Theoretical Computer Science 409 (2008) 450–460

Contents lists available at ScienceDirect

Theoretical Computer Science

journal homepage: www.elsevier.com/locate/tcs

Edge-fault-tolerant Hamiltonicity of pancake graphs under the conditional fault model

Ping-Ying Tsai [b], Jung-Sheng Fu [c,*], Gen-Huey Chen [a]

[a] Department of Computer Science and Information Engineering, National Taiwan University, Taipei, Taiwan, ROC

[b] Department of Computer Science and Information Engineering, Hwa Hsia Institute of Technology, Taipei, Taiwan, ROC

[c] Department of Electronic Engineering, National United University, Miaoli, Taiwan, 36003, ROC

ARTICLE INFO

Article history:
Received 21 March 2008
Received in revised form 27 August 2008
Accepted 5 September 2008
Communicated by D.-Z. Du

Keywords:
Cayley graph
Conditional fault model
Fault tolerance
Hamiltonian cycle
Pancake graph

ABSTRACT

The conditional fault model imposes a constraint on the fault distribution. For example, the most commonly imposed constraint for edge faults is that each vertex is incident with two or more non-faulty edges. In this paper, subject to this constraint, we show that an n-dimensional pancake graph can tolerate up to $2n-7$ edge faults, while retaining a fault-free Hamiltonian cycle, where $n \geq 4$. Previously, at most $n-3$ edge faults can be tolerated for the same problem, if the edge faults may occur anywhere without imposing any constraint.

1. Introduction

The performance of an interconnection network (network for short) highly relies on its interconnection topology. In recent decades, a lot of network topologies have been proposed in the literature [1,8,16]. Among them, the pancake graph [1,14] is suitable to serve as the topology of a large-scare parallel and distributed system, because of its scalability and other favorable properties, e.g., regularity, recursiveness, symmetry, sublogarithmic degree and diameter, and maximal fault tolerance. The pancake graph, which belongs to the class of Cayley graphs [1], was introduced (and named) from the famous "pancake problem" whose answer is exactly the diameter of the corresponding pancake graph (see [14]).

It was shown in [15] that the diameter of the pancake graph is bounded above by $3(n+1)/2$, where n is the dimension of the pancake graph. It is still an open problem to compute the exact diameter of the pancake graph. In [28], an $O(n \log n)$ time broadcasting algorithm for the pancake graph was proposed. Besides, an algorithm that can exchange the contents of any two sub-pancake graphs with constant time was also proposed. In [27], the pancake graph was shown to be super connected, i.e., for all $1 \leq k \leq n-1$, there are k node-disjoint paths between every pair of nodes so that they contain all the nodes of the pancake graph. In [5], the problem of job allocation and job migration on the pancake graph was studied.

An *embedding* of one (guest) graph G into another (host) graph H is a mapping f from the node set of G to the node set of H [26]. Each edge of G corresponds to a path of H under f. The *dilation* of f is the maximal length of the paths in H that are the images of edges in G under f. The *congestion* of f is the maximum number of edges in G whose corresponding paths in H contain the same edge in G. The *load* of f is the maximum number of vertices in G that are mapped to the same vertex in H.

In [11], a complete binary tree of height $\sum_{m=2}^{n} \lfloor \log m \rfloor$ was embedded into the pancake graph with load 1, congestion 1, and dilation 2. When the complete binary tree has up to about 2/3 faulty nodes, they can be replaced with non-faulty ones

* Corresponding author. Tel.: +886 37381527; fax: +886 37 362809.
E-mail address: jsfu@nuu.edu.tw (J.-S. Fu).

0304-3975/$ – see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.tcs.2008.09.015

of distance four at most apart. In [13], a particular embedding of the hypercube into the pancake graph was investigated, where each node of the hypercube was mapped to a set of nodes of the pancake graph.

The ring network is one of the most fundamental networks for parallel and distributed computing. Many simple and efficient algorithms were developed before on ring networks for solving various algebraic problems and graph problems (see [2,26]). If a network can embed ring networks, then it can execute these ring algorithms as well. In [22,24], the pancake graph was shown to contain cycles of all possible lengths.

Since faults may occur to networks, it is significant to consider faulty networks. Many fundamental problems such as diameter [9,25], routing [4], broadcasting [31], gossiping [10], and embedding [3,6,7,17–19] were solved on various faulty networks. Two fault models were adopted before. One is the *random fault model* [4,7,9,10,17–19,31], which assumes that the faults may occur anywhere without any restriction. The other is the *conditional fault model* [3,6,25], which assumes that the fault distribution is subject to some constraints, e.g., two or more non-faulty links incident to each node.

In this paper, adopting the conditional fault model and assuming that two or more non-faulty links are incident to each node, we show that a pancake graph with up to $2n-7$ link faults contains a fault-free Hamiltonian cycle, where $n \geq 4$ is the dimension of the pancake graph. That is, we show an embedding of a ring network into a faulty pancake graph with load 1, congestion 1, and dilation 1, where the ring network has the same number of nodes as the faulty pancake graph. If the random fault model is adopted, the pancake graph can tolerate at most $n-3$ link faults, while containing a fault-free Hamiltonian cycle (see [22]). No previous work on the pancake graph considered the conditional fault model.

The following results are all relating to cycle embedding or path embedding under the same fault model and assumption as ours, where n is the dimension of the mentioned network. The hypercube (star graph) with up to $2n-5$ $(2n-7)$ link faults contains a fault-free longest path between every pair of nodes [29] ([30]). The m-ary hypercube (star graph, alternating group graph, locally twisted cube, and crossed cube, respectively) with up to $4n-5$ $(2n-7, 4n-13, 2n-5$, and $2n-5$, respectively) link faults contains a fault-free Hamiltonian cycle [3] ([12,30,20,23], respectively), where $m \geq 3$. Further, in [21], sufficient conditions for establishing fault-free Hamiltonian cycles in matching composition networks were proposed, where instances of matching composition networks include crossed cubes, twisted cubes, locally twisted cubes, and generalized twisted cubes. All these embedding results above have load 1, congestion 1, and dilation 1.

In the next section, the topology of the pancake graph is reviewed. Some necessary definitions, notations and properties are also introduced. The main result, i.e., a fault-free Hamiltonian cycle in a pancake graph with up to $2n-7$ link faults under the conditional fault model and our assumption, is shown in Section 3. Finally, this paper concludes with some remarks in Section 4.

2. Preliminaries

It is convenient to represent a network with a graph G, where each vertex (edge) of G uniquely represents a node (link) of the network. We use $V(G)$ and $E(G)$ to denote the vertex set and edge set of G, respectively. Throughout this paper, we use network and graph, node and vertex, and link and edge, interchangeably. The following is a formal definition of the pancake graph.

Definition 1. An n-dimensional pancake graph, denote by $\wp_n$, has the vertex set $V(\wp_n) = \{a_1a_2\ldots a_n | a_1a_2\ldots a_n$ is a permutation of $1, 2, \ldots, n\}$ and edge set $E(\wp_n) = \{(a_1a_2\ldots a_n, b_1b_2\ldots b_n) | a_1a_2\ldots a_k = b_kb_{k-1}\ldots b_1$ and $a_{k+1}a_{k+2}\ldots a_n = b_{k+1}b_{k+2}\ldots b_n$ for some $2 \leq k \leq n\}$.

