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Probing for Reasons: Presentations, Questions, Phases

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PROBING FOR REASONS: PRESENTATIONS, QUESTIONS,

PHASES

by

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As chair of the candidate's graduate committee, I have read the thesis of Kellyn Nicole Farlow in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

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ABSTRACT

PROBING FOR REASONS: PRESENTATIONS, QUESTIONS, PHASES

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Master of Arts

This thesis reports on a research study based on data from experimental teaching. Students were invited, through real-world problem tasks that raised central conceptual issues, to invent major ideas of calculus. This research focuses on work and thinking of these students, as they sought to build key ideas, representations and compelling lines of reasoning. This focus on the students' and their agency as learners caused a development of the psychological and logical perspectives, and highlighted students' choices in academic and social roles. Such choices facilitated continued learning among these students.

ACKNOWLEDGMENTS

I would like to thank my parents who give me endless support and love. They have always showed me how to be grateful for and learn from great opportunities and blessings that come my way.

One such opportunity has been to work with and learn from each member of my committee. Bob Speiser, Chuck Walter, and Janet Walter have mentored me in unique and meaningful ways. Bob Speiser spent countless hours working with me, while providing wisdom that I will always treasure. Chuck Walter made this great learning opportunity possible, supported my goals, and gave much needed guidance along the way. And Janet Walter provided fresh insight and new perspective. I always felt that I could talk with her about my concerns. Thank you.

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CHAPTER I: BACKGROUND AND MOTIVATION

Motivation. Influenced by my experience as a learner, developing teacher, and research assistant, two main motivations helped shape my project: to help make mathematics more accessible for all students and to gain better insight into how pedagogy and curriculum design might support long-term, not just short term, learning goals.

Initial Questions. These motivations suggested four initial questions, which I planned to refine upon selection and conceptualization of the data:

- *What do learners do when they solve challenging mathematical problems successfully? Specifically: How do they reason? What do they write, draw, or communicate? How might they reflect on their experience?*
- *How might careful, explicit analyses of learners' problem solving offer opportunities to revisit and perhaps rethink important mathematical content*?
- *How might such studies offer insight into how more learners might come to understand and care about important mathematics concepts? How might such understanding help enrich their lives and the lives of those around them?*
- *How might such studies of learners' work and thinking help me to refine and clarify my own emerging pedagogy?*

These questions focus foremost on students. In particular, a close look at what students do when they successfully solve problems and at the opportunities that better facilitate the development of their reasoning could provide greater insight into how all students may have increased access to mathematics. This close look may also have implications for pedagogy and curriculum. With greater access to mathematics, students are free to use their mathematical knowledge to solve problems they may experience throughout life. A focus on how students reflect on their experience may help me better understand their previous work, and their continued learning.

The Course. Honors 250 is an experimental problem-based mathematics course on the mathematics of change, for students in the arts. The semester I collected data, Fall 2005, the class consisted of 14 students with varying mathematical backgrounds. Twelve of the fourteen were dance majors and the other two had closely related majors. Throughout the semester these students worked in groups on nine tasks.

Task. On September 20 and 27 the Honors 250 students worked on their second task, Estimating the Mile Record from Peter Taylor's *Calculus: The analysis of functions*. Here is the task:

At the right is a table of the men's metric world track records as of August 1976 (taken from the Guinness *Book of World Records*). At that time, the mile was still a major track event, contested by the top runners. Based on the metric data, provide the best possible estimate of what the mile record should have been in August 1976.

METRIC WORLD RECORDS (August 1976)		
x metres	*T* secs	Year set
100	9.96	68
200	19.81	71
400	43.9	68
800	103.5	76
1 000	133.9	74
1 500	212.2	74
2 000	291.4	76
3 000	455.2	74
5 000	793.0	72
10 000	1650.8	73
20 000	3444.2	76
25 000	4456.8	75
30 000	5490.4	70

Figure 1.

Note that 1 mile equals 1609.344 m.

The students were also given the following graph:

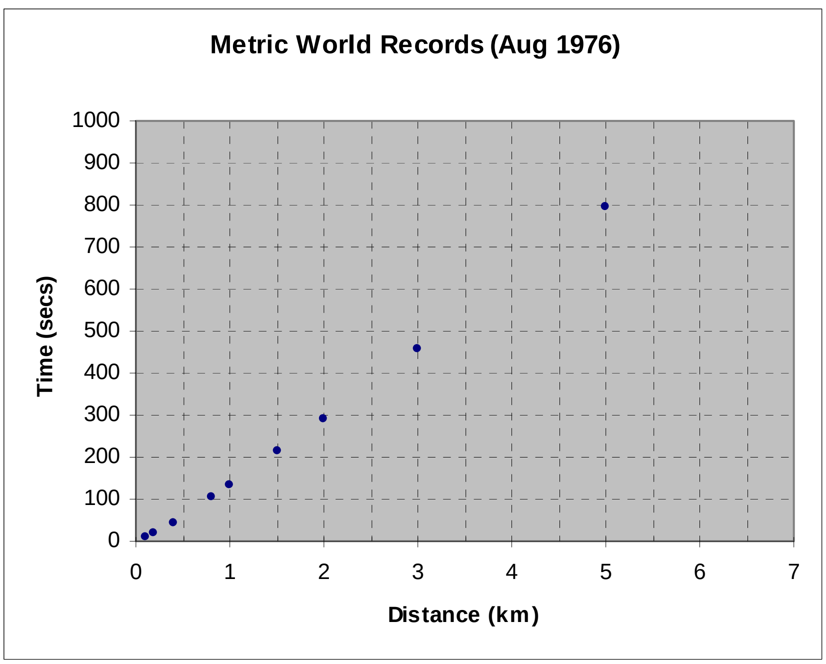

Figure 2.

Taylor (1992) gives two criteria necessary to successfully for finding such an estimate. “Whenever you are asked for an estimate, what is required is an upper bound and a lower bound and an argument that the desired number must lie between” (p. 12). The two criteria for an estimate are: 1. an upper and a lower bound and 2. justification of those bounds.

The students chose to work on this task in three main groups seated at two tables. At one table was Jerica’s group, with members: Jerica, Erin, and four other students, and Brigette (see Fig. 3). Brigette chose to work alone but collaborated often with the other groups.

Figure 3. From left to right: Jerica, Hadley (not visible), Erin, Bryan, Lindsey, Tiffany, and Brigette.

At the other table were Rachel, Ashley, Corinne, and three other students (see Fig. 4).

Figure 4. From left to right: Corinne, Shalaina, Jill, Ashley, Keely (not visible), Rachel, and Anthony (not pictured).

September 27. On September 27, the second of two three hour class sessions spent on this task, all three groups presented their estimates for the mile record. During Brigette's presentation, a question arose about Brigette's solution. Rachel offered an answer to the question and Erin began to ask questions about Rachel's reasoning. A fifteen minute discussion between Erin, Rachel, and others ensued. During this discussion, I was intrigued by the sense-making of these students, and especially by Erin's questions. In a way, I viewed Erin's questions as an informal, task-based interview. By the end of the discussion, my curiosity to better understand these students' thinking had been aroused to the point that I felt I needed to review their task journals and reflective essays. I found this entry in Rachel's task journal, written some time after September 27[1]:

[1] Rachel's journal consisted of three journal entries on three of the nine tasks the class had worked on. Although it is apparent that this journal entry was written after September 27, none of the journal entries were dated.

> I felt like I got an "answer" really fast and then didn't quite know where to go from there.
>
> After talking about it in class, it made sense to look at the whole graph instead of the closest times. The graph made more sense to me when I thought about really running. For the shorter distances you can keep up a fast pace, for longer one, you have to keep a slower, constant pace . . . so it makes sense that the graph would be concave [up].
>
> It also helped with talking as a class to discover that looking beyond the closest points can help you get closer to the actual time!

Even after Rachel had reached a solution she continued to reason and make sense of the task. Rachel's journal, as I understand this entry, sounds three major themes: 1. the centrality of representations and reasoning from and about representations; 2. bringing oneself into a picture to make sense of it; and 3. how the conversation with Erin triggered reflections that led Rachel to reason further.

Research Background. Rachel's insights provide possible directions for how to think about the data surrounding this task, but her insights also suggest how the analysis presented here might fit in with and further other research.

Rachel's first theme, the centrality of representations used by Rachel and others, ties in with work done by Davis (1984), Speiser and Walter (1997), and others that focuses on the representations students construct and how students reason from such representations. In my work a representation is a presentation "either to help convey specific details of one's thinking to oneself, or to facilitate communication with others" (Speiser & Walter, 2007), as explained in more detail as part of the analytic starting points. When Rachel said "it made sense to look at the whole graph instead of the closest times," for example, it is clear that a careful focus on how representations are made, interpreted, and used will be central to later analysis of student cognition.

