than the median

Skewness is an important defining characteristic of statistical distributions. A measure of skewness is required to fully describe any asymmetric distribution. Traditional uncertainty propagation, however, only propagates a mean and variance. With no skewness

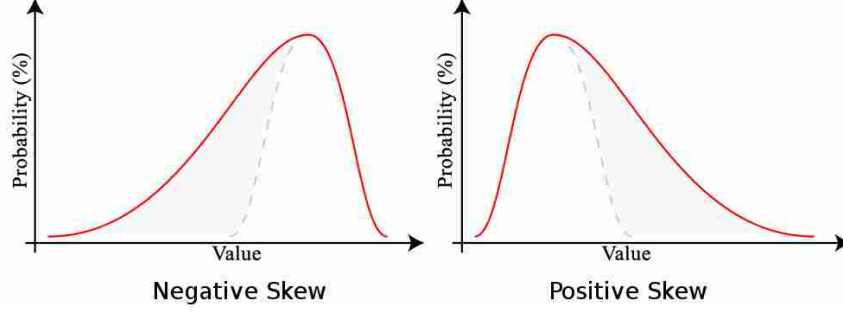


Figure 4.3: Examples of Negative (Left) and Positive (Right) Skewness

information available, skewness is neglected (assumed equal to zero) and a Gaussian distribution is assumed.

#### 4.2.2 Skewness Propagation Formula Derivation

Using a first-order Taylor series to propagate skewness through a system model results in an output skewness equal to the input skewness. It has already been demonstrated that this often does not reflect reality, even for simple nonlinear functions, and a proof is presented later in Section 4.2.4.

Consequently, a second-order Taylor series will be used to derive a formula for skewness propagation. The second central moment of output  $y$  has already been given in Eq. (3.15). The third central moment is given by the dot-product in Eq. (4.2).

$$E [(y - \bar{y})^3] \approx \begin{bmatrix} \mu_3 \\ \frac{3}{2}(\mu_4 - \mu_2^2) \\ (\frac{3}{4}\mu_5 - \frac{3}{2}\mu_2\mu_3) \\ (\frac{1}{4}\mu_2^3 - \frac{3}{8}\mu_2\mu_4 + \frac{1}{8}\mu_6) \end{bmatrix} \cdot \begin{bmatrix} \partial_1^3 \\ \partial_1^2\partial_2 \\ \partial_1\partial_2^2 \\ \partial_2^3 \end{bmatrix} \quad (4.2)$$

where  $\mu_k$  is the  $k$ -th central moment of input  $x$ , and  $\partial_1$  and  $\partial_2$  respectively represent the partial derivatives  $\frac{\partial f}{\partial x}$  and  $\frac{\partial^2 f}{\partial x^2}$ , evaluated at the mean  $x = \bar{x}$ . The third moment is a cubic function, and consequently it has four terms. Equation (4.2) has both first and second partial derivatives, because it is based on a second-order Taylor series. If a higher-order Taylor series were used, Eq. (4.2) would contain higher-order partial derivatives.

The second moment from Eq. (3.15) and the third moment from Eq. (4.2) can be used with the definition of skewness given by Eq. (4.1) to estimate the skewness in output  $y$ . This output skewness estimation is given in Eq. (4.3).

$$\gamma_1 = \frac{E[(y - \bar{y})^3]}{\{E[(y - \bar{y})^2]\}^{1.5}} \approx \frac{\begin{bmatrix} \mu_3 \\ \frac{3}{2}(\mu_4 - \mu_2^2) \\ (\frac{3}{4}\mu_5 - \frac{3}{2}\mu_2\mu_3) \\ (\frac{1}{4}\mu_2^3 - \frac{3}{8}\mu_2\mu_4 + \frac{1}{8}\mu_6) \end{bmatrix} \cdot \begin{bmatrix} \partial_1^3 \\ \partial_1^2\partial_2 \\ \partial_1\partial_2^2 \\ \partial_2^3 \end{bmatrix}}{[\mu_2\partial_1^2 + \mu_3\partial_1\partial_2 + \frac{1}{4}(\mu_4 - \mu_2^2)\partial_2^2]^{1.5}} \quad (4.3)$$

Equation (4.3) estimates output skewness using the input central moments,  $\mu_k$ . If input skewness, kurtosis, and higher-order statistics are known instead of input moments, these statistics can easily be substituted into Eq. (4.3) in place of these moments.

For the sake of brevity, the skewness propagation formula has only been derived for mono-variate functions. However, this derivation can easily be extended to multivariate functions as desired.

It should be noted that the most computationally expensive part to propagating skewness is calculating first and second derivatives. However, these have already been calculated in order to propagate variance if a second-order Taylor series was used, and consequently the additional cost to also propagate skewness is minimal.

### 4.2.3 Skewness Propagation Assumptions and Limitations

The following five assumptions and limitations apply to the method just presented to propagate skewness:

1. Equation (4.3) is based on a second-order Taylor series. Consequently, it will predict output skewness perfectly for second-order (or lower) functions. Accuracy decreases with increasing non-linearity.

2. While not a problem for most analytical engineering models, it should be noted that this method of skewness propagation requires the system model to be represented as a closed-form, twice-differentiable, mathematical equation. This is due to the dependence of this method on a Taylor series.
3. The system model  $y$  must be representable as a closed-form, differentiable, mathematical equation.
4. Taking the Taylor series expansion about a single point ( $\bar{x}$ ) causes the approximation to be of local validity only [36, 19]. Consequently, the accuracy of the approximation generally decreases with an increase in the input moments  $\mu_k$ .
5. The statistical input distribution must be known (at least up to the 6th central moment).

#### 4.2.4 Proof: Nonlinear Functions Produce Non-Gaussian Output Distributions

Consider the propagation of Gaussian error. With a Gaussian distribution, the following expressions are true:

- All odd moments ( $\mu_k$ , where  $k$  is odd) are equal to zero
- The fourth moment is equal to three times the second moment squared [39] ( $\mu_4 = 3\mu_2^2$ )
- The sixth moment is equal to fifteen times the second moment cubed ( $\mu_6 = 15\mu_2^3$ )

Consequently, Eq. (4.3) reduces to Eq. (4.4) when inputs are Gaussian.

$$\gamma_1 \approx \frac{3\sigma_x \partial_1^2 \partial_2 + \sigma_x^3 \partial_2^3}{(\partial_1^2 + \frac{1}{2} \partial_2^2 \sigma_x^2)^{1.5}} \quad (4.4)$$

where  $\sigma_x$  is the standard deviation of the input distribution. Equation (4.4) proves that nonlinear functions (i.e., the second partial derivative is non-zero) produces a skewed non-Gaussian output, even with Gaussian inputs. Consequently, the commonly-made assumption that outputs are Gaussian is generally erroneous for nonlinear functions.

### 4.3 Propagation of Kurtosis

Any statistical property that is propagated through a system model improves the accuracy of the predicted output distribution. For example, propagating both a mean and a variance is more accurate (and useful) than propagating a mean alone. In a similar manner, propagating skewness (as shown in Section 4.2) in addition to a mean and variance also improves the accuracy of the predicted output distribution.

Further improvements in accuracy can be obtained by also propagating kurtosis, the fourth-order statistic. This section defines kurtosis and excess kurtosis, and derives a formula for propagating kurtosis through an analytical system model.

#### 4.3.1 Definition of Kurtosis

The fourth-order statistic is kurtosis. Kurtosis is a measure of a distribution’s “peakedness,” or the thickness of the distribution’s tails. Kurtosis (denoted  $\beta_2$ ) is the fourth standardized moment, and is defined in Eq. (4.5).

$$\begin{aligned}\beta_2 &= E \left[ \left( \frac{x - \bar{x}}{\sigma} \right)^4 \right] \\ &= \frac{\mu_4}{\sigma^4}\end{aligned}\tag{4.5}$$

The kurtosis of a Gaussian distribution is equal to 3.

