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An Investigation of How Preservice Teachers Design Mathematical Tasks

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ABSTRACT

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The tasks with which students engage in their mathematics courses determine, for a large part, what students learn. Therefore, it is essential that teachers are able to design tasks that are worthwhile for developing mathematical understanding. Since practicing teachers seldom incorporate worthwhile mathematical tasks in their lessons, we would expect that they did not become proficient at designing worthwhile tasks while in their teacher education programs. This thesis describes a study that investigated what preservice secondary teachers attend to as they attempt to design worthwhile mathematical tasks. Three participants were selected from a course at a large private university where preservice teachers are taught and practice the skill of task design. This "Task Design" course was observed, and the three participants were interviewed to determine what they attend to while designing tasks. There were seven main characteristics that the main participants in the study attended to the most often and thought were the most important: sound and significant mathematics, reasoning, appropriateness, clarity, communication, engagement, and openness. How the participants attended to these characteristics is described. Some implications for teacher education, such as requiring preservice teachers to explain how their tasks embody certain characteristics, are given based on the results.

Keywords: worthwhile mathematical tasks, preservice mathematics teachers, task characteristics

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Chapter 1 - Introduction

Mathematical tasks are an essential part of teaching because they are the tools that teachers use to help students learn certain mathematical concepts (Simon & Tzur, 2004). Tasks "determine not only what substance [students] learn but also how they come to think about, develop, use, and make sense of mathematics" (Stein, Grover, & Henningsen, 1996, p. 459). Additionally, the tasks that students engage with in their mathematics courses "convey messages about what mathematics is and what doing mathematics entails" (National Council of Teachers of Mathematics [NCTM], 1991, p. 24). Since "what students learn is fundamentally connected with how they learn it" (NCTM, 2007, p. 18), the types of tasks teachers present to their students to engage with determine students' learning opportunities. Therefore, the tasks that teachers use in their instruction must be worth spending instructional time on, not only because they affect how a student must think, which impacts how students will learn, but also because they affect what students perceive the nature of mathematics to be.

Designing mathematical tasks is one of the most important responsibilities of mathematics teachers (NCTM, 1991; Remillard, 1999). Mathematical tasks may be more or less worthwhile for developing mathematical understanding depending on the characteristics they possess (a discussion of which is in the Theoretical Framework Chapter.) Researchers have found, however, that many teachers include tasks in their instruction that the researchers consider to be less worthwhile than desired. For example, Jacobs et al. (2006) analyzed the TIMSS 1995 and 1999 video studies in order to see to what extent the eighth-grade U.S. mathematics teachers taught or did not teach according to the documents (or the *Standards*) set forth by NCTM (1989, 1991, 1995, 2000). These researchers believed that mathematical tasks and how they were used during lessons was a large indicator as to whether teachers were exhibiting standards-based teaching. Most teachers in their study used tasks that were not complex and that could be solved using a routine procedure, which means that the tasks were exercises rather than worthwhile tasks. Many of the teachers in the study believed they were teaching in accordance with reform, but they were not. So, it is likely that many teachers currently do not design worthwhile tasks to include in instruction even if they think they do.

Since we know that practicing teachers seldom incorporate worthwhile mathematical tasks in their lessons, we would expect that they did not become proficient at designing worthwhile tasks while in their teacher education programs. In order to help preservice teachers (PSTs) improve their task design skills, we must learn how they design tasks. Knowing more about how PSTs design tasks is the first step toward the improvement of their task design skills. The purpose of this study is to describe how PSTs design worthwhile tasks in an effort to improve the task design skills of future teachers. Learning what PSTs attend to as they attempt to design worthwhile tasks is valuable because this gives teacher educators specific information they could use to work to improve how PSTs design tasks. Being able to design worthwhile tasks would be an invaluable asset to the PSTs once they start to teach because this essential skill would give their secondary students access to tasks that allow them to develop a rich understanding of mathematics.

Chapter 2 – Theoretical Framework

In this chapter, I first define a mathematical task and task design. I then describe what makes a mathematical task worthwhile. Finally, I detail a framework for inferring teachers' knowledge and beliefs. Such a framework is needed so that I can infer the PSTs' knowledge and beliefs concerning what makes a task worthwhile.

Mathematical Tasks and Task Design

A mathematical task is "a classroom activity, the purpose of which is to focus students' attention on a particular mathematical idea" (Stein, et al., 1996, p. 460). Projects, problems, constructions, applications, and exercises are examples of mathematical tasks (NCTM, 1991). There are three ways that teachers can design mathematical tasks. First, teachers can select a task from the resources that are available. These resources could include the textbook they are using, other curriculum materials, or the internet. Second, teachers can adapt already existing tasks. Often, teachers can find a task, investigate it, and make changes to the task so that it fits their needs. Third, teachers can create a task from scratch. Throughout this document, task design will refer to all three of these methods: selecting, adapting, and creating.

Worthwhile Mathematical Tasks

My view is that a mathematical task is worthwhile for developing mathematical understanding if the task is worth spending instructional time on and that a task can be more or less worthwhile depending on the characteristics it possesses. Worthwhile mathematical tasks are included as one of the standards in the *Professional Standards for Teaching Mathematics (Professional Standards)* (NCTM, 1991). The *Professional Standards* and other researchers urge teachers to take into account many considerations as they design worthwhile tasks, but the literature does not have a uniform definition of what makes a task worthwhile. In other words, the literature neither agrees on or clearly outlines which task characteristics must be present and to what degree in order to classify a task as worthwhile. There are, however, some characteristics—that the task is based on important mathematics and requires students to problem solve and reason—that are emphasized across the literature and seem to be a requirement of a worthwhile task.

Worthwhile mathematical tasks introduce students to and engage them with important mathematical ideas (NCTM, 2000). The *Professional Standards* stated that a worthwhile mathematical task should be based on "sound and significant mathematics" (NCTM, 1991, p. 25), and Hiebert et al. (1997) argued that appropriate tasks should require students to think about important mathematical ideas. Beyond the content of a task, the reasoning a task requires of students is also main focus in the literature. Although authors may describe this reasoning using different terminology, such as problem solving or cognitive demand, it is clearly vital that students are required to think in order to learn mathematics with understanding. The *Professional Standards* stated that worthwhile tasks should be intellectually engaging and require students to reason and problem solve (NCTM, 1991). Hiebert et al. (1997) argued that appropriate tasks should be real problems to students in that students do not already know how to solve them.

The concept of cognitive demand has been used by many researchers to gauge the quality of the reasoning required of mathematical tasks. The cognitive demand of a task is "the kind and level of thinking required of students in order to successfully engage with and solve the task" (Stein, Smith, Henningsen, & Silver, 2009, p. 11). Stein et al. (2009) explained that tasks of different levels of cognitive demand provide different learning opportunities for students. According to Smith and Stein (1998), in order to determine if a task is worthwhile, a teacher needs to consider how a student must think in order to complete a task. If a task only requires students to use a rehearsed procedure, for example, it is not worthwhile for developing mathematical understanding. A task that requires a high level of cognitive demand requires students to explore relationships between different representations, think about their own thinking, and solve problems they do not already know how to solve (Stein, et al., 2009). Stein et al. (2009) argued that tasks that require a high level of cognitive demand are worthwhile to include in instruction because of their potential for developing mathematical understanding.

The thinking that is required for students to engage with a task is influenced by their background knowledge and experiences. Tasks should be challenging, yet accessible (NCTM, 2000). Smith and Stein (1998) suggested that a teacher must consider the age, grade-level, prior knowledge, and experiences of their students when deciding the quality of a task. Smith and Stein explained that if a teacher gave a sixth grader a task where she must add five two-digit numbers and explain how she did it, the student would most likely use the algorithm and list the steps she went through to solve it. However, if the teacher gave the problem to a second grader who did not know the algorithm and who had manipulatives available, the student would be more likely to explore the mathematical concepts. A worthwhile task requires students to use current knowledge to solve the new task (Hiebert, et al., 1997) and “creates the need” (Confrey, 1993, p. 318) for a new concept by requiring students to construct new knowledge about a concept.

There are other characteristics that seem to be helpful in making a task worthwhile. The *Professional Standards* (NCTM, 1991) included that a worthwhile task should convey that mathematics is an ongoing human activity, require students to communicate about mathematics, help students to make connections among mathematical ideas, and develop students’ skills along with understanding. The *Professional Standards* (NCTM, 1991) also suggested that teachers consider students when designing tasks, recommending that worthwhile tasks be based on "knowledge of students' understandings, interests, and experiences" and "knowledge of the range of ways that diverse students learn mathematics" (p. 25), and that worthwhile tasks should develop students' dispositions to do mathematics. Smith and Stein (1998) also stated that tasks should require deep thinking from students, but should not be too difficult for students so that students feel they can engage with the task. My view is consistent with the literature that a worthwhile task must be based on important (or sound and significant) mathematics and require students to reason and that there are many other characteristics, such as promoting communication and being interesting to students, that help a task to be more worthwhile.

The literature has described many characteristics in discussions about worthwhile tasks, but one of the focuses of this project was to determine which characteristics the PSTs felt were important in a worthwhile task. After determining which characteristics were the most important to them, I studied the literature's views about these ideas to inform me about the characteristics in order to create literature-derived definitions for these characteristics. These literature-derived definitions are presented in the Results Chapter as I describe the characteristics the PSTs felt were important.

A Framework for Inferring Knowledge and Beliefs

PSTs have their own ideas about worthwhile tasks that are likely to differ from those of researchers and of their instructors. In this study, I collected sufficient evidence to infer the PSTs' knowledge and beliefs about worthwhile tasks, and I assumed that the PSTs' knowledge and beliefs influenced how they designed tasks. To make these inferences, I use Leatham (2006), wherein he laid out a "sensible system" (p. 92) framework for inferring the beliefs of teachers. Leatham viewed knowledge, or what we "more than believe," as a subset of beliefs, or what we "just believe" (p. 92), so this framework can help to infer both knowledge and beliefs. Leatham views teachers' beliefs as a "sensible system" (p. 92), which is "an internally consistent organization of beliefs" (p. 93). The sensible system framework assumes that beliefs influence someone's actions, although we cannot assume that a particular belief will always result in a particular action. If a researcher interprets a teacher to have a certain belief, the teacher's actions may show if the interpretation is close or not. Teachers are sensible decision makers, so if it seems like the beliefs and practice of teachers are inconsistent, then the researcher must rethink his conclusions about what the teachers' beliefs were or consider if another belief was deemed more important in that situation.

Chapter 3 – Literature Review

The purpose of this research is to describe how secondary PSTs design mathematical tasks. I will discuss below literature that concerns how teachers in general design tasks since the literature is dispersed between four different types of teachers (elementary inservice, elementary preservice, secondary inservice, and secondary preservice), and there is little research about secondary PSTs specifically. The research on inservice teachers likely gives us a best-case view of how PSTs might design tasks. The literature about elementary PSTs, however, may under represent secondary PSTs because the latter receive more training, of course, in mathematics teaching and mathematics content.

Much of the literature concerning how teachers design tasks describes teachers who were given some sort of intervention, such as professional development or a course. After these interventions, teachers tended to improve in the areas on which the researchers (and the interventions) were focused. For example, Boston and Smith (2009) reported that after professional development focused on selecting and enacting cognitively demanding mathematical tasks, secondary inservice teachers selected high-level tasks more often and improved in their ability to maintain high-level cognitive demands.

Research has described some of how teachers design tasks by describing some of the considerations that the teachers attended to (or thought about, paid attention to, or considered), but typically a full description of teachers' considerations is not the focus of the studies. The literature generally does not fully describe how teachers designed tasks or all of the considerations teachers attended to since they were often more concerned with describing how the intervention yielded particular results. The literature does, however, shed some insight into what teachers attend to as they analyze or design tasks since many studies describe the tasks that the teachers designed.

In most studies that gave some information concerning what teachers attended to, the researchers told the teachers to attend to a specific framework. These researchers tried to improve how teachers analyzed or designed tasks according to this particular framework by explicitly teaching them about the framework. These studies will be described first, followed by the few studies in which the researchers did not tell the teachers to attend to particular ideas.

Told to Attend

All of the studies which taught teachers a specific framework were either exclusively about or included a focus on the reasoning required by the students. Teachers' ability to analyze tasks according to their cognitive demand or design tasks that require high levels of cognitive demand was studied the most often. The researchers used the four levels of cognitive demand in the Task Analysis Guide (Stein et al., 2009).

Regarding analyzing tasks, both Osana, Lacroix, Tucker, & Desrosiers (2006) and Boston (2013) included a description of teachers' ability to sort tasks accurately according to their level of cognitive demand after the intervention. Osana et al. (2006) found that overall the PSTs had more difficulty accurately classifying tasks with higher levels of cognitive demand than classifying tasks with lower levels of cognitive demand. They also found that many of the PSTs were influenced by task length and considered shorter problems to have low cognitive demand. One of Boston's (2013) goals was to describe how the secondary mathematics teachers' knowledge of cognitive demand changed after participating in professional development. The teachers improved in their ability to identify low-level tasks, but they continued to struggle classifying certain tasks as high-level because they thought that the presence of a procedure meant that the task was low-level. Boston found that two characteristics (explanation and real-world context) misled some of the teachers into thinking that a task was high-level when it was actually low-level and vice versa. Some of the teachers identified high-level tasks as low-level because they did not require an explanation or did not connect to a real world context, and some identified low-level tasks as high-level if it did require an explanation or contain a real-world context. After the professional development, the teachers began using the terms representations, generalizations, connections, and open-ended in their classification rationales, and they used terms related to explanations, procedures, and computations more often in their rationales.

Several researchers (Boston & Smith, 2009; Koleza, Markopoulos, & Nika, 2011; Kosko, Norton, Conn & San Pedro, 2010; Stein et al., 1996; Thomas & Williams, 2008) taught teachers about cognitive demand in order to help them to design tasks that were of high cognitive demand, though designing tasks usually was not the sole focus of the studies. Stein et al. (1996) and Koleza et al. (2011) provided additional characteristics of the tasks that the teachers designed in addition to describing their improvement in designing high cognitive demand tasks. At the end of both studies, the teachers' tasks were more likely to have many possible solution strategies and use multiple representations. Stein et al. (1996) also reported that a majority of the middle school teachers' tasks required mathematical justifications. Koleza et al. (2011) found that the elementary inservice teachers initially struggled to raise the cognitive demand of tasks, and thought that the complexity of the arithmetic determined the level of cognitive demand.

Some (Kosko et al., 2010; Thomas and Williams, 2008) reported on teachers' ability to design specific tasks according to their framework, which included but went beyond cognitive demand, but did not offer other considerations the teachers attended to outside of the framework that the teachers were taught. Thomas and Williams (2008) attempted to improve secondary inservice teachers' ability to design performance-based tasks for urban learners that were culturally relevant and that required a high level of cognitive demand. After the year-long professional development, most of the tasks the teachers designed demonstrated most of the characteristics of performance-based tasks, some of the tasks required high cognitive demand, but most of the tasks were not culturally relevant. Thomas and Williams (2008) explained that the teachers thought the tasks were culturally relevant since they involved a real-world scenario, but they were not actually culturally relevant since they were not tied "to specific occurrences that students could relate to their current situations nor to their cultural upbringing" (p. 113). Kosko et al. (2010) reported on how well secondary PSTs were able to design tasks that imbedded the NCTM (2000) "Process Standards" and were of high cognitive demand, as well as on how well the tasks actually elicited these mathematical processes from students. They found that PSTs improved in their ability to elicit these cognitive processes at high cognitive demand, but they still struggled to discern between the two highest levels of cognitive demand.

As previously mentioned, most researchers who taught teachers a particular framework focused on or included cognitive demand. Norton and Rutledge (2007) and Crespo and Sinclair (2008) did not include cognitive demand in their framework, but they still did focus on reasoning. Norton and Rutledge (2007) utilized an iteration of the letter exchange that was conducted in Kosko et al. (2010), but Norton and Rutledge (2007) focused on improving secondary PSTs' ability to elicit the activities found in the NCTM (2000) "Process Standards" (rather than also focusing on cognitive demand). The teachers improved in their ability to elicit the cognitive activities described in the "Process Standards," especially communication and representation. Crespo and Sinclair (2008) found that elementary PSTs posed better problems if they first explored related mathematical contexts. The authors felt that in order for the teachers to be able to pose good mathematics problems, they must have the opportunity to figure out what may be problematic about a situation. At the beginning of the study, the tasks the PSTs designed were factual, simple, and clear but required little cognitive work and were not very interesting. They then taught the teachers about Vacc's (1993) framework for evaluating problems, which involves deciding if problems are factual, reasoning (not obvious), or open (where information is known by students, but many answers are acceptable) and about the aesthetics of the problems, which includes that the task is interesting, surprising, novel, or simple. The teachers' tasks, in the end, became more focused on reasoning.

Not Told to Attend

It is less common for researchers to investigate how teachers design tasks without telling the teachers to focus on something in particular. I will describe the findings of three such studies. Arbaugh and Brown (2005) used task sorts to determine how learning about cognitive demand influenced their high school teachers' thinking about mathematical tasks. They did teach the teachers a particular framework (of cognitive demand), but they described in some detail how the teachers analyzed tasks before they were taught about cognitive demand, and they did not tell the teachers after they learned about cognitive demand how to sort the tasks. For the initial sorts, many teachers sorted based on surface characteristics, such as the mathematical topics, the processes the students must complete, or the fact that the task was set in a real-world context or involved manipulatives. This finding is consistent with the observations of Stein et al. (2009) that when teachers classify tasks according to their cognitive demand, they are sometimes misled by "superficial features" (p. 7), such as when a task requires manipulatives, uses real-world situations, requires several steps, or uses representations. In Arbaugh and Brown (2005), a majority of the final sorts, however, were based on the level of thinking that was required by the students.

In Crespo (2003), elementary PSTs were involved in a letter exchange with students throughout an 11-week course, and the author described how the letter writing experience changed how the teachers posed problems. The course involved doing non-traditional mathematical tasks and collaborative problem posing, but the teachers were not taught a particular framework. The focus of the letters and tasks within them were up to the PSTs. In the beginning, the teachers' main approaches were to make the problem easier to solve, pose familiar problems, or pose problems blindly. At the end of the course, the teachers' main approaches were to pose unfamiliar problems, pose problems that challenge thinking, and pose problems to get at students' thinking. By the end of the study the teachers had changed their beliefs of what makes a task worthwhile in that they wanted to challenge students with the tasks, and they were more willing to risk confusion. The tasks at the end of the study were more open-ended, puzzle-like, less typical, less leading, required justification, more complex, or brought out students' thinking.

Remillard (1999) did not provide a particular intervention for her teachers. Rather, the author examined how two elementary inservice teachers used a new reform-oriented textbook, with one of the main goals being to describe how the text affected how the teachers designed tasks. One of the teachers tried to find tasks that made students reason, but found that the textbook did not support these efforts and so looked to the book less and less. She "invented" her own tasks that could not be solved with rote skills, could be solved multiple ways, and required students to explore or explain (Remillard, 1999, p. 324). The other teacher "appropriated" tasks from the textbook (Remillard, 1999, p. 323). This teacher first chose exercises and avoided tasks that required manipulatives or discussion but later became more likely to choose tasks that helped students to explore mathematical meaning.

Teachers' Perspectives

It is uncommon for the research about teachers and task design to focus on teachers' point of view. De Araujo and Singletary (2011) argued that teachers' perspectives were lacking in the current literature about task design. Their research is unique in that their goal was to describe secondary inservice teachers' views about worthwhile tasks and compare those views to the vision put forth in the *Professional Standards* (NCTM, 1991). They organized their teachers' conceptions about the characteristics of worthwhile tasks into three main categories: (1) related to the mathematical content (including embedding the mathematics in a meaningful context and promoting students to develop the mathematics themselves), (2) mathematical processes (including promoting problem solving, promoting mathematical communication, and making connections) and (3) practical considerations (including that the tasks are achievable and standards-based). They found that several teachers' conceptions aligned with the definition of a worthwhile task found in NCTM (1991). NCTM (1991) and the teachers each had a characteristic of worthwhile tasks that the other did not. NCTM (1991) discussed developing mathematical skills, and the teachers discussed the need to base tasks on the standards they are required to teach.

Thus, we have some information about what teachers attend to as they speak about, analyze, or design tasks, but it is difficult to determine exactly how the research about different types of teachers relates to secondary PSTs. The literature does suggest some possible ways that PSTs may design tasks, though. For example, PSTs may attend to a variety of task characteristics, such as many solution strategies, multiple representations, and the cognitive demand of the task (e.g., Stein et al, 1996; Koleza et al., 2011). The PSTs also may be misled by surface characteristics, such task length, the complexity of the arithmetic, whether the task required an explanation, or whether the task was set in a real-world context when they are attempting to determine the cognitive demand of a task (e.g., Boston, 2013; Osana et al, 2006). The PSTs may design tasks that were familiar or that they had not fully investigated or perhaps design tasks that would challenge student thinking or bring out students thinking as the elementary PSTs in Crespo (2003). The PSTs may feel that characteristics, such as that a task is set in a real-world context, requires problem solving, requires communication, is achievable, and is standards-based, are important to a worthwhile task, such as the teachers in de Araujo and Singletary (2011).

Most of the research, however, does not give a full description of what the teachers attended to while attempting to design worthwhile tasks. The research generally focused on how the teachers attended to a particular framework chosen by the researchers instead of describing how the teachers designed worthwhile tasks when the teachers could decide for themselves what they felt was important in a task. Although de Araujo and Singletary (2011) actually described teachers' conceptions of worthwhile tasks, they did not study how teachers design tasks to make them worthwhile. They argued that further research is needed in order to understand how teachers actually design tasks to make them worthwhile, which is the purpose of the current research study. Furthermore, this research is focused on preservice rather than inservice secondary teachers.

I argue that it is important to know what secondary PSTs attend to and *how* they attend to these considerations when they are not told to attend to a particular framework. Knowing how secondary PSTs design tasks when they can choose which characteristics of tasks are important to them and how they will apply those characteristics is vital if we are to intervene to help them improve their thinking about tasks. It is important to know what we can specifically do in order to help PSTs to attend to considerations in ways that will help them to make significant improvements. This can lead to more focused efforts to improve how they design tasks.

Research Question

The purpose of this research is to better understand what considerations secondary PSTs attend to and how they attend to these considerations when designing tasks. Thus, my research question is the following: What characteristics of tasks do PSTs attend to as they attempt to design worthwhile mathematical tasks and how do they attend to these characteristics?

Chapter 4 - Methodology

In this chapter, I describe the setting and participants of my study, the data that I collected and how I collected it, and how I analyzed the data. This description illustrates how I was able to answer my research question.

Setting

My study focused on three of the secondary PSTs who were enrolled in the "Task Design" course at a large private university in the fall semester of 2011. The title of the course was "Task Design and Assessment of Student Understanding." In the Task Design course, secondary PSTs were taught and practiced the skill of task design. The instructor laid out seven learning outcomes for the course. She wanted the PSTs to (1) understand certain selected mathematical concepts; (2) be able to identify concepts and procedures in a textbook unit, mathematical topic, or task; (3) be able to analyze tasks in order to anticipate what students would likely learn from it; (4) understand how tasks support students' development of conceptual understanding and procedural fluency and be able to design tasks that help students to develop their understanding of concepts and procedures; (5) understand how assessment can be used to support students' development of conceptual understanding and procedural fluency and be able to design tasks that provides information about students' understanding; (6) know how using mathematical tasks can improve their instruction; and (7) participate in the mathematics community.

The class periods included discussions of readings, the completion and discussion of tasks, and working on written assignments. Regarding the readings, most class periods included a discussion of an assigned reading. The PSTs' readings were Skemp (1976), chapter four of Black, Harrison, Lee, Marshall, and William (2003), the first ten chapters of Stein et al. (2009), and the first fifteen chapters of Schoen (2003). The instructor generally wrote discussion questions on the white board, asked the PSTs to discuss them at their tables, walked around the room and sometimes contributed to the conversations at the tables, and then led a whole-class discussion on the topic.

Regarding tasks, the PSTs frequently completed tasks and discussed them as a table and/or in a whole-class discussion. They completed all of the “Task Sort” tasks from the Middle School Task Sort from Smith, Stein, Arbaugh, Brown, & Mossgrove (2004) and also did several tasks involving periodic functions.

Regarding written assignments, frequently the PSTs were given time in class to work on their assignments. They had five main assignments: the Adapted Tasks Assignment, Project #1, the Assessment Plan, the First Pen Pal Task, and Project #2. (Most of the tasks they designed were instructional tasks, with the goal of helping students to learn new concepts. The First Pen Pal Task, however, was assigned with the purpose of assessing the students’ understanding of a concept.) For the Adapted Tasks Assignment, the PSTs were required to improve three of the Task Sort tasks (see Smith et al. (2004) for a full list of the tasks) and explain why they made the changes they did. The projects were full lesson plans (including associated tasks) written on a lesson plan template. The first section of the template had them list the “fundamental mathematical concepts (with concept descriptions),” “the requisite mathematics,” the applicable *Common Core State Standards for Mathematics (or “Common Core” standards)* (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGACBP & CCSSO], 2010), the materials needed, and an abbreviated unit sequence. The actual lesson included sections for “launching student inquiry,” “supporting productive student exploration of the task,” and “facilitating discourse and public performances.” As part of their description of the lesson plan sequence, the PSTs were asked to anticipate student thinking and to describe their potential responses to that thinking. The PSTs reviewed each other's lessons during class before they were turned in. Project #2 was similar in nature to Project #1, but also required the PSTs to include assessment questions, have three others complete their tasks, and adapt the tasks based on that student work.

In the Assessment Plan, the PSTs described the kinds of formative and summative assessment they planned to use; this was not a focus for my study. Lastly, for the First Pen Pal Task, the instructor located a precalculus teacher who was willing to have her students complete tasks that the PSTs designed. (The idea for the letter exchange came from Crespo (2003) and Norton and Rutledge (2007).) The instructor introduced the First Pen Pal Task to the PSTs as an assessment task, with the goal of providing evidence of student thinking that the PSTs could analyze. Each of the PSTs wrote a letter to a precalculus or honors precalculus student that included a task. The PSTs were told that the task should last no more than thirty minutes and were provided with the sections in the *Common Core* (NGACBP & CCSSO, 2010) that included concepts that the precalculus students had just been taught or were about to learn. The precalculus students completed the tasks and the PSTs graded their responses. The first iteration of the letter exchange (the First Pen Pal Task) was completed by all of the PSTs in the course. My three main participants also posed an additional task to their students (the Second Pen Pal Task), which the students also completed. This letter exchange allowed the PSTs more time than they would normally have when working with students in person; they had additional time to practice new skills and reflect on what they were doing (Crespo, 2003).

The instructor did not give the PSTs a framework they should follow as they designed tasks. In one instance, she explained to them that she did not give them a structure to follow since it would likely not work for many of them. The instructor expressed to the class on several occasions that teaching is personal and that the PSTs needed to find the teaching methods that were right for them. On one occasion she stated,

> Whatever you're doing in the classroom is coming from such an inner part of you, and so all I feel that I can really do that will help you is give you principles and then hope that you can figure out how to apply them within your own paradigm…. Think about what kind of teacher you want to be and then try to apply these principles to that vision.

The class would discuss tasks and offer various aspects they liked or discuss what could be improved, but the instructor usually did not tell them the "right" way. It seems like she wanted to expose them to many tasks and have them design and discuss tasks in order to figure out what worked for them. When the instructor asked the PSTs to improve tasks, she stated that the PSTs could get full credit even if they adapted the task in a way that was different than what she would have done. She also expressed that teachers need to find the balance for themselves, based on their goals, about how open and how much structure a task is going to have.

The instructor's philosophy for teaching the PSTs is consistent with how she thinks about teaching secondary mathematics students. She explained that she believed that students learn from making choices and struggling. She also stated that she felt that the teacher should not jump in before the student has had enough time to make choices in the task and struggle with the concept on his own because then the student would not have as much opportunity to learn.

Even though she did not give the PSTs a comprehensive framework of what they should consider when they design tasks, she did give them important ideas to think about. (If and how the instructor approached various task characteristics will be discussed in the Results Chapter.) Course readings included the Mathematical Tasks Framework from Stein et al. (2009), but the instructor did not require the PSTs to specifically apply the Mathematical Tasks Framework to their task design except when they adapted tasks and were told to raise the cognitive demand.

She told them that in order to make the task better, it should not still be procedural, but should be more open, richer, and move from a low to high level of cognitive demand.

The Task Design course was part of a reform-oriented mathematics education program that consisted of six core mathematics education courses: (1) Critical Review of School Mathematics (which was meant to strengthen PSTs' understanding of high school mathematics), (2) Exploration of Teaching (which is "a field-based initial teaching experience") (3) Task Design and Assessment of Student Understanding, (4) Mathematics Teaching with Technology, (5) Mathematics Teaching in the Public Schools (a methods course with some field experiences) and (6) Student Teaching. The prerequisites for the Task Design course were Honors Calculus 2, Fundamentals of Mathematics (introduction to mathematical proof), Elementary Linear Algebra, and the first two mathematics education courses. It was typical for mathematics education students to take the Task Design course in their junior year.

I studied PSTs in the Task Design course because I wanted to see how PSTs design tasks when they are able to focus on task design. Later in their program, such as when they student teach, they have a variety of other responsibilities apart from task design. There is not another time in their program where they devote such a large portion of time and effort to designing tasks. This allowed me to learn a great deal about what they attend to while they design tasks. Additionally, I had a goal to determine some implications for how PSTs are taught task design. The hope is that PSTs in the future will be able to learn how to design tasks more effectively, so that they will be able to practice this skill as they continue in their program and student teach. That way, they will be more likely to be proficient task designers by the time they start to teach.