Clearly, $\wp_n$ has $n!$ vertices and is regular of degree $n-1$. $\wp_1$ is a vertex, $\wp_2$ is an edge, and $\wp_3$ is a cycle of length six. Fig. 1 illustrates the topologies of $\wp_3$ and $\wp_4$. Intuitively, if $(a_1a_2\ldots a_n, b_1b_2\ldots b_n) \in E(\wp_n)$, then there exists $2 \leq k \leq n$ so that $a_1a_2\ldots a_n$ and $b_1b_2\ldots b_n$ can be obtained from each other by reversing the leftmost k bits. When $b_1b_2\ldots b_n = a_ka_{k-1}\ldots a_1a_{k+1}a_{k+2}\ldots a_n$, $(a_1a_2\ldots a_n, b_1b_2\cdots b_n)$ is referred to as a *k-dimensional edge*. We use $N^{(k)}(u)$ to denote the neighbor of a vertex $u \in V(\wp_n)$ that is connected to u by a k-dimensional edge, and $E^{(k)}(\wp_n)$ to denote the set of all k-dimensional edges in $\wp_n$. For example, refer to Fig. 1 again, where $N^{(4)}(2143) = 3412$ and $E^{(2)}(\wp_3) = \{(123, 213), (321, 231), (132, 312)\}$.

For convenience, we use $\langle u\rangle_i$ to denote the ith leftmost digit of a vertex u, i.e., $\langle u\rangle_i = u_i$ if $u = u_1u_2\ldots u_n$, where $1 \leq i \leq n$. It is not difficult to see that the subgraph of $\wp_n$ induced by the set of vertices u with $\langle u\rangle_n = r$ forms a $\wp_{n-1}$, denoted by $\wp_n^{(r)}$, where $1 \leq r \leq n$. Thus, $\wp_n$ contains n $\wp_{n-1}$'s, i.e., $\wp_n^{(1)}, \wp_n^{(2)}, \ldots, \wp_n^{(n)}$. Fig. 1 illustrates $\wp_4^{(1)}$, $\wp_4^{(2)}$, $\wp_4^{(3)}$, and $\wp_4^{(4)}$ that are embedded in $\wp_4$. We use $\tilde{E}_{p,q}(\wp_n)$ to denote the set of those n-dimensional edges in $\wp_n$ that connect $\wp_n^{(p)}$ and $\wp_n^{(q)}$, where $p \neq q$. For $I \subseteq \{1, 2, \ldots, n\}$, we use $\wp_n^I$ to denote the subgraph of $\wp_n$ induced by $\bigcup_{r\in I} V(\wp_n^{(r)})$. Pancake graphs are vertex symmetric, but not edge symmetric (see [24]).

A path (cycle) in G is called a *Hamiltonian path* (*cycle*) if it contains every vertex of G exactly once. G is *Hamiltonian* if it has a Hamiltonian cycle, and *Hamiltonian-connected* if it has a Hamiltonian path between every two distinct vertices. $\wp_n$ is known to be Hamiltonian for $n \geq 3$ (see [24]) and Hamiltonian-connected for $n \geq 4$ (see [22]). In the rest of this paper, we let $F(\subseteq E(\wp_n))$ denote a set of edge faults in $\wp_n$, $\deg(u)$ denote the *degree* of a vertex u, which is the number of edges incident to u, and $\delta(G) = \min\{\deg(u) | u \in V(G)\}$ be the minimal vertex degree of G. Moreover, we use P_{v_0,v_t} to denote a path from vertex v_0 to vertex v_t, and $P^{(H)}_{v_0,v_t}$ to denote a Hamiltonian path from vertex v_0 to vertex v_t in the mentioned network.

The following are some properties of $\wp_n$ that are necessary in order to prove our main result in the next section.

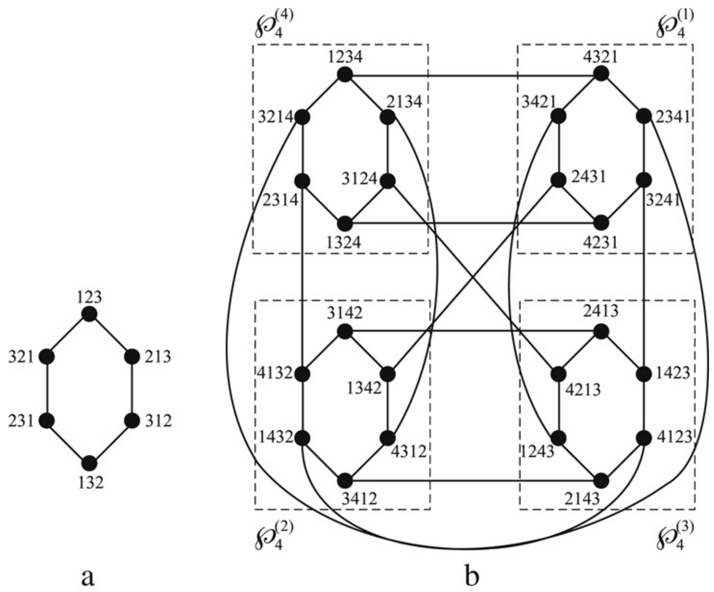

Fig. 1. Topologies of (a) $\wp_3$ and (b) $\wp_4$.

Lemma 1 ([22]). $|\tilde{E}_{p,q}(\wp_n)| = (n-2)!$ *for all* $p, q \in \{1, 2, \ldots, n\}$ *and* $p \neq q$*, where* $n \geq 3$.

Lemma 2 ([22]). $\wp_n - F$ *is Hamiltonian if* $|F| \leq n-3$*, and Hamiltonian-connected if* $|F| \leq n-4$*, where* $n \geq 4$.

Lemma 3 ([22]). *Suppose that* $u, v \in V(\wp_n)$ *and* $\langle u\rangle_n \neq \langle v\rangle_n$*, where* $n \geq 5$*. For any* $I \subseteq \{1, 2, \ldots, n\}$ *and* $|I| \geq 2$*, there exists a* $P^{(H)}_{u,v}$ *in* $\wp^I_n - F$ *provided the following two conditions hold:*
(C1) $|\tilde{E}_{i,j}(\wp_n) - F| \geq 3$ *for all* $i, j \in I$ *and* $i \neq j$;
(C2) $\wp^{(r)}_n - F$ *is Hamiltonian-connected for all* $r \in I$.

Lemma 4 ([22]). *Suppose that* $u, v \in V(\wp^{(r)}_n)$ *and* $u \neq v$*, where* $r \in \{1, 2, \ldots, n\}$ *and* $n \geq 4$*. If* $d_{u,v} \leq 2$*, then* $\langle N^{(n)}(u)\rangle_n \neq \langle N^{(n)}(v)\rangle_n$*, where* $d_{u,v}$ *is the distance between* u *and* v.

Lemma 5. *Suppose that* $e_1, e_2 \in E(\wp_4)$ *and* $e_1 \neq e_2$*. There exists a Hamiltonian cycle in* $\wp_4 - \{e_2\}$ *that contains* e_1.

