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Discovering the Derivative Can Be "Invigorating:" Mark's Journey to Understanding Instantaneous Velocity

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DISCOVERING THE DERIVATIVE CAN BE “INVIGORATING”:

MARK’S JOURNEY TO UNDERSTANDING INSTANTEOUS VELOCITY

by

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This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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As chair of the candidate's graduate committee, I have read the thesis of Charity Ann Hyer in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

DISCOVERING THE DERIVATIVE CAN BE “INVIGORATING”

MARK’S JOURNEY TO UNDERSTANDING INSTANTANEOUS VELOCITY

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Masters of Arts

This is a case study using qualitative methods to analyze how a first semester calculus student named Mark makes sense of the derivative and the role of the classroom practice in his understanding. Mark is a bright yet fairly average student who successfully makes sense of the derivative and retains his knowledge and understanding. The study takes place within a collaborative, student-centered, task-based classroom where the students are given opportunity to explore mathematical ideas such as rate of change and accumulation. Mark’s sense making of the derivative is analyzed in light of his use of physics, Mark as a visual learner, the representations he used to make sense of the derivative using Zandieh’s (2000) framework for representations of derivatives, and his conceptions of the limit over time. Classroom practice allowed Mark to exercise his agency and explore tasks in ways that were personally meaningful. The findings in this study contribute new details about how calculus students might solve tasks, develop strategies, and communicate with each other.

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Chapter 1 – Introduction

In a day of calculus reform, we need studies that help us learn how students think and develop understanding of basic ideas in calculus so that we can develop more efficient teaching methods and enrich our curriculum. Scher (1996) describes the relative lack of articles addressing students' conceptualizations of key calculus ideas:

> The output of the calculus reform movement, for example, has included many articles on curriculum for use in computer laboratories. By comparison, there are far fewer articles that examine how students conceptualize such key calculus topics as rate of change and accumulation. Developing theories and models of how students come to these understandings is critical for the design of new curriculum. (p. 11)

Case studies of individual students are of great importance in understanding how students in general think, act, and develop understanding (Thompson, 1994). Many studies point to failures, struggles or misconceptions students may have, but there is a lack of case studies highlighting student success stories (Speiser, private conversation, February 24, 2007). How would it be if calculus students saw the beauty and meaning behind the definition of the derivative? How would it be if normal students succeeded in forming deep understanding about basic calculus concepts?

Mark, a first-semester undergraduate calculus student who found discovering the derivative "invigorating," is such a student. This research is a case study that examines how Mark conceptualizes the derivative as instantaneous velocity and how his understanding develops over time.

Mark was in many respects a very average student. Mathematics educators would be able to recognize students like him among their own students. Yet in the circumstances in which this research took place he was able to demonstrate outstanding ability. Mark stood out because of his determination and apparent enjoyment of the class as well as his tendency to express himself often and well verbally, making him an ideal candidate for study.

Bandura (1989) describes some of the attributes Mark displayed when speaking of a person with a strong sense of self-efficacy. The setting in which Mark was working fostered the enhancement of self-efficacy:

> Development of resilient self-efficacy requires some experience in mastering difficulties through perseverant effort. If people experience only easy successes, they come to expect quick results and their sense of efficacy is easily undermined by failure. Some setbacks and difficulties in human pursuits serve a useful purpose in teaching that success usually requires sustained effort. After people become convinced they have what it takes to succeed, they persevere in the face of adversity and quickly rebound from setbacks. By sticking it out through tough times, they emerge from adversity with a stronger sense of efficacy. (p. 5)

Mark was enrolled in an experimental, student-centered calculus course designed to allow students freedom to develop their own ideas and understanding while working with others and collaborating as a class. Students had time to explore ideas and were not readily handed formulas or methods for solving problems. Mark took days, even weeks developing ideas of rates of change, limits, and instantaneous velocity and we shall see that his sustained effort paid off. Given the freedom to explore the idea of instantaneous rate of change in a real-life context, Mark used creativity, group collaboration, and previous knowledge in physics, mathematics, and philosophy to make sense of the derivative.

Research questions

Research in mathematics education has repeatedly tried to answer questions such as: How do students learn? What is the best approach to teaching calculus (or any branch of mathematics)? How do students develop understanding that they not only retain, but also can apply in different situations? What motivates students to high achievement? While I do not attempt to answer these questions in their entirety, the case study that I have done regarding Mark's making sense of the derivative will add to a general body of research in mathematics education. Specifically the research in the study will focus on the following research questions:

- How does Mark make sense of the concept of derivative while working on the cat task[1]?
- What are ways that the classroom structure influences Mark's learning?

The study provides an in-depth, thorough analysis of Mark's journey to understand the derivative, the setting that allowed him to do so, and events and ideas that contributed and influenced his choices and decisions in the sense making process. I will show how making sense of the concept of derivative is a complex process that involves sustained inquiry and that discovering the derivative in a task-based, student-centered setting yields positive results. In the following chapters I will present the learning theories I use as a theoretical lens, literature regarding studies of undergraduate students' understanding of calculus concepts, specific definitions of calculus concepts such as the

[1] The cat task involves a series of photos of a cat running in front of a grid taken at intervals of 0.031 seconds (Muybridge, 1887/1957), and the object of the task is to find the cat's instantaneous velocity at frames 10 and 20. The task will be discussed further in the literature review.

derivative and limit that will be used in the study, a thorough analysis of Mark's developing understanding of the derivative, and present implications for future research in classroom design, curriculum, and calculus reform.

Chapter 2 – Theoretical Perspective

In this chapter I will outline the theoretical perspectives on student learning that frame this study. My perspective revolves around the following major ideas: (1) knowledge and understanding are not transferred directly from one person to another, but assimilated through the individual's lens of experience and built by the individual, (2) social interactions help students build understanding as they communicate with each other, (3) learning is a complex process that takes time, and (4) agency plays an important role in student learning.

Transfer and Building Knowledge

Knowledge and understanding are not transferred directly from one person to another, but are created based on experience (Ashton, 1992; von Glasersfeld, 1995). Students must construct their own understanding. While much research exploring the transfer of knowledge is related to teacher-student relationships (von Glasersfeld, 1995), it is important to remember that knowledge and understanding do not transfer directly from *any* one person to another, including peer-to-peer relationships (Zandieh, 2006). What is heard or seen is assimilated through an individual's personal lens that is formed based on previous experience. As a result of their experiences, students form concept images (Vinner & Dreyfus, 1989) of mathematical ideas. A student's concept image is an individual notion of a given concept in mathematics built by the student based on the student's experience. Concept images change over time as the student learns more and encounters different ideas.

Because students learn through experience, it is important to afford students ample opportunity to have experiences that will best help them explore and form correct ideas with deep understanding. Classrooms that allow students to explore meaningful tasks, discuss, and present ideas to one another create atmospheres in which deep conceptual knowledge can flourish. Such classrooms can allow for such exploration of mathematical ideas (Speiser, Walter, & Maher, 2003; NCTM, 2000).

Social Aspect

While each individual creates his or her own knowledge and understanding of concepts, the social plane influences individual construction. Social interactions help students build understanding. Students negotiate terminology and create meanings through social interaction (Schnepp & Nemirovsky, 2001). Mathematics involves ideas built on concrete experience but also involves abstract notation. Lacking in either is going to decrease a student's ability to do mathematics (Davis & Maher, 1997). It is in the social plane that notation is clarified and used as a means of communicating one with another. Students learn conventional notation as well as creating their own notation, but until the community of learners understands that notation, it cannot be used as a means of communication. Notation is an important part of understanding and communicating about mathematics. Individuals contribute to social norms and ideas and community discourse affects individual thinking. Terms are defined, notation is refined, and ideas are clarified through social interaction.

Related to the reflexive relationship between the social plane and the individual is the idea of public and private presentations (Raman, 2003; Speiser & Walter, 1997; Walter & Gerson, 2007). Private presentations are those presented to oneself, convincing oneself, thinking through ideas, and forming mental arguments. Public presentations are when students engage in public discourse with their peers, teachers or as a class. One facilitates the progress of the other—the private presentations are brought to the public domain, modified, returning to the private thoughts, and public presentations affecting private presentations. What we know changes as we act and think and our understanding changes “as we reflect, communicate, and perhaps restructure what we know” (Speiser et al., 2003, p. 25). Being able to communicate and express one’s ideas is important in the development of student understanding.

Critical Events and Key Ideas

Learning is a complex process that takes time. I would like to compare the process of learning to a river and its tributaries. Like all metaphors, this one is not completely isomorphic but it does convey important information about learning. A river starts out small at its source and builds as water from tributaries are added to it. These tributaries mix with the water already in the river and all the water becomes one moving mass. Sometimes there is a fork in the river and one branch gets more water than the others, but further down the river the braches may meet up again, usually making the river larger than it was before the fork. Along the way, water seeps into the ground or is evaporated into the air and at the same time more little streams of water find their way into the river. The river is dynamic, changing and advancing as it flows along.

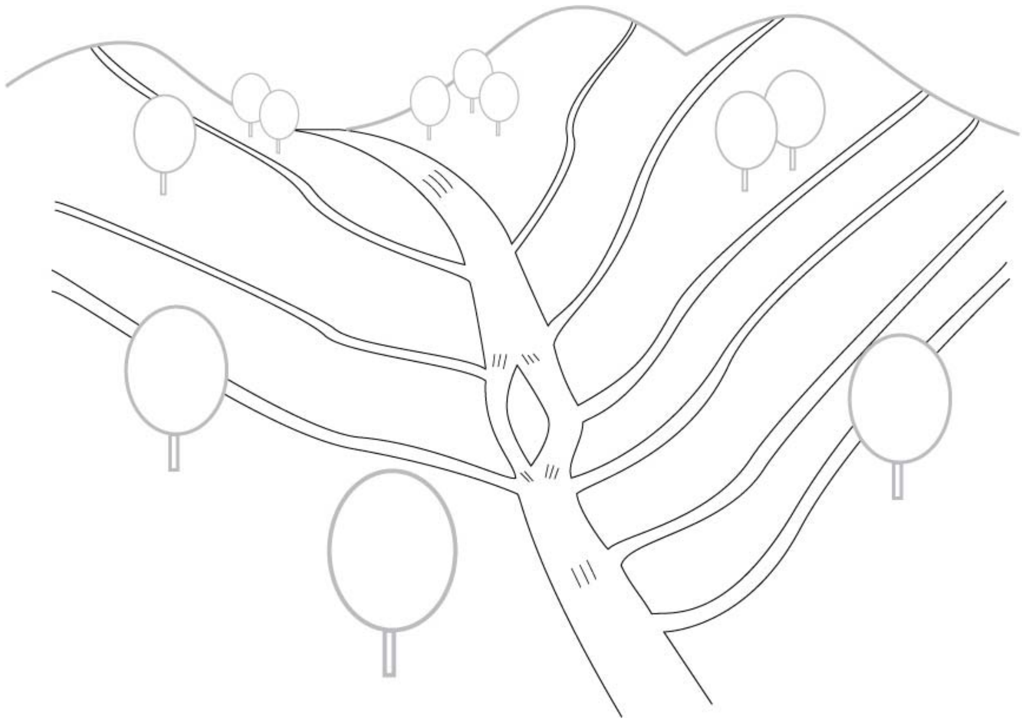

Figure 2.1: River analogy

Learning is also a dynamic process, ever advancing through our experiences (Rasmussen, Zandieh, King, & Teppo, 2005). Knowledge and understanding start out small, but as one continues to learn, tributaries of knowledge are added to the growing body as the learner encounters new ideas or has new experiences. Gained knowledge and understanding mingle with that which is already there, becoming a complex body of concept images, some compartmentalized (Vinner & Dreyfus, 1989), some connected. Sometimes ideas are left behind or discarded like the water that evaporates or seeps away. However, the body of knowledge and understanding grows larger and larger over time as the learner grows in experience. Sometimes, just like the fork in the river, there is a branching of ideas and the learner devotes time to a particular idea. Often, previous ideas that were not given as much attention have a place again and the river is even stronger than it was before. Once ideas solidify, they become strong currents in the river of understanding that determine the course of the river.

As previously stated, learning is a complex process that takes time. Over time students are able to develop mathematical understanding if they are afforded opportunities to explore and construct sound conceptual understanding. "To build, explain, and justify mathematical conclusions in challenging problem situations (like the cat task) requires *time, freedom, and diverse personal experience.*" (Speiser et al. 2003, p. 25). For a student to arrive at a deep conceptual understanding of a mathematical concept, the student needs time for the ideas to solidify. Time is a necessary element in the development of understanding. I believe that over time there are certain episodes or critical events that have a significant impact on the direction and development of student understanding. These critical events could be compared to a bend in the river or the entering of a tributary. There are also threads of ideas, which I will call key ideas, which weave their way through the process. Key ideas stay in the stream of water directing the path of the river as much as objects on the outside may. Both the critical events and key ideas play significant roles in student thinking. I will describe each of these in more detail below.

Critical events. While students explore ideas in task-based learning situations, there are episodes that provide insight into the development of students' ideas. Maher and Martino (1996) have called these episodes "critical events." While Maher and Martino used critical events mainly as an analytic method, those events are events that help frame a student's understanding and therefore part of a perspective on how students learn. I believe that students have moments where ideas "click," where they connect ideas or have a "light bulb turn on." There are also certain discussions students have with each other or with instructors that play a significant role in the development of the students' understanding. All of these are examples of critical events and these critical events occur after a student has thought about a concept, explored it, and asked questions (whether to himself or otherwise).

Key ideas. Along with moments of discovery or insight into student learning there are recurring themes that appear while observing students in learning situations. Raman (2003) examined students' development of proof and found that heuristic, procedural, and key ideas all played roles in proof. She defined a key idea as "an heuristic idea which one can map to a formal proof with appropriate sense of rigor. It links together the public and private domains, and in doing so gives a sense of understanding and conviction" (p. 323). In other words students create ideas on their own based on heuristic experience and key ideas are those ideas that map what a student is thinking to the public domain or that of formal mathematics. It is important that these ideas come from the student.

I see key ideas not only as the link between the public and private domains, but the ideas that guide and direct a student and become the focus of his or her attention. There are significant ideas that a student has prior to creating and during the creation of a justification, which play important roles in the creation of a proof or in the understanding of a concept. Without a key idea to guide, a student will never traverse down a certain path, but if that idea has come to the mind of the student, then, and only then, will the student pursue a novel path. These ideas may or may not come from certain events or timed periods in which the student is observed but could come from far back in the student's background. In referring to the river analogy, they will come in as different tributaries and combine with ideas already there. For this reason I believe both key ideas and critical events in the classroom are important to consider when studying students' learning.

The Role of Agency

Students need freedom, the capacity to exercise their personal agency, to build understanding. In this paper personal agency will be defined as the ability to choose and act for oneself.