Participants

Three of the PSTs from the Task Design course were selected to be the three main participants for this study. Choosing only three participants gave me a good understanding of what these three participants thought about task design. Also, since I had five data collections, more than a few participants would have been beyond the scope of my project.

In order to select these participants, the entire class was given a survey (see Appendix A) at the beginning of the course. The survey collected background information about the PSTs and assessed their initial knowledge about mathematical tasks. The surveys were coded and analyzed in order to choose three participants that differed from each other in as many meaningful ways as possible and whose responses were representative of the variation in the class. I also looked for participants who were articulate since I needed subjects that could explain their thinking well. The three main participants that were chosen consented to be part of the study and consisted of two white females, Carly and Jamie, and one white male, Ryan. (These names are pseudonyms.)

On their surveys, these three participants differed in several meaningful ways. First, several responses suggested that their background knowledge and experience was varied. The participants varied in their confidence in their knowledge of college-level mathematics; both Carly and Ryan stated they were very competent in college-level mathematics, while Jamie explained that it takes her some time to understand the material. They varied in their tutoring history; Carly had tutored many students, Ryan had tutored in high school, and Jamie had never tutored. Also, Jamie focused more on meaning and definitions than the other two participants when asked about the knowledge a student would need to solve the task in question six.

Second, the participants seemed to have a different initial knowledge of tasks. The characteristics they listed that described a good task varied. They also gave varying responses when asked which of two tasks was better and if a task was good or not. For example, their responses were different on which task they would choose and why on question five. Carly would choose Task B since it does not lead students to a specific solution path, Jamie would choose Task A since it would build off of what students know and would be engaging, and Ryan would combine the two tasks, so that the new task would give them something "concrete," but also "stretch" them to think.

My three participants all had the same instructor for the Exploration of Teaching course (the semester prior to taking this class). In that class they had read the *Professional Standards* (NCTM, 1991), which gave them an introduction into what worthwhile tasks are. They had also conducted nine observations of secondary classrooms, and in one of the observations, they had been asked to pay attention to the tasks the teacher used.

I expected that my participants' task design skills would change as the semester progressed, but how they changed was not the focus of my study. I was able to collect a body of data to inform what the participants persistently attended to as they designed tasks. I also recognized that it most likely would not be possible to delineate exactly how the Task Design course and the instructor impact how they design tasks, but my goal is to describe how my participants design tasks, not determine the exact factors which lead to their beliefs. Through my data collection, I sought to ascertain what the PSTs thought was important about tasks – even if what they thought was important was different from what they thought that the instructor valued.

Data Collection

I collected several different types of data: observations, field notes, written work, and interviews. I attended the Task Design course and videotaped the class throughout the semester. I did this in order to learn what the PSTs were taught in class and how my participants spoke about tasks in class in order to see if this was consistent with how they spoke about tasks in the interviews. The room where the class was held was equipped with two built-in video cameras. One of the video-cameras was positioned so that it could swivel and capture the entire classroom. The other was positioned directly above a classroom table, which is where my three participants were seated. So, I was able to collect data from my group of participants and also collect data from part of the rest of the class at the same time. I took field notes during my observations, recording what the instructor and PSTs were doing and the ideas that were discussed. The field notes helped me to get a more accurate portrayal of what was happening during the observations. I also made copies of the three participants' written work throughout the semester, including notes, class work, homework, and exams.

The primary source of data for the study was five sets of interviews with the three participants. These interviews were spread throughout the semester and each lasted about an hour. The interviews were meant to get at what the PSTs attend to as they design or discuss tasks. Four of the interviews were done individually, and one of the interviews was a group interview with all three of the participants. The last four interviews focused specifically on tasks the participants had designed. Because I was able to do some initial analysis during data collection, I added questions to the interview protocols[1] for each participant to attempt to reveal more of their understanding of tasks. Also, I conducted interviews based both on tasks they designed for the instructor and tasks they designed for secondary students to complete, since I hypothesized that the PSTs may focus on different characteristics if the task were going to be solved by an actual student instead of only be graded by their instructor.

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[1] For each of the interviews protocols in the appendix (except for the group interview), I have included the protocol that was used for one of the participants. An asterisk is used to denote questions that were unique to that participant's protocol. The questions that are slightly different for each participant, but were asked of all participants in that interview, do not have an asterisk (e.g., what would a student already understand about ______ (a certain mathematical concept)?)

In the first interview, the participants sorted the Task Sort tasks twice, each time according to a task characteristic of their choosing (see Appendix B for the interview protocol). The instructor had the PSTs complete and discuss these tasks in the first few weeks of the class and before I had conducted the first set of interviews. The purpose of including this activity in the first interview was to learn what the participants' initial ideas were about tasks and to learn about how they view the characteristics they used to sort the tasks with. I also asked questions about students, such as the knowledge a student would need to know to complete the task, in order to give more information about what they attended to while analyzing the task.

The second interview centered on Project #1 (see Appendix C for the interview protocol). I analyzed their projects and created sample student work before conducting this interview. In the interview, I asked the participant to walk me through what they thought about when they designed the task and asked questions to provide information about their motivations for making the choices they did. The participants were then given the sample student work and asked to analyze it and describe if and how they would like to adapt the task after seeing the sample work.

In the third interview, the three main participants were interviewed as a group (see Appendix D). They were asked to work together to design a task that would fit in a geometry unit in high school with the goal for students to understand and be able to justify the characteristics of quadrilaterals. After they designed the task (the "Quadrilateral Task"), they were asked questions about why they chose this task and about what students would need to know to do the task and what they would learn. The group was also given a list of seven characteristics to rank in order of importance. The list of task characteristics included the characteristics that appeared to be the most important to each of the participants in the first two interviews. I chose to have participants work as a group in this interview because I thought that being in a group may have helped them to be more descriptive than they would have been if they were designing a task alone in an interview since they needed to explain themselves to each other. They also may have felt pressured and struggled to design a task by themselves; the group interview allowed the participants to share ideas.

The fourth interview was centered on the letter writing exchange (see Appendix E). After the participants had designed their First Pen Pal Tasks and the secondary students had completed them, I analyzed the tasks and conducted the interviews. I asked questions about why they designed the tasks the way they did. They were also presented with the precalculus students' work on their tasks and asked to assess their responses. Finally, they brainstormed a follow-up task (the Second Pen Pal Task) based on the students' responses to the First Pen Pal Task. After the interview, they posed their follow-up task, which the secondary students completed. This interview gave me some insight into how my participants designed tasks that were going to be posed to students.

In the fifth interview, the participants were asked to complete and discuss two tasks and were also asked questions about Project #2 (See Appendix F). The first task, the Irrational Number Task (from Cooney, Sanchez, Leatham, & Mewborn (2001)), was one where the student learns the fundamental mathematical concepts by solving the task, but the fundamental concepts are not obvious just from looking at the task. The second task, the Factoring Task, was a straight-forward exercise, where the fundamental mathematical concept is clear in the instructions of the task. The PSTs completed these tasks and were asked questions about what they thought about them. The PSTs were then given a list of task characteristics they had previously mentioned and asked to describe if and how they were related to each other. Lastly, they were asked questions about what they attended to while designing Project #2, which I had analyzed prior to the interviews.

Data Analysis

Data were analyzed as much as possible during the fall semester of 2011 in order to continually inform data collection, but were analyzed in more depth after the data had been collected. The surveys provided some of what my participants initially thought about tasks. I was able to ask them about statements in their survey responses that were unclear and about the characteristics they felt were important for tasks to have in the first interview. After each interview, I noted what the participants were attending to or any interesting connections they had made and attempted to infer their knowledge or beliefs about tasks, which helped me to design subsequent interviews to confirm my inferences and discover more information concerning their beliefs about a good task.

I also performed an initial analysis of the projects and the First Pen Pal Task before doing those interviews. In each case I first completed the task and identified the important mathematical ideas that were inherent in the task. I looked to see if the task seemed to be appropriate to the level of student who would do the task, and I anticipated how students would respond to the task. I also inferred what the participant attended to while designing the task by looking at the features of the task. Since this initial analysis was done as I collected data, I was able to ask the participants questions in order to conform or refute my inferences concerning what they attended to. I will illustrate how I came up with additional questions for the interviews using some of my initial analysis of Jamie's first project (see Appendix C for Jamie's interview protocol). For an example of a practical concern, after completing the paper-ripping task, I noticed that if students counted each piece of paper individually, it would be very time consuming. Also, if students tore the papers the way Jamie described in her instructions, it would not be possible to tear the papers eight times. So, I wondered if Jamie had fully investigated the task, if she realized the time it would likely take students, and if she planned on actually having students make the full ten tears or if she anticipated them generalizing from fewer tears. For an example of a task feature concern, a predominant feature of her task was that students were doing the physical activity of tearing paper. I inferred that she wanted this task to be hands-on and that she may feel that students learn by doing physical activities. She even stated in the lesson plan that the task was meant to be engaging. So, I wanted to know what she meant by engaging, how this task was engaging, and how important it was to her for a task to be engaging. (The questions I added to address these issues are italicized in Appendix C.)

After I collected the data, I analyzed it more fully. I first transcribed and coded all of the interviews. I did not want to miss anything that the participants attended to while designing tasks. So, when coding the interviews, I interpreted everything that the participants were speaking about and assigned specific codes to each statement. I ended up with 125 specific codes. When I would come up with another code, I would go back through the data and make sure that I did not miss any instances where that code should have been applied. I would apply a code whether the participants were explicitly describing a certain characteristic or whether they attended to the idea, but did not discuss the characteristic explicitly. For example, the code that a task "can be solved many ways" was used whether a participant explicitly stated that a worthwhile task should have many solution strategies or if they described more than one way their task could be solved. There were considerations that the teachers in the literature attended to that informed me by giving me some ideas on what I may see in my data, but I did not create codes for these considerations until I saw them come up in my data. I was open-minded as I coded and looked for unanticipated considerations.

By the time I had conducted and initially analyzed the first two interviews, it became clear that there were seven broad characteristics that were the most important to the participants. During subsequent interviews, I was able to ask the participants questions regarding these characteristics. After I finished coding with the 125 codes that came out of the data, I used the literature to help me to form literature-derived definitions of the seven broad characteristics. For each of the seven characteristics, I then went through the list of 125 codes and found the ones that seemed to be related to that characteristic. Some of the codes fit into more than one characteristic. As I learned more about how the participants viewed certain characteristics, I sometimes realized that their view included codes that I had not yet included for that characteristic. So, I would go back and add and analyze the coded statements from those codes for each of the participants.

Most of the 125 specific codes were captured by the seven broad characteristics. The few specific codes that did not fit in the seven characteristics were used for other purposes. The specific codes that did not fit into one of these characteristics were "definition of a mathematical task," codes related to the participants' previous experience with the task or opinion about the task ("done a similar task before," "initial opinion about task," "liked task," "liked topic for task," "task worked," "understood task," and "task worked out nicely") and the task types ("assessment task," "instructional task," and "introduction task"). I used the "definition of a mathematical task" code in order describe their definition of a mathematical task in general. I used the participants' experience with the task codes and the task types codes in order to give me more information about the tasks they designed. This information gave me details about why they designed particular tasks certain ways.

The field notes were compiled and acted as a catalogue of the important exchanges in the observations, which helped me to find and transcribe select portions of the classroom observations. In the compilation, I included the main activities the PSTs were involved in, any discussions about what the instructor explicitly taught about task design, and ideas about teaching and task design that seemed to be taken-as-shared. There were several portions of the field notes that required more analysis than could be done only with field notes (such as when the instructor explicitly taught the PSTs about fundamental mathematical concepts). These select portions were transcribed in order to be more thoroughly analyzed. After coding the transcripts of the interviews, it was clear which characteristics were attended to the most frequently and were the most important to the participants, and I looked in the field notes, the transcripts from the classroom observations, and the textbooks to see if and how these characteristics were discussed in the course. This process informed me about what could have influenced the participants' views. I also looked for statements in the field notes and the transcripts from the classroom observations that seemed to confirm or refute my inferences about the participants' views that I had made from coding and analyzing the interviews.

Previous to the interviews I conducted an initial analysis of the tasks that the participants designed, but I used the tasks they designed as part of subsequent analysis as well. These tasks were analyzed according to each of the seven broad characteristics that the participants attended to the most frequently and thought were the most important. After having coded and developed a better understanding of how the participants viewed a particular characteristic, I was able to analyze the tasks more thoroughly for each of these important characteristics. I looked to see to what extent the task displayed that characteristic so as to infer if and how the participants attended to the characteristic while designing the task. I also assessed whether the tasks were consistent with what they professed was important in a task. In addition, I analyzed the participants' other class work and homework besides the tasks they designed and looked for confirming or disconfirming evidence of the participants' views of task design. I focused on their class notes, their Assessment Plans, their Midterm Exams, and their Final Exams.

The final stage of analysis involved writing data stories for each participant. In order to create these stories, I analyzed how my participants viewed the seven characteristics of tasks that came up the most frequently and were the most important to them. For each of these important characteristics and for each of the participants, I analyzed both the explicit and implicit coded statements that were related to a characteristic. I analyzed the explicit statements about a particular characteristic in order to see what each participant would say about a characteristic when they were explicitly speaking about that characteristic. I then analyzed the implicit statements to see how each participant actually spoke about or treated a characteristic while they were discussing their tasks and not discussing task characteristics specifically. I compared the explicit and implicit analyses along with how each of the characteristics showed up in the participants' written tasks in order to give a full picture of how each of the participants viewed each of the characteristics. Lastly, I compared this description to my analysis of the participants from the field notes and transcripts of the classroom observations and to my analysis of the participants' other class work and homework to confirm or refute my inferences about each of the participants.

Chapter 5 – Results

The participants' main definition of a mathematical task was that a mathematical task is a problem or activity that builds students' knowledge about mathematics, but they defined a mathematical task in slightly different ways. Ryan described it in a very general way as "any problem that you'd use mathematics to try to find a solution." To Jamie, a mathematical task is "some kind of activity that helps to build students' knowledge or proficiency with the subjects in mathematics," which she indicated included when students are building proficiency by practicing procedures. Carly stated that a mathematical task is "an activity that facilitates students' either learning a new concept or making extensions on a previously-known concept." To Carly, a task could not involve just practicing procedures, however. She distinguished between homework problems and tasks; she explained that a task has to involve expanding the students' knowledge and that homework problems can involve expanding knowledge or practicing a procedure you have already learned. Neither Ryan nor Jamie excluded practicing procedures from their definition of task. This means that Ryan and Jamie felt that the goal of a task could be for students to practice a procedure while Carly, on the other hand, did not think that a question that only required practicing a procedure was considered a task at all. That said, this slight variation in the definition of a mathematical task did not have a great impact on the results. The participants, including Carly, discussed the role of practicing procedures at various times throughout the interviews, and I will bring up the difference in their definitions when it is relevant in this chapter.

What made a mathematical task worthwhile is captured by seven task characteristics the participants spoke about the most frequently and felt were the most important for a task to have. The characteristics were 1) sound and significant mathematics, 2) reasoning, 3) appropriateness,

4) clarity, 5) communication, 6) engagement, and 7) openness. After the first two interviews, I identified these characteristics as ones that seemed to be the most commonly discussed and the most important to the participants. After coding all of the interviews, it became clear that these seven characteristics captured the majority of what the participants attended to regarding task characteristics.

In this chapter, these characteristics are roughly ordered based upon the importance of the characteristics for the participants. Sound and significant mathematics and reasoning were both the most important characteristics for all of the participants, so these characteristics are discussed before the others. The importance of the other characteristics, however, varied somewhat by participant. So, the order of the five remaining characteristics values the relative importance that the participants expressed. Appropriateness was important to all of the participants, so it is discussed next. Clarity and communication follow because clarity was necessary to a task being worthwhile to one of the participants and communication was almost necessary to another participant. Engagement follows as it was very important to one of the participants. Openness comes last because, although it was fairly important to all of the participants, it was the least important characteristic to them out of these seven important characteristics.

For each of the seven characteristics, I first present the literature-derived definition for the characteristic. As mentioned previously, I was able to create these seven categories after analyzing what the participants attended to. I then used the literature to inform me about these characteristics, so these literature-derived definitions appear within the sections for each of the seven characteristics. I then describe how the Task Design class treated the characteristic. Next I describe how each of the participants viewed the characteristic and how important that characteristic was to him or her. Finally, I compare the analyses of the three participants in order to give an overall description of how they attended to each characteristic.

Sound and Significant Mathematics

Each of the participants qualified the type of mathematics that must be in a worthwhile task. Although the participants used a variety of terminology to describe this type of mathematics, I will use the term "sound and significant mathematics" since it is a term commonly used in the field (NCTM, 1991, p. 25). In the *Professional Standards*, NCTM (1991) stated that worthwhile tasks must be based on sound and significant mathematics. Hiebert et al. (1997) agreed that tasks should require students to think about important mathematics. Although there are several terms that are used to describe the quality of the mathematics in tasks that are often used in the mathematics education literature, it is not common for authors to define them. For example, one of the texts of the Task Design course, Schoen (2003), stated that important mathematical ideas must be embedded in tasks, but did not describe how to determine if a piece of mathematics is important. So, it is necessary to consider the term and construct a definition.

I view sound and significant mathematics and fundamental mathematics (FM) as synonymous. My view is that sound and significant mathematics is related to fundamental mathematical concepts (FMCs) in that sound and significant mathematics as a whole is made up of fundamental mathematical concepts (FMCs). FMCs are specific descriptions of mathematical concepts, then, that are sound and significant. How can one determine if a task contains sound and significant mathematics, then? Leatham, Peterson, Stockero, & Van Zoest (2013) argued that in order to determine if an instance of observable evidence of students' mathematical thinking is significant, one must first articulate the mathematics of the instance. I would argue that this process also is necessary in order to determine whether that mathematics is sound. So, one must first determine the mathematical concept(s) embodied by a task in order to determine if that mathematics is sound and significant.

The instructor of the Task Design course explained to the PSTs how they should articulate the mathematical concept (or FMC) in a task. The instructor addressed what a FMC is because the PSTs in the class struggled as a whole to articulate the FMCs of their first project, and they would also need to list the FMCs for their second project. The instructor explained that FMCs are "really specific descriptions of the mathematics that students are going to learn in this lesson." The instructor gave several examples of broad descriptions that were not adequate and showed the PSTs how they could change the broad description into one or more specific descriptions. For example, she stated that it would be too broad to say the FMC of the instructor's example lesson was to "understand what like terms are." Instead, she explained that one of the FMCs of the lesson would be a specific description of what terms are: "expressions consist of numbers and variables combined with binary operations add, subtract, multiply, and divide. The terms of an expression are separated by add or subtract or are the beginning or ending of the expression." Copes and Shager (2003) also stated that teachers must first articulate the specific important mathematical concept they are trying to teach in a lesson.

Both the instructor and the texts stated that care should be taken that the students will actually learn the mathematics after engaging with the task. The instructor stated that a task must be designed in order to help students to develop an understanding of the mathematical concept even though it is easy to design a task that is fun, but does not have very much mathematics. She communicated that a task is not worthwhile if the students do not actually learn the concept at hand. Kahan and Wyberg (2003) explained that in order to teach mathematics through problem solving, the teacher must identify the important mathematics to be learned and then design a task where students will likely learn that concept.

However, the instructor and the texts did not discuss the "fundamental" part of FMCs, meaning they did not describe what it means for a mathematical concept to be fundamental. The instructor referenced good or important mathematics in a general way. For example, she did tell the PSTs that she may have them analyze a task on the Midterm Exam by identifying the important mathematics in the task, but did not explicitly tell the PSTs what it means for a task to be based on important mathematics. I take the view of FMCs that the instructor taught her students as part of my definition. I will take a FMC to be (in part) a specific description of what students should learn from the task. Since, however, a mathematical concept of a task must be fundamental in order to be a FMC, one must also determine if that concept is fundamental (or sound and significant). I thus consider each part of the term "sound and significant" individually in order to construct a definition of sound and significant.

Regarding soundness, a mathematical concept clearly needs to state a mathematical idea that is accurate and needs to be complete in that it does not leave out part of the idea that is essential. We must also, however, consider how the mathematical concept is embodied in the task to assess soundness. NCTM (1991) described three criteria for assessing the mathematical content of a task. Two of these criteria relate directly to the soundness of the mathematics of a task. The first criterion that teachers should consider is if a task represents the mathematics in it appropriately. For example, in a task where students are required to calculate the mean of a set, a teacher should consider if it makes sense to calculate the mean in that situation and if the task helps the student understand the concept instead of just the procedure. Second, teachers should consider if a task accurately conveys what doing mathematics entails. For example, if students mostly complete tasks where they practice learned procedures, this may convey to students that mathematics is primarily about getting right answers. If students are usually required to complete tasks where they are required to reason in order to determine how to approach the task, however, this may convey to students that doing mathematics is primarily about critical thinking. I argue that the mathematics in a task likely will not be sound if it does not represents the mathematics appropriately or does not accurately convey what is entailed in doing mathematics.

A mathematical concept in a task is significant if it is worth learning. Leatham et al. (2013) defined the mathematics in a student's comment within a lesson to be significant if it was worth the instructional time to pursue in a particular class and at a particular time. They explained that the mathematical significance of the articulated mathematics can be determined if it is appropriate and related to a central goal that those students are learning. The first criterion, that the comment is appropriate, will be discussed in the Appropriateness Section. Even though the participants were not designing tasks for a particular group of students, they usually identified a grade-level that the tasks were designed for, so this definition (specifically the second criterion of centrality) can be used here in a less-contextualized way. Thus, one can determine if the mathematical concept in a task is significant if it is worth learning, meaning that it is related to a goal that would likely be central for the learning of the level of students who would do the task. Leatham et al. (2013) pointed out that one can use experience or other resources, such as the *Common Core* (NGACBP & CCSSO, 2010) and the *Principles and Standards for School Mathematics* (NCTM, 2000), as a guide in making decisions about which mathematical concepts are related to central goals. So, we can determine, using these documents and our experience, if the mathematical concept of a task is an idea that is essential and central to a well-developed understanding of mathematics, if the idea is nice to know, but not vital, or if the idea is completely unrelated to the main goals we want students to learn and thus need not be taught.

A way to help determine if a task is both sound and significant is NCTM's (1991) third criterion for assessing the mathematical concept of a task. This criterion applies to and will be part of how I view sound and significant mathematics. The third consideration they outlined was that teachers should consider if and how a task helps students to develop their skills. NCTM (1991) suggested that it is preferable for a task to simultaneously build the understanding of the concept and increase students' automaticity. A task would not be sound it if artificially separated a concept and an accompanying procedure. A task would also lose significance if students are not helped to see how concepts and procedures are related and helped to see beyond just the procedures.

In sum, to determine if a task contains sound and significant mathematics, I take the view that one must first complete the task and then articulate the specific piece of mathematics that the students should learn from the task. Care must be taken that students will likely actually learn that mathematical concept by engaging with the task. Then, we must determine if that mathematical concept is fundamental (or sound and significant) and thus a FMC. I view that FMCs are specific descriptions of mathematics that are sound and significant and that sound and significant mathematics as a whole is made up of FMCs. So, the literature-derived definition I will use of FMC is a specific description of the piece of sound and significant mathematics that students should learn from the task. If we deem the mathematical concept and how it is embodied in the task to be fundamental (or sound and significant), then the task does contain sound and significant mathematics. The literature-derived definition of sound and significant mathematics that I will use is that the mathematics of the task is accurate, complete, and worth learning. Thus there are several questions that one must also consider in order to determine if the mathematical concept in a task is sound and significant:

1. Is the mathematical concept accurate and complete?
2. Is the mathematical concept worth learning? That is, is the concept emphasized as being important in standards documents? Is the concept central to a well-developed understanding of mathematics?

There are also several questions that should be considered to see if the mathematical concept is embodied in the task in a way that is sound and significant:

1. Does the task represent the mathematics in it appropriately?
2. Does the task accurately convey what doing mathematics entails?
3. How does the task help students to develop skills? Does the task simultaneously build the understanding of concepts and procedures or separate the concept from the procedure?

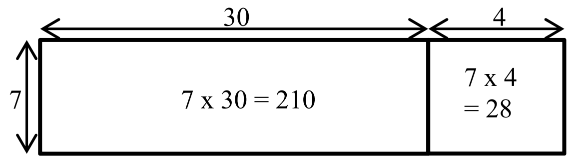
Carly. As Carly designed tasks or talked about the tasks she had designed, she often qualified the type of mathematics that needed to be in the task. She used various words interchangeably to describe the kind of mathematics that should be in a task, including good, important, useful, worthwhile, valuable, or sound and significant. I will use the term "good math" to describe the mathematics that must be in a worthwhile task since she used that phrase so frequently. As she spoke about good math, she would often elaborate on what she meant. She explained that the mathematics must be "valuable to the students' learning" and that the mathematics in a task is good if it will help students to learn other mathematics in the future and is related to the goal of what students need to learn.

Carly asserted that a task cannot be worthwhile unless it has good math in it. She explained that "if it doesn't have valuable mathematics in it, it's not gonna be a good task. No matter what else is going on in the task, without that, it's no good." According to Carly, this characteristic is the most important characteristic to assess in order to determine if a task is worthwhile. For the most part, however, it seems that Carly believed that all mathematics is good math. When asked if a task was a good task or what characteristics the task had, she usually would state that the given task had good math. For example, for Project #1, she explained that "you have to know how squares and square roots work and how they're related to each other in order to get past - I mean to go further in math, and so you just have to know." There was only one instance in the interviews when Carly implied that a task did not have good math in it. Task J asked students to mentally compute a two- or three- digit number multiplied by a one-digit number using an area model (see Figure 1). As she spoke about the task, she explained that she may not actually use this task because it does not lead to any other concepts. When asked later in the first interview what characteristics were important for a good task to have, she said that a task needs to have good math, which is why she "felt a little less strongly" about Task J since good mathematics "needs to be things that are useful, things that are going to help them to get to other mathematics. You know, it just needs to be something that actually is related to the goal." So, since this task did not seem to fit in the secondary mathematics curriculum, it is likely that Carly thought it did not have good math.

Task J

Manipulatives/Tools: None

One method of mentally computing 7 x 34 is illustrated in the diagram below:



Mentally compute these products. Then sketch a diagram that describes your methods for each.
a) 27 x 3
b) 325 x 4

Figure 1. Task J from the Middle School Task Sort from Smith et al. (2004).

Carly explained that the FMC of a task is what one needs to think about or do in order to solve the task. She seemed to think of FMC and good math as different ideas since she described them in different ways, and she did not suggest that a FMC is only a FMC if it will help the students to learn something later or if it important to students' learning. Thus, she may have thought of a FMC more as the mathematical concept in the task, without attending to the fact that it needs to be fundamental as well. However, there is a term that Carly used occasionally, "bigger picture" or "big" mathematical ideas, which she may have associated with FMCs. She explained that "bigger picture" mathematical ideas are the conceptual ideas that allow students to do the smaller cases on their own. In one instance, she was describing what the FMCs were in the Irrational Number Task and then said, "So, the big mathematical idea would be expanding the concept of real numbers." If she felt that FMCs and big mathematical ideas were synonymous, then FMCs would need to be concepts that help students to learn other concepts in the future.

Carly seemed to conceive of a FMC as a general topic, instead of a specific description, since she usually stayed at the topic level in her responses when she was asked what the FMC of a task was. For example, when asked what the FMC of Project #2 was, she said that it was "finding area," and when asked what the FMC was of the Factoring Task, she said that it was "factoring." Also, in Carly's first project, the FMCs she listed were mostly at the topic level and described what students would learn about instead of the actual idea they would learn. One of the concepts she listed was "Geometric representation of perfect squares and how that connects algebraically: connections to area and side length of a square." There is one case in the interviews where she was more specific when asked what the FMC was, but she then made the comment more general, as if she thought that a general topic was what was being asked for. She stated, "That irrational numbers are not just numbers that are written - that can't be written as fractions. So, the big mathematical idea would be expanding the concept of real numbers." Some of the FMCs she listed for her final draft of Project #2 were specific. The rough draft of the project contained general descriptions of what students would do in the task, such as "use what students already know about area to find the area of more difficult shapes." On the peer review, Jamie stated that the instructor wanted concept descriptions, such as describing the definition of area. Through working with her partner, one of the FMCs in the final draft were "Area is the amount of space in a 2-dimensional closed shape. Example: A square with side length 3 feet has an area of 9 square inches because that is how many square units are within that shape." Two of the other FMCs were unclear, but there was an obvious attempt to be more specific.

Even though she likely conceived of a FMC as a general topic, she was aware of more specific concepts inherent in the task that students could learn. On several occasions, she gave a statement that resembled the FMC of the task when asked a question other than "What is the FMC of the task?" As Carly described how a student may do Project #1, she described what a student may think in order to find the square root of a number if that number is the area of a square. She explained,

> I know that the area is going to be this side times this side and… then they have the same number multiplied together. If this is a perfect square, then the square root is just one of those - the side lengths.

The FMC of this task is that the length of a side of a square is the square root of the area of that square. The FMC consists of the ideas that Carly is discussing in the quoted statement. So, Carly is capable of describing the ideas that would be part of a FMC, but she did not extract the ideas from her description of student reasoning and clearly articulate them in the interview.