Proof. Now that $\wp_4$ is vertex symmetric, we assume that e_2 is incident to vertex 1234, without loss of generality. When $e_2 = (1234, 2134)$, there are three Hamiltonian cycles in $\wp_4$ that contain all other edges of $\wp_4$. When $e_2 = (1234, 3214)$ or $(1234, 4321)$, there are four Hamiltonian cycles in $\wp_4$ that contain all other edges of $\wp_4$. These eleven Hamiltonian cycles are listed in Appendix. □

Lemma 6. *Suppose that* $s, t \in V(\wp_n)$*,* $s \neq t$*, and* $\langle s\rangle_1 = \langle t\rangle_1$*, where* $n \geq 4$*. For every* $(x, y) \in E(\wp_n)$ *with* $\{x, y\} \cap \{s, t\} = \varnothing$*, there exists a* $P^{(H)}_{s,t}$ *in* $\wp_n$ *that contains* (x, y).

Proof. We prove this lemma by induction on n. This lemma holds for $\wp_4$, which can be verified by the aid of a computer program (see [32]). Then, supposing that this lemma holds for $\wp_k$, we construct a $P^{(H)}_{s,t}$ in $\wp_{k+1}$ that contains (x, y) in the rest of the proof, where $k \geq 4$. Notice that the two conditions (C1) and (C2) of Lemma 3 are satisfied for $\wp_{k+1}$ (let $n = k+1$ and $F = \varnothing$), as a consequence of Lemmas 1 and 2. We first consider the situation of $(x, y) \notin E^{(k+1)}(\wp_{k+1})$. So, we have x, $y \in V(\wp^{(\alpha)}_{k+1})$ for some $1 \leq \alpha \leq k+1$. Two cases are discussed below.

Case 1. $\langle s\rangle_{k+1} \neq \langle t\rangle_{k+1}$. When $\langle s\rangle_{k+1} = \alpha$ or $\langle t\rangle_{k+1} = \alpha$, a desired $P^{(H)}_{s,t}$ in $\wp_{k+1}$ is shown in Fig. 2(a), where $\langle s\rangle_{k+1} = \alpha$ and $I = \{1, 2, \ldots, k+1\} - \{\alpha\}$ are assumed. Two vertices $v_1 \in V(\wp^{(\alpha)}_{k+1}) - \{s, x, y\}$ and $u_2 \in V(\wp^I_{k+1})$ with $\langle v_1\rangle_1 = \langle s\rangle_1$, $\langle u_2\rangle_{k+1} \neq \langle t\rangle_{k+1}$, and $(v_1, u_2) \in E^{(k+1)}(\wp_{k+1})$ can be found. The induction hypothesis assures a $P^{(H)}_{s,v_1}$ in $\wp^{(\alpha)}_{k+1}$ that contains (x, y). Lemma 3 assures a $P^{(H)}_{u_2,t}$ in $\wp^I_{k+1}$. The two Hamiltonian paths together with (v_1, u_2) form a desired $P^{(H)}_{s,t}$ in $\wp_{k+1}$.

When $\langle s\rangle_{k+1} \neq \alpha$ and $\langle t\rangle_{k+1} \neq \alpha$, a desired $P^{(H)}_{s,t}$ in $\wp_{k+1}$ is shown in Fig. 2(b), where $\langle s\rangle_{k+1} = \beta$ and $I = \{1, 2, \ldots, k+1\} - \{\alpha, \beta, \gamma\}$ are assumed. An edge $(z, w) \in E(\wp^{(\gamma)}_{k+1})$ is first determined such that two edges (v_1, z), $(w, u_4) \in E^{(k+1)}(\wp_{k+1})$ exist with $v_1 \in V(\wp^{(\beta)}_{k+1}) - \{s\}$, $u_4 \in V(\wp^I_{k+1})$, and $\langle u_4\rangle_{k+1} \neq \langle t\rangle_{k+1}$. Then, as a consequence of Lemma 1, there exist $(v_2, u_3), (u_2, v_3) \in E^{(k+1)}(\wp_{k+1})$ with $u_2, v_2 \in V(\wp^{(\gamma)}_{k+1}) - \{z, w\}$ and $u_3, v_3 \in V(\wp^{(\alpha)}_{k+1}) - \{x, y\}$. There exists a $P^{(H)}_{s,v_1}$ in $\wp^{(\beta)}_{k+1}$. Since $\langle u_2\rangle_1 = \langle v_3\rangle_{k+1} = \langle u_3\rangle_{k+1} = \langle v_2\rangle_1$ and $\langle u_3\rangle_1 = \langle v_2\rangle_{k+1} = \langle u_2\rangle_{k+1} = \langle v_3\rangle_1$, the induction hypothesis assures

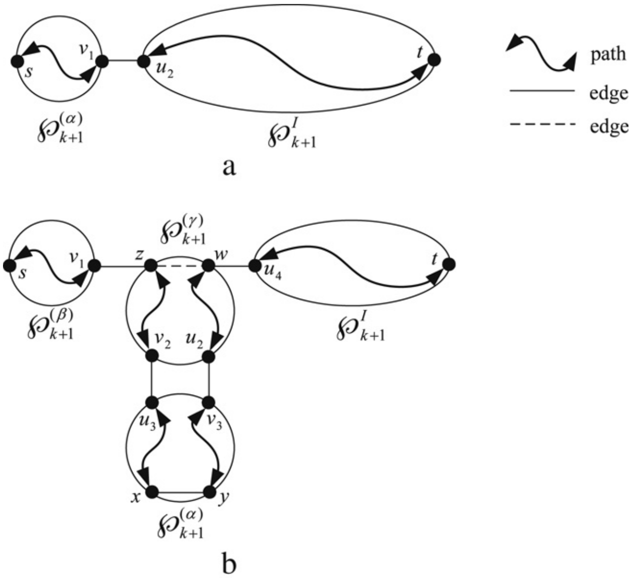

Fig. 2. A $P_{s,t}^{(H)}$ in $\wp_{k+1}$ that contains (x, y) when $(x, y) \notin E^{(k+1)}(\wp_{k+1})$ and $\langle s\rangle_{k+1} \neq \langle t\rangle_{k+1}$. (a) $\langle s\rangle_{k+1} = \alpha$. (b) $\langle s\rangle_{k+1} \neq \alpha$ and $\langle t\rangle_{k+1} \neq \alpha$.

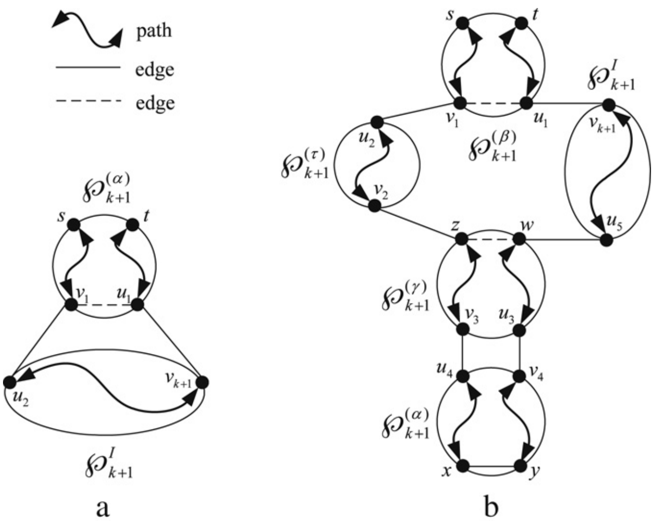

Fig. 3. A $P_{s,t}^{(H)}$ in $\wp_{k+1}$ that contains (x, y) when $(x, y) \notin E^{(k+1)}(\wp_{k+1})$. (a) $\langle s\rangle_{k+1} = \langle t\rangle_{k+1} = \alpha$. (b) $\langle s\rangle_{k+1} = \langle t\rangle_{k+1} \neq \alpha$.

a $P_{u_2,v_2}^{(H)}$ in $\wp_{k+1}^{(\gamma)}$ that contains (z, w) and a $P_{u_3,v_3}^{(H)}$ in $\wp_{k+1}^{(\alpha)}$ that contains (x, y). By Lemma 3, there exists a $P_{u_4,t}^{(H)}$ in $\wp_{k+1}^{I}$. The four Hamiltonian paths, breaking (z, w), together with (v_1, z), (w, u_4), (v_2, u_3) and (u_2, v_3) form a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$.