#### 4.3.2 Definition of Excess Kurtosis

In statistical analysis, “excess kurtosis” (denoted  $\gamma_2$ ) is often used more than kurtosis. In practice, the term “kurtosis” more often refers to excess kurtosis instead of the fourth standardized moment. To avoid confusion, this thesis uses the definition of kurtosis presented above and defines excess kurtosis as the fourth cumulant divided by the square of the second cumulant, as indicated in Eq. (4.6). Cumulants (denoted  $\kappa_i$ ) are statistical properties similar to moments, where  $\kappa_2 = \mu_2$ ,  $\kappa_3 = \mu_3$ , and  $\kappa_4 = \mu_4 - 3\mu_2^2$ . While it can be more convenient to

use cumulants instead of moments in some statistical analyses, the propagation of uncertainty using a Taylor series is much simpler and easier using central moments.

Since a Gaussian distribution has a kurtosis of three, the “minus 3” in Eq. (4.6) causes a Gaussian distribution to have zero excess kurtosis.

$$\begin{aligned} \gamma_2 &= \frac{\kappa_4}{\kappa_2^2} \\ &= \frac{\mu_4}{\sigma^4} - 3 \end{aligned} \tag{4.6}$$

Figure 4.4 shows the excess kurtosis of several common types of distributions.

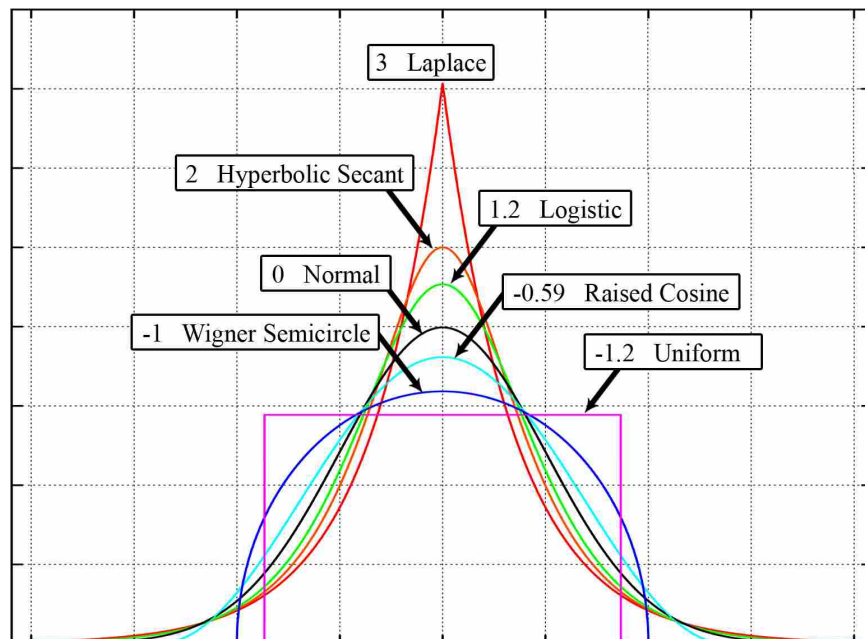


Figure 4.4: Excess Kurtosis of Various Common Statistical Distributions

### 4.3.3 Kurtosis Propagation Formula Derivation

A second-order Taylor series will also be used to propagate kurtosis through a system model. The third central moment of output  $y$  has already been given in Eq. (4.2), and the fourth moment is given in Eq. (4.7).

$$E [(y - \bar{y})^4] \approx \begin{bmatrix} \mu_4 \\ 2(\mu_5 - \mu_2\mu_3) \\ \frac{3}{2}(\mu_2^3 - 2\mu_2\mu_4 + \mu_6) \\ \frac{3}{2}(\mu_2^2\mu_3 - \mu_2\mu_5 + \frac{1}{3}\mu_7) \\ \frac{1}{16}(6\mu_2^2\mu_4 - 3\mu_2^4 - 4\mu_2\mu_6 + \mu_8) \end{bmatrix} \cdot \begin{bmatrix} \partial_1^4 \\ \partial_1^3\partial_2 \\ \partial_1^2\partial_2^2 \\ \partial_1\partial_2^3 \\ \partial_2^4 \end{bmatrix} \quad (4.7)$$

The excess kurtosis  $\gamma_2$  in the output distribution  $y$  is given by Eq. (4.8).

$$\gamma_2 = \beta_2 - 3 \quad (4.8)$$

where kurtosis  $\beta_2$  is given by Eq. (4.9).

$$\beta_2 = \frac{E [(y - \bar{y})^4]}{\{E [(y - \bar{y})^2]\}^2} \approx \frac{\begin{bmatrix} \mu_4 \\ 2(\mu_5 - \mu_2\mu_3) \\ \frac{3}{2}(\mu_2^3 - 2\mu_2\mu_4 + \mu_6) \\ \frac{3}{2}(\mu_2^2\mu_3 - \mu_2\mu_5 + \frac{1}{3}\mu_7) \\ \frac{1}{16}(6\mu_2^2\mu_4 - 3\mu_2^4 - 4\mu_2\mu_6 + \mu_8) \end{bmatrix} \cdot \begin{bmatrix} \partial_1^4 \\ \partial_1^3\partial_2 \\ \partial_1^2\partial_2^2 \\ \partial_1\partial_2^3 \\ \partial_2^4 \end{bmatrix}}{[\mu_2\partial_1^2 + \mu_3\partial_1\partial_2 + \frac{1}{4}(\mu_4 - \mu_2^2)\partial_2^2]^2} \quad (4.9)$$

An estimate of the kurtosis of an output distribution can be obtained using Eq. (4.9) and a known input distribution. The input central moments  $\mu_k$  can be estimated using any appropriate population sampling technique.

#### 4.4 Example: Solar-Powered Unmanned Aerial Vehicle Propeller Thrust

Consider the power and propulsion system of the solar-powered unmanned aerial vehicle (UAV) shown in Figure 4.5.



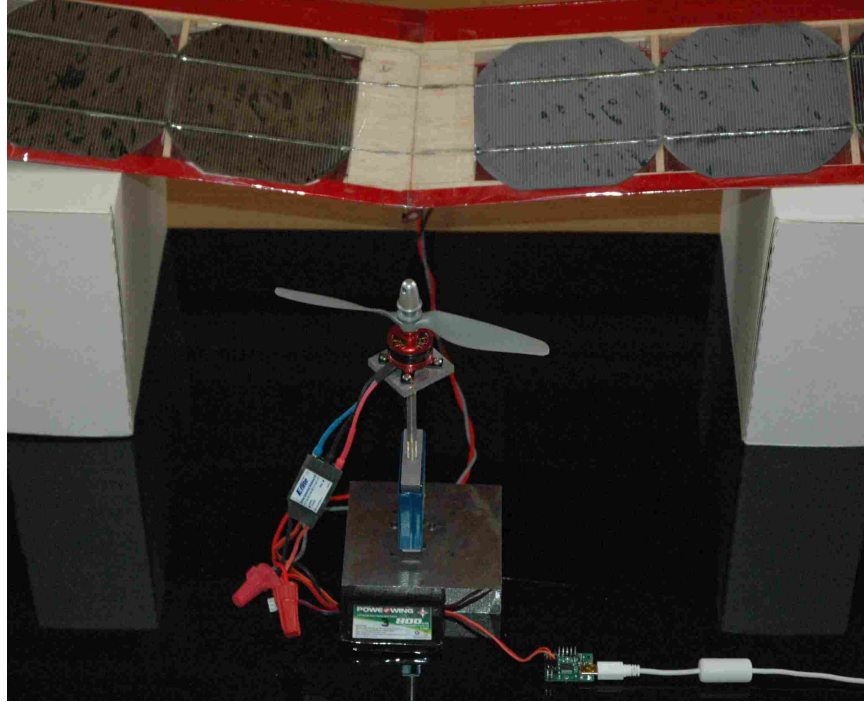


Figure 4.5: Power and Propulsion System of a Solar-Powered Unmanned Aerial Vehicle (UAV)

Larson (see Reference [21]) and the author of this thesis have previously created a compositional system model for this system (as shown in Figure 4.6), deterministically propagated a max/min error envelop through the model, and validated the results using real data. As this thesis focuses on statistical, non-deterministic error analysis, a full description of this model and the deterministic approach used to propagate error will not be replicated in this thesis. However, the propeller component of this system and its corresponding data will be used to demonstrate the increased accuracy obtained from propagating higher-order statistics through a system model.