> The exercise of agency is what makes mathematical thinking possible. We distinguish between a perspective in which learners' development of agency is fostered by the teacher (Cobb & Yackel, 1998), and our view that personal agency in learning is omnipresent and its existence is not dependent on teacher intervention. However, the enactment of personal agency in productive mathematical inquiry can be constrained or encouraged by teacher intervention. (Water and Gerson, 2007, p. 209)

While the teacher may influence a student one way or another by comments or encouragement, it is ultimately the student who decides how to develop his or her knowledge. This is true in any classroom setting, but as Walter and Gerson indicated, the teacher can encourage or constrain a student's exercise of agency in problem solving situations and so the classroom set up can influence the extent to which a student can and will exercise personal agency in problem solving. Allowing students the opportunity to choose strategies for solving problems or tasks helps those problems and tasks become personally meaningful to the student (Castle and Aichele, 1994) and makes students better problem solvers (Siegler, 1996).

Agency plays an important role in defining a student's belief in his or herself. One's belief of one's ability to control one's own situation has a lot to do with resulting ability to control one's situation (Bandura, 1989). When a student has success their self-confidence increases. When that success stems from choices the student has made, the confidence that student has in his or her choices also increases. Not only do students' choices affect their resulting understanding, but recognizing their own power of choice will affect the degree to which the student will direct his or her own thinking.

Chapter 3 – Literature Review and Definition of Terms

Research has shown time and time again that many high school and undergraduate students do not have a firm understanding of calculus concepts (Baker, Cooley, Trigueros, 2000; Dudley, 1993; Zachary, 2004). Andrew M. Gleason said the following in his forward for Taylor's calculus book (Taylor, 1992):

> It is widely agreed that calculus instruction has taken a wrong turn. While more and more students take calculus courses, fewer and fewer emerge from that experience with a useable knowledge of the subject. Far too often students learn only to carry out the technical manipulations of calculus with no regard for their meaning, and many never even finish the course. (p. vii)

Not only do fewer students emerge with a usable knowledge of calculus, but also students professed neither to understand nor to like it (Dudley, 1993). "For most students [calculus] was not a satisfying culmination of their secondary preparation, and it was not a gateway to future work. It was an exit" (Dudley, 1993, p vii). Research findings in mathematics education point to the difficulty students have solving problems "in context" (c.f., Caldwell & Goldin, 1987; White & Mitchelmore, 1996). For many students there is a gap between their symbolic manipulation skills and their conceptual understandings of the material (Zachary, 2004). Most students are left with formally proved statements without a clear intuitive sense of why such relationships exist (Schnepp & Nemirovsky, 2001). A complete understanding of *why* is lacking even if students can see some steps that seem to make sense.

Both conceptual and procedural competence in mathematics are necessary for a student to develop proficiency in mathematics (National Research Council, 2001; Roddick, 1995; Zachary, 2004). Being able to perform computations does not imply an understanding of mathematical meaning, the recognition of structures, or the ability to interpret the results. Rules can be followed without much conceptual development having taken place (Goldin & Shteingold, 2001). Therefore, more emphasis needs to be placed on conceptual understanding and students need opportunities to develop strong conceptual understanding in calculus classes.

Calculus Classrooms

Numerous classroom experiments have been conducted to improve student understanding of calculus concepts. I would like to address a few that share similar characteristics to this study. Habre and Abboud (2006) studied a reformed calculus class at the Lebanese American University in Beirut, Lebanon. Students thought about ideas conceptually, used multiple representations, visualization and graphing calculators but were not to use calculators on the required traditional departmental exam at the end of the semester. Conceptually, “students showed an almost complete understanding of the derivative, particularly the idea or the instantaneous rate of change and/or the slope of a curve at a given point” (p. 57). Problems presented to the students encouraged visualization of calculus. Despite the strong emphasis the class placed on visual representations, the students for the most part, still maintained a very algebraic vision of functions as formulas. Students found that this type of classroom set up required high levels of thinking and work, and felt like they arrived at a deeper understanding of the subject. However, Habre and Abboud were not pleased with the ending results of this course. They felt like too many students dropped the course early on, didn’t answer questions on the final exam well, or failed the course.

Using physics to solve calculus problems has also been a topic of study among research in mathematics education. Marrongelle (2004) conducted research in an integrated calculus and physics (ICP) course to examine how students used physics to help them solve problems involving average rate of change, derivative and integral concepts. In this study, students created contexts for calculus problems using physics. The curriculum development team for the course felt that students' experience with motion detectors would better prepare them for formal definitions of limits, derivatives and antiderivatives, so the students explored derivatives and antiderivatives in the first week using motion detectors in a physics laboratory. Students predicted graphical behavior of velocity and acceleration given a position graph and other similar situations, which were followed by class discussions about average velocity and average acceleration. Class structure allowed the students to explore ideas on their own and formulate conjectures. However, instructors still played a large role in the students' development of ideas. After discussing average velocity of the motion detectors, the physics instructor asked what would happen if the students considered smaller and smaller time intervals. With concrete physical experience the students were then given abstract calculus questions such as finding the average and instantaneous rate of change given the graph of an arbitrary function, $f(t)$. They worked in groups and discussed their findings. Marrongelle reports that students varied in their reliance on the physical experience but that the physical experience was important for each student's understanding.

While many students use understanding gained through physics to solve calculus problems, often times students who are concurrently enrolled in calculus and physics courses will build understanding of concepts concurrently as well, and thus use what they learn in their physics class in their calculus class and vise versa. Sometimes students have not developed a strong grasp of a concept in either class, but will be developing it in both arenas. Such a situation occurred with a group of calculus students (Schnepp & Nemirovsky, 2001) who had been introduced to terms in their physics class a couple weeks prior to using technology to explore rates of change in their calculus class. Because they had heard terms like average rate, velocity, and derivative in their physics course they were using them often in their calculus class. However, their interpretations of these terms were not carefully thought out and were often inconsistent. The teacher intervened asking the students what they meant by certain phrases. Discussion ensued and "through this classroom conversation, the students were not introduced to technical terms from *without* but refined their fluent use of everyday language" (Schnepp & Nemirovsky, 2001, p. 102) to gain precision and logical consistency.

Cat task. Lomen and Lovelock were the first to use the Murybridge time-lapse photographs of a cat running along a grid in a calculus classroom to explore motion. Their students had a basic understanding of derivatives before exploring the cat's motion and the photographs were used as means to explore and discuss average and instantaneous velocities while using technology (Cushing et al., 1992). Similar research conducted primarily by Speiser and Walter (1994, 1996) accompanied by Maher (Speiser et al., 2003) and Glaze (Speiser, Walter, Glaze, 2005) explored students' collaborations on what they termed the cat task. Using the same photographs, Speiser and Walter asked their students to find the cat's velocity at frames 10 and 20. Their studies differed from Lomen and Lovelock's because the students in their classes were not solely exploring the photos using technology, but were free to use creativity and any other resources the students may have had to explore instantaneous velocity, relationships between distance, velocity, and acceleration over time, and other calculus concepts. The cat task was introduced after the students had already explored instantaneous velocity and derivative related concepts. They found that students who do not typically consider themselves proficient in mathematics were able to make sense of these ideas and feel ownership of the mathematics they explored. Students used their own physical motion to represent the cat's motion in solving the task. These students, unlike the students in Habre's study, were not required to pass a departmental exam. Rather, the purpose of the research was to explore students' representations and development of understanding of key calculus concepts, which proves valuable for future implementations of physical representations of motion and exploratory, task-based learning.

The above studies differ from each other in being student-centered or teacher-centered, classes where the majority of students are math and science majors versus those that are not. We see that there are success stories among non-mathematics and non-science majors in task-based learning. Is such success possible with mathematics and science major students? Could the success obtained by utilizing physics concepts be obtained in a student-centered classroom instead of one that was teacher-centered?

The study presented in this paper, while similar in various ways to the studies described above, is unique in that it is a study involving a student with a background in mathematics and science in a task-based, student-centered classroom. While Mark utilizes his knowledge of physics and other fields to help him solve calculus tasks, he does so by his own volition. The cat task was also used before students had been formally taught anything about instantaneous velocity.

Definition of Terms

Representations. In order to learn more about student understanding much research has been conducted concerning students' representations (NCTM, 2001). Representations have been defined as presentations to either oneself or others (Speiser & Walter, 1997), "a tool to think of something which is constructed through the use of the tool" (Hähkiöniemi, 2006), and a mapping from one domain to another whose correspondence preserves structure (Cuoco, 2001), among others. Studying how students both create and use representations provides insight into student's concept images (Vinner & Dreyfus, 1989), and how students build understanding of mathematics (Cuoco, 2001; Speiser, Walter, & Glaze, 2005).

Visualization. I would like to consider "visualization" as a certain genre of representations.

> One could argue that visualization and visual thinking should be one of the central elements in calculus reform. Conceptually, the role of visual thinking is so fundamental to the understanding of calculus that it is difficult to imagine a successful calculus course which does not emphasize the visual elements of the subject. This is especially true if the course is intended to stress conceptual understanding, which is widely recognized to be lacking in many calculus courses as now taught. (Zimmermann, 1991, p. 136)

In the 1980's (and even a little before) there was a big push towards using a visual approach, as technology seemed to open a whole new dimension to mathematics education (Zimmermann, 1991). Researchers felt that technology opened "seeing mathematics in the mind's eye" to all students (Zimmermann, 1991). Thus most literature found using the word "visualization" speaks mainly of using technology to visualize graphs, motion, changes in functions, etc. During this time focus was on what the student produced—graphs, diagrams and the like. However, historically mathematicians who “visualized” to help them understand and to develop ideas in mathematics did not just limit the idea of visualization to graphs.

In this paper visualization will be defined as representations that elicit a visual image. This includes images both mental and drawn, language that invokes a visual image (including body language such as gesture). Thus the term visualization is almost as broad a term as representation, but is limited to those representations that invoke visual images. Since it is impossible to entirely determine the mental images a student creates, a researcher must rely on the student’s discourse, hand gestures, and what the student writes or draws to determine how the student uses visualization in the learning process.

Derivative. When considering representations of the concept of the derivative in calculus, one must consider students’ representations of rate of change, limits, and general representations for the derivative. Zandieh (2000) developed a framework for exploring and analyzing student understanding of the concept of derivative. She found four major categories of representations of the derivative: (1) graphically as the slope of the tangent line to a curve or the slope of the graph under magnification, (2) verbally as instantaneous rate of change, (3) as physical speed or velocity, and (4) symbolically as the limit of the difference quotient. Representations may be a combination of the above categories and within these categories there are variations. Students may use such representations of the derivative without understanding the process underlying the object. In such cases the representation is called a “pseudo-object” (Zandieh, 2000). As students learn more and more about the derivative they are able to represent the derivative in ways that they had not previously considered.

Zandieh reasons that for a student to fully understand these different representations of derivative, the student must first reify the processes of slope (or ratio), limit, and function. Consider the symbolic representation of the derivative in terms of the process-object layers a student goes through to understand and use the limit of the difference quotient. The process of finding rise over run needs to be reified and considered as an object for the limiting process to act on the rise over run ratio. As the distance between the points through which the slope passes becomes increasingly smaller, the value of the limit, or instantaneous velocity is reached. After the limit is found through the previous process, reifying the limit then enables the student to define each value of the derivative function. The derivative function acts as a process using the limit of the difference quotient (as a reified object) to determine all derivative values in the domain of the function. Finally the derivative function itself can be treated as an object when compared to other functions (Zandieh, 2000).

I believe that a student needs to reify the slope ratio in order to take the limit of the ratio, but I believe that a student can find values of the derivative function without necessarily thinking of the limit as an object. A student can think of the limit as a process by which the derivative values are obtained.

Limits. In order to understand the definition of the derivative a student must first have an understanding of limits (Hähkiöniemi, 2006). Davis and Vinner (1986) addressed misconceptions that many beginning university mathematics students tend to exhibit. Their study supports other research regarding students' concept images (Vinner & Dreyfus, 1989) of limits: that the limit is a bound that is approached but cannot be reached (Hähkiöniemi, 2006; Tall, 1991; Williams, 1991). While this conception of limits may be sufficient for some cases, it is not true in all cases and is distinct from the formal epsilon definition of limits. Tall (1991) found that the idea of the geometric limit was not an intuitive concept for students and that perhaps

> Students have difficulties because of the language, which suggests to them that a limit is 'approached' but cannot be reached. They have difficulties with the unfinished nature of the concept, which gets close, but never seems to arrive. They have even more difficulties handling the quantifiers if the concept is defined formally (p. 110).

Perhaps classroom discourse can be misleading for students. The contexts in which students tend to discuss limits lead them to think that limits are bounds that are approached but cannot be reached. Then when students encounter the delta-epsilon notation used to formally describe a limit, they have difficulty both conceptualizing and utilizing the definition.

One reason students may form misconceptions of limits is that it takes time to build conceptions of limit and thus, limits cannot be taught or learned in a short period of time. "One cannot put anything as complex as *limit* into a single idea that can appear instantaneously in complete and mature form" (Davis & Vinner, 1986, p. 300). It is very possible that a student will have a partial concept (or even incorrect concept) of limits in given situations (Zandieh, 2006), but over time as the student continues the learning process, he or she will build other ideas and create a more accurate definition of limit. Students need to be given experiences that prepare them for concepts such as limits and the derivative quotient.

Metonymy. Zandieh's (2006) research on metonymy will play a significant role in this study; therefore, I will give a more lengthy explanation of her work. Metonymy is defined as "the substitution of the name of an object closely associated with a word for the word itself" (Harmon & Holman, 2006). A part of the object can be substituted for the object itself and this is called part-whole metonymy. A few examples are "the crown" representing the king of a country, "wheels" representing a car, and "heads" representing people. Both students and members of the mathematics community often use metonymy when speaking about derivatives. Zandieh gives many examples of ways in which students use metonymy while talking about the derivative and discusses the strengths and weaknesses of such usages. Often students will use one of the representations of the derivative, such as instantaneous velocity, to represent the whole framework of the derivative—a part-whole metonymy. Depending on the student's underlying knowledge, this may be an appropriate way of communicating, or it may be a result of the student's lack of understanding. Zandieh found that many students referred to the derivative as one of the representations (see Zandieh, 2000) but were not able to tie their chosen representation of the derivative back to any other representation. They tended to compartmentalize representations of the derivative depending on the context of a given problem or situation. Thus students had difficulty communicating if their individual concepts of derivative differed and they did not have a strong enough understanding to see that their representations were isomorphic. On the positive side, when a student could make the connection, he or she would often use one representation, such as the geometric slope of the tangent line, to understand another representation, such as the symbolic limit of the difference quotient. Being able to use different metonymic representations depending on the context can be useful to students because some contexts lend themselves to certain representations more than others.

Two common metonymic short cuts are using the phrase “the derivative” to represent both the derivative function and the derivative value (Zandieh, 2006). Although commonly used and accepted the shorter phrase “the derivative” can cause students confusion because they may be thinking of the derivative value when they should be thinking of the derivative function or vice versa.