It appeared that Carly usually attended to a specific concept students could learn (even though she did not identify the specific concepts as the FMCs) when she designed her tasks. In Project #1, the Quadrilateral Task, and Project #2, she identified something specific that she wanted the student to learn and attempted to design the task so that the students would learn that concept. For example, for her first project, she wanted students to come up with their own definition of a square root. So, she thought of something that the students already knew (how to find the area of a square) that could be applied to square roots since "if you think about the area of a square, the side lengths are always gonna be the square root of the area." She referenced what she wanted the students to learn here, that if a number is the area of a square, the side length is the square root of that number. She then went on to describe the task where students practice finding the area of a square given the side length and vice versa and are introduced to the square root notation to represent how one finds the side length given a certain area. She clearly tried to get students to understand this specific idea in this task. In the Pen Pal Tasks, however, Carly probably did not attend to a specific FMC as she designed the tasks. As she and Jamie designed the First Pen Pal Task during the Task Design class, they tried to decide on a topic for the task, and then Carly suggested that they list several situations and have the students decide which are exponential and which are linear. They then listed six contexts they could use for this task and actually wrote the questions after class. Carly stated in the fourth interview that she wanted the students to learn the differences between exponential and linear situations. However, she stated that she never actually articulated the explanations for this task, and she never explained in the interview the differences she was looking for. When Carly assessed the student's work on her task, she was vague. For example, she said that that the student gave a "good explanation," but said there was "stuff missing." Since Carly did not articulate the differences between exponential and linear situations, she would not have been able to choose questions for the task that would bring out certain differences, so she likely chose questions based on the fact they were either about exponents or linear functions. It is an open question if she would have built the First Pen Pal Task upon the differences between exponential and linear situations if she had articulated them.

Carly did not include that a FMC should be sound and significant in her definition, and she probably was more concerned with finding a mathematical concept that one could learn when she was designing her tasks instead of attending to whether the concept was sound and significant. When Carly was directly asked what characteristics she wanted a task to have, she would state that she wanted the task to have good math. However, it was not common for her to bring up if the task had sound and significant mathematics while she was describing how she designed the task. Regarding soundness, there is little evidence that she attended to whether the FMCs for her tasks were sound. In the group interview, however, Carly thought that in order to find the degree measure of one of the angles in a regular polygon, she should divide 360 degrees by the number of sides the polygon has. She asked the group if that was correct, and they spent some time deciding on whether that was correct. So, they were concerned about the mathematical concept of the task being correct. Regarding significance, she did not seem to generally choose a mathematical concept because it was significant. When the group was asked what was difficult about designing the Quadrilateral Task, Carly stated that it was difficult to decide what was important for students to know, but that "generally speaking… you can take that out of the [*Common*] *Core*." However, it was not apparent that they were looking for a concept that was important as they designed the task; it appeared that they were looking for any concept that a student could learn. She stated that she chose the task for Project #1 because it was a task she had used before in a previous course and that she chose to write the First Pen Pal Task based on exponents because it is a topic she understood well. Carly did state for Project #2 that she and her partner looked through a textbook for an idea, saw the irregular shapes, and thought, "Oh yeah. That's an important thing. You have to do lots of problem solving when you do that, and it's kind of fun to figure out."

Jamie. Jamie did not use a single term frequently to describe the kind of mathematics that a worthwhile task should have. She used the term important mathematics occasionally, so I will use this term to describe the kind of mathematics that Jamie felt must be in a good task. When she spoke about important mathematics, she explained that tasks (1) should be based on mathematics the students need to know or should teach the student a concept that the teacher wants them to learn and (2) should not just be for fun. When Jamie brought up that tasks should not "just [be] something to do for fun," she meant that the task should not just be a fun activity; "it had to have a purpose teaching the kids a concept." Overall, Jamie seemed to see important mathematics as the mathematical concepts that students need to know. She stated on two occasions in the interviews that the mathematics in a task was important because the students would need that knowledge in the future. For Project #1, she explained that she thought "exponentials are going to come up in every math class that they have after this, and so it's really important for them to understand what they mean." So, it could also be that Jamie believed that the mathematics must be useful in the future to be important.

To Jamie, the most important characteristic of a worthwhile task is that it is based on important mathematics. She stated that a task will not be a good one unless it is "based on a mathematical principle that you want the students to learn." Additionally, when asked about "sound and important mathematics" in the group interview, she agreed with the others that having sound and important mathematics is the most important characteristic of a good task because "there's no point in the task unless it's based on something that they need to know." At no time did Jamie express that any of the tasks that she completed or designed did not have important mathematics in them. On her Final Exam, however, she argued that "if the concepts covered in tasks and lessons are unimportant, they are hard to connect to other concepts and are not valued by the students," so potentially not all concepts are important.

A FMC, for Jamie, is what the students will learn from the task or what they do in the task. When asked how she knew what the FMC of the Irrational Number Task was, she explained,

> Well, I was just trying to think about like what understanding the students could gain from doing this task, and so, essentially, that's what you try to build a task around. 'Cause, okay, what do I want my students to learn? And so let me find a task that can lead them to that - that end goal.

On the Factoring Task, she explained that the FMC was factoring because "that's what both of the problems were making you do." Jamie did not state that a FMC needed to be important mathematics, but she did describe them both in a similar way; she brought up for both that it is the mathematics the teacher wants the students to learn. So, it is possible that Jamie thought that the fact that there is a FMC in the task means that the task has important mathematics, but it is also possible that the FMC of the task to Jamie does not necessarily have to be "fundamental."

It is unclear whether Jamie viewed the FMC as a specific concept that students can learn from the task or as a general topic because she varied in how she answered the question, "What is the FMC of this task?" When asked what the FMC was of the Irrational Number Task, she said,

> Maybe just the meaning of irrational numbers, and I don't know. Do you want just one? I think it gets at fractions less than one. Maybe - Because - or like a property of fractions less than one: that their square is less than the fraction is itself. Just kind of extending it to irrational numbers as well, possibly.

She did not state specifically what the meaning of irrational numbers is, so that was general, but she was specific about the property of fractions less than one that students could learn. For the First Pen Pal Task, she stated that the FMC was "to be able to learn a little bit how to model contexts," "be able to write an equation for a situation that they're given, and then also just to be able to recognize what type of situation it is." In this case, she was not specific about how a student would be able to recognize if a situation was exponential or linear or how they would write the equation from that. So, this is less specific, but more specific than just saying the FMC was linear and exponential situations. When asked what the FMC was in the Factoring Task, however, she was very general when she said it was "factoring." The written FMCs in her projects also varied in specificity. For Project #1, her FMCs were: "(1) Conceptual meaning of exponents (e.g. repeated multiplication) and (2) Exponential rules and why they work" followed by a list of exponential rules. She did give an indication that the conceptual meaning of exponents is repeated multiplication, which is fairly specific, but she does not describe why the exponential rules work. In Project #2, which she did alone, she was specific in her description of three concepts: reduction, enlargement, and scale factor. For scale factor, she listed the following: “The ratio of the length of the scale drawing to the corresponding length of the actual object is called a scale factor.” Although she did not use the term FMC, she did state in the last interview that doing the lesson plans helped her realize the importance of writing “a specific definition for this concept that I’m teaching.”

While Jamie designed tasks or described how she designed the tasks in the interviews, she seemed to usually focus on a specific concept students could learn, even if she did not identify that concept as the FMC. In Project #1, the Quadrilateral Task, and the Second Pen Pal Task, she was focused on specific concepts students could learn and made decisions about the task in order to help students to learn the particular concept. For example, in the group interview where the group designed the Quadrilateral Task and before they chose the FMC, she often asked questions to help focus the group on what they wanted the students to learn, such as “What part of it do we want them to learn?” and “Do we want to talk about all the characteristics of a given type of quadrilateral, or do we want to compare one characteristic of each of the quadrilaterals as a lesson?” Once the group decided that they wanted students to justify why the sum of the interior angles of a quadrilateral is 360 degrees, she often asked the group questions or made suggestions that provided evidence that she was making an effort to help students who may do this task understand the FMC. For example, when Ryan suggested that they have protractors available, she stated,

> That might take away their sense of what it means to prove something. Because now, they’re just going to believe it because they measured it and now they might come up with a proof, but they understand it because they measured it.

She wanted the students to understand how to prove that sum of the interior angles of any quadrilateral is 360 degrees, and she clearly attempted to design the task so that it accomplishes that goal.

In two cases, however, Jamie did not appear to attend to a specific concept students could learn as she designed the task. For the First Pen Pal Task and for Project #2, she did not appear to have a specific concept in mind at all based on how she spoke about the task. Rather, she likely had a topic in mind as she designed the task. Jamie and Carly designed the First Pen Pal Task together, as referenced under Carly's description. There is also no evidence for Jamie that she picked certain questions within the task in an effort to help students to come to an understanding of the differences between linear and exponential situations. Jamie also did not articulate those differences or provide rationale for why she picked certain questions and likely chose the questions because they fit the general topic of exponential or linear. For Project #2, it also does not appear that she attended to trying to help students to get to an understanding of a particular concept. Jamie first chose the activity that she wanted the students to do where the students would draw a floor plan of the classroom, then decided what she wanted students to learn that related to the floor plan. She then chose to have them learn about scale factors, but was not specific about what her FMC was in the interview. She then looked online for what she could add to this activity that was related to the topic of scale factor. It is interesting that Jamie wrote specific FMCs in the write-up of Project #2, but did not provide evidence in the way that she spoke about the design of the task that she used those FMCs to help her to design the task.

If asked directly which characteristics she wanted a task to have, Jamie did state that she wanted the task to have important mathematics, but it was uncommon while she was talking about designing tasks or designing tasks for her to actually attend to the importance aspect of the FMC. Regarding soundness, Jamie occasionally attended to the accuracy of the tasks. In the group interview when Carly made an incorrect statement about the sum of the interior angles of polygons, Jamie worked to find a way to prove what the sum of the interior angles of a pentagon was; she clearly was concerned in this case with accuracy of the mathematical concept. Also in the group interview, Jamie showed that she attended to whether the task represented the mathematics in it appropriately, which was a rarity for any of the participants in the interviews. As referenced above, Ryan suggested that they provide protractors, and Jamie explained that she was concerned that doing that might "take away their sense of what it means to prove something." Jamie did not always do the tasks that she designed, though, which shows that she may not have always been concerned with the accuracy of the individual questions in the tasks. With regard to significance, she did not provide any evidence that she chose the FMC of the task because it was worth learning. Jamie chose the task for Project #1 because she had seen another teacher give it, and she thought that it taught the meaning of exponents well; she chose the topic for the First Pen Pal Task because it was a topic she understood. She did, when asked, state that the mathematical concept was important, but there's no evidence that she chose the mathematical concepts because they were important.

Ryan. Ryan almost exclusively used the term "fundamental mathematics" (FM) or "fundamental mathematical concepts" (FMCs) when talking about the kind of mathematics that a task should have. Of these two terms, Ryan used the term FMCs much more often. To Ryan, a FMC is a new "culminating" concept that students need to learn. It is likely that he saw FM and FMCs as synonymous since he, in several instances, used the terms interchangeably. For example, when he described how he finds the FMC of a task, he explained how he analyzes tasks to find the FM. However, when he was asked how the FMC came out of Project #2, he explained that the task led up to a "culminating point of coming up with the quadratic formula," but also that there was FM throughout the lesson since they were learning about completing the square the entire time. So, he may have seen FM and FMC as different; the FMC may be the main point that students should learn, and a task might contain FM if it is about an important general topic.

The most important characteristic of tasks to Ryan is that it has FM or is based on FMCs. He stated that a good task "should be based on fundamental mathematical concepts or making connections within your knowledge base." In addition, in the group interview, he agreed that "sound and important mathematics" is the most important characteristic of tasks.

Ryan saw FMCs as what the students would learn from the task and the important or culminating concepts that a task leads to (that likely must be new to the student). Ryan did not directly state this definition of FMC, but he gave insight into his view in three ways. First, Ryan stated that the FMC is what the students are doing in the task and what they are learning from it. When asked how he knew what the FMC was in the Factoring Task, he explained that he tries to determine what the student would be "getting out of it." He then stated, "That's kind of how I analyze any task to find out the fundamental mathematics. What am I doing in this task and what am I getting from it? What am I learning from it?" Second, Ryan stated that FMCs are the important concepts that the task leads to. When asked about the characteristics that he wanted his First Pen Pal Task to have, he explained that it is based on FMCs and that "it can't just be something that is fun, but it actually has to be leading toward important mathematical concepts." While speaking about the Irrational Number Task, he communicated that a FMC is a "culminating or a pivotal mathematical concept that you're learning." Third, in the last interview, he implied that the FMC that the task leads to must be new to the students. He contrasted tasks with a FMC and tasks that make mathematical connections in a way that it is likely that he believed that a task where students connect concepts they already know does not have a FMC in it. When asked what the FMC of the Irrational Number Task was, he stated, “Does this really get at necessarily a fundamental mathematical concept? Or is it just helping you make connections?” He expressed that it did not lead toward “any specific mathematical concepts. It’s more leading towards problem solving itself and the different ways to problem solve.” When he was asked what was necessary for a task to have in order for it to be a good task, he explained that it must either have FMCs or make mathematical connections, which supports the idea that he probably believed that if students are reasoning with and connecting ideas they already know, the task does not have a FMC.

Ryan may have conceived of a FMC as a general topic since he never articulated a specific description in the interviews when directly asked what the FMC of a task was. When asked what the FMC was of the First Pen Pal Task, he stated that it was “exponential growth and decay. That’s. I don’t know. I think that’s the core of it.” For the Factoring Task, he was little more specific, saying “factoring polynomials,” but specifically quadratics “and the idea of difference of squares,” but still very general. Even though he implied that the Irrational Number Task may not have a FMC, he stated that the FMC could either be “explanation of irrational numbers” or “an exploration in squaring function and when you square something when it becomes smaller than itself.” He was more specific here when talking about the squaring function, but still did not explain in which situation the number you square becomes smaller than the original number. Even though Ryan did not articulate specific FMCs in the interviews, when talking about the FMC for both Projects, he described looking in the *Common Core* and implied that the *Common Core* contains a list of FMCs. For example, for Project #2, he stated that he wanted the task to have FMCs and then immediately described how the “*Common Core* is good in a lot of ways because it gives you very specific goals and things to cover.” In fact, for Project #1, the concepts he listed are taken directly from the *Common Core*. For example, the first one is “formulate questions that can be addressed with data and collect, organize, and display relevant data to answer them,” which is clearly not a specific description of what a student should learn. Overall, Ryan usually talked about FMCs in a way that implied that he thought that FMC is a general topic, but his references to the *Common Core* imply that he may have believed that FMCs could also be specific concepts. The second project, which he worked on with a partner, is the most specific, even though the concept descriptions included opinion-like statements. For example, the first part of one of the concepts listed was the following: “Using the method of completing the square on the general quadratic equation gives ax^2 + bx +c=0 gives a derivation of the Quadratic Formula.”

Even though Ryan only gave general topics in the interviews for the FMCs of tasks, he is aware of more specific concepts students could learn from the task. For example, when he was asked what is important to know about scatterplots within the discussion of Project #1, he stated,

> I think the most important thing is that it’s a way of analyzing when you’ve got, you know, two quantitative variables. There’s a way to analyze if there’s a relation between them. I think that’s really, like, the underlying idea behind it.

This is definitely what a student could learn from the task, even though it is a fairly general idea here that scatterplots are a way of analyzing if there is a relationship between two quantitative variables. Also, after completing the Irrational Number Task, he explained that he had to think about when multiplying two numbers together yields a smaller number and that that only happens with fractions. He continued,

> I was reasoning through it, and I decided that it would probably have to be between negative 1 and 1 in order to be—. Well, actually, it depends on what you define as smaller, if that makes sense, because the negative ones will technically be bigger when you square them. So, it has to be between zero and 1. Irrational numbers between zero and 1.

One of the FMCs of this task is that an irrational number between zero and 1 will have a square that is smaller than itself. So, Ryan is capable of giving statements that resemble a FMC, but he did not extract the specific ideas from his description and articulate them concisely.

Ryan appeared to usually attend to a specific concept as he designed or talked about designing tasks, but he sometimes attended to only a general topic. In Project #1, the Quadrilateral Task, and Project #2, he likely did attend to a specific concept in order to design the task. For Project #1, Ryan's goal was to help students to understand that and how they can use scatterplots to analyze data with a linear or nonlinear relationship. As he described how he designed the task, he did show that he attempted to design the task in a way to help students to reach an understanding of how they could analyze the data using a scatterplot to see if it had a linear association. He explained that he would have students put their shoe size and height in a table and then ask, "How could you look at it to see if there's a relationship?" He expected students to come up with different ideas of how to analyze the data, but he would ask questions to lead student to see that they can plot the data. He then would ask students what they notice in the plotted data in order to bring out how we can use a scatterplot to see if two sets of data have a linear association and if they have a positive or negative correlation or any clusters or outliers.

Ryan did not, however, appear to attend to specific concepts for the Pen Pal Tasks. For the First Pen Pal Task, Ryan stated that he wanted the students to compare and contrast exponential growth and decay, but he never articulated the similarities and differences between them in the interview, and it is likely that he never articulated them at all. When asked, he stated that he did not go through and complete the task himself, but he did look at the answer key that came with the original task. He did, however, attend to the more general topic of "compare and contrast exponential growth and decay" as he designed the tasks. He explained that he wanted the first section of the task to be about exponential decay, the next one to be about exponential growth, and the last question to be about comparing the two. There is no evidence that the task was constructed in a way so that Ryan could help students to understand any specific similarities and differences between exponential growth and decay functions, such as the fact that an exponential growth function, when written in its standard form, would have a base that is greater than one while the base of a decay function would be between zero and one.

When asked directly which characteristics he wanted a specific task to have, Ryan would say that he wanted it to have FMCs, but he did not appear to attend to whether a task contained sound and significant mathematics while he was designing the task. With regard to whether a concept was sound, he may have been concerned with the accuracy of the concept, but there is only one case where he showed such attention. In the group interview (as referenced earlier), when Carly asked whether an idea about the interior angles of polygons was incorrect, he was the one to state that it was incorrect. He was not always concerned with the accuracy of the questions in a task, however, since he did not usually go through and complete the task himself after designing it. With regard to the significance of the mathematical concept in the task, while discussing how he designed the tasks, he never stated that he chose a particular mathematical concept because it was significant. Instead, Ryan chose the mathematical concepts for other reasons. For example, for Project #1, he explained that he enjoys statistics, looked through the *Common Core* concepts about statistics, and then chose to do the task on scatterplots because he remembered a task another PST had presented on scatterplots in a previous course. Also, he stated that he thought that exponents would be the easiest topic to write a task on for the First Pen Pal Task. However, he did cite that he looked in the *Common Core* to help him to choose what to do his two projects on. He may have believed that the *Common Core* has the concepts that are important for students to learn, but he spoke about the *Common Core* in a way that suggested that he likely believed that it "gives you very specific goals and thing to cover." Ryan probably did not attend to whether the mathematical concepts in the tasks were significant while he was designing them.

Conclusion. The participants stated that tasks must contain sound and significant mathematics in order to be worthwhile. They believed that sound and significant mathematics is the mathematics that students need to know, which is consistent with the literature-derived definition, but it is missing the fact that the mathematics must also be sound. Carly and Jamie both added that the mathematical concept should help students to learn more mathematics in the future. Carly and Ryan discussed how the *Common Core* can help you to determine the mathematics that should be in a task. This makes sense since for both of the projects and the First Pen Pal Task, the instructor told the students to look in the *Common Core* in order to determine the focus of the task. In general, they all would probably say that most tasks had sound and significant mathematics in them. Carly implied, however, that a task that does not fit a secondary mathematics goal would not have good math in it, and Ryan implied that a task where students connected two ideas they already know would not have a FMC in it.

The participants believed that the FMC is what students have to do, what they have to think about, or what they learn from doing the task. What the students will learn from the task is consistent with the literature-derived definition of FMC, though it lacks the requirements that the concept must be "fundamental" and specific. Ryan is the only participant who used the term FMC frequently on his own; the term FMC was how he described the type of mathematics that should be in a task. So, he did explicitly connect FMC to important concepts in a task. For Carly and Jamie, it is unclear how "good math" and "important mathematics" are related to how they understand the term FMC. It is unclear as to whether FMCs must be fundamental for Carly and Jamie or if they just need to be mathematical concepts. With regard to how they spoke about the FMCs, Ryan and Carly both likely conceived of FMCs as general topics. It is unclear how Jamie viewed FMCs in terms of the specificity required since she varied in how she spoke about FMCs; she did sometimes give a specific FMC when asked. Carly and Ryan are capable of describing ideas that resemble a specific concept that students could learn when they described how students would reason through the task. In the interviews, however, they did not recognize that these ideas could form the FMC, and they did not articulate a specific concept description.

While the participants designed tasks or described how they designed the tasks in the interviews, they did seem to usually focus on something specific that a student would learn to help them to design the task, even if they did not label that concept as the FMC. It is promising that the participants may be attending to a FMC while they design their tasks even if they cannot or do not articulate one. None of the participants attended to a specific concept that a student would learn for the First Pen Pal Tasks, though, perhaps because they were not required to write an entire lesson plan for these tasks that included how students may interact with the task. They probably designed their tasks based on a mathematical concept that one could know rather than a mathematical concept that is worth learning. There is little evidence to show that they attended to whether the mathematical concept in the task was sound and significant as they designed their tasks, and it is was common for the participants to choose a task because they had seen it before or to base a task on a concept because they felt that they understood it. That the participants do not normally attend to the importance of the mathematical concept in the tasks is especially apparent while the group designed the Quadrilateral Task, where they seemed to be looking for any concept that fit the requirements of the task and that they understood well enough to write a task about.

Reasoning

In order for a task to be worthwhile for developing mathematical understanding, the task must require students to reason. A task requires students to reason if the task is a true problem for the students. Hiebert et al. (1997) felt that tasks should be real problems to students in that students do not already know how to solve them. When students do not already know how to solve a task, they must use their previous knowledge and problem solving skills in order to solve it. NCTM (1991) stated that a worthwhile task should "call for problem formulation, problem solving, and mathematical reasoning" (p. 25). They argued that reasoning in a task is important because "the kind of thinking required" along with "the mode of activity" and "the way in which students are led to explore the particular content all contribute to the kind of learning opportunity afforded by the task" (NCTM, 1991, p. 27). They included that teachers should use tasks that foster students problem solving and reasoning abilities as they are developing an understanding of mathematical concepts and procedures.

The type of reasoning required in order to complete a task was clearly the characteristic that was most focused on in the Task Design class. This characteristic was the core of the texts that the PSTs read and discussed in class. Hiebert and Wearne (2003) argued that problem solving is the way that students should learn mathematics and that learning through problem solving helps students to develop a deep understanding of mathematics. They asserted that tasks should be "just within the students' reach," allowing them to struggle (Hiebert & Wearne, 2003, p. 6). The first chapter of Stein et al. (2009) is about how mathematical instructional tasks vary in the cognitive demand they require and what it means for tasks to have a low or high level of cognitive demand. The authors stressed that the learning goals need to be articulated because tasks with different levels of cognitive demand can meet different goals. They argued that students need to engage regularly with tasks with a high level of cognitive demand in order to develop deeper understandings of concepts and processes.

The instructor of the class stressed the ideas in the texts and often discussed cognitive demand and struggling as a way to learn mathematics. The instructor often communicated that tasks are better if they require a high level of cognitive demand. For the Midterm Exam, she stated that she wanted students to adapt a task to make it better. She explained that if the task was still procedural, it would not get full credit; they should move it from a low to high-level cognitive demand. She did state that there is a purpose to low-level tasks, but that it is important to build understanding first. In preparing the PSTs for the Final Exam, she stated that the levels of cognitive demand in Stein et al. (2009) and the way the authors discussed them were important to know. She did, however, tell the PSTs that the (First) Pen Pal Task did not have to be higher-level since it was a task designed for assessment. The instructor also emphasized the importance of maintaining the cognitive demand of a task. The idea that a student should struggle during the task came out several times. She explained that students learn by struggling and that the PSTs should be careful to not take away the agency of the students and so must be careful about how much information they give them.

Carly. For Carly, a task requires reasoning if the students are not told or led to the solution method or the solution method is not obvious and so have to figure out the mathematics themselves. When Carly attended to student reasoning, she often used the term problem solving or other phrases that were part of her definition of problem solving, such as students need to figure it out themselves, put it together themselves, or create or build it themselves ("It" probably refers to either the solution method or their understanding). When asked what problem solving meant, she stated, “They’re not being given it; they have to take these little pieces, and they’re forced to do the putting together themselves.” This is also how she described if reasoning was required in the tasks she designed. In Project #2, she felt that students needed to do a great deal of problem solving since they need to figure out how to apply their knowledge of the areas of basic shapes in order to find the areas of the irregular shapes. She also tried to present each problem in different ways, such as subtracting instead of adding different areas to find the total area, so that the students have to think about each question. Carly contrasted tasks that required problem solving with those that give students what to do step by step and feed students the material. She explained that Project #1 did not have very much problem solving because it was a “one-way task,” where the teacher tells students what to do step by step, such as giving students several side lengths, asking them to make a square with those side lengths using M&Ms, and asking students to find the area of each and record the information. She wished that this task required more thinking, but stated the students are not told that squaring and square rooting are inverses or how to find the square root of two, and students are supposed to use their knowledge to create a definition of square roots.

It is very important to Carly that a task requires students to reason. When asked which characteristics a worthwhile task must have, she included in her response that a worthwhile task must include “some sort of problem solving.” She described how if a task gives students what to do step by step, which is not problem solving, a student may be able to do every part, but it is not likely that the students would “connect everything and put it all together and learn this big thing.” She felt that a task must not feed the students the material, but instead needs to have the students work through the material themselves.

Carly also communicated the importance she placed on reasoning during several conversations. These conversations fall into two main ideas. First, she communicated the importance of reasoning by expressing how she wants to spend class time helping students to figure out the mathematics. She stated several times that students need to practice procedures "to get it in your head" and "make sure it's stuck," but that practice is for homework, not for tasks used in class. She explained that class time should be spent "more valuably, like, kind of, figure things out a little more rather than practicing it." In fact, her definition of a mathematical task excluded "homework problems" where students practice procedures. When asked what a typical day in her classroom would look like, she explained that she used to want to be at the board, giving the students the information, but now she wants the students "to be doing the thinking." She also stated that task-based instruction is the most effective way to teach and implied that it is because students have to reason. She stated that with this type of teaching, students have "drawn these connections and created this understanding" and are "left with the mathematics that they built themselves."

Second, she believed that developing reasoning skills can be and are important instructional goal for tasks. Besides the mathematical concept goal, Carly stated that the Quadrilateral Task had another goal for developing problem solving skills, including skills for proving, analyzing work, looking at observations, and generalizing. She also stated that the Task Design class helped her to recognize that it is okay to have a goal of the task be developing problem solving skills and mathematical habits in addition to the mathematical concept goal. She added that reasoning skills are "important to recognize as a goal and to consider how you're going to achieve that." Carly believed that the discipline of mathematics values "problem solving…, the skills to reason…, logic, and what makes an argument valid mathematically."

Since she believes that the discipline values reasoning, it would follow that she thought it was important for her to incorporate into her tasks.

Carly attended to the reasoning required while discussing all of her tasks as well, so the characteristic of reasoning was important enough to her to bring up frequently. She thought that a few of the tasks in the task sort and Project #1 led students to the solution method, which was not desirable in her opinion. For example, for Task A[2], she explained,

> When it already says in here 'find the perimeter' and it defines it as 'the distance around,' then I don't see how that leads to doing anything but just counting the sides if that's the way you understand the problem.

While discussing how she designed the First Pen Pal Task, she stated that she tried for the last two questions to make it so students would not be led to solve the questions a certain way. She explained that she put "solve" first in the instructions "solve and set up an equation" because she wanted students to solve the task any way they chose and not feel like they *had* to set up an equation to solve it.

Overall, however, Carly's tasks varied in the level of reasoning they required. The solution method is obvious in Project #1 and the Pen Pal Tasks; Adapted Task A[3], the Quadrilateral task, and Project #2 require a good deal of reasoning, but lead students to a degree; the other two adapted tasks both require reasoning and do not lead students. All the tasks she designed, except for two of the adapted tasks, included at least some scaffolding of some sort to help the student, which decreased the reasoning required. She was able to raise the reasoning required in each of her adapted tasks, so reasoning was likely a focus for her as she adapted those tasks. Carly lowered the reasoning required on Project #2 by adding some questions that are supposed to help the students with the first question in case they have trouble. The first question

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[2] Task A can be found in question five in the survey, which is in Appendix A.
[3] The adapted tasks are from the Adapted Tasks Assignment.

is comprised of a triangle, a rectangle, and a semi-circle. Since question number 2 is a circle, and question #3 is a triangle, it seems like Carly is trying to give students how they are supposed to break up the first question and making the solution method more obvious.

Jamie. To Jamie, a task requires reasoning if it cannot be solved using a known solution method. When she attended to reasoning, Jamie usually used the terms reasoning, problem solving, and mathematical rigor, or used phrases, such as a task cannot be solved using a known procedure or by rote, that the student is not led to the solution method, and that the student must really think to figure it out. She described reasoning, problem solving, and mathematical rigor in the same way, so they were probably synonyms for her. A task that requires reasoning (or problem solving or mathematical rigor) has no formula that could be used to solve it and cannot be solved by rote. Jamie actually sorted the tasks in the task sort according to their level of mathematical rigor and described the mathematically rigorous tasks as tasks that could not be solved in a rote way or where students had to reason or problem solve. She also felt that mathematically rigorous tasks usually can be solved more than one way and that having students make connections between their previous knowledge and new information helps to make a task rigorous and not too procedural. She explained that tasks that are not mathematically rigorous are simple, do not require a lot of mathematical thinking, or that there is a procedure she can use to solve them. For Jamie, "problem solving involves organizing their ideas and being able to, like, think a step ahead or think what's going on or think of things that might be beneficial for you to do or to or to look at." One of the instances she felt required reasoning in Project #1 was when students see that the papers are getting too small to rip and count and so that would "kind of force them to think about, like, what's happening every time they tear the paper."