Skemp (1978), on the other hand, takes an epistemological approach. Skemp focuses on different kinds of knowledge and justification, making a distinction between *instrumental* and *relational* understanding. Skemp's conceptualization of instrumental and relational understanding will be important as I describe and analyze the data, this paper reports on, over time.

In relation to Rachel's theme of bringing oneself into a picture to make sense of it, Ochs et al. (1994) report research physicists using representations as "interpretive frames for understanding the dynamics of both inanimate physical entities and those who are visually, verbally, gesturally, and/or imaginatively journeying through a two-dimensional display." (p. 168). This journeying is not seen by or imagined as an outside observer but is a personal, lived experience, acted by the imaginer. The idea of imaginatively journeying seems especially important for Rachel when she wrote, "The graph made more sense to me when I thought about really running. For the shorter distances you can keep up a fast pace, for longer one, you have to keep a slower, constant pace."

Finally, Rachel found that the conversation with Erin triggered reflections that led her to reason further. Speiser, Walter, and Glaze (2005) found that representations provided students with insight about mathematical reasoning, but also found that representations provided insights with long term implications. In fact, students changed their perceptions of mathematics and themselves as thinkers. Martino and Maher (1999) used representations to understand the long-term development of mathematical reasoning in students and then sought to understand how teacher questions promote student justification. “Teacher questioning that is directed to probe for student justification of solutions has the effect of stimulating students to re-examine their original solution in an attempt to offer a more adequate explanation, justification and/or generalization” (p. 75). In this data segment it is a student who chooses to ask meaningful questions that promote justification. So we must not only seek to understand this student’s questions, but also her choice to ask such questions. Walter and Gerson (2007) call such choice-making, exercising personal agency.

> As students exercise agency, they become more persistent in working on challenging tasks. Rather than disengaging when they find an answer, learners pursue individual and collaborative interests and determine the success of their endeavors by their achievement of understanding over time. Indeed, empowering learners to exercise agency in mathematics learning “is not just a matter of changing the culture of the mathematics classroom” but is a matter of letting the “ethos of the mathematics classroom match that of the context of the genesis of mathematical knowledge” (Ernest, 1998, pp. 258–259) (p. 210).

Erin, Rachel, and other students exercised personal agency in this sense during the September 27 discussion and while making sense of the task, with important long-term consequences.

Developed Questions. After living with the data of the September 27 videotape, Brigette's overhead transparency, and both Rachel's and Erin's journals, specific questions that seem most relevant to these data are:

- *What did these learners do to successfully solve this task? Specifically: What did they draw or communicate? How did they reason? What does successfully solving this task mean for these students? How did they reflect on their experience?*

These questions not only focuses on what Rachel did as her argument developed, but also Erin's role in the discussion to support the development of Rachel's argumentation.

Analytic Starting Points. To pose such questions to my data, I will take a particular approach to problem solving, which is central to a larger research program (Speiser, Walter & Glaze, 2005; Speiser, 2004), of which my work here is a component. As a student I have participated in this larger research program for several years through data collection and selection, and participated in discussions when ideas emerged and were elaborated upon.

The problem solving approach used here emphasizes two perspectives, called logical and psychological, as explained below. Mathematics, here, is seen in terms of motivation, choice, and goal directed action. While I will begin with these perspectives, I leave room for change and/or further development as my analysis proceeds.

> In this study, mathematics will be something that one *does*. Specifically, one solves a problem. We approach our students' problem solving from two perspectives. On the one hand, we take what we will call a *logical* perspective, where we focus on the learners' reasoning. We shall build especially from the approach to problem solving urged by Davis (1984), who placed central emphasis on representations. For us a representation is a presentation, either to help convey specific details of one's thinking to oneself, or to facilitate communication with others. From the logical perspective, learners' explorations are seen to proceed from the construction of one or more presentations of the problem situation to solutions to be justified through reasoning based how the given presentations have been structured. On the other hand, we can take what we will call a *psychological* perspective. Here we build on prior research about students' discourse and presentations (Speiser & Walter 1997, 2000, 2004; Speiser et al., 2003) where we have emphasized especially how students choose to present their thinking to themselves and others. Here we work first in the context of the group's collaboration on the problem right at hand, but then widen the context to consider further data, when available, that may help us to connect a given subject's work to long-term goals, values and past experience (Speiser & Walter, 2007, p. 15).

Chapter II: Data Sources and Methodology

Data sources. Honors 250 met one semester, for three hours once a week. I filmed class sessions using one digital camera, with mounted shotgun microphone, on a rolling tripod, to capture conversations and written student work in progress. As I videotaped, I made on-the-spot decisions about who and what to tape in order to support detailed analysis. I took field notes to locate critical events relevant to my initial research questions. After deciding to focus on the September 27 discussion, for reasons mentioned above, I located other pertinent data. The data used in this analysis will come from four sources. These sources are: videotaped data from the September 27 class period, Brigette's overhead transparency, Rachel's journal, and Erin's reflection paper.

Methodology. All four data sources have been used to compile a *composite narrative* (Speiser, Walter & Maher, 2003; Speiser & Walter, 2007). Portions of the September 27 videotape of the data have been selected and grouped into three "phases", described in Chapter III (p. 13), to facilitate analysis of the posed questions. Each phase was chosen in light of the analytic starting points and Skemp's (1978) terms: instrumental and relational. The goal in selection and in grouping selected portions of the videotape into phases was to answer my research questions. In particular, from a psychological perspective, to investigate the development of Rachel's argument over time, the development of Erin's questions throughout the discussion, and Erin and Rachel's long-term development as learners. From the logical perspective: to explicate the logical underpinnings of each of Rachel and her classmates' arguments, as well as the underpinnings of Erin's questions.

<u>CHAPTER III: DATA AND ANALYSIS</u>

Data. On September 27, after each group made significant progress toward a solution, the students continued work in what I have divided into three phases. The character of the mathematics used and discussed in each phase seems to change from phase to phase. To conceptualize such changes I have found it helpful to use the terms *instrumental* and *relational* as described by Skemp (1978). Relational understanding is “knowing both what to do and why” (p. 9), whereas, instrumental understanding is “rules without reasons” (p. 9).

I have used these three phases to build important background information for readers as well as organize the events for analysis. In affect I will analyze all three phases, but the analysis of *Phases I* and *II* serves a different purpose than the analysis of *Phase III*. The analysis of *Phases I* and *II* provide background for the reader to help the reader understand these students’ motivations mathematically and socially, but also to focus attention on the relational emphasis I will describe in *Phase III*.

Recall the initial questions I posed to the data: *What did these learners do to successfully solve this task? Specifically: What did they draw or communicate? How did they reason? How did they reflect on their experience?* Thinking in terms of instrumental and relational, the first question is instrumental with a focus on what the learners *do*. But in order to more fully answer this question I have posed more specific relational questions. Note that the instrumental and relational emphases of these questions complement one another.

Phase I. During *Phase I*, Ashley and Jerica gave initial presentations to help each student become more aware of the different ideas, strategies, approaches, and perspectives discussed within each group to find estimations of the mile record. To estimate the mile record, each group found upper and lower bounds, which the students called high and low estimates.

9:22 *Jerica.* It was really interesting at our table because we came up with the same answer for one of the [pause] estimates, but we found it in a totally different way so we were [pause], it was really exciting to discover that so I'm going to present what half of the group or what part of the group figured out first.

We will call the different methods Jerica refers to as the *similar triangle method* and the *slope method*. The initial presentations of each method were given by Jerica and Ashley, respectively.

Similar triangle method. Jerica began by marking two points on the board (see Fig. 5). These points correspond to the world records for the 1500 m and the 2000 m races, as first seen by the students in the given graph (see Fig. 2).

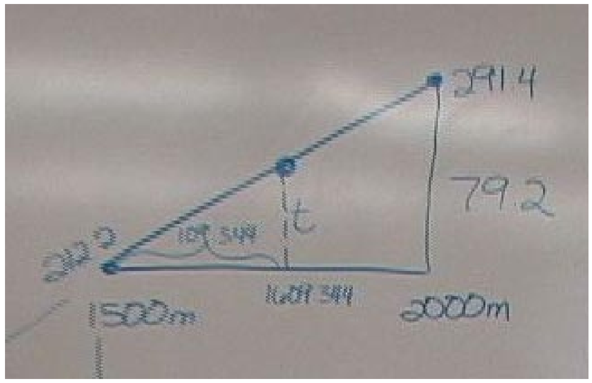

Figure 5. Jerica's similar triangles.

Using these two world record points, Jerica constructed a right triangle. Then she plotted a point to represent the mile world record and drew the vertical segment, t. Jerica constructed a new, smaller, triangle with leg, t, and shared base with the larger triangle. These two triangles are similar so Jerica used equal proportions of the legs of similar triangles to solve for t (see Fig. 6).