#### 4.4.1 The Analytical Model of Thrust

The thrust  $T$  that the propeller produces is given by Eq. (4.10)[48].

$$T = C_t \rho \omega^2 D^4 \quad (4.10)$$

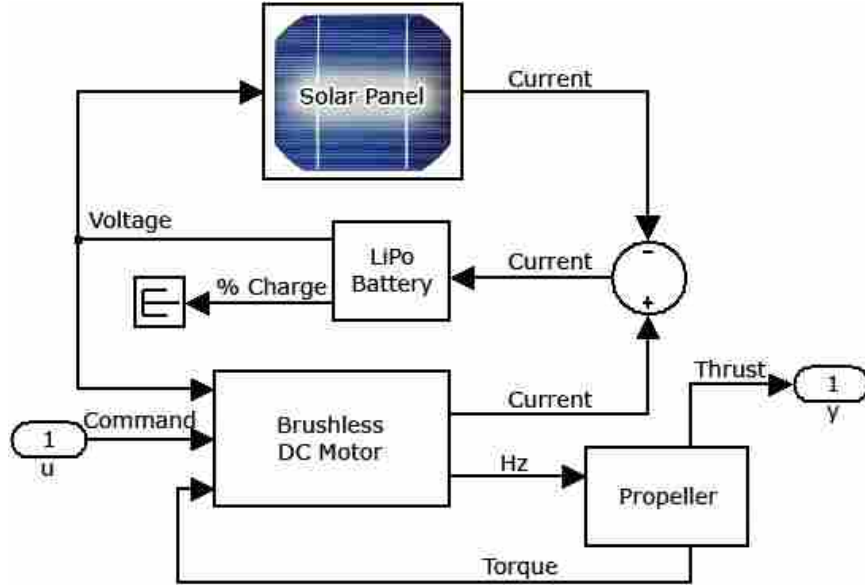


Figure 4.6: Compositional System Model of a Solar-Powered Unmanned Aerial Vehicle (UAV)

where  $C_t$  is a unitless coefficient of thrust calibrated to be  $3.458e-3$ ,  $\rho$  is the density of air assumed to be constant at  $1\text{kg/m}^3$ , and  $D$  is the propeller diameter measured to be  $0.1778\text{m}$ . The angular velocity of the propeller,  $\omega$ , is provided in  $\text{rad/s}$ , and is determined by the motor speed, which in turn is determined by a motor throttle command. Since a given throttle command does not consistently produce an exact motor speed, there is some variation in the angular velocity of the propeller. This distribution in angular velocities is described in Table 4.2 and shown in Figure 4.7.

Table 4.2: Statistical Properties of the Angular Velocity Distribution of a Propeller

Statistical Property	Value
Mean ( $\bar{\omega}$ ):	526.7
Variance ( $\sigma_\omega^2$ ):	9.383
Skewness ( $\gamma_1$ ):	-1.289
Excess Kurtosis ( $\gamma_2$ ):	4.896

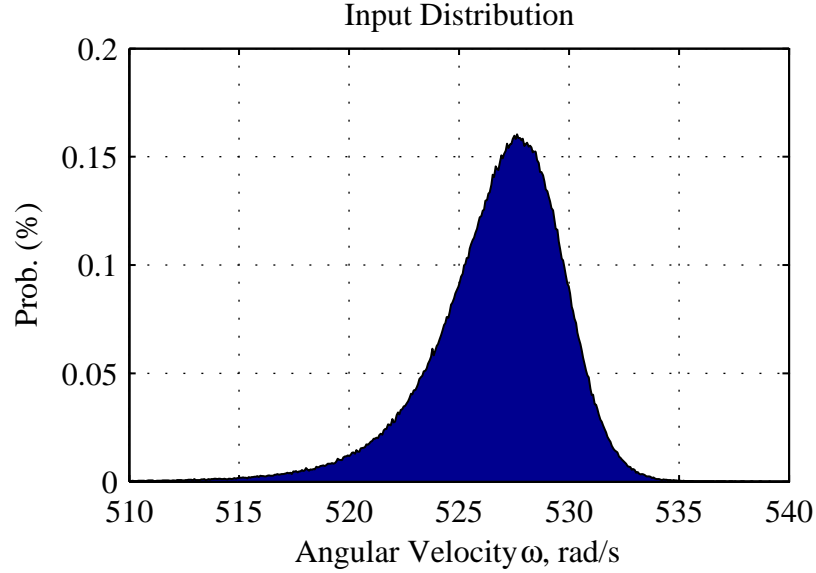


Figure 4.7: Distribution of the Angular Velocities of a Propeller in a Solar-Powered UAV

#### 4.4.2 Statistical Component Model Output Prediction

Based on this distribution of  $\omega$ , the first eight central moments were calculated using Eq. (3.2). These moments, given in Table 4.3, are used to propagate statistical properties from the angular velocity distribution to predict a distribution for thrust output by the propeller at a given motor throttle command.

Table 4.3: Central Moments of the Distribution of Angular Velocity,  $\omega$

Central Moment	Value
1st Moment ( $\mu_1$ ):	0
2nd Moment ( $\mu_2$ ):	9.355
3rd Moment ( $\mu_3$ ):	-36.34
4th Moment ( $\mu_4$ ):	6.617e2
5th Moment ( $\mu_5$ ):	-1.063e4
6th Moment ( $\mu_6$ ):	2.726e5
7th Moment ( $\mu_7$ ):	-8.678e6
8th Moment ( $\mu_8$ ):	3.308e8

The second-order prediction of the mean thrust output  $\bar{T}$  was calculated to be 0.9587N using Eq. (3.13). Equation (3.15) predicts a variance of  $1.232e-4N^2$ . Typically, higher-order statistics are not propagated and a Gaussian output distribution is assumed, which is compared with actual thrust measurements in Figure 4.8. This prediction only accounts for 65% of the actual system output distribution (i.e., 65% overlap in the area under the predicted and actual probability density functions).