Case 2. $\langle s\rangle_{k+1} = \langle t\rangle_{k+1}$. When $\langle s\rangle_{k+1} = \langle t\rangle_{k+1} = \alpha$, a desired $P_{s,t}^{(H)}$ is shown in Fig. 3(a), where $I = \{1, 2, \ldots, k+1\} - \{\alpha\}$. The induction hypothesis assures a $P_{s,t}^{(H)}$ in $\wp_{k+1}^{(\alpha)}$ that contains (x, y). An edge $(v_1, u_1) \neq (x, y)$ can be determined from the $P_{s,t}^{(H)}$ such that there exist $(v_1, u_2), (u_1, v_{k+1}) \in E^{(k+1)}(\wp_{k+1})$ with $\langle u_2\rangle_{k+1} \neq \langle v_{k+1}\rangle_{k+1}$ (assured by Lemma 4). By Lemma 3, there exists a $P_{u_2,v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I}$. Thus, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ can result.

When $\langle s\rangle_{k+1} = \langle t\rangle_{k+1} \neq \alpha$, a desired $P_{s,t}^{(H)}$ is shown in Fig. 3(b), where $\langle s\rangle_{k+1} = \langle t\rangle_{k+1} = \beta$ and $I = \{1, 2, \ldots, k+1\} - \{\alpha, \beta, \gamma, \tau\}$($|I| = 1$ as $k = 4$) are assumed. An edge $(v_1, u_1) \in E(\wp_{k+1}^{(\beta)})$ with $\{v_1, u_1\} \cap \{s, t\} = \emptyset$ is first determined such that there exist $(v_1, u_2), (u_1, v_{k+1}) \in E^{(k+1)}(\wp_{k+1})$ with $u_2 \in V(\wp_{k+1}^{(\tau)})$ and $v_{k+1} \in V(\wp_{k+1}^{I})$. Then, since $|E(\wp_{k+1}^{(\gamma)})| = (k-1)k!/2$, an edge $(z, w) \in E(\wp_{k+1}^{(\gamma)})$ can be found such that there exist $(z, v_2), (w, u_5) \in E^{(k+1)}(\wp_{k+1})$ with $v_2 \in V(\wp_{k+1}^{(\tau)})$, $u_5 \in V(\wp_{k+1}^{I})$, and $\langle u_5\rangle_{k+1} \neq \langle v_{k+1}\rangle_{k+1}$ if $k \geq 5$($\langle u_5\rangle_{k+1} = \langle v_{k+1}\rangle_{k+1}$ if $k = 4$). Further, as a consequence of Lemma 1, there exist $(v_3, u_4), (u_3, v_4) \in E^{(k+1)}(\wp_{k+1})$ with $u_3, v_3 \in V(\wp_{k+1}^{(\gamma)})$ and $u_4, v_4 \in V(\wp_{k+1}^{(\alpha)}) - \{x, y\}$. Moreover, we have

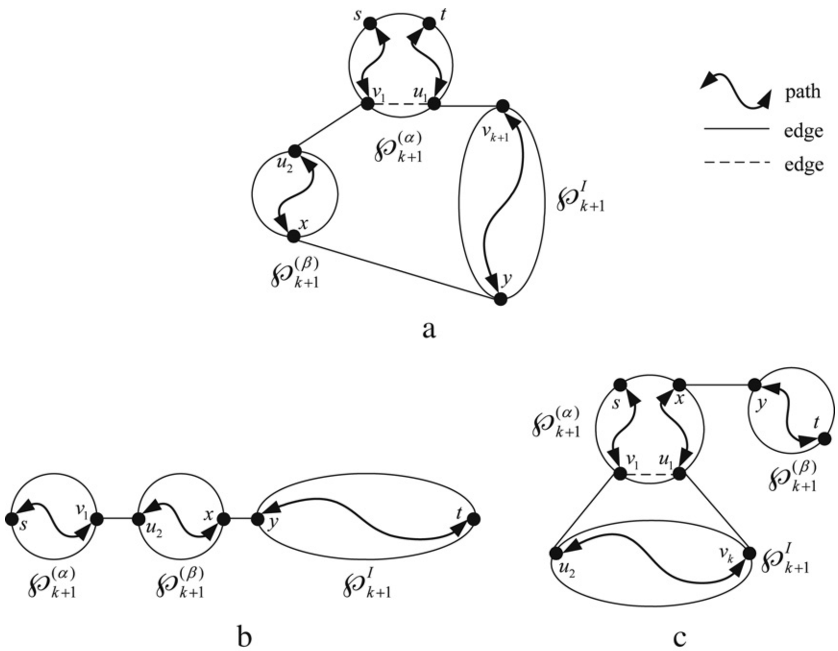

Fig. 4. A $P_{s,t}^{(H)}$ in $\wp_{k+1}$ that contains (x, y) when $(x, y) \in E^{(k+1)}(\wp_{k+1})$. (a) $\langle s\rangle_{k+1} = \langle t\rangle_{k+1}$. (b) $\langle s\rangle_{k+1} \neq \langle t\rangle_{k+1}$ and $\{\langle x\rangle_{k+1}, \langle y\rangle_{k+1}\} \cap \{\langle s\rangle_{k+1}, \langle t\rangle_{k+1}\} = \emptyset$. (c) $\langle s\rangle_{k+1} \neq \langle t\rangle_{k+1}$ and $\{\langle x\rangle_{k+1}, \langle y\rangle_{k+1}\} = \{\langle s\rangle_{k+1}, \langle t\rangle_{k+1}\}$.

$v_2 \neq u_2$, because $N^{(k+1)}(v_2)(= z) \in E(\wp_{k+1}^{(\gamma)})$ and $N^{(k+1)}(u_2)(= v_1) \in V(\wp_{k+1}^{(\beta)})$, and $\{u_3, v_3\} \cap \{z, w\} = \emptyset$, because $N^{(k+1)}(z)(= v_2) \in V(\wp_{k+1}^{(\tau)})$ and $N^{(k+1)}(w)(= u_5) \in V(\wp_{k+1}^{I})$.

A $P_{s,t}^{(H)}$ in $\wp_{k+1}^{(\beta)}$, a $P_{u_3,v_3}^{(H)}$ in $\wp_{k+1}^{(\gamma)}$, and a $P_{u_4,v_4}^{(H)}$ in $\wp_{k+1}^{(\alpha)}$ that contain (u_1, v_1), (z, w), and (x, y), respectively, can be obtained, similarly, by the induction hypothesis. There exists a $P_{u_2,v_2}^{(H)}$ in $\wp_{k+1}^{(\tau)}$. There exists a $P_{u_5,v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I}$, either when $k = 4$ or when $k \geq 5$ (assured by Lemma 3). Thus, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ can result.