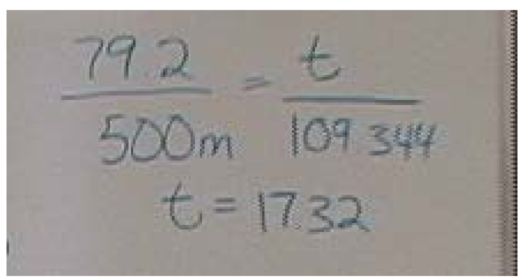

Figure 6. Jerica's equal proportions.

Jerica evaluated the expression to obtain $t = 17.32$ and added 212.2 (time of the 1500 m record) to get an upper bound or high estimate of 229.52 for the mile record (see Fig. 7).

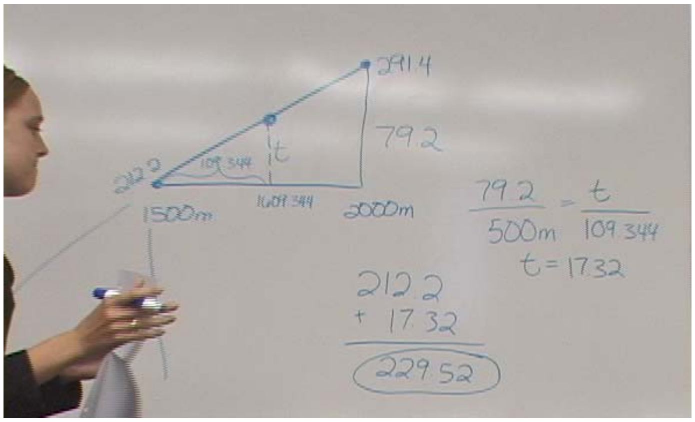

Figure 7. Jerica's presentation using the *similar triangle method*.

Slope method. To find an upper bound, Ashley's group constructed a triangle using the 1500 m and 2000 m record points as vertices like Jerica did (see Fig. 8).

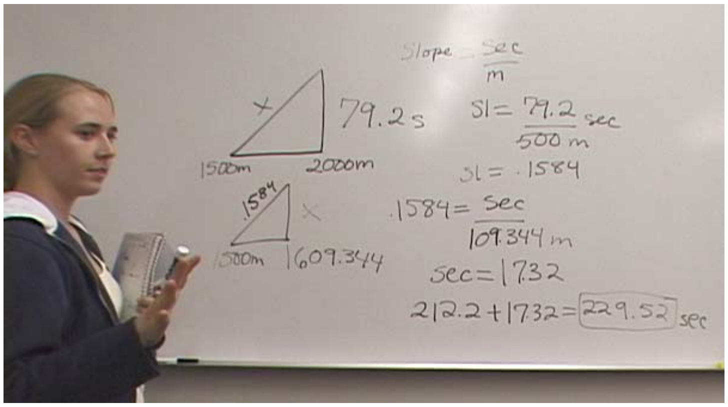

Figure 8. Ashley shows the *slope method* used by her group to find an upper bound.

Ashley used known leg lengths to calculate the slope of the hypotenuse. Using the 1500 m record point and the mile record point as vertices, Ashley constructed a smaller triangle with unknown leg length x. Ashley let the hypotenuse of the smaller triangle have the same slope as the hypotenuse of the larger triangle and set up an equation to solve for x, which she just labeled as "sec" (see Fig. 9).

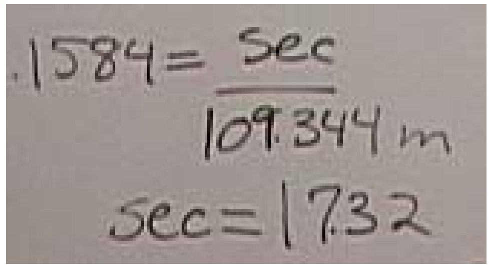

Figure 9. Ashley's equation to solve for x.

Finding sec or $x = 17.32$, she added 17.32 and 212.2 to get an upper bound of 229.52 sec.

To get a lower bound Ashley first constructed a triangle with vertices at the 1000 m record point and 1500 m record point (see Fig. 10)

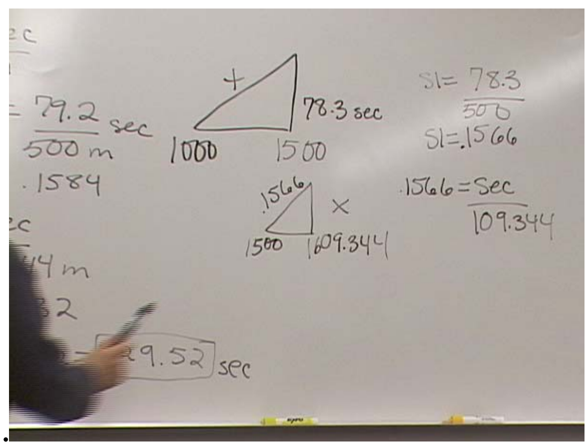

Figure 10. ***Slope method*** **to find a lower bound.**

Ashley constructed a new, smaller, triangle, assumed the slope of the hypotenuse of the small triangle to be the same as the slope of the hypotenuse of the larger triangle, and set up an equation to solve for x. The resulting lower bound for the mile record was 229.32 sec.

Jerica pointed out that there were different methods used to find upper and lower bounds for the mile record. Both Jerica and Ashley demonstrate how to use these methods. Here, I will characterize this focus on methods and the accompanying calculations, during *Phase I*, as instrumental, rather than relational.

Phase II. After Jerica's presentation [9:21–12:52], Jerica invited Brigette to show the class her method to find the lower bound. Brigette said she used the same method as Ashley, the *slope method*, and projected an overhead transparency that she had prepared a week earlier (see Fig. 11).

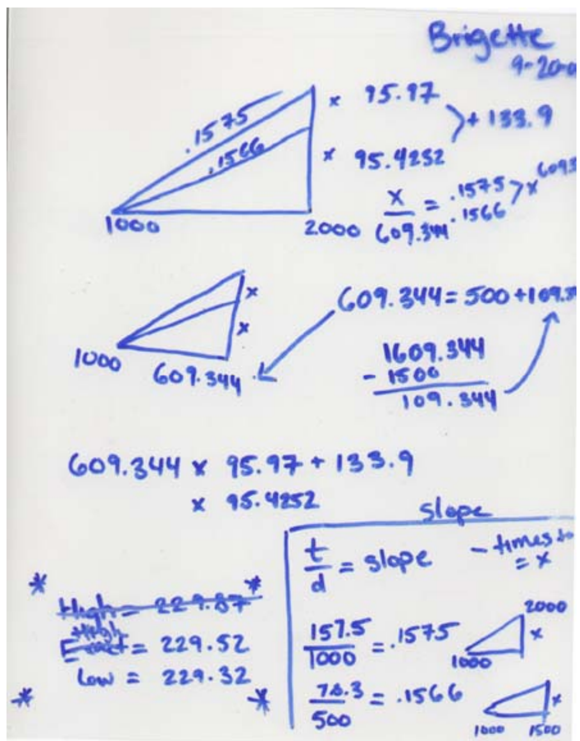

Figure 11. Brigette's overhead.

Although Brigette used the *slope method*, Ashley requested that Brigette work through her calculations because Brigette's solution seemed different. Brigette and the other students began to reconstruct Brigette's original solution. After Brigette drew a triangle with two of the vertices at the 1000 m and the 2000 m record points and presented her solution [14:15–20:06], Jerica had a question.

20:06 *Jerica*. But I have a question on that.

Brigette. What?

Jerica. Sorry, you were describing it just a little bit, but doesn’t the slope change for each one of them [hand in air tracing increasing segments]?

Brigette. What do you mean change?

Jerica. Like on the graph it [the secants] goes between the points [walks to board and draws four points and the secants between pairs of successive points, see Fig. 12].

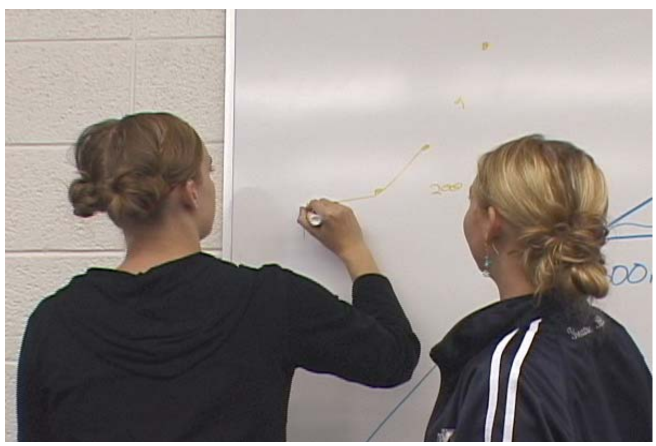

Figure 12. Jerica’s representation about different slopes.