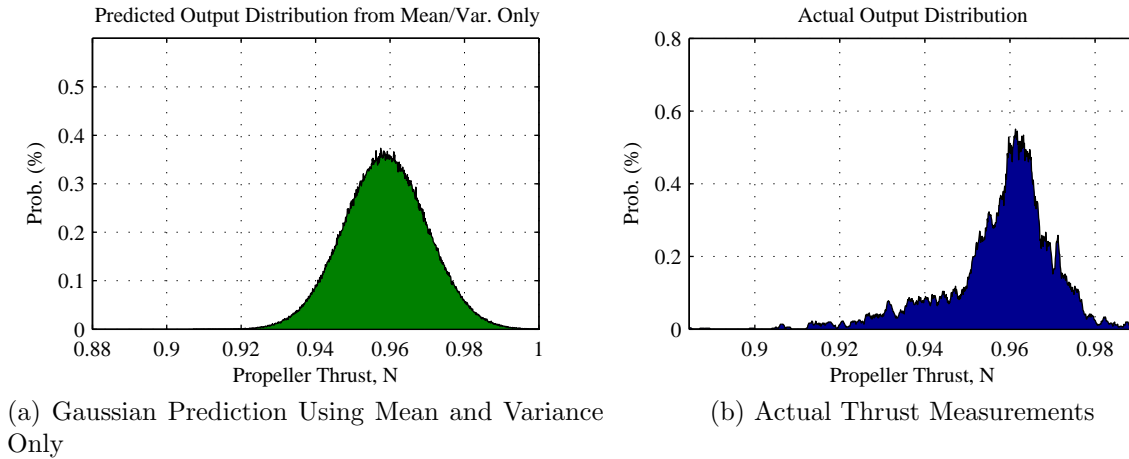


Figure 4.8: Predicted Gaussian Output Distribution of Thrust Obtained by Propagating a Mean and Variance Only (a) Compared with Actual Thrust Measurements (b)

The methods presented in this chapter to propagate skewness and kurtosis, using Eqs. (4.3) and (4.9), respectively, results in a more accurate prediction of the system output. This predicted output, shown in Figure 4.9a, accounts for 79% of actual system outputs—a 22% improvement in accuracy compared with propagating a mean and variance alone.

#### 4.5 Example: Flat Rolling Metalworking Process

Consider the manufacture of steel plates or sheets via flat rolling, where material is fed between two rollers (called working rolls). The gap between the working rolls is less than the thickness of the incoming material. As the working rolls rotate in opposite directions, the incoming material elongates as its thickness is reduced. This process, illustrated in

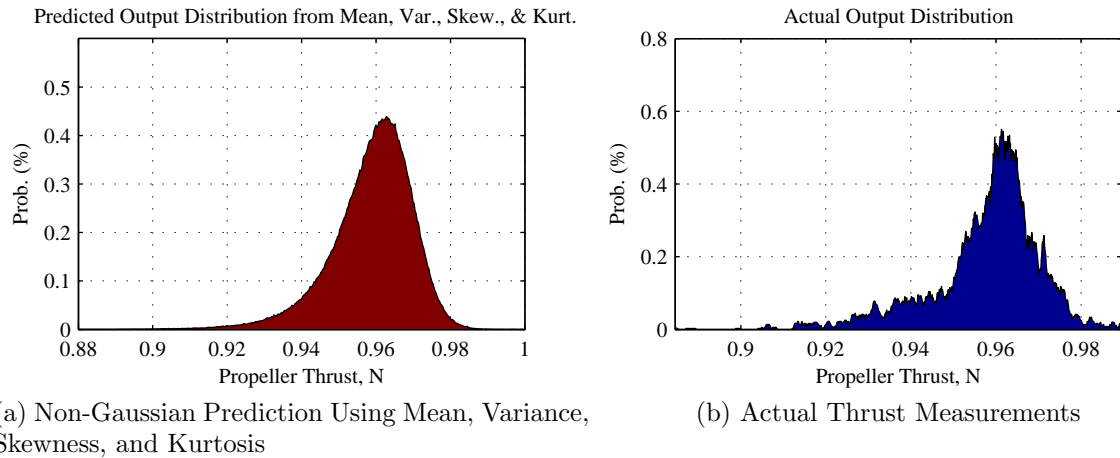


Figure 4.9: Predicted Non-Gaussian Output Distribution of Thrust Obtained by Propagating a Mean, Variance, Skewness, and Kurtosis (a) Compared with Actual Thrust Measurements (b)

Figure 4.10, can be done either below the recrystallization temperature of the material (cold rolling) or above it (hot rolling).

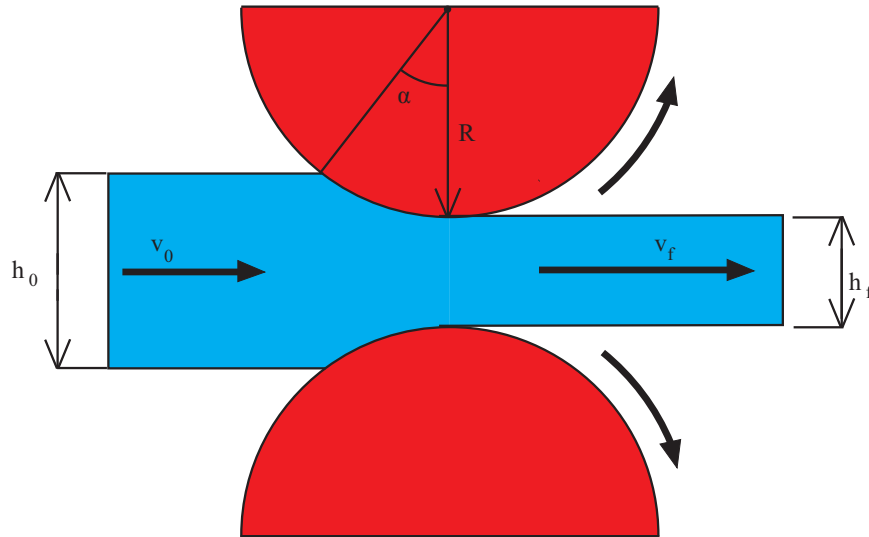


Figure 4.10: Flat Rolling Manufacturing Process Whereby Plates or Sheets of Metal are Made—Material is Drawn Between Two Rollers, which Reduces the Material's Thickness

### 4.5.1 The Analytical Model

The manufacturer desires to use its flat rolling equipment more efficiently by reducing overall rolling time for each plate. Consequently, the manufacturer desires to minimize the number of passes required to achieve final plate thickness. The maximum amount of deformation that can be achieved in a single pass is a function of the friction at the interface between the rolls and the material. If the intended change in thickness is too great, the rolls will merely slip along the material without drawing it in [49]. The maximum change in thickness attainable in a single pass ( $\Delta H_{max}$ ) is given in Eq. (4.11) [50].

$$\Delta H_{max} = \mu_f^2 R \quad (4.11)$$

where  $\mu_f$  is the coefficient of friction between the rolls and the plate, and  $R$  is the radius of the rolls. In this example, the radius of the rolls is measured to be 1m. Determining the coefficient of friction in a metalworking process is more difficult, however. The conditions surrounding friction in a metalworking process are very different from those in a mechanical device [49], as shown in Table 4.4.

Table 4.4: Friction Conditions in Mechanical Devices and Metalworking Processes

<b>Typical Mechanical Devices</b>	<b>Metalworking Processes</b>
-Two surfaces of similar material and strength	-One very hard tool and one softer material
-Elastic loads and no change in shape	-Plastic deformation occurs in material
-Wear-in cycles produce surface compatibility	-Each set of rollers makes a single pass on material -Contact area constantly changes under deformation
-Low-to-moderate temperatures	-Often elevated temperatures
-Friction force depends on contact pressure	-Friction force depends on material strength

Furthermore, lubrication is often used both to reduce friction and consequent tool wear, and to act as a thermal barrier to help regulate tool temperature [51]. All these

factors and others (e.g., rolling speed, material properties, surface finishes, etc.) combine to create variation in the friction experienced in the flat rolling metalworking process. This variation can inhibit the manufacturer’s ability to specify a gap width (and the resulting change in material thickness,  $\Delta H$ ) for each pass.

While many empirical and mathematical formulas have been presented as methods to predict the coefficient of friction in flat rolling processes, these will not be addressed in this thesis. For the purposes of this example, it is sufficient to assume that some appropriate technique has been employed to determine the distribution of friction coefficients. This distribution is described in Table 4.5 and shown in Figure 4.11.