There are some uses of metonymy that are acceptable by the mathematical community and others that are not. For example, referring to the derivative as “the slope” is appropriate and acceptable in the mathematical community while “the derivative is the tangent line” is not (Zandieh, 2006). Another example is condensing the idea that the derivative is the instantaneous rate to simply “the derivative is the rate,” is acceptable, while “the derivative is a change” is not (Zandieh, 2006). Zandieh’s students showed a high tendency to use the misstatements. However, they would often use the correct, more complete statements intertwined with their use of the misstatements, indicating that their understanding was at least partially correct. Another problem that could ensue is that students may use a correct metonymic representation, such as “derivative is the rate,” but fail to understand the underlying process. It is important to be aware of students’ use of metonymy and their underlying connections in order to analyze student understanding.

Productive disposition and self-efficacy. Productive disposition (National Research Council, 2001) and self-efficacy (Bandura, 1989) are terms used to characterize students’ attitudes and beliefs about mathematics and their capacity to learn and do mathematics. One’s attitude and belief in oneself strongly affects one’s motivation. A person with a productive disposition in mathematics would have the following characteristics:

- Seeing math as both useful and worthwhile,
- Seeing sense in mathematics,
- Believing that steady effort in learning mathematics pays off,
- Seeing oneself as an effective learner and doer of mathematics,
- Believing math is understandable not arbitrary,
- Believing that with diligent effort mathematics can be learned and that the student is capable of learning, and
- Believing that being good at math doesn't mean you have a special math gene.

When students' understanding is limited to memorized procedures their confidence in their own ability to do mathematics decreases. Therefore, a productive disposition is acquired as students are *able* to make sense of the mathematics. It is also interesting to note that those who believe that a certain "math gene" determines a student's mathematical ability tend to be more performance oriented while those who believe that their effort contributes more to mathematical ability tend to be more learning oriented.

Self-efficacy is one's perception of one's own capability (Bandura, 1989). A person's level of self-efficacy will determine how well the person deals with failure, how much persistence a person will have to continue working, even in the face of adversity. Someone who has a strong sense of efficacy will believe that he or she can control his or her own situation (Bandura, 1989). As the sense of self-efficacy increases, motivation increases.

Chapter 4 – Data Sources and Methodology

Class Setting and Participants

Data was gathered from the first semester of a task-based, student-centered, experimental honors calculus course during the winter of 2006 at a private university in the western United States. Although it was labeled as an honors course, any student could register for the course. The class was team taught by two members of the mathematics education faculty who believed that students needed to be provided with a different approach to learning calculus than the standard lecture format. Various tasks were chosen by the instructors that were meant to elicit conceptually important calculus ideas such as rates of change, velocity, acceleration, and their relation to position. The 22 students enrolled in the class worked in groups of four or five at tables around the room with minimal intervention from the instructors. Students had freedom to approach the tasks in any way they felt helpful and to justify their reasoning with one another. Presentations of student's ideas were shared with the rest of the class. The class met three times a week for 2-hour blocks.

My research focuses on Mark, one of the students in the class and his work with his group which consisted of five students: Kam (a good friend whom Mark had known in high school), Josh, Chris, and Sarah. Kam was a bright engineering major taking calculus for the fist time and concurrently enrolled in physics. Josh was an economics major who had taken calculus quite a few years previously. Chris had previously taken calculus and considered himself good at mathematics. He remembered many calculus procedures, but did not know why they worked. Sarah was also a first time calculus student and had recently changed her major to math education. Mark had taken a calculus course in high school but after taking a two year break and returning to college, he had forgotten a lot of mathematics and had enrolled in an intermediate algebra course and had subsequently taken college algebra and trigonometry. As an engineering major he was concurrently enrolled in physics and chemistry courses. Mark is of particular interest because he is a typical undergraduate calculus student: he is an engineering major with a background in mathematics and science, but is still a fairly average student.

Data Collection

Data was collected through transcripts of video data, collected student work, field notes, and interviews. Video from seven hours of class time, spanning from Wednesday to Wednesday, as well as an hour long interview, and interactions Mark had with the professors before and after class, were transcribed and all student work was collected, copied and saved for records. Students were asked to show all their work and include scrap paper, transparencies used for group presentations, and anything else that might help the instructors know what they were thinking and doing while solving a task. Due to lack of clear camera shots or hard copies of student work in progress, many of the figures in this paper have been redrawn in order for the reader to have a better view. The replica's I have included are as much like the original drawings as possible and labels have been added where students verbally provided labels.

After the semester was over I interviewed Mark. Before the interview, I gave him a task that he had not seen before (see appendix 1) to see how well he performed in order to assess his knowledge retention of the concept of derivative. When we met I asked him to explain to me how he had worked through the task and also asked him many questions relating to the work he had previously done in the class relative to the data analyzed for this paper. All background information on Mark was obtained during the interview.

The analysis for this paper will focus on the second task, the cat task (see Speiser & Walter 1997). The students had worked on the Desert Motion task (diSessa, Hammer, Sherin, & Kolpakowski, 1991; Sherin, 2000), previous to the cat task and had developed some ideas about displacement, velocity, and acceleration, but were still negotiating how these concepts affected graphical representations, their definitions, and the mathematical relationship between them. The word derivative had been used but not defined and not always used correctly. Students were still forming ideas about motion, terminology, and notation.

Analytic Methods

From the onset of the study grounded theory seemed the most reasonable method of analysis. Grounded theory allows the data to guide the analyzing process. The researcher collects data and allows the theory to emerge from the data. As data is analyzed codes are chosen, data is reviewed again and the codes become progressively concise—this method is called open coding (Strauss & Corbin, 1998). Coding becomes a reflexive process between the data and the codes themselves. In this project I used a combination of open coding and Zandieh’s framework for analyzing students’ understandings of derivatives.

I examined the video and transcript and coded for all of the instances where Mark communicated about the derivative (or relating ideas such as slope or instantaneous velocity) either to his peers, instructors, or to himself. Discourse includes verbal and bodily communications as well as written communication, so I naturally considered gestures, pauses, word choice, and voice intonation. Further questions from the end of semester interview helped solidify assertions I made regarding Mark's discourse. I chose *critical events* (Maher and Martino, 1996) that showed insight into Mark's decisions or showed Mark's way of thinking. Within the events I noticed that some of the ideas that influenced Mark were not events but recurring themes. These themes also became codes and I paid careful attention to the way these ideas wove themselves into Mark's discourse. I have focused on Mark, his comments, actions, and ideas, and only included the comments, actions, or ideas of his classmates as they directly relate to Mark.

Once I had collected all data where Mark communicated about the derivative or ideas relating to the derivative I categorized these instances in the following categories or codes:

- Connections to other classes or fields of study
- Of or relating to limits
- Of or relating to slope
    - Finding two points close together
- Derivative as slope of the tangent line
    - Derivative as the tangent line (only)
- Representations used by Mark
    - Graphs,
    - Diagrams
    - Use of visualization

    - Zandieh's derivative framework—categorizing each of Mark's representations of the derivative as one, multiple or none of Zandieh's representations
    - References or uses of notation
  - Displays of productive disposition and self-efficacy

Time codes are included to help the reader obtain a feel for the time elapsing and the amount of time it takes for Mark to move from one stage of understanding to another. The time begins at time 00:00:00 (hours: minutes: seconds) on Wednesday the 18th of January when the students are first given the cat task and continues until Wednesday the 25th of January. Time codes are given at the beginning and end of each section of transcript quoted in the text and referred to as the narrative is built.

Chapter 5 – Data Analysis

Before we begin with the analysis of the data, it is important to know relevant details about both Mark and the calculus class. Knowledge and understanding are built on previous experience, so when studying student understanding, background information is important. Mark's previous experiences in high school and other university courses, as well as work he and his group have done in class prior to the cat task, contribute to the building of his understanding.

Mark's Background Before the Calculus Class

Mark is a good student and had taken calculus in high school about five years previously. However, he had been very busy that year in high school and as he described it, he was "a very mediocre student that was more concerned with sleeping, swimming and girls than school" and slept through most of his calculus class. He learned procedures—"enough to get by"—but never really understood "what was going on." Mark had also taken physics classes in both high school and university and really enjoyed them. He was interested in knowing how and why things worked, and especially in "blowing stuff up."

In both his physics classes previous to taking this calculus course, the physics had not required the students to do anything beyond basic algebra and books had specifically stated that the equations they used were derived from calculus, but showing where they came from was "beyond the scope of the text." Mark wondered where the equations came from and how to derive them. In previous math and physics courses Mark had been learning about average velocities and average accelerations and had wondered how or if you could find instantaneous velocity or instantaneous acceleration. He had attempted to read his high school calculus text but had not understood the mathematics underlying limits. Below is Mark's own description of his history with average and instantaneous rates of change. The reader will note Mark's unanswered questions and the foundation being laid for making sense of the definition of the derivative as the average velocity taken over increasingly smaller intervals.

> Mark: Um, just basically experience with physics. Like I've taken conceptual physics before . . . Like at first the limit was a foreign concept to me . . . So, what we did deal with is we dealt a lot with average acceleration, so I had the concept in my mind of average acceleration and I also had in my mind that the closer the points got. I actually had a discussion with one of my buddies who was actually a chemistry major . . . He never actually answered my question. I don't think he actually knew the definition of the derivative . . . So I just asked him, "Well, how do you get instantaneous velocity?" Because, I mean, like, the closer and closer you get, you need an infinitesimally close point to get for the instantaneous velocity or acceleration.

Mark said he had thought about average accelerations and instantaneous velocity during previous classes. Average velocity was found by finding the slope between two points, so Mark had figured that to find instantaneous velocity one would need to find points that were infinitely close together. Since the book had been confusing, Mark asked a friend who was good at math and the friend had said it was virtually impossible to find two points infinitely close together. He had previously had such questions that had gone unanswered and was now facing them again in this class.

Immediately Prior Mathematical Activities

At the start of the analysis the class had been in session for about two weeks. In the first two weeks the class had worked on the desert motion task (diSessa, et al., 1991; Sherin, 2000). During presentations of the desert motion task, ideas emerged in students' discussions that the slope of the position graph was velocity and the slope of the velocity graph was acceleration. Mark found this so important he wrote it down and later referred to it in the data I will be discussing. The word derivative had been used by students with previous experience in calculus or physics, but the class as a whole had not defined the term. At the beginning of the analysis, Mark was still unsure what a derivative actually was.

During the cat task Mark worked with four other group members whose names will come up during the analysis: Chris, Kam, Sarah, and Josh. (Sarah is absent during approximately half of the time and her influence in group discourse is minimal.) The focus of the analysis will be on Mark, but his interaction with his group members is an important part of his development of understanding. Mark interacted the most with Kam, who he had known for years. Although Mark tended to need validation from others, and looked to others for encouragement and direction, he sought to make sense of ideas for himself before accepting them and also worked alone making sense of tasks himself before collaborating with others.

In the following analysis I will present a narrative over the eight and a half hours of collected video data. The reason I am presenting the data in a narrative is to show the complexity of the learning process. Outlining the events as they happen chronologically provides the most straightforward means for the reader to follow the complexity of how a student makes sense of the derivative, a difficult and fundamental calculus concept. Secondly, I want the reader to get a feel for the time it takes Mark to make sense of the derivative and the stages he goes through to achieve understanding. The data presented is how a fairly typical student understands the derivative in a manner different than that of traditional textbooks. Pay careful attention to how the classroom set-up affects Mark's decisions and directions and the time Mark was allowed to think through ideas.

<u>Wednesday, 18 January 2006</u>

When the group first receives the cat task, Mark takes approximately 14 minutes to make sense of the problem on his own, ignoring the conversation around him. Once he has figured out what the question is asking he declares, "Oh so between each frame is .031 seconds past. I now understand the question! Great! [*pause, then rhetorically*] She wants the instantaneous speed, right?" A couple of things are important to note here. First, Mark takes time by himself to make sense of the task. Also, knowing what he was looking for seems to give him enthusiasm about solving the problem. Once he knows what the object is, he is ready to form strategies and gather tools to help him solve the task.

Mark's first approach towards the task is a graph of the cat's displacement over time. He works steadily and meticulously on his graph, only occasionally looking up to see what others are talking about. The graph helps Mark organize the data and see general trends in the cat's motion. He is able to determine that the cat was increasing speed each frame and could tell when the cat's speed was increasing relative to other time frames.

The time it took for students to introduce the terminology "derivative" indicates that neither Mark nor any member of his group recognized immediately that the derivative is tied to instantaneous speed. After almost an hour [50:30] of working mostly independently, Mark takes an interest in the discussion around him and engages in a conversation stemming from the data points that Josh has plotted on his calculator. The transcript below shows the students' first uses of the word derivative while working on this task. Remember that students who had previous experience with calculus or physics had introduced the word derivative, but that the class as a whole had not defined the term. Interestingly, the group only mentions derivative and moves on:

50:32 Mark: Position over time. Now if I remember, the slope of the position line is velocity.
50:35 Josh: So if you take the derivative, which means the slope of that line.
50:38 Mark: There you go derivative at each point.
Josh: If you take the derivative of that line, you're going to get the velocity graph.

The conversation moves away from the derivative then about 30 seconds later Mark pipes up, "So now, how do you find the derivative of a point? Because what we're trying to find is." Then he trails off and continues to work on his graph.

I anticipated that the group would discuss possible approaches to finding the slope of the position graph at frame ten in order to find the instantaneous velocity, but instead the group's conversation moves away from the derivative and Mark turns back to his graph. The fact that Mark does not pursue this idea leads me to believe that he has not comprehended the significance of the slope of the position graph at a point and its relation to instantaneous velocity.

The above dialog between Mark and Josh is an example of when one student's perspective does not transfer directly to another. Josh shows many indications of thinking of the derivative as a separate graph *derived* from the first graph, while Mark thinks of the derivative point by point along the curve. Josh is thinking of the derivative as a "the derivative function" while Mark thinks of derivative as "the derivative value" (Zandieh 2006). While each student's view is correct, their individual understanding does not directly transfer to the other. In line 50:35 Josh uses the phrase "slope of that line." Looking at further dialog throughout the data, Josh is not using "line" to refer to a linear curve, but rather any curve, a "line graph." Mark thinks of the derivative as the slope of the tangent line, so the strong association that Mark has with "line" and "derivative" is the tangent line, which is linear.

In light of Zandieh's (2000, 2006) derivative framework and research in metonymy, Mark is looking at "parts" of the whole idea of derivative, namely, the derivative as instantaneous velocity and geometrically as the slope of the position graph at each point. Throughout the task Mark continues to demonstrate this "point-by-point" view of the derivative, which is a more process-oriented approach. It is important to note here that this point-by-point view of the derivative is applicable and meaningful in many calculus problems. In the current task the students are trying to find the speed of the cat at frame 10 and thus a point-by-point concept of the derivative is a logical approach.