Brigette. Right, but this [points to the 1000 m–2000 m secant] is only between two points.

Jerica. But even though you’re a thousand [labels 1000 below first point on the left] and two thousand [labels 2000 below the third point from the left] wouldn’t it still be two different slopes [draws the secant line from 1000 to next point and then from that point to 2000 point]?

Brigette. No, only if you're finding the slope between this and this [points to 1000 and 1500] but the slope between here and here [points to 1000 and then 2000] is one slope. . .

Jerica. So you did this [draws the secant line between 1000 and 2000, see dotted line in author's drawing, Fig. 13].

Brigette. Yeah.

Jerica. Oh, okay.

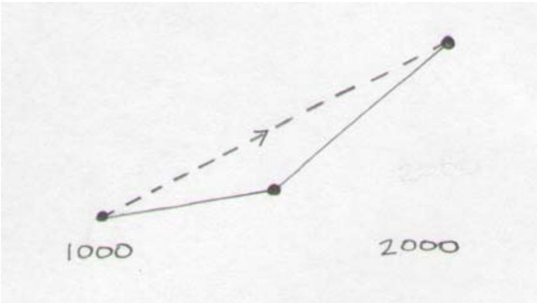

Figure 13. Author's drawing of Jerica's new 1000 m–2000 m secant line.

With Jerica's representation and added secant line, between the 1000 m and 2000 m record points, on the board, Brigette added to the representation and gave the following explanation (see Fig. 14).

20:43 *Brigette*. And so you guys found this slope [traces the 1000 to 1500 secant]. So if you extended this line out this way [draws a line extending the 1000 to 1500 secant right of the 1500 point] and I found this slope [extends 1000 to 2000 secant right of 2000 point] and [draws a vertical line just to the right of the 1500 point] the mile is somewhere in here [places a dot on the vertical line below the 1500 to 2000 secant, but above the 1000 to 1500 secant]. So that's where I got my high and my low. Does that make sense?

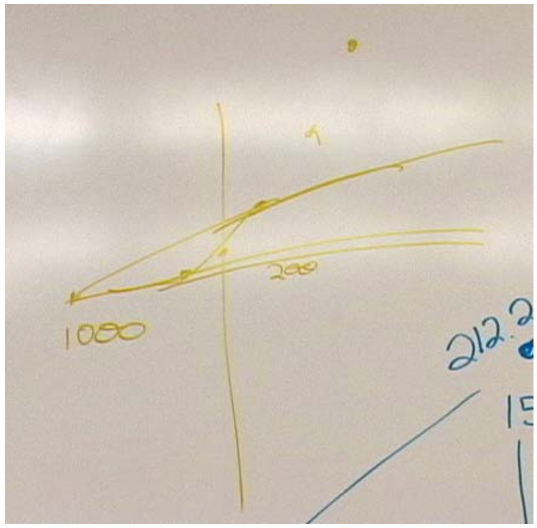

Figure 14. The final look of Jerica's representation with Brigette's addition.

These representations will be discussed in greater detail in analysis below.

Rachel had her own questions for Brigette regarding Brigette's estimates (see Fig. 15), which she asked a few minutes after Brigette and Jerica's discussion [20:06–20:43].

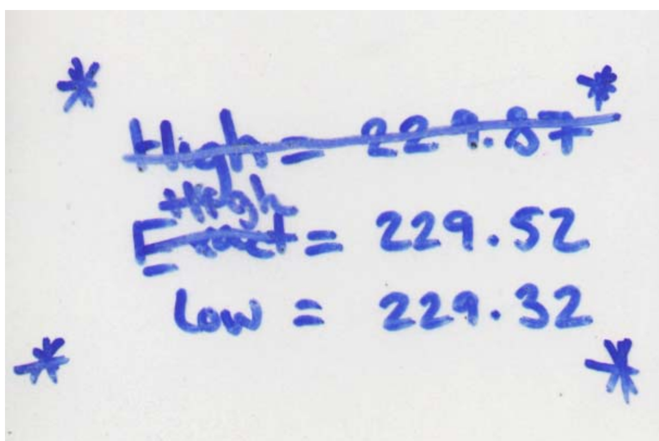

Figure 15. Brigette's three estimates.

24:47 *Rachel.* Why did you cross out your original high?

Brigette. Because my original high, um [pause], was higher than it needed to be. Because this [goes to the board and circles around the original "exact" estimate projected onto the board, see Fig. 15] is still a high estimate.

Rachel. Right.

Brigette. And so now I have an even closer. . .

Rachel. But how did you know that your high was too high? [pause] Or maybe that your low was…do you know what I mean?

Brigette. Umm [pause], well 'cause I drew out, like I had all my different points and different slopes going out like this [goes to the board to the left of the projected overhead and traces secant lines all starting at the same point]. Kind of like I did here [walks to representation seen in Fig. 14] and my original high [takes pen cap off and retraces 1000 to 2000 secant line] was higher than my, what I had called my exact. So then I just got rid of it [High = 229.87 sec, see Fig. 15] and used that [Exact = 229.52 sec, see Fig. 15] as my high.

Rachel. What slope did you use, what two points did you use for your original high?

Brigette. Umm [pause walks back toward overhead transparency and looks at paper in hand] I think I used [pause]. I don't know.

Familiar with her own groups' solution, Rachel then offered an explanation of why Brigette had two upper bounds.

27:10 *Rachel.* Yeah, because when she threw out the points [or the secant that gave a High = 229.87, see Fig. 15] just barely like when she was saying what you did before. Was this last week so it's just hard to remember?

Brigette. Yeah, like I did it all last week and I was. . .

Rachel. Like between one thousand and two thousand you'd be even higher [arms in air]. You have your point in the middle so you had two lines above your point and just one under. The one thousand to two thousand would be even higher than one thousand to fifteen hundred so I think that's what you were trying [pause] to say.

Jerica and Brigette's discussion and Rachel's explanation, during *Phase II*, mark a transition from the seemingly instrumental thinking of *Phase I* to the relational thinking examined more closely in *Phase III*. This third phase spans data from a 15-minute discussion between Rachel, Erin, and several others. Indeed, Skemp's description of relational characterizes very accurately the explanations and questions by Rachel, Erin, and others in *Phase III*.

Phase III. In order to provide a clearer explanation, Rachel went to the board, erased Jerica and Brigette's representation (shown in Fig. 14), and drew her own representation.

What did they draw? Rachel's representation (see Fig. 16) was the only drawing of *Phase III*. From the psychological perspective, Rachel's representation is a critical element of her presentation to herself and the other students, as well as the presentations of others.

27:43 *Rachel*. So theoretically it should be in a curve ever so slightly [draws four points starting at the left and each one curving higher, see Fig. 16]. So say this is 1000 [labels the first point 1000], this is the mile, no this is 1500 [labels the second point from the left]. I'm going to erase this one [erases third point from the left], this is two thousand [labels last point at very right] and then theoretically the mile
28:09 should be around here [picks a spot between 1500 and 2000 closer to 1500 and draws a new point] following the curve [connects dots in the air following a curve]. So this is the mile [labels the new point with the word “mile” on the board]. What point am I missing?

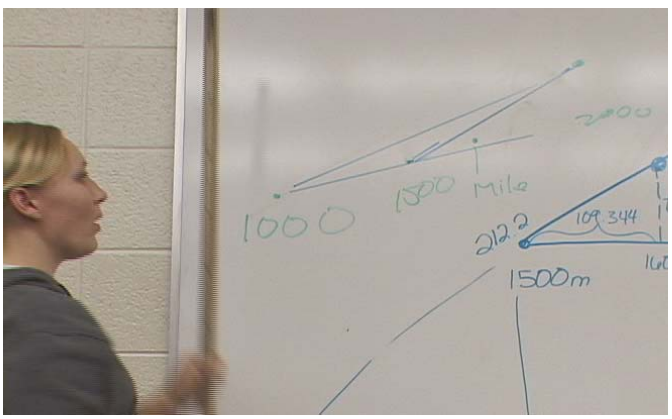

Figure 16. Rachel and her board work.

To begin construction of her representation, Rachel drew four points and labeled the first two points, from left to right, as the world records for the 1000 m and the 1500 m. She then erased the third point and continued to label the last point as the 2000 m record point. She plotted only three of the nine world record points plotted on the original graph given to the students (see Fig. 2). Indeed, Rachel included only a selected portion, of the original graph, in her representation. Also, the data points of the original graph appear to be nearly collinear (see Fig. 2), but the three record points that Rachel drew do not maintain the same nearly collinear configuration as the original graph. The locations of the points were exaggerated possibly to make clear that the points are not in a straight line. In fact, before Rachel drew the points, she said, “theoretically it should be in a curve ever so slightly.”