Table 4.5: Statistical Properties of the Coefficient of Friction Distribution

Statistical Property	Value
Mean ( $\bar{\mu}$ ):	0.35
Variance ( $\sigma^2$ ):	9e-4
Skewness ( $\gamma_1$ ):	0.7
Excess Kurtosis ( $\gamma_2$ ):	0.2

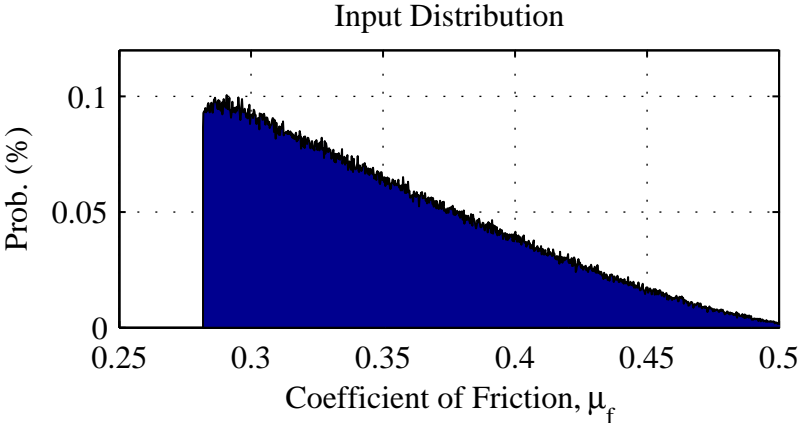


Figure 4.11: Distribution of the Coefficient of Friction in a Flat Rolling Metalworking Process

### 4.5.2 Statistical System Model Output Prediction

Based on this distribution of  $\mu_f$ , the first eight central moments were calculated using Eq. (3.2). These moments, given in Table 4.6, are used to propagate statistical properties from the friction coefficient distribution to predict a distribution for the maximum change in thickness attainable with a single pass.

Table 4.6: Central Moments of the Distribution of the Coefficient of Friction,  $\mu_f$ .

Central Moment	Value
1st Moment ( $\mu_1$ ):	0
2nd Moment ( $\mu_2$ ):	8.99e-4
3rd Moment ( $\mu_3$ ):	1.88e-5
4th Moment ( $\mu_4$ ):	2.58e-6
5th Moment ( $\mu_5$ ):	1.45e-7
6th Moment ( $\mu_6$ ):	1.45e-8
7th Moment ( $\mu_7$ ):	1.21e-9
8th Moment ( $\mu_8$ ):	1.2e-10

The second-order prediction of the mean maximum reduction in thickness  $\Delta\bar{H}_{max}$  was calculated to be 12.5cm using Eq. (3.13). Equation (3.15) predicts a variance of 0.136cm<sup>2</sup>. Typically, higher-order statistics are not propagated and a Gaussian output distribution is assumed. This Gaussian prediction is compared with actual system output in Figure 4.12. This prediction only accounts for 53% of the actual system output distribution (i.e., 53% overlap in the area under the predicted and actual probability density functions).

The methods presented in this chapter to propagate skewness and kurtosis, using Eqs. (4.3) and (4.9), respectively, results in a more accurate prediction of the system output. This predicted output, shown in Figure 4.13a, accounts for 93% of actual system outputs—a large improvement over propagating a mean and variance alone.



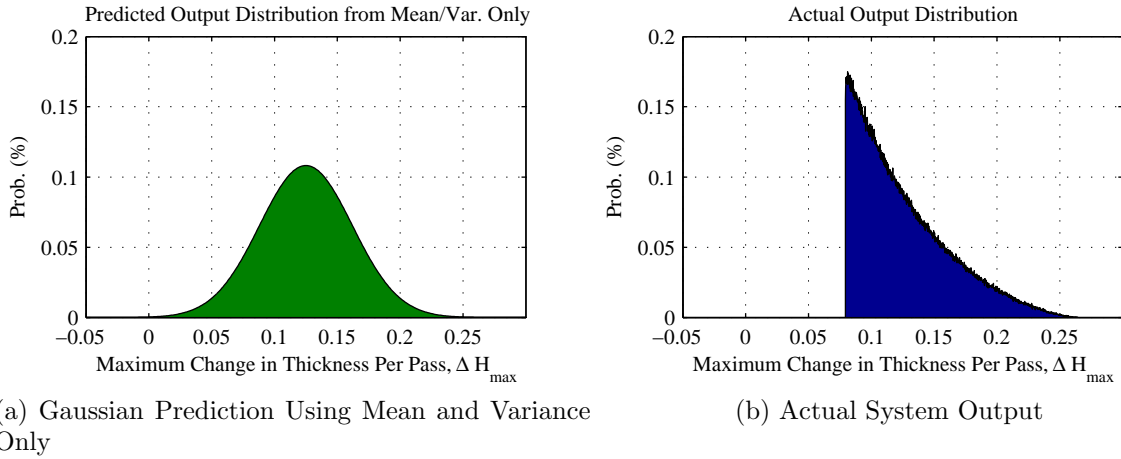


Figure 4.12: Predicted Gaussian Output Distribution Obtained from Propagating a Mean and Variance Only (a) Compared with Actual System Output (b)

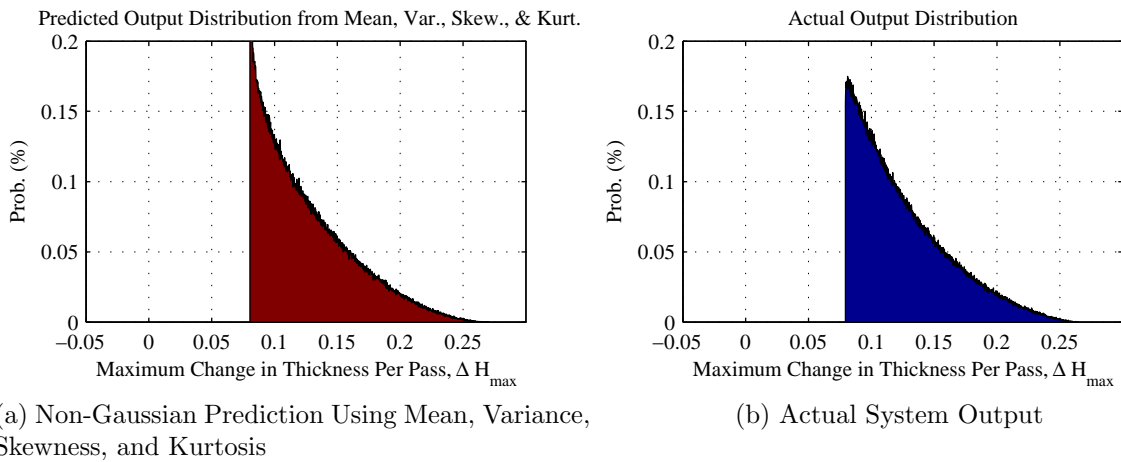


Figure 4.13: Predicted Non-Gaussian Output Distribution Obtained from Propagating a Mean, Variance, Skewness, and Kurtosis (a) Compared with Actual System Output (b)

### 4.5.3 Ramifications of Neglecting Higher-Order Statistics

Using the traditional approach where only a mean and variance are propagated and a Gaussian distribution is assumed, the manufacturer would have only been able to reduce the material thickness by a maximum of 3cm per pass, in order to achieve a 99.5% success rate. However, using the methods presented in this chapter to also propagate higher-order statistics, the manufacturer can confidently reduce the material thickness by 7.9cm per pass.

This reduces the number of passes required to achieve the desired plate thickness by over two and a half times.

As this example clearly indicates, the benefits of propagating higher-order statistics through a system model can be substantial.