Mark had taken notes during the previous task and had made a note that "the slope of a position graph is the velocity." He had felt it was very important at the time and continued to rely on this information [50:32]. Despite Mark's use of the word slope and derivative in nearly the same breath, Mark is not yet able to attempt a symbolic approach to the derivative because he has not yet reified the idea of slope in connection with the derivative. Nor has he shown any evidence of looking at average velocities over shorter and shorter intervals. In fact, he has not mentioned average velocity at all. It was not until much later that Mark reified the idea of rise over run yielding the slope and was able to work with the slope as a mathematical object connecting the instantaneous speed with average speeds by taking limits. Mark reaffirms his disconnect with slope and derivative in the interview when I was asking about his understanding of the definition of the derivative:

> Mark: At the time I don't think I even realized it was the slope. Like they gave you an equation, they gave you the slope. I don't think I even realized that … So I knew what slope was and I knew that I was looking for the slope of a line that passes through there was. But I had no idea that the slope of the line was, at that point, was the derivative. Yeah, I just didn't, like, know what a derivative is. They'd always say, find the derivative and I'm like "Okay, what's the derivative? I don't know what I'm finding."

More than once Mark clearly stated that the derivative was the slope of a point, but he did not think to work with the rise over run quotient as an object as the two points become closer and closer to one another. He seemed to have the pieces of the puzzle before him, but was not putting them together. It would be easy for a teacher or an observer to think that Mark had made connections regarding various aspects of the derivative that, in reality he had not. Even though Mark says derivative and slope of the tangent line in almost the same breath, he is not yet connecting them in the way that an expert would be connecting them.

At this point Mark’s understanding of the derivative is vague. He knows that the slope of the position graph is the velocity and the slope of the velocity graph is acceleration, but does not fully comprehend what that means. He also seems to think of the derivative as the slope of a point but does not know how to find the slope of a point. All of these questions drive him to explore and think deeply about motion and the relationship between position, velocity, and acceleration over time. The time and energy he spends working on the task now contribute to the rich understanding at which he arrives later.

After class on Wednesday, a few comments made by his instructors help prepare Mark for the notation used in the definition of the derivative as the limit of the difference quotient. Kam asks the instructors if a previous homework problem involving *f*(*b*) and *f*(*a*) was trying to find the tangent or instantaneous change.[2] The professors reply that the notation is preparing them for the notation they will use for the definition of the derivative. Mark says he is relieved because he did not make the connection to tangent lines at all. The following conversation is very important because it is where Mark connects the notation from the homework problem to the definition of derivative. We also see how important limits are for Mark.

58:22	Ins 2:	That's kinda getting you ready for notation. Um, when we start working with derivatives. It will help you with notation.
	Mark:	‘Cause if we were supposed to find the tangent line, I totally missed that.
	Kam:	No, I was just curious ‘cause I don't
	Ins 1:	But you are right. The notion of *f* of *b* minus *f* of *a* over *b* minus *a* has some real significance when you get into
	Mark:	Derivatives?
	Kam:	That's like instantaneous velocity and instantaneous
	Ins 1:	When you find the limit, yeah.
	Mark:	When you learn how to do—that's the thing like
	Ins 1:	Definition of derivative it's going to be critical to know that notation.
59:09	Mark:	Yeah, *f* of *b* minus *f* of *a* over *b* minus *a*. I remember slightly that thought . . . like I could always understand the concept of it, but I could never understand limits correctly. I never understood limits, like I would read it in textbooks and stuff and I'm like, ok, I understand it, but I don't understand how the math behind it works.

Mark repeats this notation to himself and later uses the notation *f(b)* and *f(a)* when explaining the derivative even though the part that he reads in the book uses $f(x_0)$ and *f(x)*. Mark had not previously made any connection to the notation $\frac{f(b)-f(a)}{b-a}$; however,

[2] The homework problem shows a couple of graphs whose curves pass through the points (*a, f(a)*) and (*b, f(b)*) and asks the student to identify horizontal and vertical distance between the points to obtain $\frac{f(b)-f(a)}{b-a}$.

when Instructor 1 says that it will be critical to know that notation for the definition of the derivative, Mark's memory is jogged. He had seen similar notation before, but it was not significant because as he said in the interview, he never truly understood what it meant. Now he knows that this particular notation is significant and stores this bit of knowledge away until he can make further sense of it.

Instructor 1 also mentioned limits [59:00], which become an important idea for Mark. As we see later on the following week, Mark goes home and reads about derivatives and limits, trying to figure out what the significance is of this notation and what a limit is and does.

As the conversation continues Mark explains his existing understanding of limits. In order to understand the conversation here the reader must know that while working on the Desert Motion task Mark had said something about needing to use limits and had gotten very positive feedback from the instructors. In the desert motion task Mark did not know how or why to use limits and the topic had not been pursued. Here they are referring to that time in class where Mark mentioned limits:

59:31 Ins 1: See I was really, really impressed when you last time it's like, "limits, yeah!"
Mark: I was like Oh my gosh! I get it. Part of that was because I had to write this research paper on Zeno's Paradox. Which is perfect until you take into account limits, like he has good premises.

Mark knew limits were important before, but now he is even more confident that he eventually needs to be able to be working with limits. His conception of limits is tied to his understanding of Zeno's paradox, which involves the idea of getting close but never quite there. Taking the limit is acting as if "there" is reached. This view of limits is how Mark later makes sense of what is happening to the secant lines as the distance between *a* and *b* approaches zero. To further show how Mark understood Zeno's paradox here is a clip from the explanation he gave during the interview:

> [*Mark first describes the race between Achilles and the tortoise*.] The other one was the racehorse and how you can't ever cross between point *a* and point *b* because you have to go through midpoint *c*, midpoint *d* and you always have to cut it half and half and half. And so how I think about limits is that point at the end is the point that you can never reach. That's your limit. Like the point exists, but you can't ever reach it. But we can know about it. We can know what it is and that's how I think about limits.

Mark considers the limit to be a point that the function will get close to but never reach. His concept of limits is very common for beginning undergraduate students (Davis & Vinner, 1986; Tall, 1991; Williams, 1991). Williams (1991) found that a large majority of undergraduate students believed statements defining the limit as unreachable to be true, and many of them believed that defining the limit as "a number or point the function gets close to but never reaches" to be the best way to describe limits. Again we see metonymy at play because Mark is referring to one possible aspect of a limit. However, this is not a complete part-whole representation because not all limits are approached boundaries that are never reached.

While Mark knows that limits play an important role in finding derivatives, he does not know the nature of that role yet. Nor does he understand what a limit is and does with mathematics. He is curious to find out.

<u>Friday, 20 January 2006</u>

Mark spends the first part of class working on plotting the 26 data points on his graph only occasionally commenting in response to conversations going on around him. One such comment shows Mark's focus on the tangent line as the means to finding the instantaneous speed of the cat. While Josh and Chris are discussing how to create a line of best fit on the calculator, Mark comments, “This guy is dying to get a line $y=mx+b$.” Then more to himself, “Slope? Derivative? The derivative of a point.” Mark is not talking about the same line that Josh and Chris are. While Josh and Chris are trying to find a curve to fit the points on the calculator, Mark is thinking of the tangent line he hopes to find. To Mark “line” is the linear curve, or tangent line, showing velocity while a regression line that Josh and Chris are looking for is not necessarily linear.

It is interesting that when Mark later figures out how to find the derivative he seems to find it important to find the equation for the tangent line (even though this is not necessary to find the velocity of the cat at frame 10). He seems to tie $y=mx+b$ to the slope and the derivative and thus the instantaneous velocity of the cat. Mark’s behavior is consistent with Walter and Gerson’s (2007) findings where students are looking for representations of slope and mistakenly say slope *is* $y=mx+b$. It is also consistent with Zandieh’s (2006) findings. Two-thirds of her calculus students sometimes mistakenly referred to the derivative as the tangent line (even though they also referred to the derivative as the slope of the tangent line) and a third of those students consistently made the error. Zandieh (2006) suggests that one of the reasons this may be is that the tangent line itself is “the most obvious image or endpoint of this graphical process” (p. 11)

> The slope is implicit in both graphical images, but the tangent line is explicit, visible, and thus more easily remembered. Even without the idea of a limiting process a student may remember a single image, a curve with a line tangent to it, when asked what a derivative is. (p. 11)

In such instances, the “loudest,” or most explicit, visual image is not necessarily the correct one, and for this reason relying on visualization alone could be misleading.

Chris asks to see Mark’s graph. Mark shows him the general trend—the cat speeding up. Then the following dialog takes place in which Mark clearly explains the purpose of his graph and the direction he wants to take with it [1:43:35]:

1:43:35 Chris: So what we're trying to find is that point right there [*Chris points to frame 10*]? (see Figure 5.1)

Mark: Yeah, we're going to be trying to find the derivative or the slope of this point [*sketching a tangent line*] like the point that passes straight through there. Like if we were to draw a line passing through that dot.

Kam: That would be the velocity.

1:43:55 Mark: We want to find the slope of that line and that would be the instantaneous velocity. So folks any ideas on how to do that?

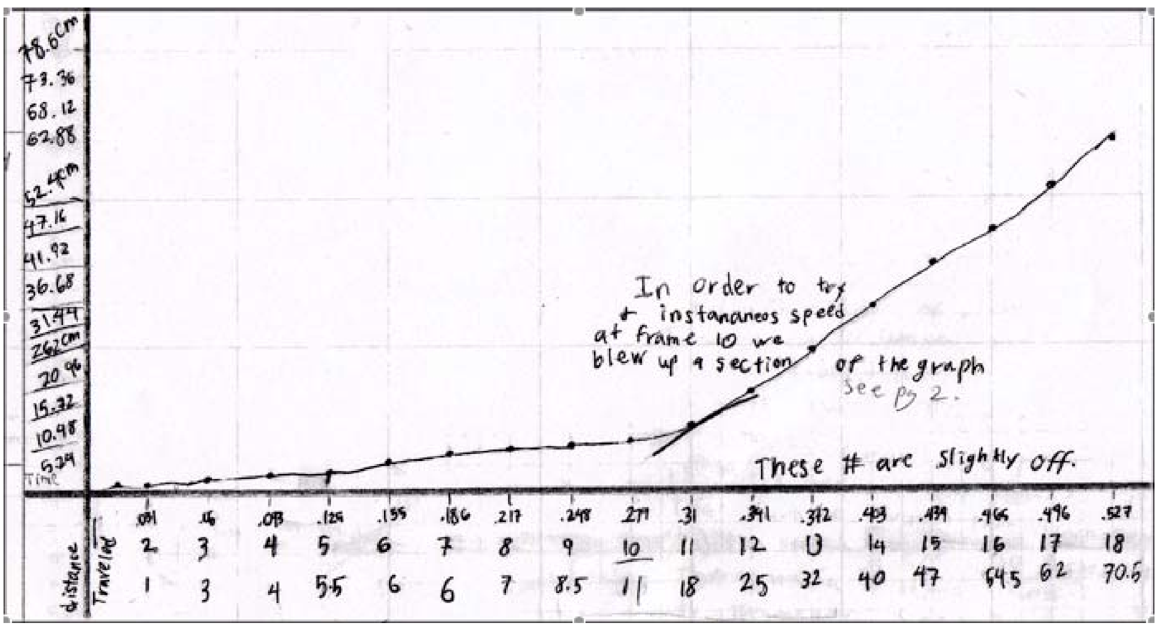

Figure 5.1: Mark’s graph with sketched tangent line

This is the second time Mark has asked the group for help. On Wednesday Mark had asked, “So now, how do you find the derivative of a point?” [51:29]. When Mark asked this first question he didn’t really expect an answer and almost asked it as much to himself as anyone. His second question shows that he is more confident and specific in how he’s going to find the derivative at the point of interest. He has made progress in his understanding in the time between the questions. He wants to find a tangent line to the curve at frame 10 knowing that would yield the instantaneous velocity. His question now is how to draw such a line accurately.

Here we also see that Mark is looking at the *slope* of the tangent line and not just the tangent line itself. This is one of many instances where Mark's correct discourse is tied to his explaining the *why* behind his choice of action.

Mark uses visual arguments and graphs to communicate ideas. Much of Mark's sense making relies on visualizing ideas. He describes himself as a visual learner. When explaining to Chris he gestured to his graph and also uses visual language, "Like if we were to draw a line passing through that dot." [1:43:40] Mark's whole argument is visual and Chris accepts Mark's strategy.

In order to find the slope Mark knows that you need two points. He also knows that to find the slope of a point, or at least a close approximation, you need to have two points very close together—or as he says, "infinitesimally close together." Finding these points and the slope between them are the focus of Mark's thinking—a key idea for Mark. He makes several comments, similar to the one below explaining why he wants to find two points infinitesimally close together while working on this task:

1:43:57	Kam:	Should we just like get [the graph] bigger. You know
	Josh:	Just keep zooming in?
	Mark:	That's the idea. You get like two points that are infinitesimally close together then you find the slope between those two infinitely close together points. That's the whole idea between limits and stuff. But I never understood a derivative to begin with, like in high school, so [*trails off*]. But we would still—if we want to blow up this section of the graph right here [data points around frame 10] you know, like we'd still need to know with our archaic way of graphing things we'd still need to know multiple points in between there and we don't have multiple points in between the frames.
	Chris:	Well, no, no we don't.
1:45:45	Mark:	That's why we started with this graph to begin with because that's what we knew where frame was, so [*trails off*]

The discrete data and Mark's idea of limits with Zeno's Paradox make it seem impossible to find two infinitely close together points, yet he knows that a derivative yields instantaneous speed so it must be possible. How do you get very close, infinitesimally close, data when the data is discrete? The graph can be an estimation of the data they do not have, but if they "zoom in" too much, their data will not be sufficient to even make a graph. As Mark pointed out somehow they need more data and they are using this graph to estimate data they do not have. He sees the possible errors that could be made while graphing and wants something more exact. However, not being able to think of any other way to do it, Mark suggests, "We could arbitrarily just draw a line and give our best estimation between it" [1:46:18]. They decide to do an estimation of a tangent line on Mark's graph, but to make the estimation "easier," or to estimate the tangent more accurately, Mark decides to "blow up" the graph like Kam suggested and just graph frames 8-12. The data was discrete so technically the only points they knew were the cat's positions at each frame. However, Mark and his group decide that the cat would probably have pretty fluid motion and so a curve passing through the points should generate a good estimation of the cat's motion between the discrete data points.