Jamie also believed that the level of reasoning that a task required was dependent upon the student's background knowledge and where the task is placed in the curriculum. This idea came out often and throughout the interviews. It is clear that this was an important part of her view of reasoning since she brought it up frequently, including while she defined terms, such as problem solving or mathematical rigor, and while she discussed the reasoning of tasks. While describing problem solving, she explained,

> If I know how to add numbers, for example, and I'm given this sheet of addition problems, that's not necessarily problem solving to me because I just have this formula in my mind for the answers. I just know how to do it. I don't have to think about it…. Whereas, if I'm unfamiliar with something, then I have to really think about what I'm doing and why I can do the things that I do.

It is very important to Jamie that a task require students to reason. She argued in the last interview that a worthwhile task must not be able to be solved by rote and must require students to reason. When asked which characteristics a worthwhile task must have, she stated, "A task has to have some aspect of it that maybe students don't automatically understand or know, so that they really have to like reason about the task that they're given in order to solve it." Jamie also showed that she believed reasoning was important since she stated that students must reason in order to learn mathematical concepts. She argued that if a task is "too procedural," students will perform the procedure without thinking about what is happening in the task, "but, when they actually have to dissect what's going on to figure it out, that's when they're really learning the mathematics behind it." She exhibited this belief on several occasions as she discussed tasks. As she brainstormed ideas for the Second Pen Pal Task, for example, and since the student had missed a particular exponential problem, she considered creating a question that gave the student a solution strategy, but she worried that such a question would limit the student's understanding. She wondered whether writing a question that would have scaffolding to help students diagram the situation “would be helpful or if it would just be more of a process like, ‘Okay, I could sit here and draw lines forever, but am I really understanding what’s going on?’”

Jamie felt that reasoning skills were important enough to be instructional goals of tasks. She attended to reasoning skills as instructional goals several times and felt that students need reasoning skills to understand mathematics. She specifically brought up developing problem solving skills as a goal of Task K (see Figure 2), her First Pen Pal Task, and Project #2. She also believed that the discipline of mathematics values mathematical concepts and also “good thinking and reasoning,” which is consistent with her belief that students must develop reasoning skills.

Task K

Manipulatives/Tools Available: Calculator with scientific functions

Penny's mother told her that several of her great-great-great-grandparents fought in the Civil War. Penny thought this was interesting and she wondered how many great-great-great grandparents that she actually had. When she found that number, she wondered how many generations back she'd have to go until she could count over 100 ancestral grandparents or 1000, or 10,000, or even 100,000. When she found out she was amazed and she was also pretty glad she had a calculator. How do you think Penny might have figured out all of this information? Explain and justify your method as clearly and completely as possible.

Figure 2. Task K from the Middle School Task Sort from Smith et al. (2004).

The frequency with which Jamie attended to reasoning and the reasoning required by the tasks she designed supports her statements that reasoning is important to her. Jamie attended to reasoning in every interview. Also, all of the tasks that Jamie designed by herself (except for one of the adapted tasks) required reasoning. In these tasks, students were not given the solution method and would likely need to try various approaches before they could solve them. In Project #2, students are told to make an accurate floor plan of the classroom, but not told how, which requires students to grapple with the idea of scale factor even before the class discusses the term. Also, she “tried to incorporate different ways of thinking” into each of the questions on the worksheet, so students could not do it “one rote way.” The tasks that Jamie designed with others required less reasoning than those she designed by herself. The Quadrilateral Task included some scaffolding (that Jamie argued against including) but still required reasoning, and the Pen Pal Tasks require the least amount of reasoning because the solution method is obvious for some of the tasks.

Even though she valued reasoning in tasks, she also felt that students must practice procedures, so they are proficient. She felt that tasks with different levels of mathematical rigor should be given. She explained,

> If you can't multiply really quickly or you can't factor, like it just makes doing math in the future really difficult, so they do need time to do that. But you definitely don't want that to be all you do because you want the understanding to be there as well.

She was not sure if procedures or concepts were more important; she felt that students need a conceptual understanding in order to build on what they have already learned, but students also must be able to perform the procedures.

Ryan. To Ryan, a task requires reasoning if the task cannot be solved using a known solution method and so the student must reason in order to solve it. When attending to reasoning, he often used the terms exploration, problem solving, or intellectually engaging or used phrases, such as requires students to come up with his own ideas, really think, reason, use deep thought processes, use different methods, struggle to solve the task, and connect what they already know in order to learn something new. Ryan used the term exploration the most often, but it appeared that exploration, problem solving, and intellectually engaging were basically synonymous to him. He described an exploratory, problem solving, or intellectually engaging task as one that does not present the solution method to the student and cannot be solved using a known solution method or formula. Even though he used exploration and problem solving in the same way throughout the interviews, there was a distinction that Ryan made between exploration and problem solving in the last interview. When asked if the two were synonymous, he struggled to distinguish the two, but ended up expressing that problem solving is part of the exploration process. He basically explained that in his mind, there is a sphere of core knowledge with something outside of the sphere that has not been experienced before. He explained that exploration "is kind of like going out of that common knowledge" and that problem solving "involves using the knowledge you know in the exploration" in order to make connections and solve the novel problem.

A large part of his view of reasoning is the idea that a task should not give the solution method. He used several different phrases to get across this idea; he described tasks as telling or not telling the students how to do it, that tasks could be solved with a memorized formula or method or not, the task gave the student a process to follow or not, and the students have done or not done a similar task before. This main idea was focused on the most in the last interview, where Ryan described what he meant by various characteristics, such as problem solving, exploration, and intellectually engaging. For example, he explained that Project #2 had problem solving because "there's things in here that students wouldn't necessarily know how to do right off the bat. There's not really a formulaic way of doing everything in here."

It was important to Ryan that a task requires students to reason. When asked what characteristics a worthwhile task must have, he included that a worthwhile task should either expand or solidify knowledge, and he explained expanding knowledge involves exploration in order to learn something new. He did state, though, that a worthwhile task could involve solidifying knowledge, which includes practicing procedures. Even though he stated a worthwhile task could only solidify knowledge, he was not completely satisfied with how the Factoring Task only solidified knowledge and wanted to "expand" the task by including some exploration of ideas. So, he ended up including a question that requires students to figure out how to solve a quadratic equation using their knowledge of factoring.

Ryan's belief that reasoning is an important characteristic of tasks is supported by three other main discussions or ideas. First, in the first interview when he was asked to sort the tasks according to an important characteristic, he first chose intellectual engagement (even though he later changed his mind and sorted according to the number of solution methods).

Second, he valued problem solving as a goal of tasks. He believed that the discipline of mathematics values basic problem solving skills and persistence, and he implied that reasoning skills can be the goal of a task. In one instance, he did not feel that there were specific mathematical concepts that came out of the Irrational Number Task, but that teachers could use it to assess students' reasoning skills. He explained that he could see the worth of this task if the teacher's focus was to teach problem solving skills.

Third, Ryan expressed that exploration is the best way to teach and learn mathematics. Ryan stated that the best way to teach math is by "guided exploration." Ryan gave an example of this that when students make observations as they are exploring, the teacher can give the formal notation or definition. He argued that when mathematics is traditionally taught, where teachers tell students what they are supposed to do,

> You don't get the understanding underneath it and you don't relate it so that when you come to another problem that's slightly different, you kind of like give up because this looks different. I don't know how to do it.

He also stated that students best learn through problem solving and that it is "really important" that students are stretched and required to think as they do tasks because that is "how you really start to understand the mathematics." He explained that before taking his mathematics education courses, he thought he understood certain procedures, but he realized that he did not truly understand them and that completing tasks in his courses helped him to gain an understanding of the procedures.

Ryan attended to reasoning often and throughout the interviews, which is consistent with the importance he placed on it. In general, Ryan's tasks required varying levels of reasoning, though. In one of the tasks he adapted for the Adapted Task Assignment, students were not given the solution method; in the Quadrilateral Task and Adapted Task J the students were led a small amount; there was more leading involved in Project #1; students were given much or all of the solution method in Project #2, the Pen Pal Tasks, and in another adapted task. Ryan did increase the reasoning required in all three of the tasks in his Adapted Tasks Assignment.

Even though Ryan believed that a task should not give students the solution method, he believed that not all mathematical concepts could be taught without giving students some information. He explained, "There are some concepts in math that you have to give some lecture to, but you don't want to give them all the information. You want to let them kind of experience it, and there's a balance there." He felt that he had to give some information to students for both of his projects. For Project #2, he stated that he wanted exploration in the task, but that with completing the square, he needed to give some information because he did not feel that students would discover this method. In the lesson, the teacher shows students how to complete the procedure using an area model, and the students are asked to solve similar questions. Then, the teacher explains to students how to complete the square algebraically, and the students are asked to solve similar questions again. The questions the students are asked to do are slightly different than the examples they are given, but they require very little reasoning. For both of Ryan's projects, there is a great deal of scaffolding, but, for Project #1, students are required to do some thinking themselves. The students are not immediately told to use a scatterplot, but instead asked how they could analyze the data in order to determine if there is a relationship. Given the way Ryan spoke about concepts and procedures, he seemed to believe that his goal in teaching mathematics was for students to learn procedures conceptually.

Conclusion. All the participants explained that a task requires reasoning if students are not given the solution method or if the task cannot be solved by a known solution method. They used various terms that were essentially synonymous to them, such as problem solving, mathematical rigor, and exploration. They wanted students to figure out how to solve the task themselves. They all felt that a task should not have a solution method that is obvious. A large focus of reasoning for Jamie was that the level of reasoning that a task required was dependent upon the student's background knowledge and where it is placed in the curriculum. Even though cognitive demand was a focus of the Task Design course, it was rare for the participants to use the term cognitive demand during the interviews. They did, however, attend to ideas from Stein et al. (2009) that were discussed in the course, such as a task not being able to be solved using a routine procedure.

Carly and Jamie both stated that a worthwhile task must require reasoning. Ryan seemed to prefer that tasks require reasoning, but he felt that a worthwhile task could involve reasoning or "solidifying knowledge," which includes practicing procedures. Each of the participants brought up that developing reasoning skills can be the instructional goal for a task, but Carly and Jamie expressed that developing reasoning skills are important goals. Each of the participants communicated that students learn mathematics better if they have to reason. The participants all attended to reasoning often and throughout the interviews, and all of their tasks varied in the level of reasoning required. Neither of Jamie's projects led the students through the task, but Carly designed a task where the teacher leads the students through much of the task, and Ryan designed a task where the teacher explains how to do a procedure. Ryan also expressed that some topics in mathematics need to be taught using some lecture, but the teacher should not give all of the information and students should do some exploration.

Appropriateness

I use Leatham et al.'s (2013) definition of appropriateness, applied to tasks, for the literature-derived definition of appropriateness. One of their criteria to determine if a student's comment within a lesson was mathematically significant was whether the idea in the comment was "appropriate for the mathematical development level of students with background similar to those in the classroom" (Leatham, et al., 2013, p. 14). Appropriateness has the following two requirements: (1) that the idea is accessible to the students, meaning that they have the necessary background knowledge to engage with it and (2) that the idea is unlikely to have been mastered already by students at a particular level, so that students are able to learn from the task (Leatham, et al., 2013). In order to determine if an instructional task is appropriate, then, one must decide if the mathematical idea in the task is likely to be accessible to the students who will complete the task, but not an idea that they have already mastered. Assessment tasks are exempt from the second requirement because it is expected that a student would have already mastered the concept, but the task should still be accessible in order to be appropriate.

It is not possible to accurately gauge the appropriateness of a task without knowing who will complete the task. In the Task Design class, the PSTs usually did not know the specific students who would complete the tasks they designed, but they specified the grade-level of the students who would complete the task in all of their tasks, except for the Adapted Tasks Assignment. Although knowing the particular students who would complete a task would be ideal, knowing the grade-level of the students who will do the task does allow one to gauge the appropriateness of the task to some degree.

While the *Professional Standards* (NCTM, 1991) did not use the term appropriateness in their definition of a worthwhile task, it did state that worthwhile tasks should be based on "knowledge of students' understandings" (p. 25). As they elaborated on worthwhile tasks, they stated that teachers must consider their students along with the mathematical content in order to gauge the appropriateness of a task. Part of considering their students requires teachers to think about what their particular students "already know and can do, what they need to work on, and how much they seem ready to stretch intellectually" (NCTM, 1991, p. 27). The idea that the task must be based on what the students know and how much they seem ready to be challenged fits perfectly with how I will view appropriateness in that the mathematics in the task must be accessible to the students who will complete the task, but not something that the students have already mastered. Smith and Stein (1998) also stressed the importance of the teacher considering the age, grade-level, prior knowledge, and experiences of their students when deciding the quality of a task.

In the Task Design class, the instructor did not explicitly teach about making sure that a task is appropriate for students. While speaking to the participants at their table or in whole class discussions, however, the instructor commented about the possible background knowledge of students multiple times and sometimes in conjunction with the notion of the idea not being mastered. For example, once when the instructor came to the participants' table, Carly and Jamie asked her for ideas for the Second Pen Pal Task. The instructor's suggestions were based on the students' background knowledge. She suggested that since Carly and Jamie now knew whether or not the students understood exponential functions, they could ask more about exponentials, ask a question about the rate of change of exponential functions, or ask them to compare two exponential functions with different bases in order to get at logarithms from the "backdoor."

Appropriateness was not the focus of either text, but both argued that tasks should be built upon students' prior knowledge. Hiebert and Wearne (2003) asserted that tasks used when teaching through problem solving should be built on students' prior knowledge and be "just within students' reach, allowing them to struggle to find solutions" (p. 6). Stein et al. (2009) argued that the teacher must consider the students' age, grade-level, prior-knowledge, and experiences in order to determine the cognitive demand of a task. They also explained that one factor that contributes to the maintenance of the cognitive demand of a task through a lesson is that the task is built upon prior knowledge.

Carly. Although Carly only rarely used the term "appropriateness," she did attend to both of the ideas within the literature-derived definition in the interviews. Regarding the first requirement, Carly argued that all students should be able to engage with tasks at their own level, so different students should be able to do the task with the different levels of knowledge. She explained, "[The task] makes sense if I think about it this way… but if you also understand this concept, you get more out of it, but, you know, you can still do it at the lower level." She felt that the task should not begin or stay at a level that is beyond what the students are capable of and that a task should be able to be "entered from multiple ways because you're not going to have every student be at the same level." When Carly spoke about instructional tasks in general, she stated that "you're going to start with something that they know... and then it's going to expand on that." Regarding the second requirement, in the first interview, she communicated that if the student already had mastered the concept in the task, the task would not be useful.

During the task sort, she brought up this idea for three of the tasks. For example, for one of the tasks, where students were required to divide several decimals using long division, she explained that she and the other PSTs had already learned how to divide decimals, so "not a lot came out of it," but the task could be useful for students if the students were "still getting acquainted with how to do this." She argued that if the task was given "at the right time, it could be more useful" since it could "teach them why the algorithm works."

To Carly, it is important that a task be appropriate. As previously mentioned, Carly felt that all instructional tasks should start with a concept the students know and help them to learn something else. She stated that it is very important that, in a task, "you can start where you are, then get to where you're wanting to go," which implies that the student should be able to do the task with the mathematical knowledge they have and be able to learn something new. She also stated that a task must expand the student's knowledge (though a homework assignment does not necessarily have to), which also means that the concept should not have been mastered. Although not explicitly connected to tasks, Carly stated on her Final Exam that learning with understanding requires that students build on prior knowledge. She explained that "students already have ideas, past experiences, and understanding of certain concepts and so these should be utilized in order to more effectively teach them new skills."

That appropriateness is important to Carly is supported by the fact that she attended to at least one of the requirements of appropriateness each time she discussed one of her tasks. She did so by describing what they would need to know in order to complete the task and/or by describing the new concept that the student would learn. She attended to the first requirement while discussing the Quadrilateral Task and the First Pen Pal Task and attended to both requirements while discussing her projects. For example, Carly attended to the background knowledge required on the Quadrilateral Task; she explained that the students would need to know that the sum of the interior angles of a triangle is 180 degrees and that a right angle is 90 degrees. For each of the projects, she explained what she thought the students would have already learned before the task and how this task helps them to learn something new using that previous knowledge. For Project #1, she stated that it was "a good way to use… previous knowledge in order to develop an understanding of a new concept." Specifically, she explained that the students would already know how to find the area of a square, and the task uses this knowledge in order to help students to learn the meaning of square roots in the context of the area and side length of a square (where side length of a square is the square root of its area). When Carly's tasks were evaluated using the literature-derived definition of appropriateness in order to see if they seemed to be appropriate for the students who would likely do them, it showed that Carly sometimes does not accurately consider students' prior knowledge. The Quadrilateral Task and the Pen Pal Tasks are appropriate. To illustrate, the First Pen Pal Task would be appropriate because it was to be given to a high school precalculus student, and it is likely that a student at this level would be familiar with both linear and exponential functions. While these students may have already mastered being able to identify if a situation is linear or exponential, the instructor of the Task Design class told the PSTs to design an assessment task, so the fact that the concept may have been mastered is acceptable.

Her projects would not be completely appropriate, but for different reasons. She appeared to have underestimated the students' background knowledge in Project #1 and overestimated it somewhat in Project #2. For Project #1, it is likely that an eighth grader would have the necessary background knowledge in order to engage with the task, but the ideas in the task may have already been mastered. In this task, Carly planned to introduce the notation and concept of square roots, but, by eighth grade, a student should already be familiar with these ideas. Carly had stated in the interview that "the square root shouldn't have been a new concept," but also stated that the students were going to learn new notation, which seems to be inconsistent. However, the task also connects the procedure of taking the square root of a number to the relationship between the area of a square and its side length, and this connection has possibly not been explored by the students. For Project #2, however, it is likely that the sixth or seventh grade students who would do this task would not have the necessary background knowledge to do all of it. Sixth graders would have been learning how to find the area of rectangles, triangles, and circles, so assuming this task was given after students had mastered those concepts, most of the task would be accessible. However, question five on the task requires students to use the Pythagorean Theorem in order to find a missing height in order to find the area of the figure. A student may not be taught the Pythagorean Theorem until eighth grade, so this part of the task would not be accessible to the target student population. Interestingly enough, Carly did include the Pythagorean Theorem as one of the pieces of requisite mathematics on her hand-written rough draft of the project, but it did not make it onto any of her typed versions.

Jamie. Jamie did not usually use the term "appropriateness," but she attended to both requirements of the literature-derived definition. She felt that a task should be built on what students already know, but not be so simple that they do not need to think in order to do the task. She explained,

> I think it's important to connect the tasks that you give them to what they already know, so that they have the skills in order to complete them, so that they aren't too difficult. And then, also, you don't want them too simple to where they're not getting anything out of it, unless you're practicing some sort of skill that they need to become proficient at or very quick at solving.

Jamie essentially argued that the students must have the necessary background knowledge in order to complete the task and must not have mastered the concept (unless it is a task with the goal of proficiency).

Jamie clearly felt that the appropriateness of a task was related to the reasoning required. She stated on her Midterm Exam that a task with a high level of cognitive demand should have "connections to the mathematical background of students," so they are "forced to pull from previous math knowledge to complete a task" and it should be one that is "just at the reach of the students" in that they must "apply what they already know to stretch a little further." Also, while sorting the tasks in the task sort according to their level of mathematical rigor, she brought up several times that the mathematical rigor of a task depends on the background knowledge of the students. A task could only be mathematically rigorous for students if they had not yet mastered the FMC of the task. For example, while discussing Task J, she explained that if students did not know the algorithm for multiplying 325 by 4, this task could help them to learn a way to do it. In addition, when asked what she would do if a student said he was not good at math, one of the things she indicated she would do was give him a task he could complete but that would be challenging for him.

The idea of appropriateness is very important to Jamie. On three separate occasions Jamie explicitly stated how important it was for a task to fit the two requirements of appropriateness. For example, after Jamie had sorted the tasks in the task sort according to two characteristics, I asked her if there were any other characteristics that she felt were important for a task to have. As previously mentioned, she explained that a task should be based on what the students know, so that they have the background knowledge to complete the task. The task should also not be too simple so that the students do not learn anything, unless the students are practicing a skill. As another example, when asked which characteristics a worthwhile task must have, appropriateness (along with reasoning) came up first. She explained that "a task has to have some aspect of it that maybe students don't automatically understand or know, so that they really have to like reason about the task that they're given in order to solve it." When asked how she would know if a task did this, she responded that "you just have to look at what the students have learned before and just try to build a task that's using something different from what they had before." These statements support that she believes a task needs to be built on the current knowledge of the students, but also not be something they already know.

The importance Jamie placed on appropriateness is supported by the frequency with which she attended to appropriateness throughout the interviews. She attended to at least one of the requirements of appropriateness for both of her projects, the Quadrilateral Task, and the First Pen Pal Task. In Project #1 and #2, she attended to both requirements. For example, for Project #2, she was asked why the task was appropriate for 7^{th} graders besides the fact it is in the *Common Core*. She explained that they had already learned about ratios, multiplication, and areas, so "they have the tools to do the task." Also, she stated that she wanted students to learn what a scale factor is and that she did not think that they would already know that. She appeared to attend to the first requirement for the Quadrilateral Task and First Pen Pal Task. For example, on the First Pen Pal Task, she expressed that precalculus students would already have some experience with linear and exponential relationships and know how to graph them and manipulate their equations; she added that "they probably have experience with most of the concepts of the task." Since she wanted this to be an assessment task, it makes sense that she did not express concern that these concepts would have already been mastered.

Most of the tasks that Jamie designed are appropriate, which is consistent with the importance she stated that she placed upon the characteristic. The Quadrilateral Tasks, the Pen Pal Tasks, and Project #2 are appropriate. For example, for Project #2, the lesson plan stated that this lesson should come after students have learned about proportions. A seventh grader would likely still have a limited understanding of ratios and proportions, then, but not have mastered what a scale factor is. However, she seemed to have underestimated the eighth graders who would do Project #1, so it would probably not be completely appropriate. Eighth graders should have the necessary understanding of multiplication in order to do the task, but Jamie planned on introducing exponential notation, and an eighth grader would have already been familiarized with it. However, they still likely do not have a developed understanding of how an exponent represents repeated multiplication or an appreciation for how quickly a number can grow in an exponential situation. The task should be adapted to account for the notation already being known. She discussed in the interview how she hoped that students would not yet know the notation for exponents, but she also mentioned that they would be familiar with squares and maybe square roots. If she thought that they knew about squares, it is interesting that she would design a task where she planned to introduce the notation to them. It could be that she did not carefully attend to the students' current knowledge while designing the task.

Ryan. Ryan attended to both requirements of appropriateness, even though he did not use the term appropriateness. He felt that tasks should allow students to use what they already know to help them learn something new. He explained, "It has to be at the level that they already know things that they can use to solve this and that as they solve this, it's going to lead into further knowledge in mathematics." If a task helps students to have "further knowledge in mathematics," they will have encountered a concept they had not previously mastered. He occasionally used the phrase “stretch them, but don’t break them” to communicate that students must have the necessary background knowledge in order to engage with a task. He argued that “you want to stretch them” so that they need to think, which is “how you really start to understand the mathematics,” adding, “but you don’t want to break them.”

Ryan thought that both requirements of appropriateness were important for a task to have. After he sorted the tasks in the task sort, he was asked if there were any other characteristics that are important for a task to have, and he described both requirements of appropriateness. As previously mentioned, he explained that a task needs to allow students to use what they already know, which is consistent with the first requirement. Ryan stated in the last interview that a worthwhile task must “expand your knowledge or solidify knowledge in some form…, so you understand something better than you did before.” That a task should help students to “understand something better” is consistent with the second requirement, that the primary concept behind a worthwhile mathematical task should not have been already mastered. Ryan also stated that a typical day in his mathematics class would include that he would present the students with a task that “builds off of what they were doing last time, but leads them to new mathematics.” Since he wants the tasks he gives to be appropriate, it must be quite important to him.

In fact, all of the tasks that Ryan designed are appropriate, which supports his statements that he believes that appropriateness is important. For example, Project #1 is planned for eighth graders, and they would have the necessary background knowledge about linear equations in order to do the task; it is planned to be right after a unit on linear equations. They also probably will not have used scatterplots to analyze bivariate data. (The only task that may not be appropriate is Project #2 because there is no specific grade-level given. If the task is given at the right time in high school, however, after students have the necessary background knowledge about quadratic equations and before they have mastered how to derive the quadratic formula by completing the square, this task would be appropriate.)

Ryan attended to appropriateness in all but the third interview, which is consistent with the importance he placed on the characteristic. When he attended to appropriateness while discussing tasks, he usually spoke about what the students would need to have already learned and/or what they were going to learn from the task. For example, Ryan explained that there is a large amount of background knowledge that a student would need to know in order to do the Irrational Number Task, such as what an irrational number is, what squaring means, and what less than means, and they would need to problem solve in order to use this knowledge in order "try and understand something that you've never seen before, such as what irrational numbers have squares that are smaller than itself." In the tasks he designed, he attended to both requirements while discussing Project #1 and the First Pen Pal Task, but mostly attended to the second requirement for Project #2. For example, Ryan attended to both requirements several times while discussing Project #1, and the essence was that the students will use their background knowledge on graphing and linear equations in order to learn about scatterplots. For example, when asked how the mathematics will emerge from the activity, he explained that students would their previous knowledge so that "they can begin to see how things that they've learned in the past, like graphing and looking at patterns – things that they already know – how that applies to what a scatterplot is," so "we just happen to add to" what they already know and "help them make connections." Ryan's discussion of the First Pen Pal Task stood out, though, because his main concern was that he did not know enough about what the precalculus students had previously learned. He stated that the PSTs were told in general what they had learned, but he did not know exactly what the students will have learned by the time they received the task, but would like to know in order to base the task on that. He felt that the task might be a good one depending upon if it was given at the right time and what they had just learned. He explained that this was meant as an assessment task, and they should have already learned about exponential functions, but if they had not learned exponential growth and decay, it could help them to learn that.

Conclusion. The participants all attended to the idea of appropriateness, all arguing that tasks should be based on students' current knowledge and should help them to learn about a concept they had not yet mastered. It was important for all of the participants that a task be appropriate, but Jamie and Ryan even argued that a task *must* be appropriate to be worthwhile. The participants usually attended to appropriateness while they designed tasks and talked about designing tasks. When the participants attended to both requirements, they usually explained what the students would need to know or what they would already know before doing the task and how that would be used to learn some new concept. Their professed belief that appropriateness is important usually did show up in their tasks; most of their tasks likely would be appropriate to give to students at the specified grade-level. Both Carly and Jamie probably assumed that students knew less than they probably would for their first projects; the tasks are still mostly appropriate since students would likely not have mastered all of the concepts in the tasks, but they could be adapted to better take into account students' knowledge. The only task that was very likely inappropriate is Carly's second project because one of the questions in the sixth or seventh grade task requires knowledge of a concept that usually is not taught until eighth grade.

Clarity

A clear task is written in such a way that students understand what is expected of them in the task. If a task is unclear, there is some unintentional ambiguity that may impact how students will complete the task since students may not understand what is expected of them. An unclear task differs from an open task in that in an open task, the solution method may not be obvious, but the students know what their goal is.

Clarity is not an idea that was explicitly taught in the Task Design class, but it did come up in the whole class discussions about a few of the Task Sort Tasks. For example, for Task K, Ryan mentioned that he was not sure if he was supposed to add the generations to find the number of ancestral grandparents. Also, many of the PSTs naturally found an expression to represent the number of grandparents for a certain generation, but the instructor pointed out that the task did not ask for it, and that if they wanted the students to find the pattern, the instructions of the task would need to be changed. So, the instructor brought up that a task should clearly indicate what the teacher wants the students to do. The instructor also stated that the Adapted Tasks Assignment should include an analysis describing why they made the changes they did from a mathematical, pedagogical, or clarity perspective, but did not expound on what a "clarity perspective" is. Clarity was not a focus of either of the texts, as would be expected. However, Stein et al. (2009) stated that one way that a task with high cognitive demand could decline in its cognitive demand through the lesson is if the task expectations were "not clear enough to put students in the right cognitive space" (p. 16).

Carly. To Carly, having a task be clear means that students understand what they are expected to do from the way that the instructions are worded or organized. She explicitly stated what she meant by clear instructions in the group interview; while the group discussed the importance of the clarity of the instructions of a task, she explained that it is important that students understand what they are expected to do.

When attending to clarity, Carly often attended to the wording of the instructions by including how the wording could impact how the students would think about the mathematics. For example, for the First Pen Pal Task, she explained that she and Jamie wanted the instructions to read “solve” before “set up the equation,” so that students could solve it however they want to; “if they choose to solve it without setting up an equation, it doesn’t kind of eliminate that possibility.” She also considered switching the order of whether she asked for the relationship or the explanation first for the six questions since that “would communication something different about what I was looking for.” Also, in Project #2, she mentioned that she and her partner intentionally did not ask students to find the area in the instructions of the task. Instead, they wanted to give the students a real-world context and let them figure out that they needed to find the area.