Rachel picked a spot to draw the mile record point which she said is theoretically about where the mile should be. The words “should be” indicate that Rachel did not just pick any location for the mile record point, but she used some criterion to decide where a reasonable location was for the mile record point. Rachel mentioned a criterion of “following the curve.” Psychologically, the order in which Rachel first drew the three given record points and then mentioned and drew a reasonable location for the mile record might emphasize that the exact location of the mile record point is not known.

Rachel. Do we have another point? [No] Okay so we have, we have, let me do it this way. So we found, we both found fifteen thousand to two, or fifteen hundred to two [pause]. Oh dear [her marker wobbles] I *can* draw a straight line, just not right now [laughter]!

Speiser. Would it help to have a ruler?

28:35 *Rachel*. Maybe, *yes* that would be delightful! Oh perfect,

Speiser. It’s already stained with green marker.

Rachel. Oh good. Okay so this line will be here just very so slightly [draws secant line between 1500 and 2000, see 1 in Fig. 17] above [pause] I still can't draw a straight line [laughter]. And then from ten [pause] a thousand to fifteen hundred [draws secant line between 1000 to 1500, see 2 in Fig. 17] it'd be ever so slightly below [pause] so then doing it from here to here [Rachel raises her hand up and then down], which is what she was doing [draws secant line between 1000 and 2000, see 3 in Fig. 17], it would be even *more* above [pause].

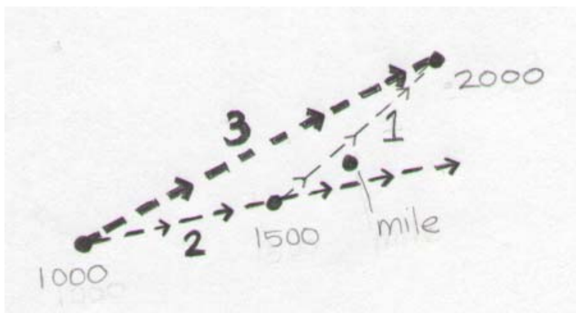

Figure 17. Author's drawing specifying order in which Rachel drew the three secant lines.

Ashley. Oh.

Rachel. So that's why this number was [points to the secant line between 1000 and 2000] even higher because you had two numbers that were higher to you just crossed out the top number and you were thinking that this one [points to the 1000 point] was that one [points to 1500 point]. Does that make sense? And I didn't know how to explain that to you because I was confused as to what was going on.

Recall that first, Rachel erased Jerica and Brigette's representation (shown in Fig. 9) and used the cleared white board space to draw her representation (shown in Fig. 14). After comparing the two representations, I found them to be practically the same.

Thinking psychologically about Rachel’s presentation, I note this comparison because it seemed odd that Rachel would erase the first drawing to redraw a similar representation (see Fig. 18). In this drawing I chose to build an interpretation of both Jerica and Brigette’s and Rachel’s representations, in which my interpretations of the two representations juxtapose one another to compare both drawings.

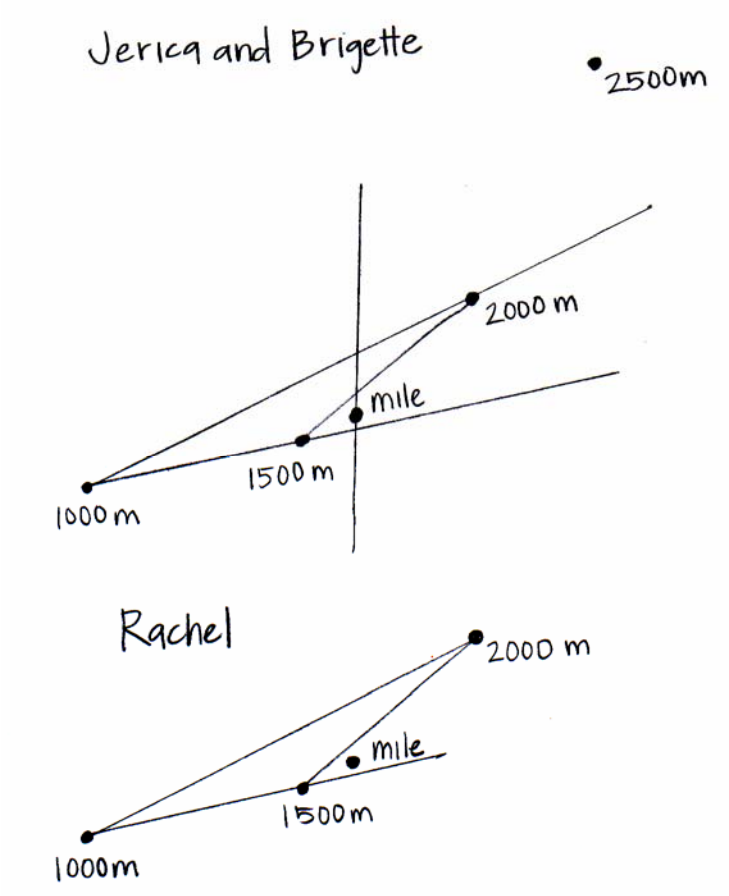

Figure 18. Comparison of interpretive drawings of Jerica and Brigette’s representation (see Fig. 14) and Rachel’s representation (see Fig. 16) (Drawing from author’s notebook).

Rachel’s drawing differs from Jerica and Brigette’s in three major ways: (a) Rachel’s drawing does not include the 2500 m record point, unlike Jerica and Brigette’s, (b) the secant line through the 1000 m and 1500 m record points is the only secant to extend past one end point (the 1500 m), whereas Brigette extended both the 1000 m–1500 m secant and the 1000 m–2000 m secant line past the right record point, and (c) Rachel's drawing does not have a vertical line going through the mile record point. These three differences will be referred to in later analysis of Rachel's presentation.

Psychologically, Rachel demonstrated how these students used standard representations, like the original graph (see Fig. 2), in personal and creative ways, which they developed themselves. They then reasoned from their representations.

What did they communicate? After Rachel's construction of her representation, Erin began to ask a series of questions about Rachel's representation. Walter (2004) explains: "mechanisms for question generation may be important antecedents in the development of mathematical understanding, and identification of those mechanisms depends on reflective analyses of chronologies of various events" (p. 26). Thus, understanding Erin's questions during *Phase III* might provide important insights.

29:23 *Erin*. So you think, you think that's the most correct way to find it, is to go the slope from one thousand to fifteen hundred and the fifteen hundred. . . ?

29:28 *Rachel.* Well you can still do it the way of comparing triangles [referring to Jerica's presentation , based on Fig. 5] because you're basically doing the same thing, but you could compare. . . like you can still do the comparing triangle things for this slope [points to secant line between 1000 and 1500] because you're [pause] you have [pause] one triangle that you know two coordinates [moves yard stick in the air in shape of a triangle] and another triangle that's going to be [pause] exactly proportional, but you're missing one of them. So you can compare it in the same way and get what we got for the second triangle [pause] and get [pause] the same number.

29:55 *Erin*. I, I have no idea what you just said [Rachel and Erin laugh] I'm sorry [pause]. Well I, I mean all those three slopes are different [stretches arms out toward Rachel's drawing on the board with caliper shaped hand see Fig. 19].

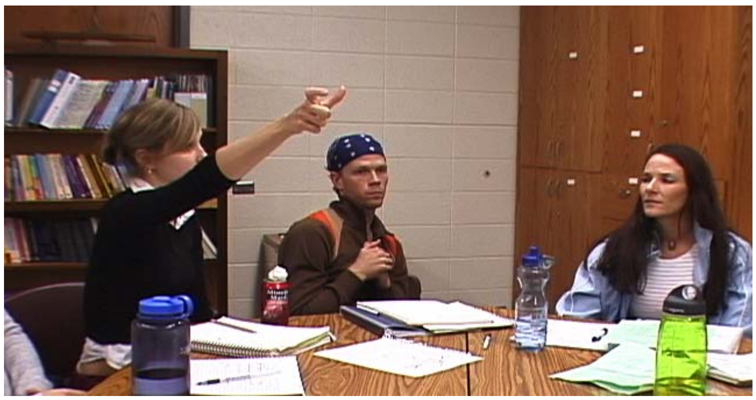

Figure 19. Erin with caliper hand gesture.

Rachel. Right.

30:11 *Erin*. So I'm just wondering, I mean I'm not saying you're wrong like you're probably right, I'm just wondering why. . .

Rachel. Why it works?

30:17 *Erin*. Why you think that the slope from one thousand to fifteen hundred [supports her arm in the air that points with caliper shaped hand toward representation shown in Fig. 16, see Fig. 19] is a better slope to use than one thousand to two thousand.

30:24 *Brigette*. Because if. . .

Erin. Like if you're comparing two slopes.

Brigette. . . . A low estimate

Rachel. Because between these two slopes [points to two secant lines in her drawing]. . .

30:28 *Erin*. ‘Cause you’re sure that the mile is on. . . ?