#### **4.6 Comments on Propagation of Higher-Order Statistics**

The variance in a system's output can easily be predicted using a first-order Taylor series and knowledge of the input variance. However, having only a mean and variance and lacking any additional information about the output distribution, system designers often make the erroneous assumption that the output is Gaussian. This chapter has shown how inaccurate that assumption can be, even for very simple functions. By following the methods shown in this chapter, system designers can more fully describe an output distribution by also propagating higher-order statistics, such as skewness and kurtosis, through a system model.

While sufficient for many physical systems, the approach to higher-order statistical error propagation presented in this chapter may not work for all types of system models, such as state-space models, Laplace transforms, and differential equations. Additional work is required to adapt the method presented for use with these types of models.



## CHAPTER 5. CONCLUSION

Error propagation and uncertainty analysis is a complex but important part of the system design process. Accurately predicting a statistical probability density function for system outputs can mean the difference between a successful system design and one that fails to meet design objectives. The methods provided in this thesis allow the system designer to accurately propagate fully-described statistical distributions through a system model without incurring significantly greater computational costs.

The correction factors presented in this thesis for the efficient and accurate propagation of variance have only been determined for trigonometric, exponential, and logarithmic functions. Additional work is required to determine correction factors for other nonlinear, closed-form functions.

The author has based this research on the assumption that a closed-form, analytical equation is necessary to use the methods presented in this thesis. However, it is the author's belief that this same approach can be applied to any model (closed-form or not) where outputs can be obtained from given inputs and derivatives can be numerically evaluated. While non-closed-form models are outside the scope of this research, the author still believes the correction factors presented in Chapter 3 can be determined for non-closed-form equations if the truncation error is predictable, outputs are attainable, and derivatives can be determined. Additional work is required to determine what characteristics the model must have in order to have a predictable truncation error.

If these methods could be used with models that are not closed-form, it would greatly increase the quantity and variety of models this research applies to. This would allow the designer to use these methods for more real-world engineering problems, including finite-element models, dynamic models, transfer functions, state-space models, differential equations, software packages, "black-box" models, and others.

The formulas for propagating skewness and kurtosis that have been developed in this thesis have produced very accurate estimates of actual output skewness and kurtosis in the examples used in this thesis. However, it is possible that these estimates may not be perfectly accurate for other nonlinear system models. If this is true, then additional work is required to determine if correction factors can be determined that achieve higher-order accuracy in skewness and kurtosis propagation without increased computational cost.

Lastly, future work could make the correction factors and skewness and kurtosis propagation formulas more accessible to design engineers. This could involve creating a table to be published online and/or included in textbook appendices.

## REFERENCES

- [1] Murphy, B., 2009. “Early verification and validation using model-based design.” *EDN*, **54**, July, pp. 39–41. 1
- [2] Poolla, K., Khargonekar, P., Tikku, A., Krause, J., and Nagpal, K., 1994. “A time-domain approach to model validation.” *IEEE Transactions on Automatic Control*, **39**(5), May, pp. 951–959. 1, 5
- [3] Hamaker, H. C., 1995. “Relative merits of using maximum error versus 3(sigma) in describing the performance of laser-exposure reticle writing systems.” *Proc. SPIE*, **2440**, May, p. 550. 1, 5
- [4] Vardeman, S. B., 1994. *Statistics for Engineering Problem Solving*. ISBN: 0-534-92871-4. PWS Publishing Company, Boston, MA. 1, 4, 13, 14, 18
- [5] Koch, P. N., 2002. “Probabilistic design: Optimizaing for six sigma quality.” In *43rd AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference*, Paper Number AIAA-2002-1471. 3
- [6] Glancy, C., 1999. “A second-order method for assembly tolerance analysis.” In *Proceedings of the 1999 ASME Design Engineering Technical Conferences*, no. DAC-8707 in DETC99, ASME/DETC. 3, 10
- [7] Kiureghian, A. D., 1996. “Structural reliability methods for seismic safety assessment: A review.” *Engineering Structures*, **18**(6), June, pp. 412–424. 4
- [8] Thanedar, P. B., and Kodiyalam, S., 1992. “Structural optimization using probabilistic constraints.” *Structural and Multidisciplinary Optimization*, **4**, pp. 236–240 10.1007/BF01742750. 4
- [9] Melchers, R. E., 1999. *Structural Reliability: Analysis and Prediction*. Ellis Horwood Series in Civil Engineering. John Wiley & Sons, New York. 4
- [10] Parkinson, A., Sorensen, C., and Pourhassan, N., 1993. “A general approach for robust optimal design.” *Journal of Mechanical Design*, **115**(1), March. 4
- [11] Chen, W., Wiecek, M. M., and Zhang, J., 1999. “Quality utility - a compromise programming approach to robust design.” *Journal of Mechanical Design*, **121**, June. 4
- [12] Chen, W., Sahai, A., Messac, A., and Sundaraj, G. J., 2000. “Exploring the effectiveness of physical programming in robust design.” *Journal of Mechanical Design*, **122**(2), June, pp. 155–163. 4

- [13] Su, J., and Renaud, J. E., 1997. “Automatic differentiation in robust optimization.” *AIAA Journal*, **35**(6), pp. 1072–1079. 4
- [14] Taguchi, G., 1992. *Taguchi on Robust Technology Development: Bringing Quality Engineering Upstream*. Asme Press Series on International Advances in Design Productivity. ASME Press, New York, August. 4
- [15] Messac, A., and Ismail-Yahaya, A., 2002. “Multiobjective robust design using physical programming.” *Structural and Multidisciplinary Optimization*, **23**(5), pp. 357–371. 4
- [16] Oberkampf, W. L., DeLand, S. M., Rutherford, B. M., Diegert, K. V., and Alvin, K. F., 2002. “Error and uncertainty in modeling and simulation.” *Reliability Engineering & System Safety*, **75**, March, pp. 333–357. 4
- [17] Halton, J. H., 1960. “On the efficiency of certain quasi-random sequences of points in evaluating multi-dimensional integrals.” *Numerische Mathematik*, **2**, pp. 84–90 10.1007/BF01386213. 4
- [18] Hammersley, J. M., 1960. “Monte carlo methods for solving multivariate problems.” *Annals of the New York Academy of Sciences*, **86**, May, pp. 844–874. 4
- [19] Hutcheson, R. S., and McAdams, D. A., 2010. “A hybrid sensitivity analysis for use in early design.” *Journal of Mechanical Design*, **132**(11), November, p. 111007. 4, 17, 38
- [20] Owen, A. B., 1998. “Latin supercube sampling for very high-dimensional simulations.” *ACM Transactions on Modeling and Computer Simulation*, **8**(1), January. 4
- [21] Larson, B., Anderson, T. V., and Mattson, C. A., 2010. “System behavioral model verification for concurrent design and modeling.” In *13th AIAA/ISSMO Multidisciplinary Analysis and Optimization Conference Proceedings*, no. 9104 in AIAA-2010, AIAA/ISSMO. 5, 42
- [22] Evans, J. W., Zawadzki, R. J., Jones, S. M., Olivier, S. S., and Werner, J. S., 2009. “Error budget analysis for an adaptive optics optical coherence tomography system.” *Optics Express*, **17**(16), August, pp. 13768–13784. 5
- [23] Hamaker, H. C., 1995. “Improved estimates of the range of errors on photomasks using measured values of skewness and kurtosis.” *Proc. SPIE*, **2621**(198 (1995)), March. 5
- [24] Oschmann, J., 1997. Gemini system error budget plan, January. 5
- [25] Samet, H., 2005. *Foundations of Multidimensional and Metric Data Structures (The Morgan Kaufmann Series in Computer Graphics and Geometric Modeling)*. Morgan Kaufmann Publishers Inc., San Francisco, CA, USA. 6
- [26] Hayes, B., 2003. “A lucid interval.” *American Scientist*, **91**(6), November-December, pp. 484–488. 6
- [27] Box, G. E., and Tiao, G. C., 1992. *Bayesian Inference in Statistical Analysis*. ISBN: 9780471574286. Wiley, Hoboken, NJ, April. 6