In addition to attending to how students would do the mathematics based on the clarity of task, Carly also recognized that the lack of clarity of a given task could be good or bad. It could be good, according to Carly, because the students could do the task differently without being concerned whether their answer was the right one. In Task K, it was unclear if students should add all of the generations together or just include the number of people in one generation to find the number of ancestral grandparents. She initially thought that was bad, but explained that “maybe that worked out better” since there was not one definite correct answer, and “you just sneak in a little extra mathematics that way.” Carly believed that a lack of clarity could also be bad since students could interpret the task in a more straightforward way than the teacher had hoped. For Task A, the students were asked to find the perimeter of a one, two, and three-square chain "using the side of a square pattern tile as a measure," and the students were provided with square pattern blocks. It was ambiguous as to whether the length of the side of an actual pattern block should be used as the unit of measure or if the squares drawn on the task represented the pattern blocks that should be used. She explained that if students are intended to use the actual square tiles to measure, the problem requires more thinking because it involves fractions; otherwise, students just need to count the number of sides.

It is fairly important to Carly that the task be clear. She expressed that clear instructions are important, but that it is more important for tasks to be clear if the teacher is not going to be present. This is because if the teacher is present for the launching of the task, she can clear up any confusion herself. For example, when asked what was difficult about designing the First Pen Pal Task, Carly explained,

> We really had to make sure that everything was just right because we weren't going to be able to fix it if it wasn't. Whereas in a classroom, if your instructions are a little wordy, it might confuse them and kind of get you off-task for a little bit, but you'd be able to recover from it 'cause you could fix it yourself.

Since Carly thought that she would be able to fix any clarity problems that her tasks had during class, it makes sense why Carly usually did not complete the tasks as part of her process for designing the tasks. Carly not completing the tasks supports the relatively-low level of importance she placed upon clarity. It would be difficult for her to attend to clarity and to see if the student would likely understand what is expected if she did not complete the task she had designed.

Clarity was important to Carly, but definitely not the most important characteristic to her. This may be why her written tasks showed a range of clarity. Project #1 was very unclear, the Quadrilateral Task was clear, and the other tasks were mostly clear, but did have some ambiguities. For Project #1, it was difficult to determine if students would likely understand what was expected of them. This is because the lesson involved mostly verbal instructions, and most of the lesson plan was unclear about what the teacher needs to specifically do or say in the launching and completion of the task. For example, in order to launch the task, she started by saying, “Given various side lengths, the students will graphically represent such squares using their M&Ms and graph paper,” but did not state exactly what the teacher should tell the students to do with the M&Ms and graph paper or which numbers the teacher should use for the side lengths. Thus, it is likely that the task would not be clear for the students since it is not clear for the teacher unless the teacher adds specificity to the lesson plan. In the other tasks, besides the Quadrilateral Task which was clear, there was at least one of the sentences in the instructions in the tasks that were unclear, but the lack of clarity was not severe. For example, in her adaptation of Task A, she added several questions to the end of the task, including, “Did you expect the perimeter to double?” Since she had asked the students to find the perimeter of five different chains, this statement is unclear as to which two chains students should compare.

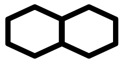
Even though Carly’s tasks varied in clarity, clarity was important enough to her that she attempted to adapt Task A in the Adapted Tasks Assignment and Project #2 (after the peer review) to be clearer. To illustrate with Task A, she thought that it was unclear whether students should use the side of the square in the drawing or the side of the actual pattern tile in order to find the perimeter. The original task stated “using the side of a square pattern tile as a measure, find the perimeter…,” and she changed it in her Adapted Tasks Assignment to “using the side of a square pattern tile as one whole unit, measure the perimeter…” In her analysis, she wrote, “Clarifying that a square pattern tile is a whole unit, takes away the confusion that led to merely adding the sides, which produced a general formula that was not equal to the perimeter.” It is interesting that she thought that simply changing “as a measure” to “as a whole unit” would make it clearer to the student that the manipulative was to actually be used to measure the squares drawn instead of just represent the squares drawn, but she did obviously care about trying to make the task clearer.

Jamie. Jamie described clarity as making sure the question asks what is intended so students understand the instructions and what is expected of them in the task. She stated that "you really have to think about the way you word things. That way, students are answering exactly what you want them to."

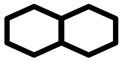
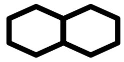
The writing of tasks was problematic to Jamie, and she expressed multiple times that it was difficult for her to word tasks clearly. While talking about how she worded the tasks she designed Project #1, the First Pen Pal Task, and Project #2, she stated at least twice in each of those interviews that it was difficult for her to word her tasks in a clear way. She also brought up her difficulty in wording Task K (see Figure 2) and Task P (see Figure 3) in her analysis in the Adapted Tasks Assignment. When asked how the peer review affected how she revised Project #2, she stated that the peer review helped her to write some of the questions better, but she still felt like she would need to "change them again and again; just it's really hard to explain what you're asking for with the language." In fact, in the second interview, she stated that she preferred to verbally give instructions to make sure that students understand what is expected. She expressed that it is easier for a teacher to see if students do not understand what is expected when the teacher is explaining the instructions of the task to them than if the teacher hands out a piece of paper with the task and asks them to read it. Since Jamie felt unsure in her ability to word written tasks clearly, it makes sense that she would feel more comfortable giving verbal instructions since she felt that it would be easier to assess whether students understood the instructions that way.

Task P

Manipulatives/Tools: Pattern Blocks

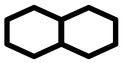
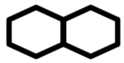
For problems 1-3, use  as the whole or unit.

1. Find 1/2 of 1/3. Use pattern blocks. Draw your answer.

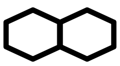
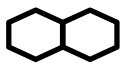
Show 1/3. Show 1/2 of 1/3. 1/2 x 1/3 =___

2. Find 1/3 of 1/4. Use pattern blocks. Draw your answer.

Show 1/4. Show 1/3 of 1/4. 1/3 x 1/4 =___

3. Find 1/4 of 1/3. Use pattern blocks. Draw your answer.

Show 1/3. Show 1/4 of 1/3. 1/4 x 1/3 =___

Figure 3. Task P from the Middle School Task Sort from Smith et al. (2004).

Jamie usually attended to how the wording of the question would impact the mathematics that would be done when she attended to the clarity of specific tasks. In fact, she attended to how the wording would impact what mathematics the students would do in all of the tasks she designed, except for one of the adapted tasks and the First Pen Pal Task. For example, while the participants were deciding which questions to ask for the Quadrilateral Task, Jamie asked questions of the group such as, "Is there a way we word it without pushing them toward a pattern, but not implying that it's the same for all quadrilaterals?" As mentioned previously, Jamie focused on the wording of Task K, but she did it in a way that was concerned with the mathematics the students would do. She stated that depending on the goal the teacher had for Task K, the fact that the task was not completely clear about if students needed to add the generations or not in order to find the number of ancestral grandparents "could be a good or bad thing." However, when she adapted it, she explained in her analysis that "because of the ambiguity of some of the statements in the task, it is hard for a student to pick out what question the task is really getting at," and she changed the task so that it led to one concept (logarithms) instead of multiple possible concepts (exponents, summations, and/or logarithms). Jamie did sometimes, however, attend to the clarity of tasks with other goals in mind. For example, she bulleted her list of questions instead of putting the questions in paragraph form in Task K in order to help students keep "from getting lost in all of the words."

To Jamie, the clarity of a task is fairly important. While the participants were ranking the seven characteristics in order of importance, Jamie was the first participant to speak about clear instructions. She stated that clear instructions are important, "but if initially your instructions aren't clear, I feel like they can be cleared up. So, I think that would be towards the bottom [in the ranking]." For Jamie, it is more important for a task to be clear if the teacher is not going to be with the students when they complete the task. When asked how designing the First Pen Pal Task was different than designing Project #1, she expressed that she would have to "think a lot about making sure to write everything out" since she would not be present when the student completed the First Pen Pal Task. Since she felt that the teacher could "clear up" any misunderstandings about the instructions, it makes sense that she usually did not complete the tasks she designed as part of the designing process. Also, the only tasks where she had an option of using verbal instructions were the projects, and she chose to use verbal instructions in both of these tasks.

Even though Jamie often said that she struggled with the wording of tasks and said she preferred to give verbal instructions, she generally worded tasks well. Most of her tasks were clear; it seemed that students would likely understand what they were expected to do. The First Pen Pal Task and Project #1 were mostly clear, but had some ambiguity. In the First Pen Pal

Task in one of the questions, it stated, "There is already 1 gallon of mix made for the game," though it should read "gallon of cocoa" instead of "gallon of mix." This is something that a student may overlook, but it does make the problem a little confusing since the problem asks for the amount of mix it would take to make five gallons of cocoa. Also, Project #1 has the teacher verbally state,

> You have a piece of paper that you are going to repeatedly rip in half. After you tear it once you put the two pieces back to back and rip them both in half again. Then count up the number of pieces the paper has been torn in.

Then, the teacher writes "How many pieces of paper will you have after 8 tears? After 10? Can you give a general statement or equation describing what happens for a given number of tears in the paper?" From these instructions, some students may think that they should actually complete all 10 tears, but this process can only be completed four or five times. When asked how she chose the number of rips she asked the students for, she said that she "had no thought in choosing those numbers … it was further than just, like, one or two." Some students who think they should carry out all 10 rips may waste time ripping each tiny paper individually, but others may think about what happens to the papers each time they are ripped and generalize since they cannot carry out all 10 rips by putting the papers back to back.

Ryan. To Ryan, a task has clarity if the students understand what they are expected to do. For example, in the first interview, he stated that clarity was "a very important part of any task… so the students really know what they are getting into and what they're expected to do." He also was probably referring to clarity when he used the phrase "easy to understand." Ryan had mentioned that he wanted tasks to be "easy to understand, but challenging," so I asked how that showed up in Project #1. He explained that it is easy to understand how to collect the data and "what you're shooting for and what you're looking for." Ryan usually spoke of clarity in terms of the written instructions of a task, but he did occasionally discuss the clarity of verbal instructions. When asked how clear instructions showed up in Project #1, he explained that would depend on if the teacher can help students to understand what they are expected to do with his delivery of the task.

When Ryan attended to clarity while analyzing or designing tasks, he usually simply described whether parts of the tasks were clear or not or how he would fix the task to make it clearer. For example, Ryan sorted the tasks in the task sort according to how clear the instructions were. He created three piles for the tasks; the piles were "clearly-written instructions", "could use a couple of tweaks to make it more clear and so the task can be better", and "need[s] some bigger clarification." He felt like two of the tasks, including a task which required students to divide decimals, were "pretty clear what you're supposed to do," so they went in the first stack. He felt like Task J "could be even more clear" about coming up with your own method or using the method diagrammed, so it went in the second stack. Lastly, he stated that Task K was very unclear since it did not specify if students should add the generations or not and Task A was unclear whether the students were to use the side of a square pattern tile as a measure or not, so these tasks went in the unclear pile.

Ryan also usually attended to the clarity in the tasks he designed by describing whether the tasks were clear or not and explaining how he would fix the task to make it clearer. For example, Ryan adapted the First Pen Pal Task from a task he found online and tried to make it "coherent enough so that they understand it." He explained that he "tweaked" the grammar to make the task clearer and took out some parts that he did not feel added to the task. The part he "tweaked" was that he changed "List some similarities between the two functions and list some major differences" to "List some similarities and differences between the two functions." He felt that his new version was clearer even though this question on the original task was probably just as clear.

Ryan sometimes attended to how the level of clarity of a task would impact how the students do the mathematics in the task. This happened in the Pen Pal Tasks, in the Adapted Tasks Assignment, and while designing the Quadrilateral Task. The most informative example is of the Pen Pal Tasks. In one of the questions in the First Pen Pal Task, Ryan realized that the lack of clarity impacted how the student did the task and made that question clearer. In question 2a, students are asked for the number of magic beans in the first box if we are given that box 10 has five beans "if in the decay process you lose two-thirds each time?" There are at least two reasons why this question in confusing. First, you are told that you "lose two-thirds each time," but what are you losing two-thirds of each time? The question means that you lose two-thirds of the beans in box one and then put what you have left in box two, then lose two-thirds of the number in box two and then put what you have left in box three, and so on. The only way the student would do that correctly is if she decided to follow the same basic process as question number one. Second, the language of losing two-thirds of the beans is confusing. The purpose is for students to see that losing two-thirds of the beans on each step from box one to box 10 would be the same as taking one-third of the beans on each step and putting them into the next box. Removing one-third of the beans on each step was actually what students were asked to do on question one, which was the exponential decay example. Then, the student is supposed to see that, since we are going backwards from box 10 to box one, she needs to multiply the number of beans on each step from box 10 to box one by three in order to get the number of beans in box one. This question is meant to be an exponential growth example. After seeing the student's work on this question, Ryan realized that it was confusing. After the student who did his First Pen Pal Task did not do what he expected, he pointed out,

> Well, as you can see, depending on how you word a question or how you present a task depends on the kind of answers that you're going to get. I mean like question two there, when I say that in the decay process you lose two-thirds each time, because it was worded that way, she started using two-thirds and then she's like, "Wait, how am I supposed to use two-thirds? What am I doing?" Completely missed what I was hoping she would get out of it.

After being asked to brainstorm a follow-up task to give to this student, he stated that he would want to rewrite this question because the wording might have been unclear. The Second Pen Pal Task addressed this same idea and was very clear. It reads,

> Suppose you get to box nine, and you have nine beans left. If you know that you took one-third of the beans each time to get to box nine, work backwards to find out the number of beans you started with in box one. Make sure to show your work. (Hint: If you take one-third of the beans each time going from box 1 to box nine, how much must you multiply by each time in order to work backwards from box nine to box one?)

When Ryan had a second try at this question, he probably gave the student too much guidance, but he definitely did address the clarity problems from the first task.

Clarity is very important to Ryan. He explicitly stated twice that a worthwhile task must be clear. He argued that "clarity and coherence is necessary" because "if the students don't understand what you're asking for, then it's going to be difficult." When asked to sort the tasks in the task sort according to an important characteristic, the second characteristic he chose to use was clarity. He expressed that clarity is especially important if the teacher does not give any verbal instructions to the students along with the task. Ryan attended to clarity in each of the interviews, and all of his tasks were clear (except for part of the First Pen Pal Task), which supports his statements that clarity is necessary.

Even though clarity was very important to Ryan, he usually did not complete the tasks he designed as part of his initial design process. For the First Pen Pal Task, for example, when asked if he had completed the task before giving it, he explained that he looked over the answer key from the original task, but he did not go through and do the task himself. It seems like it would be difficult for Ryan to make an accurate evaluation of how clear a task is without completing it.

Conclusion. The three participants have a similar definition of clarity; they all believed that a task is clear if the students understand what they are expected to do from the instructions in the task. Jamie, more than the other participants, felt like wording tasks in a clear way was a difficult process. Additionally, both Jamie and Ryan discussed clarity in terms of the verbal instructions that a teacher gives as well as the written instructions of the task. All three of the participants chose to do a teacher-guided activity that involved verbal rather than written instructions for their first task. Even though Jamie was the most worried about the clarity of her tasks, she was the most explicit about what the teacher should do and say, which makes it more likely that the task will be clear for the students.

Ryan believed that clarity is more important than the other participants. He felt that it is necessary for worthwhile tasks to be clear, whereas both Carly and Jamie argued that it is important for a task to be clear, but more important if the teacher will not be present for launching the task because the teacher will not be able to clear up confusions. The participants generally wrote tasks that were clear. Ryan had one task that was not clear, and Jamie had two tasks that were not clear. Carly had more issues with clarity, however, and her tasks ranged in their level of clarity. When adapting tasks for clarity, the participants usually did improve the clarity of the task. Also, they all recognized that changing the wording of instructions can impact how the students would do the mathematics in the task, although they did not always address this as they attended to clarity.

One common issue is that it was the norm for the participants to not complete a task as part of the designing process. It seems that it would be difficult to know if a student would likely understand what is expected of them in the task without doing the task.

Communication

A task can require students to communicate their ideas in two ways: verbally or in written form. Communicating thinking verbally requires students to discuss their thinking with each other in order to complete the task and will be referred to as the discussion aspect of communication. The participants did sometimes reference how a task could be used in a whole-class, teacher-led discussion. Whole-class discussion, however, is not part of the communication characteristic of the task as it will be used here since teacher-led discussion is not an inherent part of a task, and a teacher could feasibly take any task and have some sort of a whole-class discussion about it. The discussion aspect of communication, however, is part of the task since the task itself requires students to discuss with each other. So, in this section, unless otherwise noted, the term discussion will refer to the student-to-student discussion that is part of a task. The second way a task can require students to communicate their thinking is in written form. Communicating thinking in written form requires students to explain their thinking on paper in order to complete the task and will be referred to as the explanation aspect of communication.

Worthwhile tasks should "promote communication about mathematics" (NCTM, 1991, p. 25). Encouraging students to communicate their ideas is important since it benefits them, but it also benefits the teacher. NCTM (1991) stated that when students explain their thinking, it helps them to clarify their ideas and develop their understanding. They argued that worthwhile tasks must be based on "knowledge of the range of ways that diverse students learn mathematics" (NCTM, 1991, p. 25). Later, within their elaboration of this idea, they described how if a teacher knows that having students explain their thinking helps them to clarify and develop their understanding of a concept, the teacher may want to design a task that requires students to explain. Requiring students to communicate their thinking also benefits the teacher because it reveals their thinking to the teacher. Cooney et al. (2001) argued that when students communicate their thinking, it is more likely that teachers will be able to understand what their students know and can apply. NCTM (1991) also asserted that teachers should choose tasks that help them to learn about what their students understand and how they learn. Designing tasks that require students to communicate their thinking helps teachers to do this.

The characteristic of communication was not explicitly taught in the Task Design class, but it was mentioned a few times by the instructor or in the whole group discussions. Discussion came up more often and it seemed to be an expected companion to most tasks. There is a "Facilitating Discourse" section of the lesson plan template that the participants were required to use for their projects. Ryan asked the instructor what he should do for that section. The instructor explained that "the discourse is going to be whenever they're working together in groups and talking to one another" and that facilitating discourse should be somewhere in the lesson. A PST asked the instructor during a whole-class discussion how they as teachers are to discern if a student understands a concept as they are working on a task. The instructor suggested that they get students to talk about what they are doing as they are doing the task since a teacher gains much information that way as she is walking around listening to students.

Explanation came up less often during class. The instructor mentioned, though, that the type of explanations they require is a sociomathematical norm, and the class discussed their thoughts on Rasmussen, Yackel, and King (2003). This chapter is focused on the type of mathematical explanations a teacher can require during whole-class discussions and proposed that an acceptable mathematical explanation should be based on meaning, not procedures. Stein et al. (2009) mentioned that the fact that a task includes an explanation does not guarantee that it will have a high level of cognitive demand, as an explanation question on a task could ask students to simply describe the process they used or actually ask them to explain their thinking.

Carly. Carly seldom referenced the term communication in the interviews. She probably viewed communication and discussion as synonymous, with explanation as a separate, but related characteristic. While describing open-ended tasks, she explained that an open-ended task promotes communication because it has different solution methods and answers, which gives students a chance to discuss the mathematics and present their ideas. Also, in the last interview, she stated that communication and discussion are related to explanation and justification. Communication is important to Carly, and she showed this by relaying the importance of both discussion and explanation.

When Carly did attend to discussion as a characteristic of tasks she usually expressed that students' discussions should include comparing their ideas and solution methods with each other. For example, in the third interview, she described how she hoped that students would find the sum of the interior angles of different quadrilaterals and then compare what they had done with each other. Discussion was important to Carly since she stated that she wanted it to be part of her daily teaching. When asked what a typical day in her classroom would look like, Carly explained that she would first have students do the task individually, then have them discuss the task with each other and decide which methods worked best. Carly only explicitly included discussion in her two projects, but that is consistent with her statement that discussion will be part of her daily teaching since both of these are full lesson plans. In these lesson plans, she described how the students are to work in groups and discuss the task. In Project #1, she even cited a "Standard for Mathematical Practice" (Construct viable arguments and critique the reasoning of others) from the *Common Core* (NGACBP & CCSSO, 2010). She added the following commentary:

> As students make conjectures about these relationships, they will likely have some disagreements about which aspects are related and how to use this information. This will require students to come up with mathematical arguments to support their conjectures. I expect this will happen both during group work and class presentations.

When Carly attended to explanation, she usually referred to having students explain their thinking, but did include an explanation of a student's process for solving a task on two occasions. So, her main view of explanation was that explanations are descriptions of student's thinking, but she does not exclude describing the process for solving a task from her view of explanation. One instance where this view was made evident was as she sorted the tasks in the task sort into three piles according to the level of explanation required. The first pile explicitly asked for an explanation and used the term explanation. It included tasks that may ask the students to "explain [their] reasoning," but the task may also simply ask the students "to explain the process [they] used." Also, in all of the questions that required an explanation in her written task assignments, she was not looking for students to list the steps they took to complete a mathematical procedure; rather, she usually posed how and why questions. The tasks that Carly identified as requiring explanations in the interviews usually used the word explain or justify but did not necessarily have to, and half of her written assignments actually used the term explain in the explanation question. She did, however, recognize that a task can require an explanation without using the term. For example, she identified the question, "What does the square root of a number mean? How is it related to the square?" on Project #1 as a question that requires explanation and described how it gave the students "a place to explain what the definition of what we've learned that day is."

Carly occasionally used the term justification. She used the terms explanation and justification together multiple times, without distinguishing between them, and other times used the words interchangeably. She could have believed they were synonymous or that a justification is a type of an explanation, where the student is explaining his reasoning for why he thought about the task a certain way or why his method makes sense. When asked how explanation and justification show up in Project #1, she described how there is a question that asks students to explain and that, in their groups, students should give the answer they got and the reason why they thought about it that way. Carly may have been differentiating here, and describing justification as a description of their reasoning for how they got their solution.

That a task required explanation was very important to Carly. She argued its importance several times through the interviews. She even included explanation as a characteristic that is very important (though not necessary) for a worthwhile task to have and chose explanation when asked to conduct a task sort according to an important characteristic. The importance she placed upon explanation is consistent with the fact that she attended to explanation in all of the interviews and that she included some sort of question that required a written explanation in each of her written assignments, though her two projects included the explanation question within the homework assignment or assessment question.

That explanation is important to Carly is consistent with her beliefs that explanations are valuable to help students learn from the task and are valuable to provide the teacher evidence of the student understanding. Regarding students, Carly argued (both in the interviews and in the cover page to the First Pen Pal Task) that explanations are valuable for students because it helps students to solidify their thinking and because some students are not going to learn the concept until they have to explain it. She explained that there will be "different kind of learners in your class," that some of those learners "aren't really going to learn it until they've had the chance to explain it," but that explaining benefits all students since "if you can explain it and back up what you're saying, then you know what you're doing." On the cover page to the student on the First Pen Pal Task, she and Jamie explained their rationale for wanting the student to explain their reasoning, which explained how it would benefit the student. This rationale included,

> Being thorough in your explanations will help you to practice with the concepts you have been learning in your class and will hopefully give you a better understanding of the mathematics involved. The more completely you explain how you think about this mathematics concept, the more you and I can learn from this task.

Carly also clearly felt that a purpose of explanations is to show the teacher what the student understands. Carly spoke about how well the teacher could determine a student's level of understanding from a task several times throughout the interviews. For example, while she was sorting the tasks in the task sort according to the characteristic of explanation, she mentioned that the theme for the pile of tasks that did not require any explanation would be that if "the teacher wanted to look at it and see how much you understand, they would have no idea." She even spoke about students' actual responses to explanation questions and what she thought the student's level of understanding was in Project #1 and the First Pen Pal Task. For example, within the First Pen Pal Task, it asks the student to decide whether six different situations exhibit a linear or exponential relationship, explain why it is that relationship, and how they know it. Carly wanted to see if the student would "pick up on things that actually matter." While analyzing her student's responses, she stated several times that the student did not explain why the situation was linear or exponential. She explained that the student knows why each question is linear or exponential (probably because he did identify each situation correctly), but he is not telling how he knows. So, she concluded that he probably did not know how to explain it, did not know what she was looking for in an explanation, and may not be used to "providing a why and so maybe he doesn't know what the important parts of a why are." She did state that she had not completed the task before assigning it and so had not written out the explanations she was looking for, which likely made it more difficult for her to assess the student's understanding. So, she assumed a level of understanding that was not apparent to her in the student's explanations, but she was attending to the student's level of understanding while looking at his responses. Furthermore, Carly indicated that it is more likely that a student understands an idea if he can use his own language in an explanation instead of the using the language in the task.

Jamie. Jamie did not use the term communication in the interviews, but she did use it in her analysis of how she adapted one of the tasks in the Adapted Tasks Assignment. In the analysis, she expressed the importance of communication in general when she stated,

> One of the skills that we as teachers of mathematics are striving to leave our students with is one of communicating their ideas effectively, which will help them with all collaborative and problem solving situations they will find themselves in.

Jamie likely felt that communication is important since she sees teaching communication skills as a goal of teachers.

Jamie did not attend to discussion very much, but she did express that discussions between students should include sharing and critiquing ideas. She did not explicitly state how important discussion was, but she probably thought it carried at least some importance since she stated that students do benefit from discussing their reasoning with others. She also included in her Assessment Plan that students will work on their tasks in partnerships or small groups every day. Jamie only attended to discussion in the second and last interviews. This is not surprising since part of these two interviews were about each of her projects, and both of these projects are full lesson plans, which included group work for at least part of the tasks. During the group work in these lessons, she wanted students to critique each other's ideas, and she even cited a

"Standard for Mathematical Practice" (Construct viable arguments and critique the reasoning of others) from the *Common Core* (NGACBP & CCSSO, 2010) in both of these lesson plans. To illustrate, Jamie discussed what she wanted to happen in the groups while the students worked on Project #1 several times throughout the interview. She hoped the students would discuss the changes they noticed in the number of papers for each rip they had, discuss the patterns they see, try to convince each other of the general equation, and challenge each other's ideas.

Jamie's main view of explanation was that students must explain their reasoning and show their understanding in an explanation, but she did occasionally reference students explaining the process they used to solve a task in an explanation. She explicitly stated that written explanations help teachers to get at a student's understanding when she described how she and Carly "had to make sure that we had them writing out how they thought about it" on the First Pen Pal Task, so that "we could get at their understanding because we couldn't watch them do it." One instance where this view was shown was as Jamie completed the task sort. As she sorted, she attended to if the students were asked to explain and how well students' understanding would be shown by the task. She divided the tasks into three piles and stated that the first one was where the students were explicitly asked to explain, the second one was where completing the task showed the students' understanding even though the task did not call for an explanation, and the third pile did not ask for an explanation and did not show a student's understanding. The first pile that explicitly asked for an explanation asked for students to explain their reasoning or for students to explain the process they used. Even though the second pile did not ask for an explanation, she felt that students show their understanding by doing the task. For example, for one of the tasks, Jamie thought that using the base-ten blocks to compare the two numbers is "basically their explanation for why one is less than the other," "so, it doesn't say to explain, but just in doing the task, using the blocks, they end up showing their reasoning."

Jamie's view that explanation questions should show a student understanding is also supported by the fact that as she analyzed her student's responses to the First Pen Pal Task, she actually was evaluating the understanding of the student who completed the task. For example, after Jamie saw that the student incorrectly labeled number five as linear, she tried to decode the student's understanding from her explanation of "the output is proportional to the input of the overall process. The number of rabbits in a generation is always twice as many as in the generation before it." Jamie stated,

> So, the student's looking at maybe I have two rabbits, and I'm always going to get out three. Looking at that relationship versus focusing on the question we were asking. Like, the fifth generation, how many rabbits total you're going to have. But I can see maybe where she would get a linear relationship from that.

The fact that Jamie actually did attend to the student's understanding without being asked to adds evidence that she believed that explanation show what a student understands.

Jamie usually wrote explanation questions that required students to explain their reasoning with a how or why question, which is consistent with her main view of explanation. For example, Jamie focused on having the students explain why a situation showed a certain relationship in the Pen Pal Tasks, and a homework question for Project #2 asked, "If I scale a figure with a scale factor of x (where x is a rational number) how does the area change? Explain why you know this." She only wrote one question that asked students to explain the process; this happened in her Adapted Task K, where she changed "explain and justify your method as clearly and completely as possible" to "explain the process you took to get the answers and justify why they are correct." (Jamie sometimes used the term justification, but she did not clearly differentiate it from explanation. However, it seems that justification is an explanation where students explain their reasoning, as illustrated in Adapted Task K.) Also, when Jamie wrote her own explanation questions, they did not necessarily have to use the term explain; she used the term explain in four of the six assignments. So, she recognized that explanation questions do not have to explicitly ask for an explanation.

For Jamie, requiring explanations is an important characteristic of tasks and probably much more important to her than discussion. In the last interview, Jamie was conflicted about whether or not a worthwhile task must require an explanation, but she definitely preferred tasks that did require one. She first stated a worthwhile task must require an explanation, but then decided that she would try to design tasks with explanation and would choose a task with explanation over a similar one without it, but she would not avoid using a task just because it did not require an explanation. The fact that Jamie clearly believed that explanations provide teachers evidence of whether students understand the concepts in the task supports Jamie's statements that explanation is an important characteristic. Even though she felt that explanation is important, she also mentioned that including an explanation might not fit the teacher's goal if she, for example, wanted the students to practice a skill. She also felt that a teacher could verbally ask for explanations instead of writing it directly into the task. Additionally, Jamie attended to explanation in all of the interviews and included an explanation question in all of her written assignments, though four of the tasks had an explanation question within the task, while the two projects had an explanation question after the task. The fact that she attended to explanation so much and that she included it in every written task assignment is consistent with the importance she expressed that she placed on this characteristic.