Next we consider the situation of $(x, y) \in E^{(k+1)}(\wp_{k+1})$. So we have $\langle x\rangle_{k+1} \neq \langle y\rangle_{k+1}$. Two cases are discussed below.

Case 1. $\langle s\rangle_{k+1} = \langle t\rangle_{k+1}$. Assume $\langle s\rangle_{k+1} = \langle t\rangle_{k+1} = \alpha$. When $\langle x\rangle_{k+1} = \alpha$ or $\langle y\rangle_{k+1} = \alpha$, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ can be obtained, by a construction method similar to Fig. 3(a). When $\langle x\rangle_{k+1} \neq \alpha$ and $\langle y\rangle_{k+1} \neq \alpha$, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ is shown in Fig. 4(a), where $\langle x\rangle_{k+1} = \beta$ and $\langle y\rangle_{k+1} \in I = \{1, 2, \ldots, k+1\} - \{\alpha, \beta\}$ are assumed. The two edges (v_1, u_2), (u_1, v_{k+1}) with $\langle v_{k+1}\rangle_{k+1} \neq \langle y\rangle_{k+1}$ can be determined as before (see Fig. 3). The induction hypothesis assures a $P_{s,t}^{(H)}$ in $\wp_{k+1}^{(\alpha)}$ that contains (v_1, u_1). There exists a $P_{u_2,x}^{(H)}$ in $\wp_{k+1}^{(\beta)}$. Lemma 3 assures a $P_{y,v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I}$. Thus, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ can result.

Case 2. $\langle s\rangle_{k+1} \neq \langle t\rangle_{k+1}$. When $\{\langle x\rangle_{k+1}, \langle y\rangle_{k+1}\} \cap \{\langle s\rangle_{k+1}, \langle t\rangle_{k+1}\} = \emptyset$, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ is shown in Fig. 4(b), where $\langle s\rangle_{k+1} = \alpha$, $\langle x\rangle_{k+1} = \beta$, and $\langle t\rangle_{k+1}, \langle y\rangle_{k+1} \in I = \{1, 2, \ldots, k+1\} - \{\alpha, \beta\}$ are assumed. The edge (v_1, u_2) with $v_1 \neq s$ and $u_2 \neq x$ can be determined as before (see Fig. 2(a)). There exist a $P_{s,v_1}^{(H)}$ in $\wp_{k+1}^{(\alpha)}$ and a $P_{u_2,x}^{(H)}$ in $\wp_{k+1}^{(\beta)}$. Lemma 3 assures a $P_{y,t}^{(H)}$ in $\wp_{k+1}^{I}$. Thus, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ can result.

When $\{\langle x\rangle_{k+1}, \langle y\rangle_{k+1}\} = \{\langle s\rangle_{k+1}, \langle t\rangle_{k+1}\}$, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ is shown in Fig. 4(c), where $\langle x\rangle_{k+1} = \langle s\rangle_{k+1} = \alpha$, $\langle y\rangle_{k+1} = \langle t\rangle_{k+1} = \beta$, and $I = \{1, 2, \ldots, k+1\} - \{\alpha, \beta\}$ are assumed. There exist a $P_{s,x}^{(H)}$ in $\wp_{k+1}^{(\alpha)}$ and a $P_{y,t}^{(H)}$ in $\wp_{k+1}^{(\beta)}$. The two edges (v_1, u_2), (u_1, v_k) with $\langle u_2\rangle_{k+1} \neq \langle v_k\rangle_{k+1}$ can be determined as before (see Fig. 3(a)). Lemma 3 assures a $P_{u_2,v_k}^{(H)}$ in $\wp_{k+1}^{I}$. Thus, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ can result.

When $\langle x\rangle_{k+1} = \langle s\rangle_{k+1}$ or $\langle y\rangle_{k+1} = \langle s\rangle_{k+1}$, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ can be obtained, by a construction method similar to Fig. 2(a). When $\langle x\rangle_{k+1} = \langle t\rangle_{k+1}$ or $\langle y\rangle_{k+1} = \langle t\rangle_{k+1}$, a desired $P_{s,t}^{(H)}$ in $\wp_{k+1}$ can be obtained similarly. □

The main result of this paper is stated as the following theorem whose proof is shown in the next section.

Theorem 1. *$\wp_n - F$ is Hamiltonian provided $|F| \leq 2n - 7$ and $\delta(\wp_n - F) \geq 2$, where $n \geq 4$. That is, a ring network of size $n!$ can be embedded into $\wp_n - F$ with load* 1*, congestion* 1*, and dilation* 1.

3. Proof of Theorem 1

We prove Theorem 1 by induction on n. The theorem holds for $\wp_4$, which is assured by Lemma 2 ($2n - 7 = n - 3$ as $n = 4$). Then, supposing that the theorem holds for $\wp_k$, we construct a Hamiltonian cycle in $\wp_{k+1} - F$, where $k \geq 4$ and

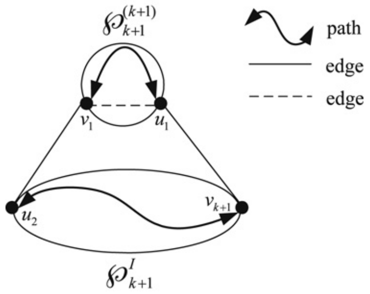

Fig. 5. A Hamiltonian cycle in $\wp_{k+1} - F$ when $|E(\wp_{k+1}^{(k+1)}) \cap F| \le k-4$.

$|F| \le 2(k+1) - 7 = 2k - 5$. Without loss of generality, we assume $|E(\wp_{k+1}^{(k+1)}) \cap F| \ge |E(\wp_{k+1}^{(k)}) \cap F| \ge \cdots \ge |E(\wp_{k+1}^{(1)}) \cap F|$. By Lemma 1, we have $|\tilde{E}_{p,q}(\wp_{k+1})| = (k-1)! \ge 2k-2 \ge |F|+3$, i.e., $|\tilde{E}_{p,q}(\wp_{k+1}) - F| \ge 3$ for all $p, q \in \{1, 2, \ldots, k+1\}$ and $p \ne q$. Four cases are discussed below, according to the value of $|E(\wp_{k+1}^{(k+1)}) \cap F|$.

3.1. When $|E(\wp_{k+1}^{(k+1)}) \cap F| \le k-4$

We have $|E(\wp_{k+1}^{(i)}) \cap F| \le k-4$ and $\delta(\wp_{k+1}^{(i)} - F) \ge 3$ for all $1 \le i \le k+1$. A Hamiltonian cycle in $\wp_{k+1} - F$ is shown in Fig. 5, where $I = \{1, 2, \ldots, k\}$. The induction hypothesis assures a Hamiltonian cycle in $\wp_{k+1}^{(k+1)} - F$. An edge (u_1, v_1) can be determined from the Hamiltonian cycle such that there exist $(v_1, u_2), (u_1, v_{k+1}) \in E^{(k+1)}(\wp_{k+1}) - F$ with $u_2, v_{k+1} \in V(\wp_{k+1}^{I})$. If (u_1, v_1) does not exist, then $|E^{(k+1)}(\wp_{k+1}) \cap F| \ge k!/2 > 2k-5 \ge |F|$, a contradiction. Lemma 2 assures that $\wp_{k+1}^{(j)} - F$ is Hamiltonian-connected for all $1 \le j \le k$. By Lemma 4, we have $\langle u_2 \rangle_{k+1} \ne \langle v_{k+1} \rangle_{k+1}$. Then, by Lemma 3, a $P_{u_2, v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I} - F$ exists.