- [28] research.microsoft.com, 2011. Basics of bayesian inference and belief networks <http://goo.gl/rSJCD>, July. 6
- [29] Box, G. E. P., and Wilson, K. B., 1951. “On the experimental attainment of optimum conditions.” *Journal of the Royal Statistical Society. Series B (Methodological)*, **13**(1), pp. 1–45. 6
- [30] Chen, W., Baghdasaryan, L., Buranathiti, T., and Cao, J., 2004. “Model validation via uncertainty propagation and data transformations.” *AIAA Journal*, **42**(7), p. 14061415. 6
- [31] Lombardi, M., and Haftka, R. T., 1998. “Anti-optimization technique for structural design under load uncertainties.” *Computer Methods in Applied Mechanics and Engineering*, **157**(1-2), pp. 19–31. 7
- [32] Jackson, P. S., 1982. “A second-order moments method for uncertainty analysis.” *IEEE Transactions on Reliability*, **R-31**(4), October, pp. 382–384. 10
- [33] Shapiro, S. S., and Gross, A. J., 1981. *Statistical Modeling Techniques*. Marcel Dekker, Inc., New York and Basel. 10
- [34] Walwyn, R., 2005. *Moments - Encyclopedia of Statistics in Behavioral Science.*, Vol. 3 John Wiley and Sons, Ltd, 111 River St. Hoboken New Jersey United States. 11
- [35] Taylor, B., 1715. *Methodus Incrementorum Directa et Inversa [Direct and Reverse Methods of Incrementation]*. Proposition VII, Theorem 3, Corollary 2. London Translated into English in D. J. Struik, *A Source Book in Mathematics 1200-1800* (Cambridge, Massachusetts: Harvard University Press, 1969), pages 329-332. 11
- [36] Mattson, C. A., and Messac, A., 2002. “A non-deterministic approach to concept selection using s-pareto frontiers.” *Proceedings of ASME DETC*, **2**(DETC2002/DAC-34125), September, pp. 859–870. 13, 16, 17, 38
- [37] Tellinghuisen, J., 2001. “Statistical error propagation.” *Journal of Physical Chemistry A*, **105b**(15), March, pp. 3917–3921. 13, 18
- [38] Lindberg, V., 2000. Uncertainties and error propagation - part i of a manual on uncertainties, graphing, and the vernier caliper Internet, July. 13
- [39] Julier, S., Uhlmann, J., and Durrant-Whyte, H. F., 2000. “A new method for the nonlinear transformation of means and covariances in filters and estimators.” *IEEE Transactions on Automatic Control*, **45**(3), March, pp. 477–482. 16, 38
- [40] Putko, M. M., III, A. C. T., Newman, P. A., and Green, L. L., 2002. “Approach for uncertainty propagation and robust design in cfd using sensitivity derivatives.” *Journal of Fluids Engineering*, **124**(1), March, pp. 60–69. 16
- [41] Hamel, J., Li, M., and Azarm, S., 2010. “Design improvement by sensitivity analysis under interval uncertainty using multi-objective optimization.” *Journal of Mechanical Design*, **132**(8), August, p. 081010. 16



- [42] Goodman, L. A., 1960. “On the exact variance of products.” *Journal of the American Statistical Association*, **55**(292), December, pp. 708–713. 18
- [43] Zhang, J., and Du, X., 2010. “A second-order reliability method with first-order efficiency.” *Journal of Mechanical Design*, **132**(10), October, p. 101006. 18
- [44] Aono, H., Shyy, W., and Liu, H., 2009. “Near wake vortex dynamics of a hovering hawkmoth.” *Acta Mechanica Sinica*, **25**(1), pp. 23–36. 28
- [45] George, R. B., 2011. “Design and analysis of a flapping wing mechanism for optimization.” Master’s thesis, Brigham Young University, Provo, Utah, June. 28
- [46] Mekid, S., and Vaja, D., 2008. “Propagation of uncertainty: Expressions of second and third order uncertainty with third and fourth moments.” *Measurement*, **41**(6), July, pp. 600–609. 34
- [47] von Hippel, P. T., 2005. “Mean, median, and skew: Correcting a textbook rule.” *Journal of Statistics Education*, **13**(2). 35
- [48] Garner, W. B., 2009. Model airplane propellers <http://www.drc.org/pdf/Model%20Propellers%20Article.pdf>, March. 42
- [49] Degarmo, E. P., Black, J. T., and Kohser, R. A., 2003. *Materials and Processes in Manufacturing.*, 9 ed. ISBN: 0-471-65653-4. Wiley. 47
- [50] Tyfour, W. R. Rolling Internet (<http://www.freewebs.com/tyfour/Rolling.doc>). 47
- [51] Lenard, J. G., 2007. *Primer on Flat Rolling.* ISBN: 0080453198. Elsevier Science. 47

## APPENDIX A. STEP-BY-STEP SOLUTION TO DUAL-PROPELLER, THREE DEGREE-OF-FREEDOM HELICOPTER PROBLEM

Consider the compositional system model for a dual-propeller, three degree-of-freedom helicopter discussed in Section 3.6. This model, given initially in Eq. (3.25), is repeated below in Eq. (A.1).

$$\begin{aligned}
 \ddot{\theta} &= C_1 C_2 + C_3 \\
 C_1 &= \frac{(m_2 L_2 - m_1 L_1)g}{m_1 L_1^2 + m_2 L_2^2 + J_y} \\
 C_2 &= \cos \theta \\
 C_3 &= \frac{L_1 k_m (u_L + u_R)}{m_1 L_1^2 + m_2 L_2^2 + J_y}
 \end{aligned} \tag{A.1}$$

Recall that variation exists in variables  $m_1$ ,  $m_2$ ,  $L_1$ ,  $L_2$ ,  $J_y$ , and  $k_m$ , and  $\theta$ . The means and standard deviations for these variables are repeated in Table A.1.

Table A.1: Design Parameters for Dual-Propeller, 3-DOF Helicopter

Design Parameter	Mean	Std. Dev.
$m_1$	0.891 (kg)	$10^{-4}$ (kg)
$m_2$	1.000 (kg)	$10^{-4}$ (kg)
$L_1$	0.850 (m)	$10^{-4}$ (m)
$L_2$	0.3048 (m)	$10^{-4}$ (m)
$J_y$	0.0014 (kg·m <sup>2</sup> )	$10^{-5}$ (kg·m <sup>2</sup> )
$k_m$	0.0546	$10^{-5}$
$\theta$	0	1°

## A.1 Variance Propagation Through Component 1

All first-order partial derivatives of component  $C_1$ , evaluated at the input means, are given in Eq. (A.2).

$$\begin{aligned}
\frac{\partial C_1}{\partial m_1} &= \frac{-gL_1(m_1L_1^2 + m_2L_2^2 + J_y) - gL_1^2(m_2L_2 - m_1L_1)}{(m_1L_1^2 + m_2L_2^2 + J_y)^2} = -5.4096 \\
\frac{\partial C_1}{\partial m_2} &= \frac{gL_2(m_1L_1^2 + m_2L_2^2 + J_y) - gL_2^2(m_2L_2 - m_1L_1)}{(m_1L_1^2 + m_2L_2^2 + J_y)^2} = 4.8085 \\
\frac{\partial C_1}{\partial L_1} &= \frac{-gm_1(m_1L_1^2 + m_2L_2^2 + J_y) - 2gm_1L_1(m_2L_2 - m_1L_1)}{(m_1L_1^2 + m_2L_2^2 + J_y)^2} = 0.5020 \\
\frac{\partial C_1}{\partial L_2} &= \frac{gm_2(m_1L_1^2 + m_2L_2^2 + J_y) - 2gm_2L_2(m_2L_2 - m_1L_1)}{(m_1L_1^2 + m_2L_2^2 + J_y)^2} = 18.2601 \\
\frac{\partial C_1}{\partial J_y} &= \frac{-g(m_2L_2 - m_1L_1)}{(m_1L_1^2 + m_2L_2^2 + J_y)^2} = 8.1501
\end{aligned} \tag{A.2}$$