Ryan. In the one instance where Ryan used the term communication, it showed that he thought of communication as verbal and involving sharing thinking. During the discussion in the last interview about how several characteristics were related, he stated that explanation and communication were related and that encouraging communication among the teacher and students is good since it forces the students to articulate their thinking out loud.

Ryan did not often attend to discussion as a characteristic of tasks. When he did, however, his focus was on students discussing solution strategies or ideas for solving a problem. Ryan did not explicitly state how important he believed discussion to be. However, when asked what a typical lesson would look like in his classroom, he included how he would want students to work on the task individually or in groups, depending on the task, followed by a class discussion. Since he included that he would want to have student-to-student discussion often, he likely felt that it was important. Also, he adapted Task J to include having students come up with their own methods and discuss them and included discussion in both of his projects, which is consistent with his statement that he would want to have discussion in most of his lessons. To illustrate how he used discussion, in both of the project lesson plans, the lesson switches back and forth between group and class discussion. The students work on a question that is posed to them, then the class discusses it. For example, in the first project, the students work in groups to figure out how to show if there is a relationship between shoe size and height and then get back in groups again to analyze the class data that has been collected. Ryan also attended to the student with a social disability in both of his projects; for example, in the second project, he wrote that the student should be paired with a student willing to help someone with a disability. Ryan actually spoke about the whole-class discussions that can come from the task more often than student-to-student discussion, and he suggested that the characteristics of multiple solution methods and openness are likely to promote later class discussion.

For Ryan, having students explain their thinking probably meant or at least included showing their work or writing down in words the steps they used to solve a task. Half of his explanation questions in his written assignments actually used the term explain, so he realized that a question can require an explanation without using the term. To illustrate his view of explanation, when asked which characteristics he wanted his First Pen Pal Task to have, he included that he wanted students to explain their thought processes, but then described this characteristic as having students showing their work. He stated, “I wanted them to explicitly like have their thinking written out on the paper, so that I can make sure to see their thought processes.” Even though he referred to the students’ thinking, he went on to describe how students do not like to show their work and that he liked to do the mathematics in his head in his early grades. Then he stated, “When you get into like these upper-level math classes and things like that, really, your thought processes written out is really important. So, I needed to put, show your work.” Since he did not distinguish between explaining thinking and showing work, he may have felt that they are synonymous or, in the very least, that showing work is a way to explain thinking. The explanation question in the First Pen Pal Task asked students to “explain how your process will work if the genie gives you a different number of beans to start with.” He wanted to see if students would be able to rewrite the equation from a previous question with a different number of beans. Ryan’s focus here clearly was on if the student could describe how to carry out a procedure again. He seemed satisfied with the student’s response, which was “It will work because you always need one-third, in the exponent is a number of box you’re on, therefore, letting you plug in any number.” He stated,

> So, they’re just saying, like, you always need one-third, so you’re always dividing by three and then you can just plug in any number of beans what number you start out with and then you just divide by three to the box number.

Even though Ryan typically described explanation as explaining the process one used to solve a problem, he was capable of writing questions that asked for more than a process-oriented explanation. For example, as he worked with the group to design a task that required student to justify why a quadrilateral has an interior angle sum of 360 degrees, he posed questions that did not ask for students to explain a process, such as “What is the angle sum of the interior angles of any quadrilateral and justify how you know that?” and “Is there a way you can prove that all quadrilaterals have the same angle sum?” (He only used the term justification a few times in the third interview, but justifying seemed to mean explaining the reason why a solution method works.) In fact, four of the six written assignments contained questions that asked students to explain their reasoning. For example, he asked students to explain their reasoning on three questions which asked students to identify the type of function in the Second Pen Pal Task, and he asked “Are there any problems that might arise with this type of representation? Could you solve all order 2 polynomial equations with this area based method?” in Project #2.

Ryan explicitly stated that he felt that explanation is important. He stated, “I think [explanation is] important in any task, any assignment, any assessment. I will expect my students to actually explain their thought processes.” He also stated in the group interview that he felt that he understands concepts better when he has to justify them. He did not attend to explanation as a characteristic until the fourth interview and only briefly attended to it in the last interview. So, it seems that explanation is not something Ryan focused on as he spoke about his tasks, but he did have at least one explanation question within all of his written assignments, even though Project #1 had the explanation question in the homework instead of within the task. Since Ryan did not attend to explanation as a task characteristic very often, but stated that explanation is important to have in every task, he may have believed that explanation was more of a way of effectively using tasks than a characteristic of the tasks themselves.

Conclusion. All of the participants felt that communication as a characteristic is good to have in a task, but it was not the most important characteristic to them. The participants usually referred to communication by using the terms discussion and explanation. All the participants expressed that discussions should include students sharing their solution methods and ideas with each other, but discussion was not a main focus for any of the participants. They all did, however, include group work in their projects. They all suggested that discussion is important; both Carly and Ryan explained that they wanted discussion to be part of their daily teaching, and Jamie stated that students do benefit from discussing their reasoning with someone else. It was rare, however, for them to attend to how to promote productive mathematical discussion while discussing how they designed their tasks.

All of the participants attended to explanation more than discussion, and Carly and Jamie attended to explanation much more than Ryan in the interviews. They all stated that requiring explanations is important, but Jamie almost felt that explanation was a necessary characteristic in a worthwhile task. Carly specifically addressed that explanation was valuable for the student to help him solidify his understanding. They all included at least one explanation question in all of their written assignments, and they all recognized that a task can require explanation without using the term explain. All of the participants stated that explanations should consist of students' thinking, but Jamie seemed to be the most convinced of this since she focused on how students must explain their reasoning and show their understanding in an explanation. All of the participants spoke of explanations as a description of a process that the students used to arrive at a solution at least once, so they probably did not exclude it from being an acceptable explanation.

For Ryan, though, it is likely that having students explain their thinking meant (or at least included) having students describe a process or show their work. Even though the participants occasionally used explanation questions that only required students to list the steps of a procedure in their written assignments, they were more likely to include a question that they hoped would elicit student thinking. Ryan asked twice and Jamie asked once in the explanation question for students to write the process they used to solve the problem, but the participants normally used how and why questions that required students to explain their thinking.

Engagement

Teachers should consider how tasks engage their students. NCTM (1991) argued that worthwhile tasks should be based on "knowledge of students' understandings, interests, and experiences;… display sensitivity to, and draw on, students' diverse background experiences and dispositions;" and "promote the development of all students' dispositions to do mathematics" (p. 25). Teachers should aim to make their tasks interesting to their students. One way to create interest is to set tasks in familiar contexts, but "not always, however, should concern for 'interest' limit the teacher to tasks that relate to the familiar everyday worlds of students; theoretical or fanciful tasks that challenge students intellectually are also interesting" (p. 27). Engagement as a task characteristic includes whether students are *able* to engage with a task as well as whether students *want* to engage with a task.[4]

Engagement was not a characteristic that was explicitly taught in the Task Design course, and the participants were not taught how to make a task engaging. Although the class did discuss related ideas occasionally and the texts mentioned some of these ideas, that students were

[4] There are a number of factors, besides their mathematical background knowledge, that can impact whether or not students are able to engage with a task. Accessibility, or whether student have the mathematical knowledge to engage with the task, is the first requirement of an appropriate task (Leatham, et al., 2013) and has been addressed in the Appropriateness Section. The "able to engage" part of the Engagement Section focuses on other factors that can influence if a student is able to engage with a task.

able to engage with a task was not a focus. The instructor did not use the term "learning styles," but she stated on one occasion that "those who are visual thinkers often really respond" to using algebra tiles. She mentioned how tasks can be modified for special populations several times, such as ESL or learning disabled students. For example, our table of participants considered removing the pictures from Task P, and the instructor stated that that would be better for most students, but that the learning disabled students would need the pictures.

That students *want* to engage with the tasks was also not a focus of the course. The entire class did discuss the importance of helping students to feel successful and of creating an atmosphere of trust, so that students feel comfortable to make mistakes. The instructor stated that students like the feeling of power they get when they solve something hard. In the instructor's comments, she communicated that contexts and manipulatives can sometimes help students to understand the mathematics, but also are sometimes added just so the task has the context or manipulatives, which is not useful. While the participants were discussing Task A, the instructor asked if it is helpful to have the context of trains since there is no reason to actually take a perimeter of a train in real life. Engagement came up occasionally in the books, but was not a main idea. Schoen (2003) stated that tasks need to be engaging and problematic, but accessible to the target students. The texts also communicated the idea that interest is good, but not the most important consideration. Copes and Shager (2003) stated that the most important criterion is that the task teaches students the concept you want them to learn, however, so the task should not just be for fun. Kahan and Wyberg (2003) argued that some mathematical ideas can be taught well with real-world tasks, but others would be better taught without these real-world contexts. Stein et al. (2009) warned that real-world contexts, diagrams, and manipulatives are superficial features of a task that may make a low-level task seem high-level. They also stated, however, that one way a high-level task can decline in its cognitive demand is if the students are not interested or motivated.

Carly. For Carly, students are engaged in a task if they are thinking about the task and understanding their solution method as they work on the task. Carly explained that a teacher can use their knowledge of their students' personalities to help them to see the students who are "actively engaged" in working with the task. A teacher may be able to tell that a student who is talking about the task is thinking about and understanding what she is doing, but a more introverted student may need to be questioned further. I will now describe how she viewed whether a student is able to engage with a task, followed by how she viewed whether a student wants to engage with a task.

Carly believed that tasks should be accessible,[5] so that students are able to engage with the task. She had two different definitions of accessibility, but they both are focused on every student being able to engage with the task regardless of their differences. She first stated that a task is accessible if the student is able to find a way to start and do something on the task. She felt that her first project was accessible, for example, because students are asked to make a square using M&Ms, which is something she felt that anyone could do. She hoped that by completing that first step, they would be able to "have the tools in front of them to think about where to go next because they started on it." Carly stated that she had a second definition for accessibility, which basically was that all students should be able to engage with the task at their own level. She stated that a task is accessible if the task allows the students to think about the task the way they want to, so that "it's not set up in a way so that if you don't think about it the

[5] The participants used the term accessibility differently than how it was defined in the Appropriateness Section. In general, accessibility to the participants is that everyone can engage with the task. They had a broader view of the term than that of the literature-derived definition since the literature-derived definition focused on whether students have the mathematical knowledge to engage with the task. When using the term accessibility, the participants focused on factors other than mathematical knowledge that can impact whether students can engage with the task.

way that was intended, you don't get anything out of it." She felt that the task should not begin or stay at a level that is beyond what the students are capable of and that a task should be able to be "entered from multiple ways because you're not going to have every student be at the same level."

According to Carly, "learning styles" and familiar contexts are two ways that can impact whether students are able to engage with a task. I will first describe how Carly attended to learning styles. Carly discussed the idea of different kinds of learners or that a task should allow students to learn in a way that works for them a few times throughout the interviews. It is very important to Carly that a task fit a student's learning style because if a task does not allow students to learn in a way that works for them, they might not be able to learn from it. She stated that some learners will only learn if, for example, they explain it or if they hear it multiple ways. On her Final Exam, she adapted one of the tasks to require students to draw a graph. She stated that allowing students to see the problem in more than one way helps every student, especially those with different learning styles. Both of Carly's projects have evidence that she probably attended to learning styles while designing them since there is a visual or tactile aspect to them, and she even brought up the idea of learning styles while discussing Project #1. She expressed that she wanted Project #1 to have multiple representations since some students may think using formulas and others may prefer thinking visually, and she wants students to be able to learn in a way that works for them.

Carly did not want students to be unable to do a task because of the context it is set in, and she clearly believed that it is necessary for students to understand the context of the task. She did not believe, however, that the context must be a real-world context. She stated that having a familiar context helps students to be able to start thinking about the task because they understand the situation even if the student does not yet understand the mathematics involved. She only occasionally attended to the idea of a familiar context throughout the interviews. For example, she felt that Task K had "a context that students can understand and easily draw out to be able to start writing equations." Both of the Pen Pal Tasks and Project #2 also had contexts that students should be familiar with and understand. The First Pen Pal Task had story problems about familiar contexts, such as selling cocoa at a football game. Carly explained that this is "so that people can relate to them" and "draw on their own experiences … to maybe help them wrap their minds around it."

It was more common for Carly to attend to whether students would want to engage with a task than whether a student is able to engage with a task. Carly attended to a student wanting to engage with a task by describing the task as fun, interesting, motivating, not overwhelming, or that it makes the students feel comfortable. She specifically addressed how to tell if a task will be interesting to students. She explained that an interesting task is based on experiences and things students are familiar with, such as setting a task in the context of the school cafeteria.

Carly felt that it is fairly important that students want to engage in a task since completing tasks takes more effort for students than does being given the information, and teaching using tasks is "the most effective and beneficial way to teach mathematics." She even addressed how she may be able to motivate students to put in the effort to do tasks using a task by giving students a "smaller scale problem" and then showing them how much they understand after doing it. She also, however, believed that a student wanting to engage with a task is not the most important characteristic of tasks; tasks should not just be for fun. She explained that an important way that the Task Design class influenced Project #1 was that she should not think about "Oh, this is a fun problem that I want students to do; what can they possibly learn from it?," but rather "These are the things that I want to teach them. How can I create a way that will connect these ideas to that stuff?" She continued,

> So, I think that was a really big deal because there are a lot of things that are fun to do in math and fun connections you can make, but if you're going at it from that approach, then it's not going to be very effective.

Carly did not attend to students wanting to engage with tasks very much, which is consistent with her belief that engagement is good, but not vital to a task. She brought it up while discussing Task J, both projects, and the First Pen Pal Task. These ideas came up the most with Project #1, where she explained that she wanted the task to be fun, wanted it to be based on prerequisite knowledge so that students are not overwhelmed, and felt that students would feel comfortable playing with the candy. One could argue that all of the tasks that Carly designed could be interesting to students in some way. For example, both the First Pen Pal Task and Project #2 have real-world contexts that could be interesting to the students, and both projects require students to work in groups.

Jamie. For Jamie, students are engaged in a task if they "participate with their full attention," are diligently trying to solve it, and "are actively talking about it" if they are in groups. She defined an engaging task as a task that can "catch all the students' attention, has something for them to do, and [is] interesting to them." I will now describe how she viewed whether a student is able to engage with a task, followed by how she viewed whether a student wants to engage with a task.

Jamie believed that tasks should be accessible to all students, so that they are able to engage with the task. Jamie defined an accessible task as one where every student can start the task and do something on it. She also occasionally used the term approachability, which she used as a synonym for accessibility. In the last interview, she explained that accessibility meant that "every student can, like, start the task, or like, can work on the task because it's really daunting to be given something if you have no idea how to even start or what to really do." She continued, "So, if it's more accessible, then it just allows the students to succeed or learn something from the task." When asked what she would want students to be able to start or do on the task, she explained that it depended on the task. However, she gave the example that "every student can rip a piece of paper in half and count how many pieces it is" on Project #1, but if a student had a better mathematical understanding, they would "get something else from the task as well and be able to approach it a different way." When asked how to make sure a task is accessible, she stated that "you have to look at what your students can do, I guess, and try to build some of that into the task." She also stated that the characteristic of "students being comfortable with the task" is related to its accessibility.

Two ways that Jamie mentioned that can allow a student to engage with a task are based on learning styles and on the context of the task. Regarding learning styles, Jamie stated that different students learn better in different ways, but she did not state that students would not be able to learn if it did not fit with their learning style. She explained that some students learn better by doing, some learn better by reading, some learn better by hearing, and some learn better by writing. She only brought up this idea while discussing Project #1, where she explained that she incorporated hands-on aspects to the task to help students who learn better by doing. She felt that a task she adapted on her Midterm Exam was accessible to all students since they would be able to use manipulatives to measure to find several examples, but they would still be required to think abstractly to find all of the possibilities.

It was vital to Jamie that students understand the context of a task. She stated that students should not be given tasks with situations that are difficult for them to understand, so that all students have access to the experience. Even though she did not bring up this idea very much, none of Jamie's tasks were set in a context that students would be unable to understand. Jamie's Pen Pal Tasks and Project #2 had real-life contexts that would be familiar to students. For example, Project #2 required students to make a floor plan of their classroom; the student's classroom would clearly be a familiar, understandable context. Even though students should understand the context, Jamie discussed in the fourth interview that not all tasks need to involve a real-world situation. She argued, "Sometimes [a real-world context] can be beneficial, but a lot of times as well I feel like [a real-world context] can get in the way or distract from what you're really trying to teach."

Jamie attended to the idea that a student want to engage with a task a great deal throughout the interviews. She described engaging tasks as being fun, interesting, hands-on, or motivating and felt that group work helped a task to be engaging. She expounded upon student interest by saying that an interesting task is "something that [the students] care about or, like, desire really to figure out." One way she mentioned that tasks can be engaging is by having real-world contexts. She stated that a task she adapted on her Midterm Exam would be more engaging since the problem has a situation they can relate to, so the students will pay more attention to it.

It was very important to Jamie that a task make students want to engage with it because she believed that tasks need to be engaging, so that students pay attention and are motivated to learn what you want them to. She felt that in order for students to feel motivated to put in the effort, what they are learning must be fun and they need to feel successful with what they are trying to learn. Jamie tried to help students to be motivated to do tasks, but also believed that tasks themselves could be motivating. When asked what she would do if a student said they could not do math, she stated that you could give them a challenging task that the student would be able to complete to show them that they are good at math.

Jamie expressed the importance of students wanting to engage multiple times throughout the interviews. For example, when asked to list which characteristics she thought were important in the first interview, she included that tasks “need to be engaging for students or else they’re not going to want to learn anything from it.” Jamie spoke about engagement the most while discussing her projects. For example, she described Project #1 as fun and motivating, and she included group work and the hands-on aspect of ripping paper in order to help the students be engaged. She even stated within the lesson plan of Project #1 that “because the task is designed to be engaging, the students’ attention can be maintained even when the task gets difficult.” There is, however, evidence that she tried to make it so students would want to engage with all of the tasks she designed, which is consistent with the importance she placed upon this characteristic. For example, Jamie added group work or a class activity to two of her adapted tasks, both of her projects were hands-on, and the First Pen Pal Task and Project #2 were set in real-world contexts. Jamie clearly believed that students’ feelings are an important consideration in teaching mathematics. She stated that the best way to teach math was to make sure the teacher showed the students that she likes mathematics and for the teacher to connect personally with the students and care about them in order to help students to like mathematics. She even expressed that she wants to have personal interviews with students in order to let them know that she cares about them and believes in them, which will result in students putting more effort into the class and progressing more.

Jamie believed that engagement is very important and that a task should be engaging if possible since it helps students to learn if they enjoy the tasks. However, she also felt that it would probably be impossible for every student to be interested in every task, and she asserted several times that engagement is not the most important characteristic of tasks. For example, she stated that she wanted Project #1 to be engaging, but also felt that it should not just be for fun since it had to have the purpose of teaching students a mathematical concept. In the fourth interview, when asked if she felt that she had improved in her task design skills, she acknowledged that she used to try to focus more on making the task fun, but now tries to focus more on the mathematical concepts.

Ryan. Ryan did not specifically address what it meant for a student to be engaged in a task, but he did attend to ways that student would be able or want to attend to a task. Regarding whether students are able to engage with a task, Ryan felt that tasks should be accessible, meaning every student is able to engage with them. He stated that an accessible task is a task where "all students are going to be able to get the knowledge from it, though you may need to do different things with different students." His focus with accessibility was that a teacher must consider students with disabilities and must adapt the task for them, and he argued that the teacher needs to think of as many ways as possible that a student may think about the task and think about how to help students who may struggle. For example, he stated that he thought about accessibility while designing the Quadrilateral Task since he thought about students with learning disabilities, which is why he brought up ways to help struggling students. In fact, Ryan added accommodations in both his projects for students with social or learning disabilities.

Two ways Ryan occasionally attended to students being able to engage with the task are learning styles and whether the task is non-exclusive. He stated several times that students learn in different ways; he explained that some students learn more visually, some learn more if they hear the teacher explaining or writing on the board, and some learn more when they are doing a task or touching something. He felt that it is good to have a lesson hit on these different ways of learning in order to reach more students, but he did not say that students would not learn if it did not fit their learning style. Since learning styles did not seem to be very important to him, it likely was not a large consideration as he designed tasks. Both of Ryan's projects do, though, have either tactile or visual aspects to them. For example, in Project #1, students had to measure their height and write their shoe size and height on the board.

Ryan described a non-exclusive task as a task the does not discriminate, does not go against any students' beliefs or values or who they are as a person, and does not make any student feel uncomfortable. He felt very strongly that a task should not be exclusive. When asked if non-exclusivity, a characteristic he had previously mentioned, showed up in Project #1, he stated that he did not think that the task would be discriminatory unless there were students who did not wear shoes. Ryan did not bring up the idea of students understanding the context of a task, but that could logically fit within the idea that a task should not discriminate since if a student did not understand the context of a task, that task could be seen as excluding that student. None of Ryan's tasks are discriminatory, and they have understandable contexts. For example, the First Pen Pal Task has an understandable, but fanciful, context involving a genie and magic beans.

Ryan attended to students wanting to engage with a task multiple times throughout the interviews. He only used the term "engaging" occasionally as it is being used in this section since he stated that he viewed engaging and intellectually engaging as synonymous, but he did describe Task P, the projects, and the Pen Pal Tasks as fun, exciting, enjoyable, interesting, or not overwhelming. For example, he designed Project #1 with the hope that the task would get students excited to explore other relationships in their lives, and he explained that he did not want to make the First Pen Pal Task too long because he did not want the student to get bored and not give the amount of feedback he wanted. Some ways he mentioned that would make tasks fun or exciting were that it contained math history or was based on a real-world context.

For Ryan, it was fairly important that students want to engage with a task. When directly asked how important it was that students enjoy a task, he stated that on a scale from zero to ten, he would put it at a 6. He continued,

> You should make every effort to make [a task] enjoyable in some form to get students engaged… to bring them in so that they are interested in it, but at the same time, I don't think you can make everything that you do in math exciting.

While discussing the First Pen Pal Task, he expressed that a task should not just be for fun and that it must be based on FMCs.

Even though engagement was not vital to Ryan, all of the tasks he designed have aspects that he may feel were engaging. For example, instead of teaching completing the square using algebra only, Project #2 included the Babylonian area method for completing the square, which could be interesting for students. Ryan did show concern for students' feelings about mathematics even though engagement was not a very important characteristic to him. For example, when asked what his most important goal was as a teacher, he stated that "the pivotal goal that I have I guess would be to do my best to help students see that they can do math."

Conclusion. Carly and Jamie expressed that students are engaged in a task if they are diligently thinking about and trying to solve it, but Ryan did not explicitly define what it means for students to be engaged in a task. All of the participants argued that tasks should be accessible so that all students are able to engage with them. All three of the participants brought up the idea of learning styles in the interviews, but it was not a major focus for any of them. Carly was the most extreme in saying that if a task does not fit the way a student learns, the student may not be able to learn from the task, while Jamie and Ryan both suggested that different students learn better in different ways. Both Carly and Jamie specifically argued that students must be able to understand the context of a task, and it is likely that Ryan felt the same. Ryan was the only participant who expressed that a task should be non-exclusive.

The participants used similar terminology to describe students wanting to engage with a task—words such as fun, interesting, and motivating. Jamie also felt that hands-on tasks were engaging, so it is not surprising that both of Jamie's projects were hands-on, while just the first project was hands-on for Carly and Ryan. Jamie clearly attended to students wanting to engage with tasks the most and often showed her concern for students' emotions in other ways. Carly and Ryan both likely felt that students wanting to engage in tasks is fairly important while Jamie felt that it was more important since she believed that tasks must be engaging so that students are motivated to learn what you want them to. All of the participants expressed that tasks should not just be for fun and should have the purpose of teaching a mathematical concept, but both Jamie and Ryan added that the task should be engaging if possible. They also felt that tasks do not need to be based on a real-world context, even though a real-world context can be beneficial. Real world contexts, along with manipulatives, were common ways for them to make their tasks engaging.

Openness

Open-ended tasks are tasks that can be solved more than one way and may have more than one correct answer. A good open-ended task, however, asks a question where the solution strategy is not immediately obvious to the student, so the student must figure out a solution strategy for himself. According to NCTM (1991), many good tasks can be solved more than one way and some have more than one reasonable answer. They added that "these tasks, consequently, facilitate significant classroom discourse, for they require that students reason about different strategies and outcomes, weigh the pros and cons of alternatives, and pursue particular paths" (NCTM, 1991, p. 25). Cooney et al. (2001) pointed out that when a task has only one answer, students are likely to believe that there was only one method that could be used to solve the task. They added that tasks that require students to explain their thinking are more likely to elicit a range of responses since students think in many different ways.

Openness was an important characteristic of tasks to the instructor, and that was shown in her class. On one occasion, the instructor lectured about open versus closed tasks. She stated that she believed that people in general learn by making choices and that the more open a task is, the more choices the student is able to make. She explained that an open task could be open in method, meaning the student can choose the method, open-response, meaning there is more than one acceptable answer, or both. She also expressed that an open-method and open-answer task without any graphic organizer to help the student will take longer for the students to complete than a closed task with a graphic organizer to help them. She told the PSTs that they as teachers will need to find the balance themselves of how open each task will be and how much structure they will have in each task since open tasks with less structure take more time. When the class was talking about the Midterm Exam, the instructor communicated that there are multiple ways to make a task better, but that it should not be procedural, and that they should make it "more open and more rich." She also gave the PSTs several tasks about periodic motion that she described to have varying degrees of openness and structure. Openness was not a focus of the texts, but Ziebarth (2003) discussed assessment in a problem solving class in general and stated that open-ended assessment items are most often required to elicit sufficient information about student understanding.

Carly. Carly defined an open-ended task as a task that can be solved different ways and might even have different answers. She rarely used the term open-ended in the context of tasks, but, as she was describing how several different characteristics were related, she stated, “If [a task] is open-ended, then you can have different solution methods and possibly different answers.” Carly stated that openness was related to both communication and accessibility, explaining that “the open-endedness of [a task]” helps to promote communication because students are able to discuss the different methods and answers; if a task is straightforward, then students can only check with each other if they got the right answer. She also stated that in order to find out if a task was accessible, she would try to find if there was more than one way to do it.

Carly attended to the first part of her definition of openness, that a task can be solved multiple ways, much more than the second part of her definition of openness, that a task could have multiple correct answers. The place she attended to multiple approaches as a characteristic the most was during the task sort. She mostly sorted the tasks based on the class discussion that was held in the Task Design class about each task and if the PSTs completed the task multiple ways or if they mostly did the task the same way. She did make some interesting decisions while she sorted, however, which shed light on two beliefs she had about multiple approaches. First, tasks with unclear instructions, which lead students to do them more than one way, or where students do not follow the instructions, which lead students to do the task a different way than expected, may or may not count as having multiple approaches depending on the resulting methods. For example, on Task P, some of the PSTs did not follow the instructions of the task; they were “given pattern blocks and were told exactly how to do it, but not everyone used the blocks, or they would get mixed up and accidentally do it a different way.” She put this task in the pile with multiple solution methods probably because she felt that the multiple methods led to “more understanding.” Second, even though some tasks involving computation could be done in slightly different ways, the methods are not different enough to classify them as having multiple approaches. For example, she described that there were “slightly different approaches” on one of the tasks, where students are required to find the sale price of a sweater, “like whether you looked at it as 70% or you found the 30% and subtracted it and then there were subtle differences that you could do within that, like, finding 10% and multiplying it by three or seven times.” Carly would put a task like this in the stack with tasks that could be solved one way because the approaches were too similar. Carly brought up that task had more than one correct answer a few times, but she seemed to usually be referring to the number of solution methods. Carly did not actually attend to multiple correct answers in the interviews.

Openness was important to Carly. When asked to sort the tasks in the task sort by an important characteristic, the number of approaches a task had was the first characteristic she used. She stated that she saw less value in the tasks with fewer methods compared to the tasks that elicited much discussion and many methods. When asked after the task sort why she thought it was important for a task to have many approaches, she gave two reasons. First, she explained that it allows students to learn more from each other. She gave an example of students being able to learn that using proportions and multiplying decimals could be two different ways to solve the same problem. Second, it shows students that problems can have more than one correct answer. She explained that we are “kind of hoping to establish classrooms where you listen to your peer because they have a mathematically sound argument” and so if the teacher uses tasks with more than one right answer, it gives the students “validation that their solution can also be right even though it’s different from mine; there isn’t just one right answer.”

Carly attended to openness in every interview, which is consistent with the importance she stated that she placed upon it. Her written tasks, however, had varying levels of openness, so openness was not the most important characteristic for her. Project #1 was not very open, Adapted Task A and the First Pen Pal Task had some aspects that were open, and the other adapted tasks, the Second Pen Pal Task, the Quadrilateral Task, and Project #2 could be solved multiple ways. To illustrate, for the Quadrilateral Task, the students could prove that the sum of the interior angles of a quadrilateral is 360 degrees at least two ways. Carly stated that the final justification would probably be based on the idea of dividing a quadrilateral into two triangles, but that their approaches could be different for how they figured out their final justification. She thought that students may try to break up a quadrilateral into a square and two triangles or immediately try to break it up into two triangles. Project #1 definitely was the least open of Carly's tasks because it only had one expected way that students would do the main activity; the teacher told the students what to do at each step, such as giving students several side lengths, asking them to make a square with those side lengths using M&Ms, and asking students to find the area of each and record the information. The most open aspect of this task was when Carly asked the students to find the side-length of a square with an area of two, where students could feasibly try different methods to approximate the answer, such as drawing a square with non-integer side lengths on the graph paper in order to approximate an area of two or using a calculator to find which number multiplied by itself would give about two. Carly recognized that the task could not be solved multiple ways. She stated several times that she wished it did have multiple solution strategies, but included the justification that since the task introduces a definition, it would be difficult for her to make it so it could be solved multiple ways. For example, when asked how being able to be solved in many ways showed up, she said,

> Not as much, actually. That's one thing I wish was different about this, but as an introduction, I didn't know how to make it go a lot of places because it's kind of learning a definition. A definition is a lot less varying, so I don't think there are a lot of ways that there can be different methods.