3.2. When $k-3 \le |E(\wp_{k+1}^{(k+1)}) \cap F| \le 2k-7$

Since $\delta(\wp_{k+1} - F) \ge 2$, we have $\delta(\wp_{k+1}^{(k+1)} - F) \ge 1$. Two cases are further discussed, according to the values of $|E(\wp_{k+1}^{(k)}) \cap F|$.

Case 1. $|E(\wp_{k+1}^{(k)}) \cap F| \le k-4$. We have $|E(\wp_{k+1}^{(j)}) \cap F| \le k-4$ for all $1 \le j \le k$. When $\delta(\wp_{k+1}^{(k+1)} - F) \ge 2$, a Hamiltonian cycle in $\wp_{k+1} - F$ can be obtained by the construction method of Fig. 5, where $I = \{1, 2, \ldots, k\}$. When $\delta(\wp_{k+1}^{(k+1)} - F) = 1$, a Hamiltonian cycle in $\wp_{k+1} - F$ can be obtained by slightly modifying the construction method of Fig. 5, as detailed below.

There exists a unique vertex, say v_1, with $\deg(v_1) = 1$ in $\wp_{k+1}^{(k+1)} - F$, for otherwise $|E(\wp_{k+1}^{(k+1)}) \cap F| \ge 2k-5$, a contradiction. Since $\delta(\wp_{k+1} - F) \ge 2$, there exists $(v_1, u_2) \in E^{(k+1)}(\wp_{k+1}) - F$, where $u_2 \in V(\wp_{k+1}^{I})$. An edge $(v_1, u_1) \in E(\wp_{k+1}^{(k+1)}) \cap F$ can be found such that there exists $(u_1, v_{k+1}) \in E^{(k+1)}(\wp_{k+1}) - F$, where $v_{k+1} \in V(\wp_{k+1}^{I})$. Since $\delta(\wp_{k+1}^{(k+1)} - (F - \{(v_1, u_1)\})) = 2$ and $|E(\wp_{k+1}^{(k+1)}) \cap (F - \{(v_1, u_1)\})| \le 2k-8$, the induction hypothesis assures a Hamiltonian cycle in $\wp_{k+1}^{(k+1)} - (F - \{(v_1, u_1)\})$. Since the Hamiltonian cycle contains (v_1, u_1), there exists a $P_{u_1, v_1}^{(H)}$ in $\wp_{k+1}^{(k+1)} - F$. With the same arguments as Fig. 5, there exists a $P_{u_2, v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I} - F$.

Case 2. $|E(\wp_{k+1}^{(k)}) \cap F| \ge k-3$. We have $|E(\wp_{k+1}^{(k+1)}) \cap F| = k-3$ or $k-2$, $|E(\wp_{k+1}^{(k)}) \cap F| = k-3$ (hence, $\delta(\wp_{k+1}^{(k)} - F) \ge 2$), and $|E^{(k+1)}(\wp_{k+1}) \cap F| \le 1$. First we consider the situation of $|E(\wp_{k+1}^{(k-1)}) \cap F| = k-3$, which occurs only when $k = 4$. We have $|E(\wp_5^{(5)}) \cap F| = |E(\wp_5^{(4)}) \cap F| = |E(\wp_5^{(3)}) \cap F| = 1$, $|E(\wp_5^{(2)}) \cap F| = |E(\wp_5^{(1)}) \cap F| = 0$, and $|E^{(5)}(\wp_5) \cap F| = 0$. A Hamiltonian cycle in $\wp_5 - F$ is shown in Fig. 6. Three edges $(u_4, v_5) \in E^{(5)}(\wp_5)$, $(u_4, v_4) \in E(\wp_5^{(4)}) - F$, and $(u_5, v_5) \in E(\wp_5^{(5)}) - F$ can be found such that there exist $(u_5, v_1), (v_4, u_2) \in E^{(5)}(\wp_5)$, where $v_1 \in V(\wp_5^{(1)})$ and $u_2 \in V(\wp_5^{(2)})$. And an edge $(u_3, v_3) \in E(\wp_5^{(3)}) - F$ can be found such that there exist $(u_3, v_2), (v_3, u_1) \in E^{(5)}(\wp_5)$, where $v_2 \in V(\wp_5^{(2)})$ and $u_1 \in V(\wp_5^{(1)})$. There exist a $P_{u_1, v_1}^{(H)}$ in $\wp_5^{(1)}$ and a $P_{u_2, v_2}^{(H)}$ in $\wp_5^{(2)}$. Lemma 5 assures a Hamiltonian cycle in $\wp_5^{(3)} - F$, a Hamiltonian cycle in $\wp_5^{(4)} - F$, and a Hamiltonian cycle in $\wp_5^{(5)} - F$ that contains (u_3, v_3), (u_4, v_4), and (u_5, v_5), respectively.

Then we consider the situation of $|E(\wp_{k+1}^{(k-1)}) \cap F| \le k-4$. We have $|E(\wp_{k+1}^{(i)}) \cap F| \le k-4$ for all $1 \le i \le k-1$. When $\delta(\wp_{k+1}^{(k+1)} - F) \ge 2$, a Hamiltonian cycle in $\wp_{k+1} - F$ is shown in Fig. 7, where $I = \{1, 2, \ldots, k-1\}$. An edge $(u_2, v_1) \in E^{(k+1)}(\wp_{k+1}) - F$ can be found, where $u_2 \in V(\wp_{k+1}^{(k)})$ and $v_1 \in V(\wp_{k+1}^{(k+1)})$. The induction hypothesis assures two Hamiltonian cycles in $\wp_{k+1}^{(k)} - F$ and $\wp_{k+1}^{(k+1)} - F$, respectively. By the aid of Lemma 4, two edges $(v_1, u_1), (u_2, v_2)$ can be determined from the two Hamiltonian cycles, respectively, such that there exist $(v_2, u_3), (u_1, v_{k+1}) \in E^{(k+1)}(\wp_{k+1}) - F$, where $u_3, v_{k+1} \in V(\wp_{k+1}^{I})$ and $\langle u_3 \rangle_{k+1} \ne \langle v_{k+1} \rangle_{k+1}$. With the same arguments as Fig. 5, there exists a $P_{u_3, v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I} - F$.

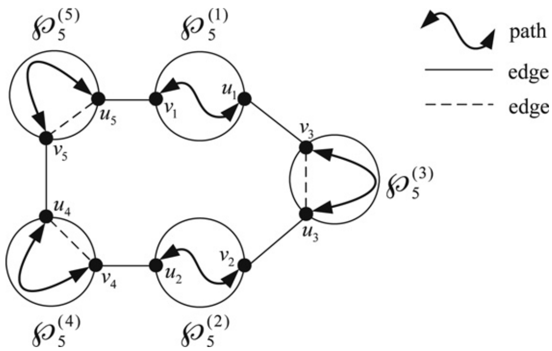

Fig. 6. A Hamiltonian cycle in $\wp_5 - F$ when $2k-7 \geq |E(\wp_{k+1}^{(k+1)}) \cap F| \geq |E(\wp_{k+1}^{(k)}) \cap F| \geq |E(\wp_{k+1}^{(k-1)}) \cap F| = k-3$.