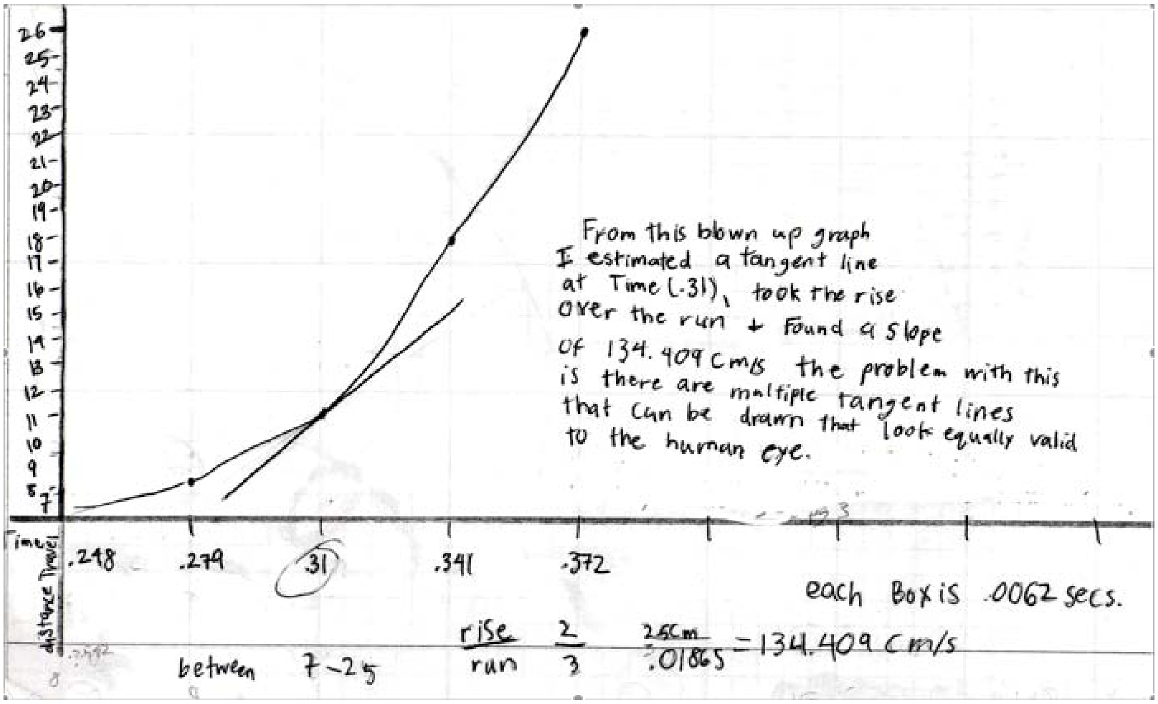

Figure 5.2 Mark's graph of the cat's motion from frames 8-12

With a carefully generated graph, Mark estimated a tangent to the curve at frame ten finding the slope of the tangent and thus an approximation for the cat's speed at that instant. Mark justifies his belief that the slope of the tangent line tells us the instantaneous speed of the cat through graphical and visual means. He thinks of the tangent line as a continuation of the instantaneous speed at that moment—as if there were a physical projectile following the graph with the given displacement over time and then some force made it continue with the same velocity at a given point in time, yielding a linear graph from there. It is through these means that Mark justifies his reasons that the slope of the tangent to the curve at frame ten would yield the cat's instantaneous velocity. We see strong indications of visual thinking and connections with physics in Mark's approaches to solving the cat task.

Mark determines the slope of the tangent line by measuring the rise over run according to the scale of his axes and estimates the slope to be 107.5 cm/sec (later he changes it to 134 cm/sec because he remeasured). This estimate falls between the estimates that Chris made at the beginning, which the group thinks is good. Mark admits it was pure estimation, and not completely satisfied he and Josh are still interested in finding something more exact.

2:23:59	Josh:	So you just kinda estimated which way the slopes would go.
	Mark:	It's totally an estimation. Like if we could plot an infinite amount of points between here then we might be able to have a perfect graph and then we might be able to, um
2:24:24	Josh:	How would you perfectly estimate the tangent line?
	Mark:	[*laughing*] Perfectly. Analyzing the tangent line.
	Josh:	How do you get it?
2:24:31	Mark:	I don’t know.

Again Mark sees the limitations of the discrete data that they have and how such data limits their estimates. He believes that an equation would help, but he still does not show any signs of knowing where to go even once an equation is obtained. Mark and his group members begin to wonder how to find the tangent line. “Perfectly estimating” the tangent line becomes an important quest for Mark. Also, we will see that Mark’s interest in finding the tangent line persists even after he has found how to find the derivative or slope of the tangent line.

All conversation regarding the tangent line is stopped when Josh announces he has an equation on his calculator [2:27:41]. Rather than first showing his group the equation, Josh shows the graphs of the plotted points, the regression line, and the average velocity graph he had made using Kam’s points, on the same axes. Mark tries to make sense of the equation asking what the variables stand for and discussing the equations and graphs with Josh. Mark’s desire to truly make sense of the mathematics is part of what leads him to the discoveries that he makes in this class.

Even with a regression equation from the calculator, Mark sees that any answer obtained with a slope from two points very close together on the regression curve is still an estimation of the cat's speed. He says, "So we might take like 39 and the next point up is like 39.00000001, but you know, you still . . . it's still an estimation" [2:35:30]. Mark also points out that drawing a tangent line to a graph by hand will also always be an estimation. Because of this variation with hand-drawn graphs, Mark feels that "like it's gonna be, it's gotta be done with numbers or mathematically in the end. Cause you can't have infinite points, you know" [2:31:40]. He is dissatisfied with any approximation throughout the process of his search for the derivative and seems to want to find a "mathematical" way of finding the slope at a point. For Mark, this "mathematical" way needs to involve limits and equations that are as accurate as possible. As he puts it, "Well, eventually you are going to get to a point where you are going to like be adding and subtracting limits. Like it's eventually going to get to that point," [1:56:11] and "It's eventually going to come down to an equation" [2:03:06]. But as yet, he does not know how to use either limits or an equation to solve the task. He is not entirely sure how a limit is used in calculus even though he knows that limits play an important role in defining the derivative. He has seen equations in physics classes and other classes he has previously taken and thinks equations need to be involved in solving the cat task.

Josh asks if you can ever get exact velocity and Mark replies that he doesn't know. Josh feels like there has to be and Mark agrees. Mark's desire to know how to find the instantaneous velocity of the cat is growing stronger and stronger. He has questions and wants to find answers. The classroom setting allows him time and freedom to explore ideas both individually and collectively. As Mark himself commented in the interview, all the struggle they had in solving the cat task made finding out about mathematical ideas such as the derivative all that more satisfying and exciting.

Josh asks how you could get exact location for points on the curve and Mark responds, “I don’t know!” He then says, “the notation of calculus is like, yeah,” [2:36:11] trailing off. Mark thinks there must be a way using the notation of calculus to get an exact answer, but he has forgotten whatever notation he had previously learned, and never really knew what it meant in the first place (see quote from interview on following page). To Mark, notation is a very important part of mathematics and while he relies on visual approaches as well as physics reasoning, he wants to be able to express these ideas in a more well-defined manner that notation makes possible. Notation is a means of communicating and expressing mathematical ideas. When notation is developed mathematical ideas can be more rigorously defined and work progresses at a faster pace.

At this point Mark and Josh’s conversation ends and they begin to listen to what Kam and Chris have been discussing. Chris used the power rule that he had previously learned in a calculus class to take the derivative of the quadratic equation that Josh found on his calculator and got that the instantaneous velocity was 134 cm/sec. Mark and Josh are interested but neither is satisfied with the strictly procedural explanation Chris gives of the power rule. In this class the students are working in contextual situations and are not satisfied with strictly procedural knowledge but seek a conceptual understanding that makes sense in context.

2:36:27 Josh: Wait. With your calculator you took the derivative?
Chris: No, I just knew it.
Josh: How do you do it?

Chris: Um, the power rule. You take this exponent and you drop it down in front. Have you had calculus before?
Josh: It's been a long time man!
Chris: Ok, so the exponent goes down and you multiply it by the cofactor and then you take *x* to the zero is 1. 1 times that is the same and then the there's no *x* so the constants go away.
Mark: Wait, could you show me what you just did?
2:36:51 Josh: But why? Why do you do that?

When Chris "explains" the power rule, or in other words, when Chris tells them the steps in the procedure used in the power rule to find the derivative, neither Mark nor Josh is satisfied with the explanation. They want to make sense of it, tie meaning to it, know where it comes from and *why* it works. They are not satisfied that Chris took calculus and remembers a rule. They want to understand that rule.

In the interview Mark talked about how he had procedural knowledge in his high school calculus class, but that he never really understood what it meant:

> I'm a high school student. I'm not going to read the math book, you know, of course not, and so like I would memorize procedures and steps. I had no idea what they meant. I would find the derivative. I didn't even know what a derivative was. I was like okay, well, I know derivative is *dx-du--dy* over *dx*, you know. Like I'd memorize the procedures, I knew enough of the procedures to get me through the class. But I didn't know what was going on, at all.

Here Mark tells us that he had memorized procedures in his high school classes and was able to get through the class. The procedures did not stick with him into college because by the time he got into this calculus class a few years later he did not remember any of the procedures and very little of the notation. In fact, seeing procedures or notation did not even trigger a memory of them. It is also interesting to see that despite Mark's ability to perform the procedures sufficiently to get through class, he says he had no understanding as to what a derivative was or what it meant. In this class the group members want to know *why* and *how* things work. They are making sense of the mathematics. Chris' procedural explanation does not help them make sense of the problem or arrive closer to a solution they are confident with, so they reject it.

Their conversation is again cut when the instructors ask the group to present their work in progress. Mark's group decides to put the range of 80.6 cm/sec to 153 cm/sec and a happy medium of the estimates: 120 cm/sec. They compare their answers to the other groups' answers, but are still not sure. At the end of class Mark shows that he still has unanswered questions:

2:51:47	Mark:	Are we totally off?
	Chris:	No, I think we're really close actually.
	Josh:	[*quietly*] close to the wrong answer.
2:52:05	Mark:	AAhhhh… My head hurts.

Class ends and they pick up again on Monday. This shows the frustration the students are feeling and the lack of closure they feel currently with the task. The lack of closure the students are experiencing makes it all the more important to find a solution they are satisfied with. It also lets them feel like they are in charge of finding a solution instead of having one handed to them—thus augmenting their ability to exercise their agency in problem solving.

Monday, 23 January 2006

Kam comes to class and says that he had an epiphany the night before and has the answer. Kam's approach to the problem does not include any of the aspects that Mark has previously been thinking about: slope of a point, tangents, derivatives, limits, etc., but rather considers that the cat's average velocity remains the same for the next 3 or 4 frames so the cat must have reached that average velocity (roughly 225cm/sec) before frame 10. Mark takes quite a while to understand what Kam means and is finally fairly convinced by Kam's visual presentation.

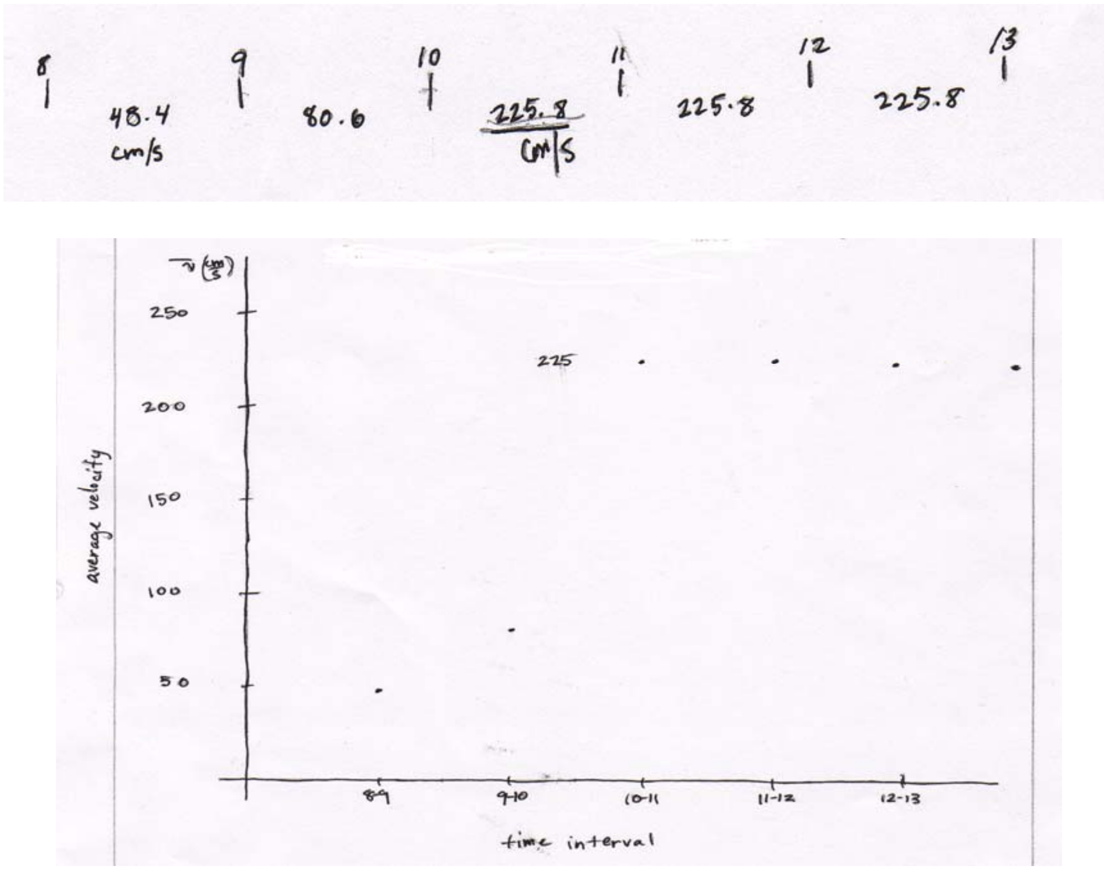

Figure 5.3: Kam’s table and graph of the cat’s average velocities

Kam explains, using a graph, how the cat would have to reach a speed higher than 225 cm/sec between frames 10 and 11 if the initial speed were lower than 225 cm/sec for the average to come out to 225 cm/sec. When Mark starts to believe Kam’s reasoning he says, “Why didn't we just see that before?! I can't believe we didn't just check that. Like it's there. Like, you know, I just checked all your numbers and it all worked. So, this graph is way off!” Mark’s graphical solution had been an estimate but at least they thought it had been a close estimate. Now that Kam’s solution seems to have reasonable backing, Mark is surprised that their answers are so different. It is important to note here that given the discreteness of the data, it is not possible to find the instantaneous velocity, only to make reasonable estimates. All the estimates they have come up with are reasonable. This instance again shows how Mark communicates visually. It is not until Kam shows how the velocity graph would have to change that Mark accepts his solution.

Even though Mark seems to be fairly convinced that Kam solution is reasonable, he still seems to think there must be a different way to solve the task—a way involving calculus. The following conversation also takes place in which we see that Mark believes they still have not used calculus:

3:20:02	Mark:	I still can't believe we overcomplicated that thing so much.
	Josh:	It was fun.
	Kam:	We learned some calculus [*smiling*].
3:20:12	Mark:	We still haven't learned any calculus yet. We're still just, a, we haven't even gotten to limits yet. So how else could we get any closer?
	Kam:	I don't think we can.
	Mark:	You'd have to get an equation of some sort.
	Kam:	Which I don't think is even.
	Mark:	You don't think it's possible?
	Kam:	I don't think so.
	Mark:	The calculator could do it couldn't it?
3:20:40	Kam:	Because it would still be guessing.