Jamie. Jamie defined an open-ended task as a task that can be solved using different approaches or can have different correct answers. She stated that an open task could be "either open in the way you approach it or open in solutions" and that "open in solutions" meant that "you could have a task that has different answers to it, I guess, that are both correct." Jamie used the term open several times, but usually just attended to the idea of multiple solution strategies or multiple correct answers.

To Jamie, an open-ended task is one that does not lead students to a particular solution strategy. For example, during the group interview and after the participants decided on the characteristic they would have the students justify, Carly had the idea of asking students to find the sum of the interior angles of several specific quadrilaterals before asking what they thought it would be for any quadrilateral. Jamie, however, suggested that they "could even leave it even more open than that" and not direct the students to specific quadrilaterals. When Carly brought up the idea again later in the interview, Jamie again stated that she felt like listing the quadrilaterals would say to the students, "Oh, look at these first and that will help you."

Openness is fairly important to Jamie, although she never explicitly stated how important it was. On her Midterm Exam, however, she stated that having several appropriate methods for solving as well as multiple solutions gives students more agency and forces them to reason, and reasoning was an important characteristic to her. Also, she repeatedly stated that she wanted both the Quadrilateral Task and Project #2 to be able to be solved more than one way. For example, when asked which characteristics she wanted Project #2 to have, she stated that she wanted students to be able to approach the task several ways. She went on to explain that there were two ways that she thought of how students could draw an accurate classroom drawing, and she felt that there probably were more ways that students could do it.

Multiple solution strategies was not a characteristic that Jamie focused on in the interviews, but all of the tasks she designed, but one, could be solved more than one way, so openness was fairly important to her. For example, for Project #1, students could actually use more than one strategy to solve the task and could have more than way to describe the general statement. When she adapted three of the tasks from the task sort, she added an open aspect to two of them (Task K from the Adapted Tasks Assignment originally was open, and she did not change that). For example, Task P requires students to use pattern blocks in order to multiply fractions, and the original task gives students what to use for the whole. Jamie decided not to give the students the whole, which allows students to do the questions more than one way and even added an additional question that asks students to come up with another whole that they could use to find one-third of one-fourth. In her analysis, Jamie explained, “I decided that by taking out the scaffolding of presenting the students with a whole that works for all of the questions helps to give them agency in the task. This question moved toward open-response and open-ended.” On the one task that was not completely open, where students were required to match the names of properties of addition and multiple with appropriate examples, she added an open aspect. She added a class activity where students would take turns writing an expression that could be simplified using the properties on the board, have another student simplify the expression, and have another name the property being used. There are many acceptable expressions that the students could choose to write on the board.

Jamie attended to multiple solution strategies more than multiple final answers, but she did bring up the idea of multiple answers a few times during the interviews. For example, Jamie was interested in finding more than one justification for why the sum of interior angles is the same for all quadrilaterals. The group had thought of one justification based on splitting the quadrilateral into two triangles, but Jamie asked if they were only going to get one solution. Later, while talking about how a student may do a task, Jamie introduced another justification based on her idea that "you can have four triangles and have 4 times 180 and then subtract 360." Also, in the fifth interview, when asked how she generally anticipates student responses, she started by saying, "I think first I'd probably start to think about like the various different correct answers you could have and like reasoning that you could have behind those correct answers."

Ryan. To Ryan, an open-ended task is a task that does not give the solution method to the student. Ryan used several different phrases to get across this basic idea; he described tasks as telling or not telling the students how to do it, that tasks could be solved with a memorized formula or method or not, the task gave the student a process to follow or not, and the students have done or not done a similar task before. Ryan was asked which characteristics he wanted the First Pen Pal Task to have, and he included that he wanted the task to be open. He explained, "I liked the organization, but I also liked that it was still very open, that the method to do it really wasn't set out for you." One of the other instances where he showed his view of openness was when he contrasted the openness of Project #1, which is "where the discussion and the exploration of the task itself really comes in," with when a teacher lectures and gives the students the information.

When discussing both of his projects, Ryan expressed that it was important that there be a balance between how open and how structured a task is. When asked which characteristics he wanted Project #1 to have, he stated that he "wanted it to be open in some senses, but also structured." He felt that he needed to have this particular task be somewhat structured in order to lead students to think about the relationships between variables, but also have open sections, where students come up with their own ideas in order to make connections between their previous knowledge and what they are learning. The lesson goes back and forth between a teacher-led class discussion and sections that Ryan described as open. For example, the teacher first asks the students if there is a relationship between their shoe size and height and then has the students record their own information on the board. The students are not immediately told to use a scatterplot, but instead asked how they could analyze the data in order to determine if there is a relationship, which would be open for Ryan since students are not given the solution method. He also felt that it is difficult to strike a good balance of structure of openness and structure. While speaking about Project #1, he stated,

> When you're teaching with a task, and you're teaching in an exploration way, you have to let – you have to make sure that you not giving too much information before they've had a chance to explore it and connect it to ideas that they already understand. And that's, I think, the most difficult part, is thinking about, where do I leave it open, where do I let them explore, do I – where do I give them guidance so that I'm leading them to where I want them to go as opposed to kind of giving them free range and letting them go crazy. So, there's like, finding – finding the right amount of scaffolding and structure without building a prison for the children to be stuck in.

Also, when asked what was difficult about designing Project #2, he explained that it was difficult to find a balance between what to lecture on and what to let students explore. He felt that "there are some concepts in math that you have to give some lecture to, but you don't want to give them all the information. You want to let them kind of experience it, and there's a balance there."

To Ryan, openness is probably fairly important. Ryan expressed that the openness in Project #1 was important because it promotes discussion and exploration and because it encourages students to make connections between what they already know to the concepts in the task. He added that students will not be able to make as many connections if they are lectured to and just told the information. Ryan attended to the open-ended nature of a task in every interview, but his written tasks varied in how much scaffolding the task included and to what extent the solution method was given to the student. The Pen Pal Tasks and Project #2 include a large amount of structure, which makes the solution method obvious or more obvious; Project #1 is heavily scaffolded, but there are open sections; Adapted Task J and the Quadrilateral Task are mostly open, but have a little hint to help the students solve it. (The other two adapted tasks did not give the solution method, and his adaptation did not change that characteristic of the task.) For example, Project #2 has a large amount of structure; the teacher walks the students through the area method for completing the square and works through the first problem, then lets the students try the next two questions, then teaches the algebraic method, etc. There is even a word problem, which states, "Find a way to use completing the square to answer the following question…" On the other hand, Adapted Task J does not give students the solution methods. It asks students to find two new methods they could use to multiply two-digit by one-digit numbers. The task, though, does give the students a little guidance by giving them an example of a method using an area model to find 7 times 34 by breaking up 34 into 30 and 4. So, Ryan likely felt that it is good to make the solution method not obvious if possible, but it clearly was not the most important characteristic to him as evidenced by the range of scaffolding provided by his tasks.

Ryan stated a few times that an open task will allow students to solve tasks multiple ways or have multiple correct answers, but it is unclear whether multiple solution strategies is part of his actual definition of openness. For example, he stated that since Project #1 had "some openness," the class has to think about the different ways to figure out if there is a relationship between shoe size and height. Also, when asked when it would be the most important to anticipate student responses, he explained that it would be more important to anticipate student responses with tasks with “more of an open-quality” because “you’re going to encounter different responses.” Ryan connected multiple solution methods with accessibility by stating that he would try to think of as many ways as he could for how students would do it in order to tell how accessible a task was.

Although it is unclear whether multiple approaches was part of his definition of open, he seemed to believe that it was only fairly important. He explained,

> I like [tasks that can be solved multiple ways] because it gives you the opportunity to show there are multiple ways of getting to the same answer…It gives you discussion. It has you really think about the math behind what you’re doing and not just using something the teacher gave you.

When asked to sort the tasks according to an important characteristic, he sorted the task according to the number of approaches they had. The piles were “more than one way to get an answer,” “not as many [approaches] you could do, but there are a couple ways you could think about it,” and “there’s one set way they want you to get the answer.” He felt that Task J, for example, would go in the second stack because most students would use the method given, but it does give students the option to come up with their own method.

Ryan attended to the number of approaches a task had in every interview at least once, and he also mentioned more than one way to complete parts of Project #1, the First Pen Pal Task, and Project #2, so multiple approaches was fairly important to him. For example, for Project #2, after Ryan and his partner found out that the Quadratic Formula could be derived two different ways using completing the square depending on the treatment of *a*, they thought it would be a good idea to compare the two methods after the students had completed the task in order to show that both methods work. All of his tasks he designed included at least one aspect that could be solved multiple ways or have multiple correct answers, but it was not usually the main characteristic he focused on while designing. It may have been the focus, though, in two of the tasks he adapted. (The third adapted task already had many possible correct answers.) For Adapted Task J, he changed the task from being told to follow the example to coming up with two different methods to mentally find a product. For another one of the adapted tasks, he added more travel options, which increased the number of combinations, which gives more possible correct answers.

However, Ryan also expressed that sometimes it is good for students to use a certain method, and he suggested that the characteristic of “can be solved in many ways” be last when the participants were ranking seven characteristics in order of importance. He also stated that many students may solve the Quadrilateral Task the same way, “which isn’t necessarily bad, as long as they understand the method they’re coming up with.” So, it seems that he felt like having multiple solution strategies is nice, but not vital.

Conclusion. Both Carly and Jamie defined an open-ended task as a task that can be solved multiple ways and may have multiple correct answers. Ryan defined it as a task that does not give the solution method to the student. He may, however, have included that an open task can be solved multiple ways in the definition. Multiple correct answers was not a large part of how any of the participants viewed openness; they attended to multiple solution strategies much more by far. Jamie believed that an open-ended task does not lead students to a certain solution strategy, and although Carly did not relate this idea to openness, she too attended to this idea.

That a task can be solved multiple ways seemed to be fairly important to all of the participants, but not necessary. It may have been more important to Carly than to the other participants. Openness was a characteristic, though, that the instructor felt was very important, so it is interesting that it was not more important to the participants. That a task does not lead students to the solution method is quite important to Ryan. Ryan brought up that it is difficult to balance how much to structure a task and how much should be left open, where students are not given all of the information. Carly was the only participant who stated that a task that she designed (Project #1) could basically only be done one way, but she was not sure how to change it. All of the participants' written tasks included at least an aspect that could be solved multiple ways or have multiple correct answers.

Openness may not be very important in and of itself to the participants, but they implied that making a task open-ended is a way of bringing about other, more important characteristics. The participants mentioned three ways openness was related to the other characteristics: (1) it is more likely that students will be able to engage in a task if it can be solved many ways; (2) tasks that can be solved many ways or have multiple correct answers promote communication between students as they work on the task; and (3) tasks that require reasoning usually can be solved many ways or have multiple correct answers.

Chapter 6 – Discussion

This current research supports the argument of Sullivan and Mousley (2001) that designing worthwhile mathematical tasks for students is a complicated endeavor for teachers. Task design clearly was complicated for my participants since there were many considerations they felt that they needed to attend to, and they seemed to struggle to attend well to one of the main characteristics (sound and significant mathematics), let alone to all seven at the same time. The majority of what the participants attended to can be captured by the seven main characteristics discussed in the Results Chapter: Sound and significant mathematics, reasoning, appropriateness, clarity, communication, engagement, and openness.

Even though I presented these characteristics separately, the participants did see relationships between these characteristic and mentioned several notable relationships. For example, in a number of different circumstances the participants expressed that tasks that are appropriate, are clear, and require communication allow or help students to reason. They also expressed that a task can be more engaging if it requires communication or is appropriate. The PSTs did not normally explicitly connect the characteristic of sound and significant mathematics to other characteristics, but the purpose of these other characteristics in general is to help students to learn the FMC of a task. The participants did state, however, that the clarity and wording of a task can impact the mathematics of the task and that a task should be engaging in order to motivate students to learn the concept. Additionally, as mentioned previously, they saw open tasks as promoting engagement, communication, and reasoning.

In this chapter, I first discuss how the characteristics the participants felt were necessary or important to a worthwhile task compare to how the literature views a worthwhile task. I then discuss the various ways in which the participants attended to these seven characteristics. Lastly, I compare the results of this study to the results of other studies about teachers and task design.

Characteristics of Worthwhile Mathematical Tasks

The literature outlines many characteristics of worthwhile tasks, but does not give a strict ranking for the characteristics that are most important in making a task worthwhile, nor does it clearly outline which task characteristics must be present and to which degree in order to classify a task as worthwhile. Specifically, the *Professional Standard*s (NCTM, 1991) lists 11 characteristics of worthwhile tasks, but does not state which characteristics are necessary or sufficient in making a worthwhile task. Several characteristics, however, are commonly emphasized throughout the literature, such as sound and significant mathematics, reasoning, and appropriateness (Hiebert et al., 1997; NCTM, 1991; Stein et al, 2009; Smith and Stein, 1998); these characteristics were also emphasized as vital or very important by the participants. Sound and significant mathematics and reasoning are both specifically included in the participants' descriptions of a worthwhile task. Both Carly and Jamie stated that a worthwhile task must have important mathematics and reasoning. Ryan's response, however, differed in that he felt that a worthwhile task needed to be based on important mathematics or mathematical connections, should involve expanding knowledge or solidifying knowledge, and must be clear. So, he included solidifying knowledge, which involved practicing what students have previously learned, and clarity in his overall definition.

Communication and engagement were technically part of the definition of a worthwhile task in the *Professional Standards* (NCTM, 1991), but it was likely that NCTM considered these characteristics to be ones that make a task more worthwhile, but are not requirements of worthwhile tasks. This seems to be consistent with the participants' explicit views about these characteristics. NCTM (1991) stated that worthwhile tasks should promote communication about mathematics. The participants felt that communication is important to have, but it was not the most important characteristic to them. That a task requires explanation was a characteristic that Jamie considered including as a requirement of a worthwhile task, however. The literature conveyed that students communicating their thinking benefits the students by helping them to clarify their ideas and develop their understanding (NCTM, 1991) and benefits the teacher by revealing thinking to the teacher and helping them to know what students know and can apply (Cooney et al., 2001). Carly expressed that explanations are valuable to help students learn from the task and are valuable to provide the teacher evidence of the student understanding. That explanations should show students' understanding was part of Jamie's definition of explanation. NCTM (1991) also argued that worthwhile tasks should be based on "knowledge of students' understandings, interests, and experiences;… display sensitivity to, and draw on, students' diverse background experiences and dispositions;" and "promote the development of all students' dispositions to do mathematics" (p. 25). The participants believed that students must be able to engage with tasks and that it is fairly important that a task be engaging. Jamie felt engagement was more important than the other participants since she believed that tasks must be engaging so that students are motivated to learn what you want them to.

Openness is not part of NCTM's (1991) description of a worthwhile mathematical task, but they did include in their elaboration of that description that many good tasks can be solved more than one way or have more than one correct answer. The participants also did not think that many possible solutions or answers was a requirement of worthwhile tasks; it was only fairly important to them overall. This characteristic may not be very important in and of itself to the participants, but they implied that making a task open-ended is a way of bringing about other, more important characteristics.

The participants attended (to some extent) to almost all of the many characteristics that teachers are encouraged to consider by the literature, but clarity was a main characteristic that was important to the participants, but was rarely discussed in the literature. It makes sense that clarity is not discussed very often in the literature since it is something that one attends to more as one is actually designing tasks as opposed to describing what is important in a task. Stein et al. (2009) mentioned, however, that one way that a high cognitive demand task could decline in its cognitive demand through the lesson is if the task expectations were “not clear enough to put students in the right cognitive space” (p. 16); this implies that unclear tasks are less worthwhile.

NCTM (1991) included in their definition that a task should “represent mathematics as an ongoing human activity” and that a task should “develop students’ mathematical understandings and skills” (p. 25). Even though these characteristics were part of their definition of a worthwhile task, it is likely that the message NCTM intended to send is that these characteristics help a task to be more worthwhile. The participants did not usually attend to the message a task would give about mathematics as “an ongoing human activity.” On one occasion, however, Jamie attended to the message the task would give about what is entailed in doing mathematics. When Ryan suggested in the group interview that students be given protractors for the Quadrilateral Task, Jamie expressed that she thought that it may take away the students’ sense of what it means to prove something if they were able to measure the quadrilaterals’ angles. Regarding understanding and skill development, the participants’ focus was that students would develop a good understanding of mathematics. The participants attended to skill development and considered it important, but it was not a large focus for them. Ryan, however, did allow for tasks where students practice procedures to be worthwhile tasks. Skill development was not even part of Carly's definition of a mathematical task at all. She felt that questions that only required students to practice procedures were "homework problems," not tasks. While Jamie did not include skill development in her definition of a worthwhile task, she did feel that it is important that students are able to apply procedures fluently and did include proficiency in her definition of a mathematical task.

Ways PSTs Attended to Characteristics

The participants attended to the characteristics in three different ways in the interviews (see Figure 4). These ways include stating the importance of the characteristic, but (1) not attending to it as they designed tasks, (2) attending to the characteristic as they designed tasks and not problematizing its application to tasks, and (3) attending to the characteristic as they designed tasks and problematizing its application to tasks. The participants problematized or viewed a characteristic's application to tasks as problematic when they acted in a way that showed that they felt it was difficult or complex to design a task that aligned with the characteristic.

1. Did not attend to the characteristic as they designed tasks
    - Sound and significant mathematics
    - Communication (discussion)
2. Attended to the characteristic, but did not problematize its application to tasks
    - Appropriateness
    - Communication (explanation)
    - Engagement
    - Openness
3. Attended to the characteristic and problematized its application to tasks
    - Reasoning
    - Clarity

Figure 4. Summary of how PSTs attended to the characteristics.

Did not attend to. It was uncommon for the participants to attend to sound and significant mathematics or to the discussion aspect of communication as they designed tasks. Sound and significant mathematics was the most important requirement of a worthwhile task according to the participants, but they seemed to take for granted that a mathematical task would be based on important mathematics. There is little evidence to show that they actually attended to whether the mathematical concept in the task was sound and significant. They seemed to design their tasks based on a mathematical concept that one *could* learn rather than a mathematical concept that is *worth* learning. In fact, they usually chose the task they designed because they had seen the task or a similar task before or because it was a topic that they felt like they understood. As long as a task had mathematics in it, they would probably say the task was based on important mathematics.

Similarly, the participants all thought that discussion was important, but they rarely attended to how tasks would encourage productive discussion as they spoke about designing their tasks. While speaking about the characteristics themselves, it was mentioned that a task with multiple possible solution strategies would promote discussion, but this was not a focus of their actual task design process.

Did not problematize. The participants attended to appropriateness, the explanation aspect of communication, engagement, and openness, but these characteristics were not greatly problematized. The participants expressed that tasks should be appropriate and would usually attend to at least one requirement of appropriateness when talking about their tasks, but they did not seem to recognize the complexity of making a task appropriate. They did not seem to do a careful analysis of the concepts in the tasks and assess whether the students would likely have all of the knowledge of the concepts necessary in order to successfully complete the task, nor seriously consider whether students may have already mastered the primary concepts the task was supposed to be helping students to learn.

The participants did not generally problematize writing questions that required students to explain. They wrote two types of explanation questions. First, they occasionally wrote explanation questions that only required students to list the steps of a procedure in their written assignments. Writing explanation questions that required students to describe how they solved a task was not problematic for the participants. Second, and more often, the participants included question that they hoped would elicit student thinking. They also did not seem to find writing explanation questions that would elicit thinking problematic. The participants did not write out the answers to the explanation questions they expected students to give. It could be that the participants almost assumed that questions would get students to explain the kind of reasoning the participants were looking for even when the participants had not carefully considered the lines of reasoning they wanted the question to elicit.

The participants believed that students must be able to engage with a task for it to be worthwhile and that it is preferable if students want to engage with a task, but they did not problematize doing either. Regarding designing tasks that students are able to engage with, Carly and Jamie specifically seemed to view making a task accessible as not very complicated. They both tried to make their first project accessible by making the activity in the task simple to perform. For example, Jamie thought that tasks were accessible if students could start it and do something on it. On her first project, she explained that any student could rip paper, so the task was accessible, but making an activity simple to perform does not necessarily mean that the students will be better able to learn (or access) the mathematics in the task. With respect to students wanting to engage with a task, the participants did not seem to think that making a task engaging was difficult to do. In order to make a task engaging, they usually would include use of manipulatives or a real-world context. In fact, all of the Pen Pal Tasks and projects either included manipulatives or a real-world context. According to NCTM (1991), a familiar context is just one of many ways to make tasks more interesting. Even though the participants stated that a real-world context is not necessary for a task to be worthwhile, they seemed to have few other ways to attend to engagement in task design. NCTM (1991) argued that theoretical tasks, such as number theory problems, can also challenge and interest students.

The participants, in general, did not problematize openness, or that a task can be solved multiple ways or can have multiple correct answers. (Ryan did problematize his definition of openness, that the task does not give the solution method to the student, but that idea is included in the next discussion about reasoning.) They attended to many solution strategies, but that did not seem to be something that was usually complicated for them to apply to a task. They generally could come up with more than one way that their task could be solved. An exception is that Carly stated that her first project could basically only be done one way, but she was not sure how to change it. So, she did express some difficulty in making a task open. Part of openness, that a task has more than one correct answer, was not a large part of how any of the participants viewed openness, and they only rarely attended to it.

Problematized. The participants problematized both reasoning and clarity. Reasoning was a characteristic the participants felt was extremely important, and they viewed designing a task that gets students to reason as complex. They focused on the reasoning of their tasks more than any other characteristic. They seemed to find it challenging to design tasks where students are required to reason to learn the concept without somewhat leading them to the solution. Ryan often spoke about the balance a teacher needs to achieve between not giving the students too much information and allowing them to explore the concept. Jamie also implied that designing tasks that require students to reason is complicated since she expressed that the level of reasoning that a task required was dependent upon the students' background knowledge and where the task is placed in the curriculum.

The participants problematized making their tasks clear. They seemed to carefully consider whether students would understand what was expected of them from the instructions in the task. Jamie, more than the other participants, expressed several times how difficult it was for her to word tasks in a clear way. Since she felt that writing clear instructions was difficult, it is not surprising that both of her projects involve the teacher giving instructions verbally instead of students being giving written instructions. They also all recognized that the clarity of a task can impact the mathematics that students will do.

Possible relationship between problematizing and knowledge and beliefs. Whether the participants attended to a characteristic or problematized a characteristic's application to tasks may give insight into their knowledge and beliefs about the characteristic. It could be that their belief about a certain characteristic is that the characteristic is not complex and is not difficult to apply successfully to tasks. Such a belief could be fueled by a lack of knowledge of the characteristic; they may not know about the intricacies of the characteristic. In this case, they would not problematize the application of those characteristics and may not even attend to them at all as they design their tasks. This seemed to be the case for most of the characteristics. For example, the participants seemed to have a limited understanding of openness and rarely attended to multiple correct answers. The participants probably had little experience completing tasks with multiple correct answers and may not have had any experience designing these tasks, which made it difficult for them to attend to this idea while designing tasks.

On the other hand, the participants may believe that the application of a characteristic is problematic, but they may not have the necessary knowledge to actually design a task that aligns with the characteristic. They may see the characteristic as complex, but may not know how to deal with the complexity. This seemed to be the case with reasoning. They felt that reasoning was very important, and they found it problematic, but their tasks varied in the amount of reason required. The tasks sometimes led students to the solution method or had a solution method that was obvious. It may be that they did not have enough background knowledge and experience to design tasks that helped students to learn a particular mathematical concept, but did not lead them somewhat to the solution method or make the solution method obvious. For example, Ryan seemed to believe in the importance of reasoning, but may have lacked the necessary knowledge to design tasks without giving student information. This may be why he felt that some concepts need to be taught using some lecture.

Clarity seemed to be the most accessible characteristic to the participants. They problematized clarity, and most of their tasks were clear. It could be that because they are accustomed to completing mathematical problems themselves, they feel like they know when the instructions of a task clearly describe what the students are expected to do. The participants did not, however, normally complete the tasks in their entirety before finishing them. It could be that they did not realize how helpful completing a task in its entirety is to making sure a task is clear. Even though they are usually able to make the initial instructions clear, in absence of doing the task the participants could not determine whether the task was clear enough to likely elicit the desired student mathematical reasoning.

It is not necessarily disappointing that the participants did not problematize all of these characteristics. It may not be possible for anyone to greatly problematize all of these characteristics without a framework to follow. It is especially encouraging that they attended to the reasoning their tasks required so much and felt that designing tasks that require students to reason is problematic.

Connections to Task Design Literature

As discussed in the literature review, in much of the research concerning teachers and task design, researchers conducted some sort of intervention and taught the teachers a certain framework with the purpose of improving teachers' task design skills according to that framework. The framework generally included a focus on reasoning and often specifically used Stein et al.'s (2009) Mathematical Tasks Framework and Task Analysis Guide about cognitive demand. Therefore it is difficult to compare this current research to the literature when the teachers were told what to attend to, and their purpose was not to describe how the teachers designed tasks in general. That said, even though the participants were not required to use a specific framework in this current research, they did attend to most of the considerations that the teachers in the other research did, such as generalization, connections, open-endedness, procedures, not routine or leading, computation, justification, reasoning, mathematical topics, complexity of the task, and bringing out students' thinking. Several studies (e.g., Osana et al., 2006; Boston, 2013; Arbaugh and Brown, 2005) cited their teachers attending to superficial features when determining the cognitive demand required by a task, such as task length, size of the numbers, number of steps, a real-world context, manipulatives, representations, explanation, and complexity of the arithmetic. While the participants attended to most of these ideas, they did not analyze tasks according to their cognitive demand, so they did not attend to them in this way. The participants did not, though, seem to try to determine the level of reasoning required by looking at such features. They did not attend to size of numbers, number of steps, or the complexity of the arithmetic.

The participants did not attend to all of the ideas that were in the frameworks of the other studies, though. The participants did not attend to cultural relevance (Thomas and Williams, 2008), cognitive demand (Stein et al., 2009), or the aesthetics of the tasks, such as whether they were surprising or simple (Crespo and Sinclair, 2008). Even though the PSTs in the Task Design course were required to read and discuss Stein et al. (2009), none of them ever brought up the construct of cognitive demand in the interviews, which is surprising. The participants were not required to apply the framework, but for some reason, it was not a useful way to talk about reasoning for them. It seemed that the terminology was not readily accessible to the participants. Many researchers who designed studies with interventions to improve teachers' task design skills were focused on improving their understanding of cognitive demand. This framework is very useful for teacher educators and researchers, so perhaps the framework is not easily understood and its utility is not seen by the PSTs without much specific instruction. Boston (2013) described how some of the teachers in her study did not use the four labels of cognitive demand from Stein et al. (2009), but used the words from the descriptions of the levels of cognitive demand. This is consistent with what my participants did in that they attended to many of the ideas from Stein et al. (2009) that were discussed in the course, such as a task not being able to be solved using a routine procedure.

Just as the elementary PSTs in Crespo (2003) first posed problems they had not yet fully investigated, my participants completed their task design process without having done the tasks. At the end of the study in Crespo (2003), however, the PSTs no longer did this. It could be that the letter exchange gave the PSTs feedback from the students, which made them see the consequences of not fully investigating the task. The participants in this current research may have needed such feedback.

At the beginning of the Crespo (2003) study, the elementary PSTs often posed routine problems or tried to make the problems easier. My participants did not normally design tasks with routine problems or try to make problems easier; they probably had a greater knowledge of what makes a worthwhile task to begin with.

De Araujo and Singletary (2011) described secondary inservice teachers' views of worthwhile mathematical tasks, and my participants attended to most of the characteristics that their teachers did, such as that the task had a meaningful context, helped students to develop the mathematics themselves, promoted problem solving, promoted communication, helped students make connections, and was achievable. The inservice teachers as well as my PSTs all emphasized that a task should require reasoning and that all students should be able to engage with the task. The inservice teachers, however, explicitly placed more importance than my participants did on a meaningful or real-world context, on promoting discussion, and on whether the task was standards-based. The inservice teachers felt that the meaningful context was important for worthwhile tasks because it allowed students to apply their knowledge to a real context and could motivate them to explore the mathematical ideas. Even though my participants stated that a task did not necessarily need a real-world context, it was a common way they used to make a task engaging. The inservice teachers included that a worthwhile task should promote discussion. While the PSTs in this current research stated this characteristic was important, it was not something they attended to very much. The inservice teachers also included that a worthwhile task is standards-based. The tasks my participants designed (except for the Adapted Tasks Assignments) were based on standards from the *Common Core*

(NGACBP & CCSSO, 2010), but they were probably less concerned about the standards than the inservice teachers since my participants were not held accountable to specific district standards.

The inservice teachers in de Araujo and Singletary (2011) did not include clarity in their definition of a worthwhile task, but clarity was an important characteristic to my participants. This could be because clarity is something that, with more experience, becomes less of a focus for teachers. Or, it could be that the teachers did not mention clarity since they were describing what a worthwhile task was instead of actually designing tasks. They may have brought clarity up if how they designed tasks was studied.