The first-order Taylor series approximation from Eq. (3.9) is used to propagate variance through this component, as given in Eq. (A.3).

$$\begin{aligned}
\sigma_{C_1}^2 &\approx \sum_{i=1}^n \left( \frac{\partial C_1}{\partial x_i} \right)^2 \sigma_{x_i}^2 \\
&\approx \left( \frac{\partial C_1}{\partial m_1} \right)^2 \sigma_{m_1}^2 + \left( \frac{\partial C_1}{\partial m_2} \right)^2 \sigma_{m_2}^2 + \left( \frac{\partial C_1}{\partial L_1} \right)^2 \sigma_{L_1}^2 + \left( \frac{\partial C_1}{\partial L_2} \right)^2 \sigma_{L_2}^2 + \left( \frac{\partial C_1}{\partial J_y} \right)^2 \sigma_{J_y}^2 \\
&\approx 3.8673\text{e-}6
\end{aligned} \tag{A.3}$$

The mean value for component  $C_1$  is -6.0152, which was determined by evaluated  $C_1$  at the input means.

## A.2 Variance Propagation Through Component 2

All first- and second-order partial derivatives of component  $C_2$ , evaluated at the input means, are given in Eq. (A.4).

$$\begin{aligned}\frac{\partial C_2}{\partial \theta} &= -\sin(\theta) = 0 \\ \frac{\partial^2 C_2}{\partial \theta^2} &= -\cos(\theta) = -1\end{aligned}\tag{A.4}$$

The second-order Taylor series approximation from Eq. (3.17) is used to propagate variance through this component, as given in Eq. (A.5). Note that the variance must be expressed in radians.

$$\begin{aligned}\sigma_{C_2}^2 &\approx \left(\frac{\partial C_2}{\partial \theta}\right)^2 \sigma_\theta^2 + \frac{1}{2} \left(\frac{\partial^2 C_2}{\partial \theta^2}\right)^2 \sigma_\theta^4 \\ &\approx 4.6396\text{e-}8\end{aligned}\tag{A.5}$$

A correction factor  $e$  is then calculated using Eq. (3.19), as shown in Eq. (A.6).

$$\begin{aligned}e &= \frac{1}{1 + 1.022\sigma_{C_2}^2} \\ &= 0.9997\end{aligned}\tag{A.6}$$

The resulting variance in component  $C_2$  is given by Eq. (A.7).

$$\begin{aligned}\sigma_{C_2,CF}^2 &\approx \sigma_{C_2}^2 e \\ &\approx 4.6381\text{e-}8\end{aligned}\tag{A.7}$$

The mean value for component  $C_2$  is 1, which was determined by evaluating  $C_2$  at the input means.

### A.3 Variance Propagation Through Component 3

All first-order partial derivatives of component  $C_3$ , evaluated at the input means, are given in Eq. (A.8).

$$\begin{aligned}
\frac{\partial C_3}{\partial m_1} &= \frac{-L_1^3 k_m (u_L + u_R)}{(m_1 L_1^2 + m_2 L_2^2 + J_y)^2} = -5.5401 \\
\frac{\partial C_3}{\partial m_2} &= \frac{-L_1 L_2^2 k_m (u_L + u_R)}{(m_1 L_1^2 + m_2 L_2^2 + J_y)^2} = -0.7124 \\
\frac{\partial C_3}{\partial L_1} &= \frac{k_m (u_L + u_R) (m_1 L_1^2 + m_2 L_2^2 + J_y) - 2m_1 L_1^2 k_m (u_L + u_R)}{(m_1 L_1^2 + m_2 L_2^2 + J_y)^2} = -4.9566 \\
\frac{\partial C_3}{\partial L_2} &= \frac{-2m_2 L_1 L_2 k_m (u_L + u_R)}{(m_1 L_1^2 + m_2 L_2^2 + J_y)^2} = -4.6744 \\
\frac{\partial C_3}{\partial J_y} &= \frac{-L_1 k_m (u_L + u_R)}{(m_1 L_1^2 + m_2 L_2^2 + J_y)^2} = -7.6680 \\
\frac{\partial C_3}{\partial k_m} &= \frac{L_1 (u_L + u_R) (m_1 L_1^2 + m_2 L_2^2 + J_y)}{(m_1 L_1^2 + m_2 L_2^2 + J_y)^2} = 103.6514
\end{aligned} \tag{A.8}$$

The first-order Taylor series approximation from Eq. (3.9) is used to propagate variance through this component, as given in Eq. (A.9).

$$\begin{aligned}
\sigma_{C_3}^2 &\approx \sum_{i=1}^n \left( \frac{\partial C_3}{\partial x_i} \right)^2 \sigma_{x_i}^2 \\
&\approx \left( \frac{\partial C_3}{\partial m_1} \right)^2 \sigma_{m_1}^2 + \left( \frac{\partial C_3}{\partial m_2} \right)^2 \sigma_{m_2}^2 + \left( \frac{\partial C_3}{\partial L_1} \right)^2 \sigma_{L_1}^2 + \left( \frac{\partial C_3}{\partial L_2} \right)^2 \sigma_{L_2}^2 \\
&\quad + \left( \frac{\partial C_3}{\partial J_y} \right)^2 \sigma_{J_y}^2 + \left( \frac{\partial C_3}{\partial k_m} \right)^2 \sigma_{k_m}^2 \\
&\approx 1.8564\text{e-}6
\end{aligned} \tag{A.9}$$

The mean value for component  $C_3$  is 5.6594, which was determined by evaluating  $C_3$  at the input means.

#### A.4 Variance Propagation Through System-Level Compositional Model

All first-order partial derivatives of the system model, evaluated at the input means, are given in Eq. (A.10).

$$\begin{aligned}\frac{\partial \ddot{\theta}}{\partial C_1} &= C_2 = 1 \\ \frac{\partial \ddot{\theta}}{\partial C_2} &= C_1 = -6.0152 \\ \frac{\partial \ddot{\theta}}{\partial C_3} &= 1\end{aligned}\tag{A.10}$$

Now that the variance and mean of each component has been determined, the first-order Taylor series approximation from Eq. (3.9) is used to propagate variance through the system-level compositional model, as given in Eq. (A.11).

$$\begin{aligned}\sigma_{\ddot{\theta}}^2 &\approx \left(\frac{\partial \ddot{\theta}}{\partial C_1}\right)^2 \sigma_{C_1}^2 + \left(\frac{\partial \ddot{\theta}}{\partial C_2}\right)^2 \sigma_{C_2}^2 + \left(\frac{\partial \ddot{\theta}}{\partial C_3}\right)^2 \sigma_{C_3}^2 \\ &\approx 7.4019\text{e-}6\end{aligned}\tag{A.11}$$

Actual variance in  $\ddot{\theta}$ , as determined from a Monte Carlo simulation with 10 million data points, is 6.5979e-6, meaning this compositional system model with a correction factor has 12% relative error. By comparison, the estimate of variance propagation using strictly a first-order Taylor series (without the compositional model or a correction factor) is 4.4853e-6 (32% relative error).

As explained in Chapter 3, this increase in accuracy comes with very little additional computational cost.