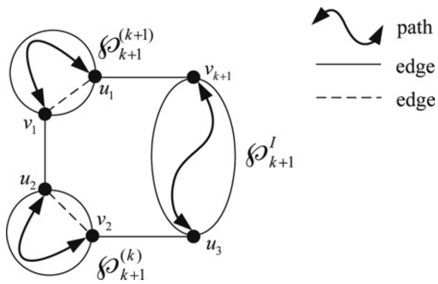

Fig. 7. A Hamiltonian cycle in $\wp_{k+1} - F$ when $2k-7 \geq |E(\wp_{k+1}^{(k+1)}) \cap F| \geq |E(\wp_{k+1}^{(k)}) \cap F| \geq k-3$, $|E(\wp_{k+1}^{(k-1)}) \cap F| \leq k-4$, and $\delta(\wp_{k+1}^{(k+1)} - F) \geq 2$.

When $\delta(\wp_{k+1}^{(k+1)} - F) = 1$, we have $|E(\wp_{k+1}^{(k+1)}) \cap F| = k-2$, $|E(\wp_{k+1}^{(i)}) \cap F| = 0$ for all $1 \leq i \leq k-1$, and $|E^{(k+1)}(\wp_{k+1}) \cap F| = 0$. There exists a unique vertex, say v_1, with $\deg(v_1) = 1$ in $\wp_{k+1}^{(k+1)} - F$. An edge $(v_1, u_1) \in E(\wp_{k+1}^{(k+1)}) \cap F$ can be found. Similar to Case 1, there exists a Hamiltonian cycle in $\wp_{k+1}^{(k+1)} - (F - \{(v_1, u_1)\})$ that contains (v_1, u_1). If $N^{(k+1)}(v_1) \in V(\wp_{k+1}^{(k)})$ or $N^{(k+1)}(u_1) \in V(\wp_{k+1}^{(k)})$, a Hamiltonian cycle in $\wp_{k+1} - F$ can be obtained by a construction method similar to Fig. 7.

If $N^{(k+1)}(v_1), N^{(k+1)}(u_1) \notin V(\wp_{k+1}^{(k)})$, a Hamiltonian cycle in $\wp_{k+1} - F$, as shown in Fig. 8(a) or (b), can be obtained, where $\alpha, \beta, \gamma, \tau \in \{1, 2, \ldots, k-1\}$. We assume $N^{(k+1)}(v_1) = u_2 \in V(\wp_{k+1}^{(\alpha)})$ and $N^{(k+1)}(u_1) = v_{k+1} \in V(\wp_{k+1}^{(\beta)})$, where $\alpha \neq \beta$. Lemma 2 assures a Hamiltonian cycle in $\wp_{k+1}^{(k)} - F$. An edge (v_4, u_4) of the Hamiltonian cycle can be found with $N^{(k+1)}(v_4) \in V(\wp_{k+1}^{(\gamma)})$ and $N^{(k+1)}(u_4) \in V(\wp_{k+1}^{(\tau)})$, where $\gamma \in \{1, 2, \ldots, k-1\} - \{\alpha, \beta\}$, $\tau \in \{1, 2, \ldots, k-1\}$, and $\gamma \neq \tau$ (assured by Lemma 4).

If $\tau \notin \{\alpha, \beta\}$, the Hamiltonian cycle of Fig. 8(a) can be obtained, where $v_3 = N^{(k+1)}(u_4)$ and $u_5 = N^{(k+1)}(v_4)$. By Lemma 2, $\wp_{k+1}^{(j)} - F$ is Hamiltonian-connected for all $1 \leq j \leq k-1$. Lemma 3 assures a $P_{u_2,v_3}^{(H)}$ in $\wp_{k+1}^{I_1} - F$ and a $P_{u_5,v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I_2} - F$, where $I_1 = \{\alpha, \tau\}$ and $I_2 = \{1, 2, \ldots, k-1\} - \{\alpha, \tau\}$. If $\tau = \alpha$, the Hamiltonian cycle of Fig. 8(b) can be obtained, where $v_2 = N^{(k+1)}(u_4)$ and $u_3 = N^{(k+1)}(v_4)$. There is a $P_{u_2,v_2}^{(H)}$ in $\wp_{k+1}^{(\alpha)} - F$, and Lemma 3 assures a $P_{u_3,v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I} - F$, where $I = \{1, 2, \ldots, k-1\} - \{\alpha\}$. If $\tau = \beta$, a Hamiltonian cycle in $\wp_{k+1} - F$ can be obtained similarly.

3.3. *When* $|E(\wp_{k+1}^{(k+1)}) \cap F| = 2k-6$

We have $|E^{(k+1)}(\wp_{k+1}) \cap F| \leq 1$ and $|E(\wp_{k+1}^{(j)}) \cap F| \leq 1$ for all $1 \leq j \leq k$. When $k \geq 5$, a Hamiltonian cycle in $\wp_{k+1} - F$ can be obtained by slightly modifying the construction method of Fig. 5, as explained below. There is at most one vertex of degree one in $\wp_{k+1}^{(k+1)} - F$. So, an edge $(v_1, u_1) \in E(\wp_{k+1}^{(k+1)}) \cap F$ can be found so that $\delta(\wp_{k+1}^{(k+1)} - (F - \{(v_1, u_1)\})) \geq 2$ and $(v_1, u_2), (u_1, v_{k+1}) \in E^{(k+1)}(\wp_{k+1}) - F$ exist with $u_2, v_{k+1} \in V(\wp_{k+1}^{I})$. Now that $|E(\wp_{k+1}^{(k+1)}) \cap (F - \{(v_1, u_1)\})| = 2k-7$, the induction hypothesis assures a Hamiltonian cycle in $\wp_{k+1}^{(k+1)} - (F - \{(v_1, u_1)\})$. If the Hamiltonian cycle contains (v_1, u_1), then Lemma 3 can assure a $P_{u_2,v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I} - F$. Otherwise, a Hamiltonian cycle in $\wp_{k+1} - F$ can be obtained, all the same as Section 3.1 (i.e., by the construction method of Fig. 5).

When $k = 4$, we have $|F| \leq 3$ and $|E(\wp_5^{(5)}) \cap F| = 2$. If $|E(\wp_5^{(j)}) \cap F| = 0$ for all $1 \leq j \leq 4$, then a Hamiltonian cycle in $\wp_5 - F$ can be obtained all the same as the situation of $k \geq 5$. Otherwise, we have $|E(\wp_5^{(4)}) \cap F| = 1$, $|E(\wp_5^{(j)}) \cap F| = 0$ for all $1 \leq j \leq 3$, and $|E^{(5)}(\wp_5) \cap F| = 0$. Let $(u_1, v_1) \in E(\wp_5^{(5)}) \cap F$. We have $\delta(\wp_5^{(5)} - (F - \{(u_1, v_1)\})) \geq 2$. If $N^{(5)}(v_1) \in V(\wp_5^{(4)})$, then a Hamiltonian cycle in $\wp_5 - F$ can be obtained by slightly modifying the construction method of Fig. 7, as explained