Chapter 7 – Conclusion

The purpose of this study was to describe how secondary PSTs design worthwhile mathematical tasks. Three participants, who were selected from the Task Design course at a large private university, were interviewed five times each to determine what they attend to while designing tasks. Sound and significant mathematics, reasoning, appropriateness, clarity, communication, engagement, and openness were the task characteristics that the participants attended to the most frequently and felt were the most important for a worthwhile task to have.

The participants believed that sound and significant mathematics is the mathematics that students need to know. Sound and significant mathematics was the most important characteristic to them, but they rarely attended to whether the mathematical concept of a task was sound and significant. The participants believed that the FMC is what students have to do, what they have to think about, or what they learn from doing the task. They also commonly described the FMCs of tasks as general topics, but they did usually focus on a specific mathematical concept as they designed their tasks.

All of the participants explained that a task requires reasoning if students are not given the solution method or if the task cannot be solved by a known solution method. Reasoning was necessary or very important to the participants. They attended to this characteristic a great deal, but the tasks they designed varied in the amount of reasoning required. Regarding appropriateness, the participants all argued that tasks should be based on students' current knowledge and should help them learn about a concept they had not yet mastered. Appropriateness was necessary or very important to the participants, and their tasks usually were appropriate. The participants all believed that a task is clear if the students understand what they are expected to do from the instructions in the task. Clarity was vital to one of the participants and fairly important to the others. This characteristic was not emphasized in the literature, but the participants attended to it a great deal.

For the participants in this study, communication included both requiring discussions and requiring explanations. Communication was important to them in general; they attended to explanation much more than discussion, and explanation was very important to one of the participants. The participants expressed that discussions should include students sharing their solution methods and ideas with each other. They also stated that explanations should consist of students' thinking, but one of the participants' definitions for explaining their thinking was (or at least included) having students describe a process or show their work.

The participants believed that it is important that a task is accessible (that all students are able to engage with a task), and they thought it was preferable that students want to engage with a task. Regarding being able to engage with a task, the participants brought up that the task must fit the students' learning styles, that the students must understand the context of the task, and that the task should be non-exclusive. Regarding wanting to engage with the task, they described an engaging task as fun, interesting, motivating, or hands-on. It was very important that students want to engage to one of the participants. Finally, two of the participants defined an open-ended task as a task that can be solved multiple ways and may have multiple correct answers. The third participant attended to these ideas, but described an open task as one that does not give the solution method to the student. Openness was the least important to them of the seven important characteristics, but they did imply that being able to solve a task many ways can help bring out other, more important characteristics.

Implications

In general, what the participants considered to be important to a worthwhile task was reasonable and most of the characteristics they attended to were supported by the literature. Their view of some of the characteristics and their ability to successfully apply the characteristics to task design, however, lead to implications of what teacher educators may want to consider addressing in their teacher education programs.

The participants varied in their views of the characteristics, suggesting a wide range of possible ways that PSTs in a given class will view the characteristics of worthwhile tasks. They may know, for example, the requirements of a worthwhile task listed by the *Professional Standards* (NCTM, 1991), but probably have different ideas of what it means for a task to align with those characteristics. PSTs can learn to design worthwhile tasks by completing, analyzing, and designing tasks where the teacher educator can elicit and build on their thinking. It is important that a teacher educator learns how the PSTs view the characteristics of worthwhile tasks and what tasks the PSTs consider to align with the characteristics, so that they can design learning activities for the PSTs to help them develop a more sophisticated understanding of how to design worthwhile tasks.

In order to elicit PSTs thinking about worthwhile mathematical tasks, teacher educators can require that PSTs state which characteristics their task is meant to embody and explain their rationale for how the task accomplishes those purposes. For example, a PST may want to design a task that requires students to explain their reasoning, but the teacher educator may find that this PST views explaining reasoning as merely "showing your work." Or, a PST may want to design an "engaging" task, and the teacher educator may find that the PSTs seem to think that only tasks with "real-world contexts" are engaging. The teacher educator may wish to require the PSTs to design tasks with particular characteristics to push on particular misconceptions or difficulties that the teacher educator has learned that the PSTs have to help the PSTs refine their ability to apply those characteristics to their task design. For example, the participants in this study seldom attended to encouraging productive mathematical discussion. So, teacher educators may wish to have PSTs design a task that promotes productive mathematical discussions as the students work on the task. It was rare for my participants to attend to multiple correct responses, so the teacher educator may wish to show how having multiple correct responses in a task creates a need in the students to communicate. In having them design such a task and requiring them to provide their rationale for how they designed a task to accomplish this purpose, the teacher educator could learn what the PSTs think about a characteristic and how they get a task to embody that characteristic.

Because the participants sometimes had different views of the same characteristics, after the teacher educators learn about their PSTs' views, they may wish to create a shared vocabulary about the characteristics. Reasoning is a characteristic that was especially important to the participants, but they may not have always been sure how to get students to reason since their tasks varied in the amount of reasoning they required. As reasoning is such a vital characteristic of worthwhile tasks, the Task Analysis Guide (Stein et al., 2009), which describes the four levels of cognitive demand, could be helpful to give PSTs a shared vocabulary. My participants were introduced to the terminology in the Task Analysis Guide, but they never adopted the framework. This framework could give PSTs a more structured way to determine how much reasoning is required from a task. If they were required to design tasks of a high cognitive demand and explain why the task had a certain level of cognitive demand, with support, they would improve in their ability to design such tasks, and they would be able to learn and use a shared language with the other PSTs in the course that may allow them to be more specific about why the tasks requires reasoning.

At times the participants did not attend to certain characteristics while designing tasks (such as sound and significant mathematics) or did not problematize applying certain characteristics (such as appropriateness). The participants did not attend to whether or not the mathematical concept of a task was sound and significant, and they needed more practice articulating the FMCs of their tasks. To address this issue, teacher educators may want to consider requiring PSTs to design a full unit's worth of tasks, instead of only assigning isolated tasks. PSTs can be instructed on how to select the most important mathematical concepts (from among more than they can "cover") and how to articulate the associated FMCs. When they plan their unit, PSTs should be required to explain their rationale for which concepts are the most important for secondary students to know. The PSTs can also be required to describe how they expect students to learn the FMC from engaging with the task. For the participants, it was common for the FMC to be the final answer to the task. The teacher educator could point out that tasks can hit a FMC in different ways. Instead of the FMC being the final answer to the task, the FMC can be learned by the student needing to construct and then use the FMC in order to solve the task. The FMC is less obvious in the second type, and none of the participants constructed a task of this type. Creating new tasks of this type may be more complicated than the PSTs are ready for, but they could be given practice analyzing such tasks to determine what the FMCs are.

My participants felt that it was important that tasks were appropriate, but they did not problematize making their tasks appropriate for students. Allowing PSTs to do a letter exchange or some other activity with a sustained connection to students would be valuable to address this issue. Teacher educators could require PSTs to carefully consider what the student already knows and which concepts would be within the student's reach. With each iteration of the letter exchange, the PSTs would get more practice inferring student thinking from the student responses and designing tasks based on that thinking. After students complete the tasks, the PSTs can use student work to assess the appropriateness of the tasks.

A letter exchange could address PSTs' difficulties in applying other characteristics in addition to appropriateness. As previously mentioned, teacher educators can require PSTs to describe how certain task characteristics are embodied in the task, but, in a letter exchange, they could also require PSTs to anticipate student responses to those task characteristics and assess those responses. In the letter exchange that was studied in Kosko et al. (2010), the PSTs were required to predict whether the students would engage in each of the five NCTM (2000) "Process Standards" and what level of cognitive demand the task would require of students. Then the PSTs would analyze the students' responses and assess whether students had engaged in what they expected and justify their assessments. Kosko et al. (2010) required the PSTs to predict *whether* students would engage in particular processes, but it would be valuable to also have PSTs anticipate *how* students will respond to all of the main characteristics they were trying to incorporate into the tasks. Actually thinking through how students may react to particular characteristics of the task along with assessing the students' responses may help the PSTs to design tasks that embody those characteristics.

Even though a letter exchange is not what a practicing teacher would do, it allows PSTs "an authentic experience" (p. 246), according to Crespo (2003), in that PSTs can practice skills that practicing teachers must have, such as posing tasks to students, assessing students' understanding, and responding to what the student did. My participants felt that the Pen Pal

Tasks were *less* authentic than their projects, which was the opposite effect I expected. They were not motivated to complete the task before they assigned it (or were finished designing it), just as with their other tasks. My participants only performed two iterations of a letter exchange, however. The participants may have felt that the letter exchange was more authentic and could have felt more motivated to complete the tasks if the exchange had been sustained over the course of the entire semester. (It is also possible that the only way PSTs will complete their tasks is if their instructor requires them to submit their solution to the task along with additional anticipated ways students may do it.)

Limitations and Future Research

The participants in this study were over half-way through their reform-oriented teacher education program, so they had learned the ideals of reform and had some experience with mathematical tasks. I hypothesize that my results would probably be similar to participants at this stage in other reform-oriented mathematics education programs, but may be more sophisticated than participants at this stage in a traditional program. Further studies in other programs are warranted in order to see how these findings may relate to other programs.

The fact that these tasks were assignments for the course probably influenced their task design since they would be trying to design a task that would be acceptable to their instructor. The instructor of this Task Design course, however, did not require students to follow a rigid structure. So, while the results may have limited generalizability, the fact that the PSTs were mostly allowed to attend to the characteristics of their choice mitigates the influence of the class on how the PSTs design tasks.

Now that we know more about what secondary PSTs attend to as they attempt to design worthwhile tasks, the next step is to determine which factors influence how the PSTs design tasks and how the factors influence what they attend to. Knowing which factors influence how PSTs design tasks can help teacher educators address the source of PSTs' misconceptions about and limitations in carrying out effective task design. Although which factors influence how the PSTs designed tasks is not the focus of this research, there seem to be at least four primary factors that influenced what the participants in this research attended to. The primary factors that probably influence PSTs' task design process are the PSTs' Mathematical Knowledge for Teaching (MKT) (see Ball, Thames, and Phelps (2008)); their beliefs about the nature of mathematics, about reform, about how students learn, and how mathematics should be taught; their goals; and their perception of the activity of task design. Hill et al. (2008) studied five teachers and found that MKT clearly affected how the teachers selected and sequenced mathematical tasks. Task selection is part of task design, but MKT also probably affects how PSTs create new tasks or adapt existing tasks. Beliefs are also likely to be a major influence. For example, Remillard (1999) argued that the ways the two teachers in her study interpreted their textbooks and subsequently designed tasks were influenced by their beliefs about mathematics and about teaching and learning. Knowing more about what influences how PSTs' design tasks can help teacher educators to develop learning activities to address specific issues. Once we know how these (and possibly others) factors influence how secondary PSTs design tasks, we can study how these factors influence how they present and enact tasks throughout their lessons. Charalambous (2011) found evidence to support a positive association between teachers' MKT and the cognitive demand required of students through task presentation and enactment, but he stated that more research is needed to provide conclusive evidence of this relationship. PSTs would also likely exhibit this same relationship.

The purpose of this research was to describe how secondary PSTs design tasks. Since it is uncommon for research to give the teachers' point of view, this research has led to a greater understanding of how PSTs design tasks and what they think is important in a worthwhile task, which is important to have if we are to improve their thinking about tasks. The results of this research has led to specific ideas that teacher educators can consider that may help them as they teach task design to their PSTs. It is important that PSTs develop sophisticated task design skills, so that when they are teachers, they are able design worthwhile tasks to include in their instruction, which would give their students access to tasks that allow them to develop a rich understanding of mathematics.

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Appendix A

Task Design Course Survey

Name:________________________________ Email: ________________________________

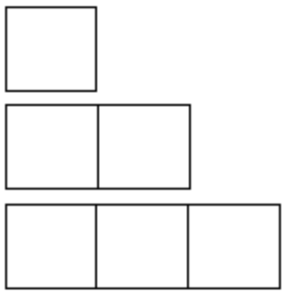
1. Briefly describe how competent you feel you are in college-level mathematics.

2. Imagine you are teaching a high school algebra class. Do you think you would ever depart from the set curriculum? If so, how/when?

3. Have you ever tutored anyone in mathematics? If so, describe the situation. For example, what was the age of the person you tutored, the capacity in which you tutored him or her, the specific subject that was tutored, the reason he or she was being tutored, and/or your reaction to this experience?

4. A mathematical task is any example, activity, exercise, or problem that is used to help students learn mathematics. What features would a mathematical task need to have for you to call it a "good" task?

5. Below are two examples of tasks about perimeter. If you were teaching a middle school mathematics class, which task would you be more likely to use? Why? List at least three reasons why you would choose that task.

Task A	Task B
Manipulatives/Tools: Square Pattern Tiles Using the side of a square pattern tile as a measure, find the perimeter (i.e., distance around) of each train in the pattern block figure shown below (from Smith et al., 2004). Train 1 Train 2 Train 3 	Manipulatives/Tools: None You need to fence in your 4 ft by 12 ft garden. You have pieces of fencing that are 3 ft long and pieces of fencing that are 4 ft long. How would you use the pieces of fencing to surround the garden? How long would the fencing be altogether?

6. Consider this task: *Solve the following quadratic equation:* $x^2 + 3x - 10 = 0$. *How could you use your answer(s) to help you sketch f(x)=*$x^2 + 3x - 10$*? Use the information you learned from solving the equation to help you draw a sketch of f(x)=*$x^2 + 3x - 10$.

a) What do you think a student would need to know to successfully complete this task?

b) What errors (if any) would you expect students to make on this task?

c) What do you think about the task in general? Is it a "good" task?

Appendix B

Interview 1 (Individual) – Task Sort Interview (Carly)

Task Sort

- *Give the tasks to the participant. Tell them that these are 16 tasks from secondary mathematics. These are the tasks that they have been doing in class. Tell them that I am going to ask them some questions about them.*

- I would like you to think of an important characteristic that you could use to sort all of these tasks. Think out loud as you sort the tasks.
- What characteristic did you use to sort the tasks? Why?
- Why were each of these tasks put in each of these piles?

- I would like you to think of an important characteristic that you could use to sort all of these tasks. Think out loud as you sort the tasks.
- What characteristic did you use to sort the tasks? Why?
- Why were each of these tasks put in each of these piles?

- How would you describe a mathematical task?
- (*Choose one of the tasks – TASK K*) What do you think about this task?
- What knowledge would a student need to know to complete this task?
- What would you expect a student to do while working on this task or thinking about while doing this task?
- What do you expect a student to learn from this task?
- What are the errors a student might make on this task?
- When would you give this task to a student? In which grade/situation would it be appropriate?
- What would the purpose for this task be?
- What mathematical topic could it be building from? What mathematical topic could it be building to?

- (*Choose one of the tasks – TASK J*) What do you think about this task?
- What knowledge would a student need to know to complete this task?
- What would you expect a student to do while working on this task or thinking about while doing this task?
- What do you expect a student to learn from this task?
- What are the errors a student might make on this task?
- When would you give this task to a student? In which grade/situation would it be appropriate?
- What would the purpose for this task be?
- What mathematical topic could it be building from? What mathematical topic could it be building to?

- Did you read about a task sort in your reading? How do you think it influenced how you sorted the tasks? Did you think about sorting with cognitive demand? The example in the book did sort using cognitive demand.
- What characteristics does a task need to have for it to be a good task? Why are those tasks so important?

General Questions

- We're done talking about these particular tasks. Now, I want to ask you some general questions.
- What do you think it takes to understand mathematics?
- How do you believe that students best learn?
- Are there any areas of mathematics that you find difficult?
- Why do you want to be a math teacher?
- How do you imagine yourself teaching mathematics? What would a typical lesson look like?
- What kind of teaching does this program advocate? Some ppl characterize this as task-based teaching. Do you think that kind of teaching is feasible? Is it useful?
- You were just assigned to design a task for your Project 1 in this class. Do you think that it is a useful assignment? Why? Do you think this Task Design class will be useful to you? What have you learned from this course so far?
- *Survey question: In question #4, you said that a good task needs to have the potential to take students from one level of understanding to a higher or deeper one. What do you mean by level of understanding and by higher/deeper?

Appendix C

Interview 2 (Individual) – About Project #1 (Jamie)

- What did you first think about when you were assigned to design a task?
- Why did you choose this task? Where did this task come from? Did you just find it, adapt it, or completely design it from scratch?
- What did you think about from start to finish as you designed the task? Could you walk me through how you designed this task?
- What features did you definitely want this task to have? (Take note of each of these features) Why did you want the task to have … (each of these features)?
- Why did you choose this mathematical topic?
- What is the purpose of this task?
- Did you notice anything that was difficult about designing this task? What did you struggle with?

- What would a student have to know to solve this task?
- How do you expect a student to solve this task? The most common way?
- Are there common errors that you might see when students do this task?
- You said students may have trouble organizing the data. How do you anticipate the students will organize the information?
- What does this task help students to learn?
- What specifically should students understand after completing this task?
- Who (type of student) would this task be appropriate for? Why?
- When would you give this task in a unit?
- What do you think an 8^{th} grader would know about exponents?

- I see that the task is about exponents. I'm going to ask you some questions to see how you understand exponents. What does a good understanding of exponents look like? What is important for a student to understand about exponents? What might a student with a good understanding be able to say or do? What are the common misconceptions that students would have about exponents?
- What procedures are related to exponents?
- I see the fundamental mathematics concepts you have written. Can you talk about how the task was designed to help students to understand that? (*Specifically, I want to know how the task was designed to get at the properties of exponents.)
- I see the performance goals you have written. How is the task designed to accomplish these goals?
- How do you see the mathematics emerging from this activity?

- *You said that problem solving skills are important in order to learn mathematics. Talk about problem solving skills in relation to this task.
- You said that some characteristics were important for a task to have. Thinking about the task you designed, how do you think this task stacks up to those characteristics? How do these characteristics show up in your task?

    - Mathematical rigor / mathematical thinking,
    - needs to be based on sound and important mathematical concepts
    - gives the students adequate practice with the skill or concept
    - be as engaging as possible
    - feel motivated and successful
    - be built from some concepts already known by the students so that they can create a connected map of concepts
    - needs to be clearly stated to facilitate an understanding of how to go about the task
    - prefaced with a background
    - students required to reason and explain
    - group discussion
- *You said that you love hands-on tasks. Talk about how that characteristic relates to your task.
- **You said that this task was engaging. What do you mean by engaging? Why do you think this task is engaging? How does that help to maintain their attention? Is it important for a task to be engaging?*
- *Previously, you said that you thought that "exponentials are difficult to explain to students and to understand." Do you still think that way? Do you think that this task helps students to understand exponents?
- **Why did you choose to have students rip 8 and 10 times?*
- **Did you want students to actually make all 8 tears or 10 tears? How much time do you think that would take? (It took me about 20 minutes to make 8 tears).*
- *Do you think that students will be able to see how the number of tears is related to the number of pieces, in order to write an equation that isn't recursive?

Give the participant my work:

- Look at the student response to the task. Can you make any sense of it? What do you think she did?
- Do you think the student learned what you wanted her to from this first task? Why?
- What do you think this student understands about exponents now?
- Did you go through and do this task yourself?
- Would you have done anything differently knowing what you know now?

- Describe mathematics as a discipline. What does the discipline of mathematics value? How do those values show up in this task?
- What is the best way to teach mathematics?
- What have you learned so far about designing tasks?
- How do you think your Task Design class affected how you designed this task?
- How do you think your instructor affected how you designed this task?

Appendix D

Interview 3 (Group) – My Participants Design a Task

Designing a Task

- *Give each of the participants some paper and writing utensils.*
- *Say:* "It is very important to verbalize everything that you are thinking throughout this interview. Make sure your voice is heard in this group interview. I would like all of you, as a group, to design one task. Imagine that you are a high school teacher and are doing a unit on geometry. The goal of this unit is for students to understand the characteristics of quadrilaterals and be able to justify those characteristics. I would like you to design a task that would fit somewhere in this unit."
- *Ask them to clarify anything that I don't understand, but don't intrude.*
- *If they start to design an introductory lesson, say:* "You've done some introductory lessons before, and I'm interested in your designing a task that falls later in the unit."

Questions Concerning the Task - ask after they are done with the task

- Why did you choose this task? Have you seen it before?
- What was your instructional goal? Talk about the importance of students being able to do/understand that.
- What task characteristics did you consider to be most important? Less important? Not important?
- What is the cognitive demand of this task? Why?
- What was difficult about designing this task?
- How did it feel to design this task together?
- What are the important concepts that students need to understand about quadrilaterals?
- Students
    - Did you think about how a student might respond to this task? How would a student respond to this task?
    - What knowledge would a student need to know to complete this task?
    - What do you expect a student to learn?
    - Who would this be appropriate for?
    - Would you give a student this task? If so, under what circumstances?
- Influences
    - Do you feel like you design tasks now differently than at the beginning of the semester?
    - How did your Task Design class affect how you designed this task?
    - How did your instructor affect how you designed this task?

General Questions for the Group

- In the past, we have talked a lot about characteristics of tasks. These characteristics have come up a lot: use students' current knowledge to help them learn something new, mathematically rigorous, requires justification, engaging, clear instructions, can be solved in many ways, sound and important mathematics.
    - I would like you to, as a group, rank these characteristics in order of importance.
    - Do you think that you could accomplish all of these at the same time?

    - What of these are easier to accomplish? Which are most challenging to accomplish?
    - Are there any characteristics that are important that are missing from this list?
- Now, I asked these questions to each of you separately, but I got different responses.
    - So, I want to ask all of you, what is a mathematical task? Together, come up with a definition.
    - What is the best way for students to learn?
    - What is the best way to teach mathematics?

Appendix E

Interview 4 (Individual) – After the Letter Exchange Experience (Ryan)

Questions that were asked in an email before the interview:

- How do you expect that the student who completes your task will do it? If you can think of more than one way the student may approach it, please explain those ways.
- Are there any errors that you would expect a student to make on this task?
- What would a student have to know to solve this task?

1. Ask about the design of the first task

- Your instructor gave you some options of topics to choose from for this task. They were matrices, exponents and logs, solving polynomials using long and synthetic division, and dealing with polynomials with complex roots and the fundamental theorem of algebra. Why did you choose these topics over these others?
- Could you walk me through how you designed this task? (Where did you get the idea? Had you seen a task similar to this before?)
- When did you pick the fundamental mathematical concept that you wanted students to learn in the process of designing the task? How did that influence how you designed the task?
- Do you feel like designing the tasks for the letters was any different than designing the task for Project 1? Did you approach it differently? How did you like the pen pal experience?
- Was this an instructional or an assessment task? (Teaching them something new versus assessing what they already know)
- What did you want students to learn OR What did you want to assess?
- What characteristics did you definitely want this task to have? To what do extent do you think the task does that? (If they don't ask about having a real-world context, ask them how important it was to them)
- What did you find was most difficult about designing this task? Was there anything else that was difficult?
  - Is there any additional information about the student you wished you would have known?
- Did you anticipate student responses before you turned in the task? Did the anticipated student responses impact the task?
- Did you solve this task yourself before turning it in?
- *Why do you have students round?
- Did you go through and try to articulate the actual explanations that students would give? What do you think the precalculus student already knew about this mathematical concept before you gave him or her the task?
- *You said in that email response to me that depending on how they did it in the beginning would affect how they did the task? How so?

2. Look at student's response

- Here is the student's work on your task. I want you to talk aloud as you look at this task.

- *What do you make of #2? What do you think about how the students described the shape of the graph as a parabola?
- Did he respond like you thought he would? Why?
- Did he learn (for an instructional task) or show that he knew (for an assessment task) what you wanted him to? How do you know that? (What evidence do you have in the student's response?)
- Now that you've looked through a student's response, would you revise this problem if you were to use it again?

3. Brainstorm the next task

- I thought it would be a great experience for you to get another chance with the student who completed your task. So, for the next few minutes, I want you to brainstorm a follow-up task that you could give this student based on the student's response on the first task. Your instructor asked you to do a reflection on the task, and she did say that posing another task could be part of that reflection. And, for this interview, I would like you to do that. The student will have a chance to complete this task. So, take a look at the student's response and brainstorm a follow-up task you could give. After this interview, I want you to finish designing it, if needed.
- What kind of mathematics would you want to pursue?
- How did the students' response affect this task?

4. General Questions

- How important is designing tasks to being a teacher? Why?
- Do you feel like you have gotten better at designing tasks so far in this year? How so?
- *In the first interview, I asked you if you had read the part in the Stein book (the purple book) that talked about a task sort. You said that you hadn't read that part, but you said that "I read the books; most of what's said in there rings true and I agree with. There are some things that I still have some issues with." I was just wondering, what are those things that you have issues with?
- *If a student said they couldn't do math, what would you do? Why?
- *Do you believe that all students can do math?
- *Are you going to teach using tasks? How do you think your students will react at the beginning of the year?
- *You have previously said that "students don't all learn the same way." Could you talk about that? Could you give some examples of how students learn differently?
- *Suppose a teacher simply gave you the definition for a new mathematical term, and then gave you examples where it applied. What would you think of that?

Appendix F

Interview 5 (Individual) – About Project #2 (Jamie)

Part I – Irrational Number and Factoring Tasks

- Give them the following task: **Write an irrational number whose square is smaller than itself. Explain why your number fits the criteria or argue that it is not possible to write such a number.** (This task is from Cooney et al. (2001)). Please complete this task.
- What do you think about this task?
- Do you think this is a good task? Why do you/ don't you like it? [Write down the characteristics that they mention on the sheet of paper].
- [If they give some characteristics] Could you explain what you mean for these? (i.e. What makes a task ______?) Why are those important?
- What would you say is the fundamental mathematical concept of this task?
- How do you know that this is the fundamental mathematical concept? Where in this task does the fundamental mathematical concept come out?

- We've talked a lot about characteristics of good tasks in previous interviews. The characteristics that you have mentioned are valuable to include in tasks. But, I was wondering, what characteristics are necessary, not just valuable, for any task to be a good one? [Write down what characteristics they say on a separate sheet of lined paper.] (If they say it depends, ask them what it depends on.)
- (Point to the characteristics of the task). Could you tell what you mean by each of these? (What makes a task ______?)
- Why are each of them necessary for a task to have?

- Give them the following task: **Factor: $\boldsymbol{x^2 - 7x + 10}$ and $\boldsymbol{4x^2 - 25}$.** Please complete this task.
- What do you think about this task?
- Do you think that this is a good task? What do/don't you like it? [Write down good characteristics that they mention on the sheet of paper].
- [If they give some characteristics] Could you explain what you mean for these? (What makes a task ______?) Why are those important?
- Would you ever use a task like this? Does it fit the characteristics that you listed that all tasks should have?
- What would you say is the fundamental mathematical concept of this task?
- How do you know that this is the fundamental mathematical concept? Where in this task does the fundamental mathematical concept come out?

- You just described quite a few characteristics. You've also brought up these characteristics before: (Hand them the list of characteristics). Could you tell me what each of these mean? (What makes a task accessible? What would a task look like that's accessible? What do you mean when you say they can do something?) Are there any of these characteristics that are related to each other?

(Jamie's list) Mathematical tasks need to…
1. give adequate practice with the skill
2. build from prior knowledge to make connections
3. clearly-stated
4. required to reason and explain
5. not too procedural
6. students are interested
7. mathematical rigor
8. needs to be based on sound and important mathematical concepts
9. engaging
10. group discussion
11. hands-on
12. should feel comfortable
13. intellectually engaging
14. open
15. accessible

Part II – General Questions

- When you are asked to anticipate student responses, how do you go about doing that?
- For which tasks do you think that it's the most important to anticipate student responses? Or it is equally-important with every task?
- How do assessment and instructional tasks differ? In their characteristics?
- How important is it that students enjoy doing a task? Do students always need to enjoy it?

- Teachers have many goals that they are trying to meet. What is the most important goal for you as a teacher?
- What does problem-solving mean to you? What makes something a problem?
- How realistic were the assignments that you had to do for this class? In other words, do you think that you would really have to do things like this as a teacher? Are there some assignments that are more realistic than others? Do you feel like you approached tasks differently if you felt like they were more similar to what you would have to do as a teacher?
- Do you think that you've improved in how you design tasks? How so?
- What can being able to design better tasks do for you as a teacher?
- Have your views changed about mathematical tasks changed this semester? Why?
- Do you think that these interviews with me have impacted how you design tasks?

- *When you got back the responses from the Pen Pal Tasks, I noticed that you and Carly switched the student responses and graded each other's. Why do you think you did that?
- *You've said that you think that both procedures and concepts are important. What is the importance of procedures in comparison to concepts?

- *You have said before that it’s difficult to tell how mathematically rigorous a task is just by looking at it. How can you tell?

Part III – Project 2

- Why did you do this task about this topic?
- Why did you choose this task? Where did this task come from? Did you just find it, adapt it, or completely design it from scratch?
- What did you think about from start to finish as you designed the task?
- How did the 3 student responses to the task affect the final task that you turned in?
- What characteristics did you definitely want this task to have? Why did you want the task to have each of these characteristics?
- Did you notice anything that was difficult about designing this task?
- Did you solve this task yourself?
- Is your project 2 an assessment or instructional task? What makes this task an instructional task?
- What did you want students to learn by doing this task?
- Where in the task does the fundamental mathematical idea come up?
- Why is this appropriate for ____ grade students? (It’s in the *Common Core*). Are there any other reasons?

- Did you find that designing this task was any different than designing the tasks for the students in the letter exchange? Why? Which (the project or the Pen Pal) did you feel like was more realistic to what you need to be able to do as a teacher?
- How do you think your Task Design class affected how you designed this task?
- How do you think your instructor affected how you designed this task?
- Could you compare your task to the Irrational Number Task for me?
- Would be okay if I wrote you an email with follow-up questions next semester?