Remembering vaguely from high school and also that Instructor 1 had emphasized limits in relation to the definition of the derivative, Mark thinks that they have not done any calculus yet because they have “not even gotten to limits” [3:20:12]—a concept that is introduced early on in most calculus classes. Because he does not feel that they have used calculus he wonders if they can get a closer answer [3:20:12]. They are working on this task in a calculus class so it seems reasonable that “calculus” should be used to solve the task. Mark thinks limits, equations, and calculus notation need to be involved somehow. We will see how knowing how to find a derivative symbolically helps Mark make sense of limits in calculus, instantaneous velocity, and the slope of a point, and finally allows Mark to feel satisfied because the symbolic definition uses notation, limits, and equations.

Josh redoes his regression equation and uses the calculator's derivative function at frame 10 and gets 137cm/sec—very close to the number Mark got from his graph. Mark is pleasantly surprised but wonders if there could be human error in the graph or in entering information into the calculator. He also wonders if 137 cm/sec could be right, but referring to Kam's reasoning he asks, "But does that mean that at some point she raises her speed above 225 for an instant, do you think?" [3:37:29] Now Mark is not sure what to believe. The coincidence of coming up with the same numbers on the calculator and graph makes it seem like it might not just be coincidence and yet Kam's reasoning about the average velocities had made sense to Mark. Mark says, "it made me feel kinda special" when Josh had gotten the same answer with the calculator that Mark had gotten with the graph. Not only did getting the same answer make Mark feel good, it spurred him to more questions—is there some other way? Were they missing something?

Sarah asks why there are different answers. In answer to her question, Mark explains how he thinks that variation both of the cat's movement and in plotting the data (human error) could cause the differences in the answers. Josh thinks that they were wrong, that the estimate of 137 cm/sec that they got from both the calculator and the graph were wrong and thinks that it could be the residuals that are accounting for the error. For now Mark says he's okay with them being wrong. His questions spur him to seek other options to solving the task.

By an hour and a half into class Mark has started to draw curves (see Figure 5.4) on his paper with secant lines going through the curves.

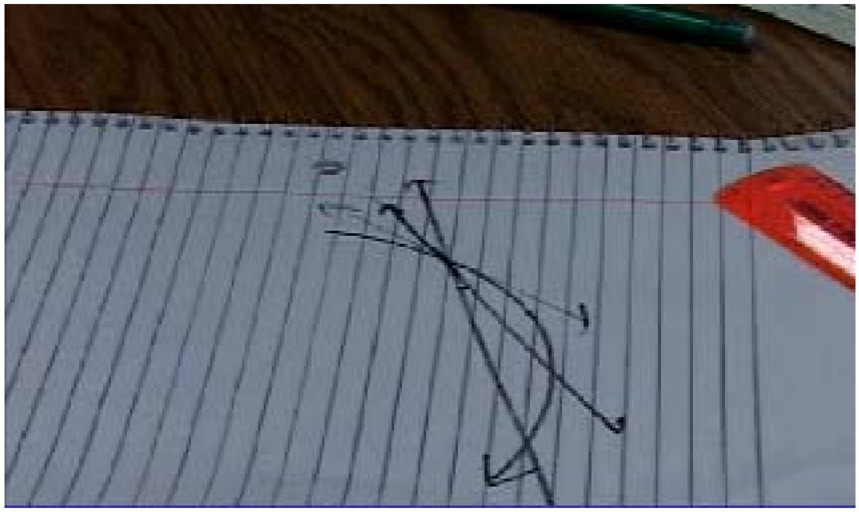

Figure 5.4: Mark's sketches on paper

He shows Kam and they speak in low voices the camera can't pick up. When the camera tries to focus in on them they say it is irrelevant. Mark continues to work with these little diagrams for the remainder of the class. By the end of the class he has spent the last half hour playing with his ideas and is frustrated. Chris asks him what he is frustrated about and he says, "I'm just trying to see—I had an idea. I was following the idea, but I think it's a dead end." He tells them he was playing with some ideas involving trigonometry because he is good at trigonometry, but they weren't going anywhere. He still wants to find something more accurate than they had before:

5:00:13 Mark: I want to know if we can get something more accurate. That's my question—if we can get anything more accurate than what we have. Can it be done? If it can, I'm going to keep working at it.

After class he talks to Instructor 1 hoping to get some direction. It is difficult to find concise enough transcript to explain Mark's ideas so I will summarize for the reader to understand then will give transcript using Mark's own words.

Mark drew the diagram in Figure 4.3. His idea was to use the data points they were given around frame 10, labeled points *a*, *b*, and *c*, and use the relationships such as law of sines or cosines with the triangles and angles formed by the secant lines in order to find the tangent line. He was hoping that he could find a trigonometric relationship to determine the angle in which to draw the tangent.

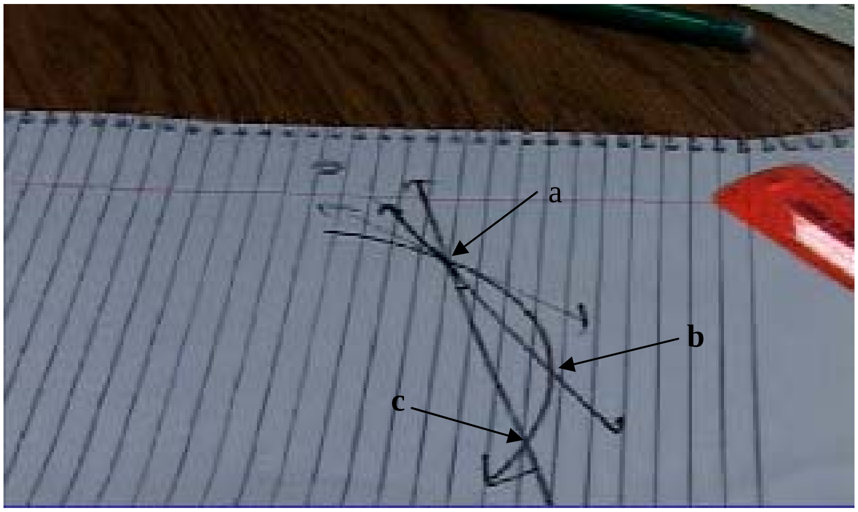

Figure 5.5: Mark's secant lines

Mark's idea and questions of his idea are given in his following explanation to Instructor 1 and Kam, which take place after class on Monday:

5:03:32	Mark:	But if we were to like draw a line between there [*between intersections a an c*] and draw a line between there [*between intersections a and b*] and then take the triangle and then take the angles between them. I'm wondering if there's a relationship between that and a tangent line that passes through that single point [*intersection a*] right here.
	Ins 1:	umm.
	Kam:	Oh, I get what you're asking.
5:03:50	Mark:	You see, that's kinda, like, I don't know, yet. That's why I'm asking, like, well, is there? Like cause we know points. Like is there a relationship between those angles and those points and a tangent curve?

Mark has taken trigonometry and was thinking about similar triangles, angles and relationships between tangents, secants and angles. He is hoping to find a way to predict the exact location of the tangent line using secants from the given data (which is discrete), angles formed between them and a tangent line at the same point by perhaps using the law of sines or cosines. He has not figured out a way to do that, but wants to know if this has been done before and if he is going in a good direction. It is interesting to see how Mark is thinking about the data he has. Since the data is discrete, the traditional calculus approach of secant lines getting closer and closer to the tangent line until when taking the limit they essentially become the tangent line, will not be possible given that they do not have a continues curve of motion. However, just having finished a class in which many of the solutions to problems could be found in relationships between angles in triangles, his approach seems logical. What a wonderful thing to have a student develop ideas like this! He is creatively thinking and exploring, building on his experience and trying to solve the task.

The following transcript helps us see that Mark really does know what he is looking for and has a basis for the direction he wants to take. Here he explains the "why" he is trying to do what he is with the triangles and angles.

5:04:05	Ins 1:	That's a really good question. And, why are you trying to find the tangent?
	Mark:	Well, because that would give you, 'cause we know that the slope of the position graph is the velocity and that was the question asked, to find the instantaneous velocity of—at this point—at frame 10.
	Ins 1:	And so what does that have to do with the tangent?
5:04:38	Mark:	Well, because, if this, if it were to have, if this curve were to have a constant velocity, it would follow that line. It would be that slope.

Mark knows that the slope of the position graph yields the instantaneous velocity. He thinks of the tangent line as a continuation of movement with the instantaneous velocity from the point at which it stems. Mark still confuses the tangent line itself with the slope subsequent to this conversation even though he says slope in this instance, again a common slip that many calculus students make (Walter & Gerson, 2007; Zandieh, 2006). However, whenever he gives *reasoning* for what he is doing, he always refers to wanting to find the *slope* of the tangent line and not just the tangent line in and of itself.

Knowing what he is looking for is an important part of Mark's making sense of the derivative, and this is a great example of Mark knowing what he is looking for and why.

As Mark, Kam and Instructor 1 continue to discuss the importance of the tangent line, Kam asks Instructor 1 about the homework problem (see footnote 1) and wondering about distances approaching zero. Mark says, "Yeah! That's why I was playing with. I was trying to figure out how to approach zero." This comment is important because later he makes sense of the limit as *h* approaches zero and even uses the notation from the homework problem. Mark is not necessarily thinking of the horizontal distances going to zero but he knows that he wants to find two points infinitesimally close together.

At the end of their discussion, Mark asks:

5:06:15 Mark: But am I onto something here?
Ins 1: You might be.
Mark: I might be.
5:06:22 Ins 1: You might be. I think you should think about it a little bit more. I think that would be really interesting and um, one thing that I'm finding in our conversation here is that you are finding a reason to do the mathematics rather than being told here is the math that you need to know and now practice it.
5:07:54 Mark: Okay, well I'm going to keep playing with that and see if I can articulate my ideas.

Mark trusts the teacher to validate his thinking. Since he hasn't entirely thought this through he wants to know whether it is worth his time to go on and presumes the instructor can tell him if he is going in a "correct" direction or not. He is not sure what direction to take, so he is also hoping for some guidance. Instructor 1 tells him, "You might be. I think you should think about it a little bit more. I think that would be really interesting."

Duly inspired, Mark goes home and reads up on limits and derivatives. When I asked him what led him to reading the texts he responded, "I get questions in my head and I start reading something." He pulled from both the university's calculus text and another text he had used in calculus in high school until he had made sense of the derivative symbolically. He had previously tried to read the section on limits and had gotten frustrated because he could not understand it. However, this time he was prepared for the reading because the ideas and diagrams that he had drawn and thought about helped him understand what had previously been difficult parts of the textbook. What he read used a different approach to finding the tangent line. The approach taken by the books is a traditional approach of showing secants that come closer and closer to the tangent line shrinking the horizontal distance between the intersections of the secant line then taking the limit as that distance goes to zero. Mark's diagram had involved secants and a tangent line and finding a relationship between them so although Mark's approach wasn't the very same as the book's some of the underlying ideas of using secants to approximate the tangent line were similar.

<u>Wednesday 25 Jan 2006</u>

Eager to show his teachers what he had learned, Mark arrives in the classroom a half-hour early to tell his professors. When they ask him to explain what he had learned he was able to do so, but struggled through parts of his explanations demonstrating that he was not just remembering, he was *reconstructing* what he had read in the texts. His work further illuminates his advancing understanding of the derivative as the limit of the difference quotient. I will present the highlights of his presentation to his teachers including episodes that show his struggles and those that show his understanding.

As he begins to explain, Mark conceptualizes the limit in two different ways: asymptotically, and as a dynamic process (Williams, 1991), retaining the idea of getting closer and closer but not quite there and then taking the limit allowing it to be "there." First, he compares the limit of the secant lines and the tangent line to the asymptote of a hyperbola, "It's kinda like the hyperbola. It'll constantly get closer and closer to zero but never actually touch zero. That's what I understand" [5:10:04]. Mark's conception of the limit also includes a dynamic movement of the intersections of the secant lines along the curve towards the point of tangency. Mark points out that even when the secant line is really close to the tangent if you go out far enough there will actually be a great difference between them. In order to get around this problem, the secant line has to be infinitely close to the tangent line:

5:11:14 Mark: But, so you can't actually ever approach that. So what they did in the formula is they arranged it in such a way that um, basically it takes into account that you almost, that you come close to infinity and then you possibly find the tangent.

This process allows him to find two points that are infinitely close together and the slope between them—just like he had hoped to do. The formula—the limit of the difference quotient—was the representation that finally allowed him to do this.

Mark used the white board to explain his understanding of the difference quotient with the instructors. Because of the glare on the white board I have recreated the diagrams that Mark drew for the reader to have a visual.

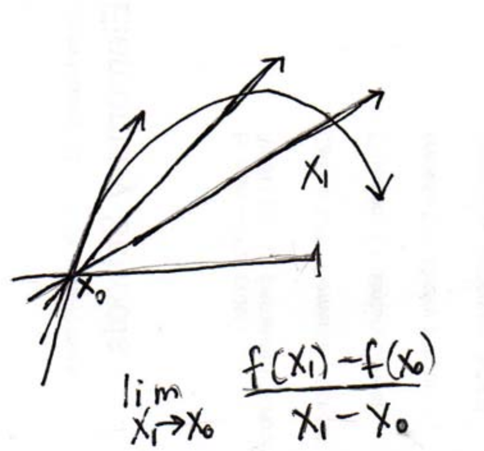

Figure 5.6: Mark's diagram showing secants and a tangent line to a curve

Mark begins using x_0 and x_1 to represent two points on the graph and writes $\frac{f(x_1)-f(x_0)}{x_1-x_0}$ on the board. It takes him about a minute and he talks to himself as he figures out what he wants to write.

Instructor 1 asks Mark what the quotient reminds him of and he says it is like the homework problem (see footnote 1). Earlier, as you may recall, Instructor 1 had told Mark that $\frac{f(b)-f(a)}{b-a}$ was important notation in the definition of the derivative. Mark is connecting the homework problem to the notation that he had seen in the book. Instructor 1 asks, "So back at your graph, what's [the difference quotient] helping us find?" and Mark responded, "That's helping us find the slope of the tangent line at the end." Mark knows that he is looking for slope. Mark and his group knew how to find the slopes of secant lines and had used secant lines in the cat task, but they did not know how to find the tangent line or the slope of the tangent line. Now Mark had a way to find what he was looking for—and just as he had anticipated, limits played a major role.

Mark then excitedly turns to an example that the book used, “They did it really cool. Like the problem they did. They did x^2 . . .” Mark begins to explain how they used “h” to indicate the distance between x_0 and x_1, and changed the notation to x and $x - h$, which he later corrects to $x + h$. He makes quite a few mistakes determining pluses or minuses and what went where. Part of his problem stemmed from the fact that he had drawn another diagram with a and b (see Figure 5.7), shown below, and referred to this diagram as he created $\lim_{x \to 0} \frac{f(x-h) - f(x)}{x-h}$.