30:32 *Brigette.* No. . . We just know that the na, the curve [traces along Fig. 16 with her hand] that's been going on with all of the other points is following this kind of pattern [traces along Rachel’s drawing with her finger].

Rachel. Like that was just [inaudible]. . .

30: 40 *Brigette*. So we're just assuming that the mile is in here somewhere [points to Rachel’s point for the mile drawn at 28:09. See Fig. 16]. That's why we're taking a high [traces 1500 to 2000 secant] and a low estimate [traces 1000 to 1500]. ‘Cause it, we can't find. . .

30:46 *Erin*. You took. . .

Brigette. . . . an exact.

30:47 *Erin.* . . . but the mile could be above your high [points to Fig. 16].

Rachel. Umm…

Brigette. Well unless it like [pause] umm [pause] totally goes out of the trend.

30:54 *Erin*. No, I'm not even saying totally out of the trend. I'm just saying between like, you have your one thou. . . , like what's wrong. I mean something's maybe wrong, I'm just not sure what it is [pause] with the slope of one thousand to two thousand?

Brigette. Nothing…

Rachel. Nothing.

Brigette….its just high, its just higher and the closer you are to what you think it is the more accurate you're going to be.

To classify Erin’s questions I coded each key event using a coding system developed by Walter. After initial coding, I found that all but three of Erin’s questions were either interrogative questions (QI) or speculative questions (QS). As Walter describes, ‘an interrogative question is asked to obtain information that is not procedural and includes “why” and “where” kinds of inquires’ (p. 39). In this case, Erin’s questions were coded as interrogative questions when she asked the other students about their logic. For example, Erin’s first question:

> 29:23 *Erin*. So you think, you think that's the most correct way to find it, is to go the slope from one thousand to fifteen hundred and the fifteen hundred. . . ?

On the other hand, a speculative question ‘is asked to suggest potential. “What if?”, “what would happen if?”, or “why don’t we try this?” questions that provide direction for further inquiry are of a speculative nature’ (p. 40). Erin’s speculative questions brought to the discussion logical possibilities that Erin seemed to be asking if the other students had accounted for or considered. For example just minutes after her first question, Erin said:

> 30:47 *Erin*. . . . but the mile could be above your high [points to Rachel’s representation, see Fig. 16].

Psychologically and logically, reasoning about their methods, these students seemed confident, given a concave up curve, that the secants from 1000 m–1500 m and 1500 m–2000 m would be below and above, respectively, the mile record point. Erin's questions go further by asking why these secants are the best to use to obtain the closest estimates for the mile record. These initial codes have revealed an underlying theme of logic in Erin's questions. For a more refined conceptualization of Erin's questions, I will develop the logical perspective into two new categories: the *logic of proof* and the *logic of agency*. On one hand, a standard dictionary definition suffices to describe the *logic of proof*: "a mode of argumentation viewed as good or bad according to its conformity or want of conformity to logical principles" (The Compact Oxford English Dictionary, 1994, p. 992). On the other hand, I define the *logic of agency* to be the process in which one imagines a range of possibilities and uses grounded, critical, and logical judgment about such possibilities. It seems clear that often such judgments must rely heavily upon one's personal experience. For example, the design and selection of representations, that make certain perspectives and observations clear, would fall into this category. In this case, one builds a representation, in certain ways, for a purpose.

These new constructs might better describe how to conceptualize Erin's questions when they are viewed as a presentation. In essence, Erin's questions are about Rachel's representation and the logic of agency behind why the secant line between the 1000 m and 1500 m record points provides the best lower bound. Erin herself uses the logic of agency to choose what questions to ask and then how to reshape those questions.

31:21 *Erin*. So but, but like [pause] you couldn't use the slope from two thousand to twenty-five hundred [caliper hand, see Fig. 18]? Do you know what I mean, like I just don't know why one thousand to fifteen hundred…

[…]

Erin. So I still don't understand why you took the slope from one thousand to fifteen hundred?

Brigette. Because we needed an estimate that would be below where the mile is assumed to be [points to the 1500 m–2000 m secant on Rachel's representation, see Fig. 16]. . .

31:54 *Erin*. So you could have taken from zero to five hundred.

Rachel. You could but the slope would have been [pause] way down low [pause].

Erin changes her questions from asking the other students why they chose the 1000 m–1500 m secant [29:23], to why they did not choose the 1000 m–2000 m secant [after 30:17], then why they did not choose the 2000 m–2500 m secant [31:21], and finally she goes to an extreme and asks why they did not choose the 0 m–500 m secant [31:54]. From the view of the logic of agency, this series of questions is an example of how Erin chose to structure her questions differently throughout the discussion, although her underlying question never seemed to change. From the view of the logic of proof, Erin must have used logic to determine whether or not the justification behind Rachel's representation was satisfactory for her. In fact by the end of the discussion, Erin said:

40:28 *Erin*. I'm just not convinced that that's like really the slope to take.

[. . .]

40:47 *Erin*. Yeah, well I still think I'd just like to do one more step in the process I was saying to see if that's really the. . . , maybe I hope I'm right I could be totally wrong, just to make sure that the slope from one thousand to fifteen hundred is really the closest slope.

Extending the new perspectives of the *logic of proof* and the *logic of agency* to all the data, including Rachel's presentation, seems to provide a new, potentially helpful look at the data. In fact, returning to the analysis above of Rachel's representation, before Rachel drew the record points on the board, she stated, "theoretically it should be in a curve ever so slightly." From the logic of agency perspective, Rachel provides a justification about the way she drew these record points in an exaggerated way. Note that it is only after Rachel had referred the global behavior of the data points that she drew the three local record points. Later, Rachel also mentioned the criterion of "following the curve." Again from the logic of agency perspective, Rachel revealed that she thought about the given discrete set of data points as part of the graph of a continuous function and that this graph is concave up throughout the region she centered on. Although this could be a habit of perception, imagining a curve was useful as these students continued reasoning about the task. Later, when Rachel drew the mile point, she seemed to rely upon logical reasoning about the global behavior of the given data points to choose where she drew.

How did they reason? With exaggerated record points, Rachel proceeded to draw secant lines between the 1000 m, 1500 m, and 2000 m points. As mentioned earlier, when Rachel drew the secant line through the 1000 m and 1500 m record points she started the line at the 1000 m record point and extended the secant well past the 1500 m record point. However, the two other secant lines (1500 m to 2000 m and 1000 m to 2000 m) began and ended at the record points. As she drew these secant lines, Rachel explained:

> 29:01 *Rachel.* And then from ten… a thousand to fifteen hundred [draws secant line between 1000 to 1500] it'd be ever so slightly below [pause] so then doing it from here to here [Rachel raises her hand from the region left of the 1500 m–2000 m secant to the region right of the same secant], which is what she was doing [draws secant line between 1000 and 2000], it would be even *more* above [pause].

Mathematically, Rachel's statement suggests that she might have reasoned about two regions within her representation, which she identified when she said, "from here to here." I have restructured Fig. 17 to emphasize these two regions which I will refer to as Region a and Region b (see Fig. 20), instead of lines and points. Region a is bound by the three secant segments and Region b is between the 1500 m to 2000 m secant above and the extended 1000 m to 1500 m secant line below.

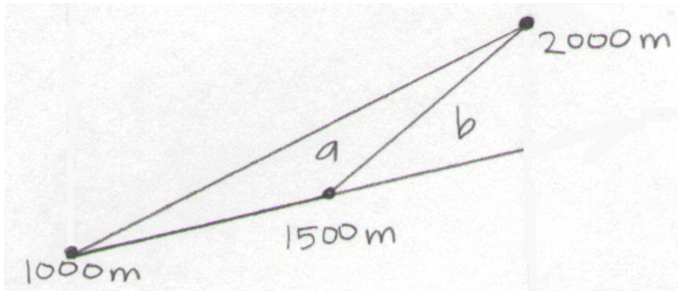

Figure 20. Author's drawing of mathematical region drawing of Rachel's representation.

Although Rachel did not explicitly mention these two regions or where the regions are bounded, it seems as though Rachel specifically looked within the interval from 1000 m to 2000 m. Indeed, there are a number of observations that suggest that Rachel worked within the [1000, 2000] interval:

- Rachel only drew the three of the nine given record points that are within the [1000 m, 2000 m] interval. As previously mentioned, Rachel did not include the 2500 m record point in her representation, unlike Jerica and Brigette.
- Rachel did not extend any secant lines past the 1000 m or the 2000 m record points, unlike Jerica and Brigette. She did, however, chose to extend the 1000 m to 1500 m secant line past the 1500 m, but still within the [1000 m, 2000 m] interval.
- Once secant lines were drawn between the exaggerated 1000 m, 1500 m, and 2000 m record points, regions were created within the bounds of the secant lines. Such regions would not have been visible areas had Rachel not exaggerated the points (see Fig. 20).
- Lastly, Rachel reasoned that the 1000 m to 2000 m secant provided an

estimate even higher than the 1000 m to 1500 m secant. Such reasoning would not hold true if Rachel worked beyond the [1000 m, 2000 m] interval.

The drawing below, Fig. 22, helps explain the last bulleted statement further. In this drawing, the extensions of the secant lines beyond the 2000m point are indicated by dashed lines and will be used to demonstrate the break down of Rachel's reasoning outside of the [1000 m, 2000 m] interval.

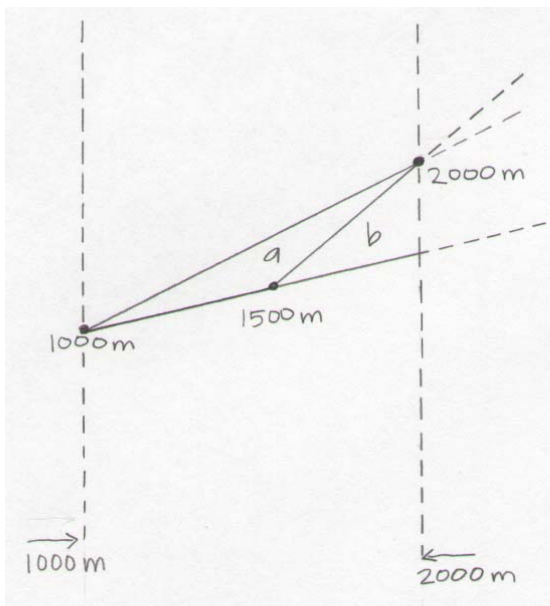

Figure 21. Author's drawing of the region model with focus on 1000 m to 2000 m interval.

According to Rachel, the slope of the 1000 m–2000 m secant line provides a bound that is higher than the bound that the slope of the 1500 m–2000 m secant provides. This drawing makes it clear that past the 2000 m record point Rachel's reasoning does not hold. In fact, use of the 1500 m–2000 m will give a bound that is higher than the bound that results from the use of the 1000 m–2000 m secant line.

Recall the third difference noted between Jerica and Brigette's and Rachel's drawings: Rachel's drawing does not have a vertical line going through the mile record point. If Rachel was thinking, as proposed above, this third difference helps to highlight the importance of Rachel's reasoning. In contrast, Brigette's vertical line was used to reason specifically about where the mile record would be, but Rachel's regions generalize Brigette's idea to find an estimate for not only the mile, but for any distance within the interval (1500 m, 2000 m). Although we cannot be sure that Rachel was reasoning from her representation in this way, her choices suggest it might be likely or possible to evoke through a few questions from another student or a teacher.

Just as these students had a period of instrumentally focused presentations which aided in building a foundation, this analysis has taken similar shape. From both psychological and logical perspectives, the instrumentally focused presentations and representations prior to *Phase III* have informed and helped shape this analysis of these students' relational thinking. Indeed, the instrumentally focused *Phase I* presentations were vital for students to share not only how they solved the task, but also how they wrote, drew, and managed proportional reasoning.

Skemp's (1978) discussion of the advantages of relational mathematics emphasizes another important feature of relational thinking: "*It is more adaptable to new tasks*" (p. 12). Skemp notes that when students are involved in relational learning "the means become independent of particular ends to be reached thereby" (p. 14). Rachel's reasoning makes possible a direct generalization to finding a time for *any* distance between 1500 m and 2000 m, not just the mile. Skemp emphasizes the ability to generalize as a particular advantage of relational thinking. Even if Rachel had not planned to reason from her representation in such a way, her reasoning, seen from a mathematical perspective, would allow the region model described above to become the basis for a more general approach. About such an approach, Skemp (1978) writes:

> In contrast, learning relational mathematics consists of building up a conceptual structure (schema) from which its possessor can (in principle) produce an unlimited number of plans for getting from any starting point within his schema to any finishing point. (I say ‘in principle’ because of course some of these paths will be much harder to construct than others.) (p. 14)

As Erin asked more questions, the relational explanations between students continued.

31:12 *Rachel*. The dots on your graph that you have are just ever, ever [arm in air with flat hand tracing in the air along a curve] so slightly curved. And so by taking the two [both hand closed like two points] closest points you're going to get closer [flat hands parallel to each other with small space between] to where it’s going to be, where the curve [bottom hand starts to trace up into a curve and top hand stays in original position] is happening. If we could even find smaller [caliper hand gesture with the distance between caliper fingers getting closer to each other] points, then you're going to get even [with small caliper fingers traces curve in the air section by section] more accurate, but those are the numbers we were given [arms in air spread wide one arm up the other arm down].

When Rachel speaks about smaller points, it is clear from the context and from her gestures that she is not suggesting points that are actually smaller. It seems reasonable that smaller points would be more given data points that are within a closer interval of the mile record. Rachel, from the logic of agency perspective, has again proposed a more accurate representation which she conveyed through gestures. Although the logic of agency highlights these gestures, the logic of proof highlights the logic she used to explain this representation. Indeed, Rachel's suggestion that given even closer data points would result in more accurate upper and lower bounds is not specific to a scenario of world record data points, but extends her thought of an imaginary curve and other points which make up the curve. Mathematically, Rachel's assertions do not necessarily hold if the curve she imagined is not concave up, but she did not explicitly say why this is true.

To sum up, the conceptualization of Erin's questions, as developed so far in this analysis, has motivate the intro two new constructs to extend the initial logical perspective: the *logic of agency* and the *logic of proof*. We have used these more specific constructs to explicate not only Erin's questions, but also Rachel's presentation. In particular the clarification of the logical perspective into the *logic of agency* and the *logic of proof* has provided a new way to conceptualize the data that highlights important connections between the logical perspectives and the psychological perspective. More specifically, Rachel's geometric understanding about the global behavior of a curve allows her to construct (logic of agency) an exaggerated representation (psychological) from which she then reasons further (logic of proof). In fact, specific features of Rachel's representation, present due to the exaggeration of the record points, allow, mathematically, for general reasoning and conclusions about any point within a certain interval and, even more broadly, about any concave up curve.

The discussion continued with more relational explanations:

32:00 *Brigette*. But then you're getting further away from this point [pointing to the mile point shown in Fig. 16, 1 shown in Fig. 22]. So the closer you can get to a slope [pointing to 1000 m–1500 m secant, see 2 in Fig. 22] that's near where you're estimating, the more accurate it's going to be. So that's why we chose this one [pointing to the secant line between 1500 and 2000, 3 in Fig. 22] ‘cause it's right, right in between these two [pointing to the 1500 point and 2000 point, 4 and 5, respectively, in Fig. 22].

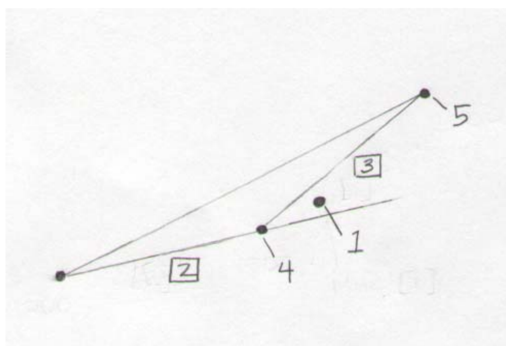

Figure 22. Author's drawing of Rachel's representation with numbered lines and points, indicating order in which Brigette referred to the representation.

32:37 *Erin*. Well right [pause] so [pause] so I'm just saying, why does the slope have to be from one thousand to fifteen hundred?

32:40 *Corinne*. Because we're trying to [pause] as far as like how humans run you gauge your quickness to how far you have to go. So like for instance, if you were going from fifteen hundred to two thousand there [pause] like [pause] to get to that two thousand they're running differently than like the whole idea between one to fiv, fifteen hundred. So you have to like [pause] think of the slow. If they're only going fifteen hundred get that [pause] slow idea and then the f…, longer idea. No, the fast idea versus the longer idea of how they move their body.

33:14 *Erin*. But that's not a longer idea that's fifteen hundred to two thousand, one thousand to fifteen hundred is the same. Like you might be more fatigued in fifteen hundred to two thousand…

Corinne. Yeah so that…

Erin. but …

Corinne. [inaudible]

Erin. So that's why you think you'd use one thousand to fifteen hundred?

Corinne's presentation introduces something new into the discussion- reasoning about, as Taylor calls it, "the animate world: the extraordinary complexity and richness of the behavior of living things." Furthermore, Corinne is using her experience to reason about Rachel's representation. In this way, Corinne connects her experience with her interpretation of the graph, much like the observations described by Ochs et al's (1994) research of working research physicists.

33:30 *Rachel*. For each race basically their pace [pause] was different. So you can take each number and find the pace of it but we're comparing [pause] two paces basically when we're taking the two numbers and doing it that way. Comparing it so that it becomes a slope instead of just their pace.