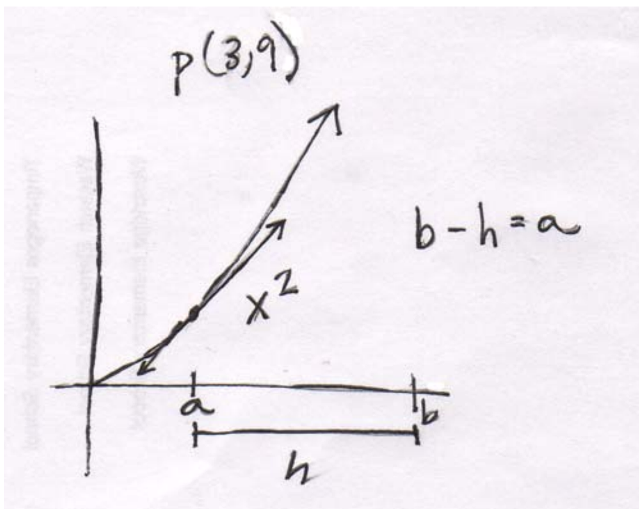

Figure 5.7 Mark’s diagram using *a* and *b*

Instructor 1 helps him correct his mistakes by asking him where each of the terms comes from allowing Mark time to think through what he is doing. She does not jump in and fix it but sits back while Mark thinks about what the notation means and where it is coming from.

Mark’s discourse indicates that he knows he is trying to find rise over run. When he carefully considers each part of the expression in terms of rise and run, he is able to correct the difference quotient and writes $\frac{f(x+h)-f(x)}{x+h-x}$. The process of correcting his mistakes took a little under 15 minutes. In this process, Mark was building and solidifying ideas.

Still needing to correct the limit (he had had the limit as *x* approaches zero), Mark was guided by Instructor 1. Again we are able to see that when Mark thinks about *why* and *what* he is doing he is able to make sense of it and correct his mistakes.

5:26:23 Ins 1: Okay, now I'm going to erase this notation [*erases as x goes to zero*] for just a minute and I want you to think about, um, about what you understood from the book and what you're trying to do here and what the notation should look like. What is it you're trying to do generally?
Mark: I'm trying to find the tangent line at point (3,9), that's what we decided.
Ins 1: Okay, when you say tangent line you mean you're trying to find the slope of the tangent line.
Mark: Yeah, I'm trying to find the slope of the tangent line is what I'm trying to find.
Ins 1: And why is that?
Mark: Because that would give us the instantaneous velocity or instantaneous rate of change.
Ins 1: Really? [*In an agreeing sort of way*]
Mark: Yes.
Ins 1: Okay, so what
Mark: Well, what I'm trying to is *h*.
Ins 1: Oh, so then what should your notation say?
Mark: Oh, as *h* approaches zero.
Ins 1: Really?
5:27:18 Mark: Yeah. So yeah, that's not as *x* approaches zero.

He readily corrects the limit to be the limit as *h* approaches zero instead of *x* approaching zero. Instructor 1 helps him by asking him to think of the big picture and what he is really looking for. He realizes that what he wants to get smaller is the value of *h* because that draws his two points closer and closer together.

Mark successfully plugged $x + h$ into x^2 and rewrote the difference quotient. Correctly interpreting $f(x + h)$ in a given function is something many calculus students struggle with, but Mark showed facility with the algebra and function notation. He wanted evaluate the expression at $x = 3$ right away, but Instructor 1 asked him to wait to do that at the end. Once the quotient was written it was reduced to $2x + h$.

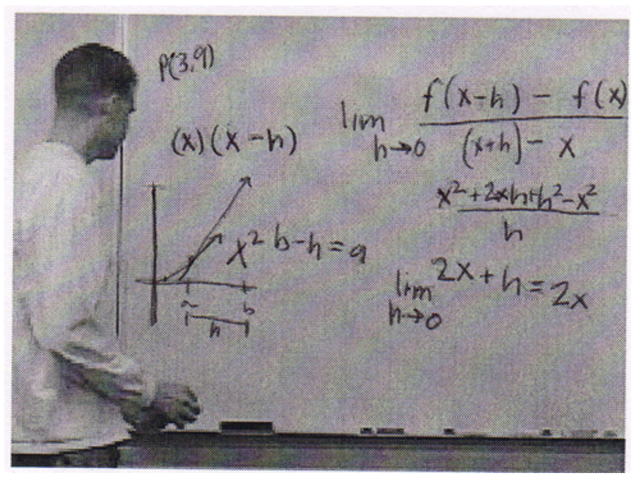

Figure 5.8: Mark at the board

5:33:32 Mark: And, they didn't give a very good proof of this, but it makes sense to me in my head. As *h* approaches zero, so then they just said that it equals *2x* and that was the derivative of *x* squared. [*Proudly showing off his work*]

Ins 1: Cool.

Mark: Yeah I was pretty excited when I. I was like, "Oh, it worked!"

Ins 1: Why, why could they say that *2x+h* is just *2x*, do you think? You said they didn't do a very good proof of it.

5:34:12 Mark: Well, in my mind, um, because it works in my mind because of this over here. [*Indicating to his original diagram with secants and the tangent line*.] That's how it works in my mind. As *x* gets smaller, as *h* gets smaller and smaller and smaller and smaller they say that zero is the limit for *h*. And it's almost as if we've actually reached the limit. Now that doesn't really actually happen in mathematics, usually. I mean like if you divide something, divide something, divide something, divide something—it doesn't ever happen. But in this instance we're almost like assuming, ok, well, yeah, now *h* is basically zero and it's the same number as *x*, sort-of. No, since *h* is zero, [*motioning towards board*] yeah. It's almost as if we reached zero. That's the way I see it.

It is interesting to see here that Mark doesn't think this "really actually happen[s] in mathematics, usually" [5:34:12]. He seems to be okay with "assuming [that] h is basically zero," when he had previously been dissatisfied with any sort of estimation. Perhaps it was the notation that made it seem more official for Mark or the fact that he read it in a book, or just the fact that it was closer than the estimation that he had come up with because h was getting infinitely close to zero—or a combination of all three that make Mark readily accept this leap.

Throughout the task Mark had continually looked for two points infinitesimally close together and thinking of taking the slope between those two points. He again shows this train of thought when Instructor 1 asks the following question:

5:35:01 Ins 1: So at the end of all that you have *2x*. But why couldn't you have let h equal zero right from the get go?
Mark: Aaahh, because, [*long pause*] cause then you're not dealing with two points. You're dealing with one point and you're assuming...and you can't really define slope off of just one point. That's kinda what I understand.

Mark's approach is a graphical/geometric, or visual approach more than an algebraic approach of not being able to divide by zero.

At 5:36:30 Mark plugs in three and gets that the slope of the tangent to the curve x^2 at the point (3,9) is six. He also gets excited about writing the equation for the tangent line and proceeds to do so, but again this takes him some time to figure out—roughly two and a half minutes with Instructor 1's questions. A little later, after Mark had wanted to find the equation for the tangent for every example they talked about, Instructor 1 asks, "Why do you care about the equation of the tangent line if you already know what the rate of change is?" [5:40:20], and Mark says, "Another piece of knowledge to have. [*laughs*] I don't know." He had been so focused on the question of how to find the exact tangent line that he continued to do so instead of simply finding the slope of the tangent line. Mark had really wondered how to find the tangent line. He had played around with the ideas of angles and triangles, but hadn't known where to go with that either. Now he understood a way to find the tangent line, answering the question he had been asking himself for a few days now. This is likely why he continually wants to find the equation for the tangent line as well as the slope of the tangent line.

Fun, exciting, and invigorating are not words commonly used by students to describe learning about the definition of the derivative. Mark even said that he didn't get his chemistry homework done because he was looking at this—a rather significant disparity from the observed norm of undergraduate calculus students. He shows both productive disposition and high levels of self-efficacy in his search for making sense of the derivative. His enthusiasm is remarkable.

After the presentations Mark has a chance to explain what he found to his group members. When he explained it to them he did not make any of the mistakes he made while explaining it to the instructors earlier and was able to say why you do each step and where all the notation came from. When he drew his diagrams for his group he is careful to label important points along the x- and y-axes and his explanation of slope flows easily from his diagram. Mark's explanation to his group members was much more concise than his explanation to his professors. When group members asked Mark questions, Mark was not flustered, but he was able to clearly answer questions.

Mark retained his understanding of derivative. In my interview with Mark at the end of the semester he was able to correctly answer a story problem involving instantaneous speed and fully explain why and how he came to the conclusion he did. He had no difficulty with the task even though he had never seen it before and was able to clearly explain why the answer he got was correct, thus demonstrating retention of calculus concepts and problem solving skills.

I also asked Mark to reexplain the definition of derivative as if he were explaining it to a struggling calculus student. He remarked that I already had it on tape three times but I told him I was asking him to see if he retained his knowledge. Mark was able to recreate his interpretation of the definition of the derivative correctly after determining what the axes represented showing retention of what he had learned. He did not falter in any of his explanations and explained it as clearly to me as he had to his classmates on the 25th of January several months prior.

Chapter 6 – Discussion and Implications

The research in this study focuses on one calculus student's development of understanding of the derivative in a task-based, student-centered classroom answering the following research question:

- How does Mark make sense of the concept of derivative while working on the cat task?
- What are ways that the classroom structure influenced Mark's learning?

The findings in this study contribute new details about how calculus students might solve tasks, develop strategies, and communicate with each other (Thompson, 1994). Based on the data presented in chapter five, arriving at a sound understanding required time for sustained inquiry and multiple degrees of partial understanding weaving together to arrive at the culminating understanding.

Analysis in chapter five provided a detailed account of how Mark made sense of the concept of the derivative while working on the cat task. In this chapter I will summarize the analysis and further connect it to the theoretical perspective and literature review to answer the above research questions. I will address the first question by reviewing the analysis in light of the river analogy given in chapter two, discussing Mark's use of physics, Mark as a visual learner, the representations he used to make sense of the derivative in light of Zandieh's framework, and his conceptions of the limit over time. Secondly, I will address the influence of the classroom setting on his learning and his display of productive disposition and self-efficacy.

The Development of Mark's Understanding

Back to the river analogy. Before even receiving the cat task, Mark had already formed some ideas that guided his thought processes. In a sense he had already begun his journey down the river of understanding the derivative with the following ideas churning in the water:

- Mark had learned about average velocity in previous math and physics classes,
- He had wondered how you find instantaneous velocity but did not know how,
- He thought finding two points infinitesimally close together would help, and
- His research on Zeno's Paradox laid the foundation for his understanding of limits.

These ideas stemmed from experience in previous classes—mainly physics courses. The second and third ideas were still questions in Mark's mind that he faced again as he tried to find the cat's speed at frame 10. Because he was already curious about how to find instantaneous velocity and had thought that finding two points infinitesimally close together could help determine instantaneous velocity, these became key ideas throughout the cat task.

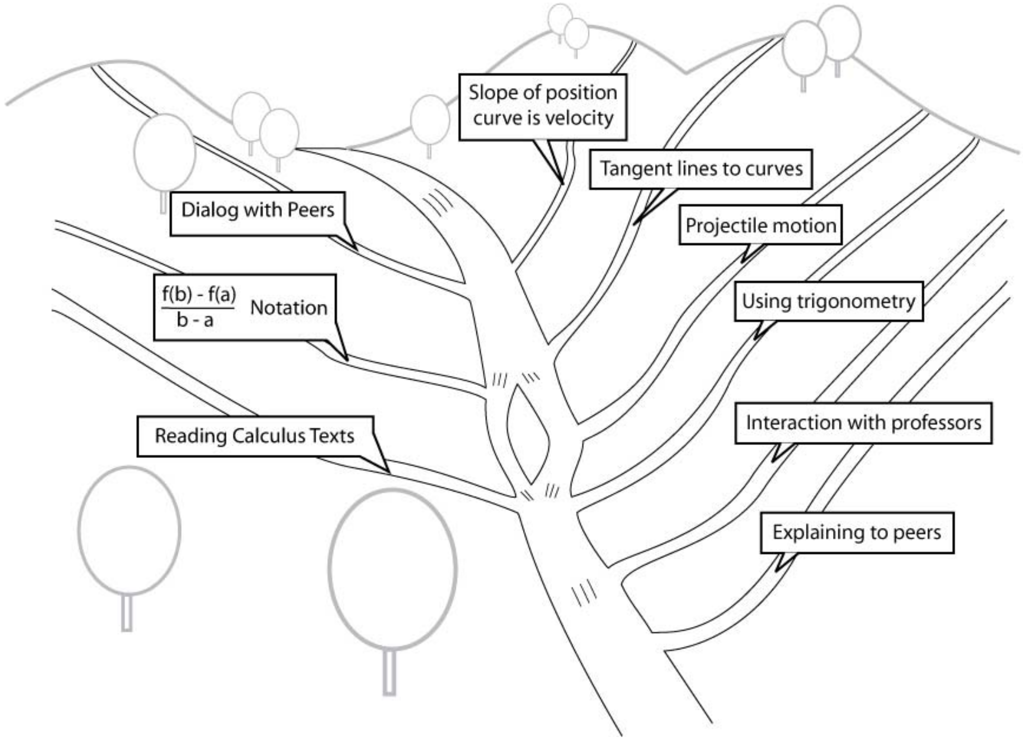

Figure 6.1 River Analogy Revisited

As he traveled down his river of sense making (see Figure 6.1), tributaries such as seeing the slope of the tangent line to the position curve as instantaneous velocity, developing important notation, trying to approximate the tangent line graphically and using trigonometry, reading the calculus books, and responding to comments and questions from instructors and peers were added to his "river" of knowledge and understanding. Entrances of the tributaries were critical events (Maher & Martino, 1996) in Mark's learning. Sometimes some of these ideas, such as the notation $\frac{f(b)-f(a)}{b-a}$, would be set aside like a fork in the river, only to meet up again later when Mark found a way to connect them more firmly to the understanding he was developing. Certain ideas such as using trigonometry to find the tangent line were discarded after Mark read calculus textbooks, but parts of the idea such as the way in which the graph was drawn remained in the river. Those that stayed were key ideas that Mark had built up heuristically and which he used to communicate his understanding (Raman, 2003).

Mark's understanding of the derivative progressed over time. Many ideas contributed to Mark's understanding—major ideas are shown in the river diagram. Questions brought up by peers, and by Mark himself, also spurred and directed Mark's sense making. His preparatory experiences, or the ideas and events contributing were vital for his subsequent level of understanding.

Use of physics. Mark built upon context-based, intuitive understandings regarding the derivative before he made sense of the formal definition with its notation and procedures. Like the students in Marrongelle's (2004) and Schnepp and Nemirovsky's (2001) studies, Mark used his understanding of physics to help him solve calculus tasks. He thought of the position, velocity, and acceleration of projectiles and how they could relate and help him understand how to use the position and velocity of the cat in the cat task. His representation of the tangent line as a continuation of constant velocity stemming from a point on the position curve [5:04:38] is related to ideas in physics.

Visualization. Mark relied heavily on visual exploration of ideas to make sense of the derivative. While the preference to communicate, justify, and think through ideas visually is Mark's preference, we can draw inferences of the positive affect visual thinking can have on any calculus student. Mark's ability to visualize helped his conceptual understanding. He was able to solve tasks, reconstruct the limit of the difference quotient, and deepen his understanding with the use of graphs and diagrams and other visual images as well as visual language when communicating with his peers or professors.

Mark often used graphs or diagrams (either diagrams he himself constructed or that others had constructed) to make sense of calculus. Modeling and representing situations with graphs and diagrams was key to Mark in both the cat task and in the Betty Kant task as well as other problems throughout the calculus semester. Mark's visual arguments helped him relate his ideas to other subject matter, determine the reasonability of his solutions and make sense of the data, and we can presume that other students would benefit from similar uses of visualization.

Along with graphs and diagrams, another example of Mark's strong use of visualization was the key idea that Mark had running throughout the task: finding two points infinitesimally close together. When he read about the definition of the derivative he saw the points, described in the text as x and x_1, as "moving" closer together. In this way the tangent being the limit of the shrinking secant lines seems to make sense. It is the shrinking of time intervals that leads to the instantaneous velocity at a given point. When asked why *h* can't be zero to start with he says, "Because then you're not dealing with 2 points" [5:35:01]. He reasoned visually and geometrically instead of giving the algebraic answer of not being able to divide by zero. When Mark explains that the limit of the secant line is the tangent, he understands why you have to go to infinity because he can visually see the problems if you don't: "If you have two points. This is just my thinking through it. Like, if you had 2 points even though this might be so minute, if you go out far enough [*motioning with hands*] that distance will still be [*hands spread out*]. Does that make sense?" He explains that even though the distance between the secant and the tangent right near the point of tangency could be very minute, if the lines are drawn out far enough the distance between them will become quite large. Here is where he sees the need of *h* going to zero so that the secant and tangent can be “infinitely close together.”

My final example of Mark’s visual thinking is his diagram of curves and triangles, trying to find the tangent line by using trigonometry (see Figure 5.5). The visual approach Mark took in trying to find how to accurately predict the angle at which to draw the secant line prepared him to understand his reading when he attempted to read the sections on limits and derivatives in his calculus books. Once he had visualized the ideas, the explanations were made clear.

Mark’s representations of the derivative in light of Zandieh’s framework. Mark’s representations can be analyzed in light of Zandieh’s derivative framework (2000). Mark’s first representation of the derivative, the graph of the cat’s motion over time, was a combination of the first and third of Zandieh’s representations: (1) graphically as the slope of the tangent line to a curve and (3) as physical speed or velocity. This representation evolved for days before he created his second representation, which was Zandieh’s fourth representation: (4) symbolically as the limit of the difference quotient.

Let’s take a closer look at Mark’s first representation. From the desert motion task previous to the cat task, he knew that the slope of the position graph was velocity and this idea became the basis for his work with the graph in finding the tangent to the graph of the cat’s position at frame 10. Mark’s conception of the tangent line—a continuation of the curve at the velocity of a single point [5:04:38], tightly connected the geometric tangent line representation with the physical speed or velocity. Since Mark was working in a context in which the rate of change was the velocity, he only referred to the instantaneous velocity of the cat and relied heavily on his physics knowledge. Thus the first and third representations presented in Zandieh's framework were combined in one and the second representation was included in the physical velocity as instantaneous velocity.

While this was Mark's first representation, the connection between slope and derivative did not come immediately. In fact, it took almost an hour into the project to even bring up the word derivative, and a few days later to really determine the meaning of the word and its relation to instantaneous velocity. The representation took time to build and refine. During the process, the group discussed finding the tangent line and questions arose about how to accurately determine the tangent to a curve. This led to a focus on the explicit image of the tangent line rather than the implicit slope of the tangent line—an incorrect metonymy for the term derivative (Zandieh, 2006).

Another metonymic issue that arose was Mark had a point-by-point conception of the derivative. In other words he considered the derivative to be the instantaneous velocity, or slope of the tangent line, at any given point, which is thinking of the whole concept of derivative by just part of it: the derivative value at a point. Until Mark explained what he had learned from reading the text to both his professors and classmates, he did not show signs of having a conception of the derivative as a continuous function which could be graphed separately from the original function. In the analysis we saw how even though Josh seemed to think of the derivative as the "derivative function" or "derivative graph", this concept image did not transfer directly to Mark or vise versa [50:32]. Each did not see the other's point of view for some time.

Mark's final representation of the derivative was the symbolic definition, which in order to understand, he had to reify the concept of a ratio that generates slope and use limits. As Mark stated in the interview he had not thought of the slope as derivative. Mark thought the instantaneous velocity could be found from the slope of the tangent line [50:32, 50:38], but did not know how to find the tangent until he was able to reify slope. In other words, he could find the slope of a line through the process of taking rise over run, but did not think of treating the rise over run ratio as an object over smaller and smaller intervals—as the symbolic definition of the derivative obliges one to do. Only when Mark read the calculus text did he begin to think of the slope as an object rather than a process.

It was not simply *reading* the text that enlightened Mark. He had tried to read the textbook more than once previous to the cat task and had not understood it. The difference was the preparation Mark had from the time spent in class working on and thinking about how to find the speed of the cat at frame ten. The importance of slope, tangent lines, and secant lines had become more explicit as Mark struggled with the ideas and communicated those ideas to others. It was through his communication that his ideas became well defined.

Mark seemed to be the most satisfied with the symbolic representation of the derivative. Limits and calculus notation were an integral part of what qualified as "calculus" for Mark. He kept trying to find something more "mathematical" and said more than once that "it" (the solution to the cat task) would eventually come down to equations and limits. Even though he accepted Kam's explanation regarding the average velocities, he still kept thinking that there had to be a way to get a better approximation. I believe part of the reason he felt this way was because he was in a calculus so he figured there was a way to solve the cat task using calculus, or by using limits and calculus notation. Consequently he continued to pursue different ideas such as trying to find the tangent line using trigonometry and eventually came to understand the symbolic definition of the derivative.

Mark's representation of limit. In order to understand the symbolic definition of the derivative a student must have an understanding of the concept of limit (Hähkiöniemi, 2006). Let us take a closer look at Mark's understanding of limit. As previously noted Mark's original conceptions of limit were strongly tied to Zeno's Paradox and the idea of getting closer and closer but never reaching a limit. This class had not defined limits in terms of delta and epsilon so it is not surprising that Mark still has a conceptual, dynamic understanding of the limit instead of the static delta-epsilon definition.

Mark's dynamic conception of limit includes an asymptotical view as well—he compares the limit to the asymptote of a hyperbola [5:10:04]. Because of his concept image, Mark believes it is not mathematically possible to "reach a limit." This view makes sense if one considers Mark's background and the connection he makes to Zeno's paradox as well as the presentation of the definition of the derivative in the text he read (Garner, 2005). Garner's text uses the traditional approach of secant lines approaching the tangent line as the distance h decreases. In this sense h is getting smaller and smaller from one side (a one-sided limit with a bound) and in light of Zeno's paradox, will never reach a length of zero. Mark also states that if h were to equal zero there would not be two points, and thus impossible to find the slope, and also algebraically one cannot divide by zero.

While Mark was able to define derivative values using the derivative function obtained by taking the limit of the difference quotient, his conception of the limit was still largely a process view at the end of the time period analyzed. He did not reify the limit, but directly calculated derivative values by going through the process of taking the limit, finding the derivative function, then leaving the limit behind and using the derivative function to find derivative values.

Influence of classroom structure

The following portion will address the second research question: "What are ways that the classroom structure affected Mark's learning?" I will discuss how Mark considered big ideas first then the smaller building blocks, how exploratory tasks created a "need" for learning mathematical concepts, the role of agency and time, and evidence of Mark's productive disposition and self-efficacy.

Most traditional calculus courses follow a text in which students first learn smaller concepts in isolation in an effort to prepare them for larger concepts. For example, students will review function notation, study a chapter on limits, and perhaps have some preparatory problems involving rates of change before learning the definition of the derivative. When texts do present the derivative, often the formal definition is given, with an explanation and students are given practice problems as homework. In contrast Mark and his classmates were given a real-life task in which they were to find the instantaneous velocity of a cat. The derivative being introduced, in this manner, as one big idea: instantaneous rate of change. Students then built the smaller ideas such as limits, average rates of change, and functions in response to a need rather than in isolation. In this manner these smaller ideas had more significance.

As shown in the previous chapter, it took *time* for Mark to develop ideas, understand concepts, and make connections. The classroom setting in which Mark was developing his conception of the derivative allowed for exploration of the underlying big ideas in calculus such as instantaneous rate of change. Instructors did not jump in too quickly, but allowed students to think through ideas thoroughly. Students had time to formulate their own theories and discuss their plausibility and in so doing deepening their own understanding.

The classroom setting also allowed students to teach each other. They listened to each other and learned from each other building their self-efficacy because they saw that they could be successful mathematics learners and teachers. Through their social interactions they were able to define terminology and notation. In their public presentations (Raman, 2003; Speiser & Walter, 1997; Walter & Gerson, 2007), or their verbal presentations to one another, students clarified their thoughts and deepened their own understanding.

Students were given a chance to explore and use their own thinking. Mark and his group created *their own* models of the situation. He was not told the best way to graph the information or even to draw a graph, but *chose* to do so and chose *how* to do so, exercising his personal agency. Students exercised their agency in choices they made, not feeling like they were constrained to one way of thinking. I do not think Mark ever would have explored his idea using trigonometry to find the tangent line if he had not had the freedom to explore the task in whatever approach seemed possible for him. His personal explorations enabled him to come to a strong conceptual understanding of the definition of the derivative.

Mark described what he thought the instructors hoped for to Chris when Chris had expressed frustration:

> I think they just basically, we just have as much time until they're satisfied that we've come to an appropriate conclusion or until they've like, or until we've exhausted all our options, and they're like okay what are you thinking? Guide your thought process.

The fact that Mark believes the teachers expect him to exhaust his possibilities may attribute to his persistence. He also sees the teachers as guides of his thinking rather than spouting founts of knowledge that he simply absorbs. Notice the phrase “an appropriate conclusion.” Mark did not say correct, right, or even *the* appropriate conclusion. This way of thinking is unusual. So many students would have believed that the teacher would just give them the answer when they had worked on it a while and not come up with the answer. In this classroom the instructors did not constrain the students’ agency in mathematical exploration, but allowed them freedom to explore their own options in their own way.

Mark shows a productive disposition (National Research Council, 2001) and a strong sense of self-efficacy (Bandura, 1989) throughout the analysis. A classroom that allowed students to explore ideas, share and discuss those ideas fostered self-efficacy and productive dispositions. Mark sees himself as an effective learner and doer of mathematics and that with diligent effort he is capable of learning—steady effort pays off. Mark believes that he has to work at math, “I don't consider myself naturally talented for the most part,” but if he works at it, he believes that he has the ability and capacity to figure out and solve mathematics problems. Mark engages in mathematics in such a way that demonstrates that he believes in himself. He contributes, listens attentively to others, asks thoughtful questions, and remains focused on the task. He does not readily absorb material, but processes it and determines whether it makes sense to him or not. He works hard on class work and homework and is persistent even in the face of difficulty. Mark sees mathematics as both useful and worthwhile. Mark showed that he knows that calculus is used to solve real world problems, that mathematics is applicable and necessary. He pointed out that methods similar to the cat task have been used to calculate physical phenomenon such as falling objects and projectiles.

Not only did Mark display enthusiasm for the mathematics, but his entire group showed similar attitudes, leading me to believe that there was something beyond Mark's character that led to such excitement over calculus concepts. I believe students' ability to explore, the struggles they had in finding answers, and the general classroom setting contributed greatly to Mark's (and his peers') energy and enthusiasm throughout this task and the rest of the semester. As Mark himself said, " I think this is a lot more exciting when I can rediscover it." Mark felt as if he had discovered something. That was exciting.

Student learning in context situations creates a "need" for solutions that make those solutions "exciting." Mark got very frustrated while working on the task—but not frustrated enough to quit. In fact, his frustration just made conjectures and discoveries all the more meaningful. When I asked Mark in the interview why students in this class seemed to have more enthusiasm about concepts like the definition of the derivative when compared to other calculus classes I've taken or taught, he said:

> The difference is like I think it's cause we got frustrated because we didn't have it. We're like, 'Aaarg! We want to use it! We want to figure out how to do it!!' And I think that's one good thing about this class. Like in all honesty, like, I think this class is a good foundational class. Much stronger foundational class than most other students. Like most other students are like that. Are like, 'Yipee. Learn the definition of derivative, throw it out next week. I won't ever retain it' . . . Like

> when you're solving a puzzle and you're trying to solve a puzzle like the cat task here, you know. It's like, well, how do I solve this puzzle? How do I find a solution to this puzzle? I want to solve this puzzle. It's almost like a game you know. And so when I, when we finally found a solution to the puzzle it was like, 'hey this is cool we found it! Look at this!'

Mark makes reference to the social interplay that helped him make sense of the task. Mark did not solve the task alone, yet his individual ideas greatly contributed to the collaborative efforts of his group. Mark's entire group was excited to solve the task and learn the mathematics. Letting students struggle for a while—and notice that Mark and his group struggled with this problem for at least a week(!)—will make the solution ever so much more meaningful.

Implications

In the literature review I presented many views of the *lack of understanding and retention* exhibited by many calculus students (Baker, Cooley, Trigueros, 2000; Dudley, 1993; Schnepp & Nemirovsky, 2001; White & Mitchelmore, 1996; Zachary, 2004). The research presented in this thesis shows that at least one student who excelled, demonstrating both procedural and conceptual understanding of the definition of the derivative, retaining his knowledge applying it to new tasks, and enjoying the whole process. Knowing how Mark made sense of key calculus concepts can help us know how other students might make sense of them.

The more we understand about how students learn mathematics, the better prepared we will be to teach mathematics. This research has implications for ways in which calculus can be taught. Students need to be encouraged to discuss and explore significant calculus ideas such as rate of change. A teacher can give tasks that require the student to think deeply about big ideas. Students can work together and present their work to their classmates. Big ideas can be a starting ground from which smaller building blocks can be assembled and have more meaning. Time needs to be allotted for students to solidify ideas and explain *why* something was true or *where* it came from. Creating such a setting can foster self-efficacy and a productive disposition in the students.

More research can contribute and support the findings in this study answering questions such as: What would have been different if Mark were not as motivated of a student? Do other students in comparable classroom environments act similarly? How well will Mark and other students do a few years in the future as a result of their experiences in this class?

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