Although Rachel still does not explicitly justify why the curve must be concave up, this explanation seems to lie close to such reasoning as Rachel made clear that she focused on the secants between record points, not average speeds.

34:19 *Corinne*. Like in a curve if you take, in a curve if you take like a little slice of the curve it’s technically a straight line. Like [pause] when if you see a curve like this but you only took like, like a minute, like a little, little thing then that's technically a straight line. So therefore you could find a cu…, it, a slope in a curve. So maybe that's what's confusing too. Because I was like oh yeah you can find little straight lines within curves and that's maybe [pause] I don't know, did that help?

Although the presentations of *Phase I* were instrumental, the depth of the relational understanding of these students in *Phase III* provides evidence that their instrumental presentations were built from strong conceptual considerations. Indeed, had not some conceptual understanding been built prior to *Phase I*, it seems unlikely that these students would have been ready to discuss and reason further in such relational depth as they did in *Phase III*. Just as the instrumentally focused presentations of *Phase I* seem to have provided the groundwork that prepared these students to think relationally in *Phase III*, it now seems that relational thinking before *Phase I* may have informed the instrumental presentations.

What does successfully solving this task mean for these students? These students could have stopped their work on this task after *Phase I* with very accurate estimates. In fact, the lower and upper bounds found using the *similar triangle method* and the *slope method* were within eight-hundredths of a second from the actual mile record set in 1975. In both *Phases II* and *III*, the other students chose to ask questions and discuss the task further. For example in *Phase II*, the students asked Brigette to review her solution with them, Jerica asked about the secant lines Brigette used, and Rachel asked Brigette why she disregarded her original "high" estimate. After upper and lower bounds were established, Erin's questions, in *Phase III*, probed for a relational justification of why the upper and lower bounds were the closest estimates to the mile record.

The American Heritage dictionary (2000) defines success as "the achievement of something desired [...]" (p. 1728). These students had directed their own inquiry, driven by their personal desire to answer questions that were meaningful to *them*. By the end of *Phase III*, Erin said:

> 40:28 *Erin*. I'm just not convinced that that's like really the slope to take.
>
> [. . .]
>
> 40:47 *Erin*. Yeah, well I still think I'd just like to do one more step in the process I was saying to see if that's really the. . . , maybe I hope I'm right I could be totally wrong, just to make sure that the slope from one thousand to fifteen hundred is really the closest slope.

Indeed, Erin concluded that the logical justification was not complete and she wanted to see "one more step in the process." Erin's criterion for successfully solving this task seems to be based on her desire for logical justification.

CHAPTER IV: DISCUSSION AND CONCLUSIONS

This section will address the last remaining developed question. This question is essential given my interest in continued learning.

How did they reflect on their experience? Given the analysis it now makes sense to bring into the discussion a final data source. In a journal entry written by Erin, on December 9, 2005 she says:

> From the start of class I have enjoyed and valued the idea of why we were doing it. In many cases I felt like a kid again, just beginning to learn math. There were all these new ideas and ways to think.
>
> There was also a comfort level. Perhaps the comfort level was due to being surrounded by other dancers, other people like me, but I do not think so. In fact, each week made me realize how much differently I think than other dancers. We could relate well to each other though and that did provide comfort. I think the real reason the comfort level was so high was the environment the instructors set up. I felt like I could say anything and it would be listened to and valued.
>
> A constant idea that kept surprising me was that math is not black and white. I learned that there are a number of ways to do things, of course

> some are better than others, but they all exist. Most importantly, I was able to learn these things for myself. I think that in previous math situations there was always a sense of rebellion. I did not like that someone was telling me the right way to do something, without an explanation why it was right.
>
> This idea of math not being black and white made it more relatable. It makes more sense that there is more than one way to do things, and so math became more real to me.

This reflection suggests Erin's awareness of her agency not only as a problem solver, but also as an active, sympathetic investigator of other students' thinking. The other students also demonstrated a similar sympathy for one another. For example, after listening to Erin's questions, from the point of view of agency, we may note a particular judgment Corinne made about her presentation to Erin [32:40]. Specifically, there are many ways Corinne could have addressed Erin's questions, but she chose to introduce physical experience to make sense of the Rachel's representation. This also demonstrates a keen awareness of how someone comes to understand something; in particular, Erin's needs as a learner. During Corinne's last statement [34:19], in *Phase III*, she said to Erin, "I don't know, did that help?" This question highlights one intent behind these students' presentations. Corinne's presentations shifted the focus from the experiential context of running to how one can find what she described as a straight line in a curve. These shifts further evidence Corinne's genuine effort and concern for the learning of a fellow student. We see another example of this behavior at the very beginning of *Phase III*, when Rachel answers Erin's first question:

29:28 *Rachel.* Well you can still do it the way of comparing triangles [referring to Jerica's presentation , based on Fig. 5] because you're basically doing the same thing, but you could compare. . . like you can still do the comparing triangle things for this slope [points to secant line between 1000 and 1500] because you're [pause] you have [pause] one triangle that you know two coordinates [moves yard stick in the air in shape of a triangle] and another triangle that's going to be [pause] exactly proportional, but you're missing one of them. So you can compare it in the same way and get what we got for the second triangle [pause] and get [pause] the same number.

Here Rachel, having worked in a group that used the *slope method*, explained to Erin that she "can still do it the way of comparing triangles," in other words the *similar triangle method*, which was the method used by Erin's group. Rachel seemed to make the effort to connect what she explained to a method familiar to Erin.

Similarly, throughout *Phase III* the students displayed an awareness of each other as they chose to accept others' representations. For example, Rachel erased Brigette's drawing and drew her own representation, and Brigette was willing to use this new representation to continue her reasoning. This interlinking of representations could only be possible given an active, welcoming interest from the students to one another and a respect for one other’s agency as learners.

This analysis, shaped by the work of the Honors 250 students and aimed to bring students’ own ways of thinking and communicating into the foreground, resulted in the extension of the starting analytic perspectives. Specifically, the clarification of the logical perspective into two new categories, *logic of agency* and *logic of proof*, has provided a more insight in the relational learning which occurred due to the agency of these learners. Erin demonstrated students’ desire for convincing reasoning in an effort to understand why mathematical conclusions are pinned by logically satisfying reasons. Erin, Rachel, and the other Honors 250 students have demonstrated the importance for students to be able to direct their own inquiries.

As quoted earlier, Rachel’s task journal, written some time after September 27 states:

> I felt like I got an “answer” really fast and then didn’t quite know where to go from there.
>
> After talking about it in class, it made sense to look at the whole graph instead of the closest times. The graph made more sense to me when I thought about really running. For the shorter distances you can keep up a fast pace, for longer one, you have to keep a slower, constant pace. . . so it makes sense that the graph would be concave [up].
>
> It also helped with talking as a class to discover that looking beyond the closest points can help you get closer to the actual time!

Rachel’s journal in combination with Erin’s reflection suggest that the student interactions of September 27, at the very least, played a significant role in how individual students would connect to later reflection and reshape their understanding over time on their own, even with and possibly due to unresolved questions. Erin’s journal highlights part of what Erin was trying to accomplish for herself through her questions on September 27. Erin’s flow breaking questions allow Erin to have a voice in the mathematics and in the classroom. With this voice, Erin seems to suggest that her experience working on this task, and others, not only gave her increased access to serious mathematical ideas, but gave her insight to who she was as a person. Rachel’s journal also gives evidence of how the lack of closure, Erin’s unresolved questions, at the end of *Phase III* added another layer of depth to the student learning. It is clear that Rachel had a sound grasp of the concepts behind her solution, but her journal entry shows that she continued to refine her solution. In this way, Erin’s choice to ask Rachel questions about the mathematical logic of Rachel’s presentation seems to have affected Rachel and broadened her approach to this task. About such a broadening, Skemp (1978) says, “but a schema is never complete. As our schemas enlarge, so our awareness of possibilities is thereby enlarged. Thus the process often becomes self-continuing, and […] self-rewarding” (p. 14). In turn, Rachel joined with Erin, thoughtfully considering Erin’s questions. These students made choices to promote one another’s continued learning. This social value allowed for better cognitive achievement.

With these data and the analysis so far we have the opportunity to see the benefits of what can happen when students are granted the freedom to explore, make choices, exercise agency in particular ways, and become cognitively aware of each other, collectively building ideas. These students did not just exercise agency, but they achieved something. These students successfully reasoned about ideas of Calculus, which they had never seen before.

Just as Erin helped broaden Rachel's approach, Erin's questions and Rachel's representation and reasoning, in the view of the analytical perspectives, has the potential to enlarge one's approaches toward students as active agents of their own learning with the desire for compelling lines of reasoning and as caring individuals in a community of learners, desirous for success in both roles.

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