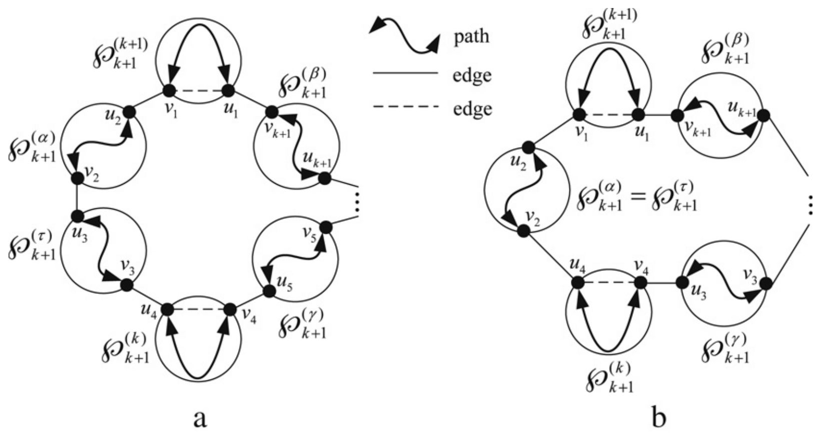

Fig. 8. A Hamiltonian cycle in $\wp_{k+1} - F$ when $2k-7 \geq |E(\wp_{k+1}^{(k+1)}) \cap F| \geq |E(\wp_{k+1}^{(k)}) \cap F| \geq k-3$, $|E(\wp_{k+1}^{(k-1)}) \cap F| \leq k-4$, and $\delta(\wp_{k+1}^{(k+1)} - F) = 1$. (a) $\tau \notin \{\alpha, \beta\}$. (b) $\tau = \alpha$.

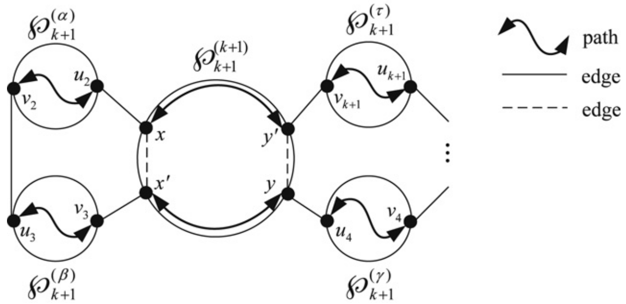

Fig. 9. A Hamiltonian cycle in $\wp_{k+1} - F$ when $|E(\wp_{k+1}^{(k+1)}) \cap F| = 2k-5$ and $|\{\alpha, \beta\} \cap \{\gamma, \tau\}| = 0$.

below. Let $u_2 = N^{(5)}(v_1)$ and $v_5 = N^{(5)}(u_1) \in V(\wp_5^I)$, where $I = \{1, 2, 3\}$. An edge $(u_2, v_2) \in E(\wp_5^{(4)}) - F$ can be found so that there exists $(v_2, u_3) \in E^{(5)}(\wp_5)$ with $u_3 \in V(\wp_5^I)$ and $\langle u_3 \rangle_5 \neq \langle v_5 \rangle_5$. Lemma 5 assures a Hamiltonian cycle in $\wp_5^{(4)} - F$ that contains (u_2, v_2), and Lemma 3 assures a $P_{u_3, v_5}^{(H)}$ in $\wp_5^I - F$. If $N^{(5)}(u_1) \in V(\wp_5^{(4)})$, a Hamiltonian cycle in $\wp_5 - F$ can be obtained similarly.

If $N^{(5)}(v_1), N^{(5)}(u_1) \notin V(\wp_5^{(4)})$, then a Hamiltonian cycle in $\wp_5 - F$ can be obtained by slightly modifying the construction method of Fig. 8(b). Let $\{\alpha, \beta, \gamma\} = \{1, 2, 3\}$, $u_2 = N^{(5)}(v_1) \in V(\wp_5^{(\alpha)})$, and $v_5 = N^{(5)}(u_1) \in V(\wp_5^{(\beta)})$. An edge $(u_4, v_4) \in E(\wp_5^{(4)}) - F$ can be found so that there exist $(u_4, v_2), (v_4, u_3) \in E^{(5)}(\wp_5) - F$ with $v_2 \in V(\wp_5^{(\alpha)})$ and $u_3 \in V(\wp_5^{(\gamma)})$. Lemma 5 assures a Hamiltonian cycle in $\wp_5^{(4)} - F$ that contains (u_4, v_4). There exists a $P_{u_2, v_2}^{(H)}$ in $\wp_5^{(\alpha)} - F$. Lemma 3 assures a $P_{u_3, v_5}^{(H)}$ in $\wp_5^I - F$, where $I = \{\beta, \gamma\}$.

3.4. When $|E(\wp_{k+1}^{(k+1)}) \cap F| = 2k-5$

We have $|E(\wp_{k+1}^{(j)}) \cap F| = 0$ for all $1 \leq j \leq k$ and $|E^{(k+1)}(\wp_{k+1}) \cap F| = 0$. There are at most two vertices of degree one in $\wp_{k+1}^{(k+1)} - F$. First, two edges $(x, x'), (y, y') \in E(\wp_{k+1}^{(k+1)}) \cap F$ are determined so that $\{x, x'\} \cap \{y, y'\} = \emptyset$ and $\delta(\wp_{k+1}^{(k+1)} - (F - \{(x, x'), (y, y')\})) \geq 2$. Since $|E(\wp_{k+1}^{(k+1)}) \cap (F - \{(x, x'), (y, y')\})| = 2k-7$, the induction hypothesis assures a Hamiltonian cycle C in $\wp_{k+1}^{(k+1)} - (F - \{(x, x'), (y, y')\})$. If (x, x') or (y, y') is not contained in C, a Hamiltonian cycle in $\wp_{k+1} - F$ can be obtained by the construction method of Fig. 5. In the rest of this section, the situation that both (x, x') and (y, y') are contained in C is discussed, where $N^{(k+1)}(x) \in V(\wp_{k+1}^{(\alpha)})$, $N^{(k+1)}(x') \in V(\wp_{k+1}^{(\beta)})$, $N^{(k+1)}(y) \in V(\wp_{k+1}^{(\gamma)})$, and $N^{(k+1)}(y') \in V(\wp_{k+1}^{(\tau)})$ are assumed ($\alpha \neq \beta$ and $\gamma \neq \tau$).

When $|\{\alpha, \beta\} \cap \{\gamma, \tau\}| = 0$, there exists a Hamiltonian cycle in $\wp_{k+1} - F$ as shown in Fig. 9, where $u_2 = N^{(k+1)}(x)$, $v_3 = N^{(k+1)}(x')$, $u_4 = N^{(k+1)}(y)$, and $v_{k+1} = N^{(k+1)}(y')$. Lemma 3 assures a $P_{u_2, v_3}^{(H)}$ in $\wp_{k+1}^{I_1} - F$ and a $P_{u_4, v_{k+1}}^{(H)}$ in $\wp_{k+1}^{I_2} - F$, where $I_1 = \{\alpha, \beta\}$ and $I_2 = \{1, 2, \ldots, k\} - \{\alpha, \beta\}$. When $|\{\alpha, \beta\} \cap \{\gamma, \tau\}| = 1$, there exists a Hamiltonian cycle in $\wp_{k+1} - F$ as shown in Fig. 10(a) if $\alpha = \gamma$ and $\beta \neq \tau$, or in Fig. 10(b) if $\alpha = \tau$ and $\beta \neq \gamma$. If $\beta = \tau$ and $\alpha \neq \gamma$ or $\beta = \gamma$ and $\alpha \neq \tau$, a Hamiltonian cycle in $\wp_{k+1} - F$ can be obtained similarly.

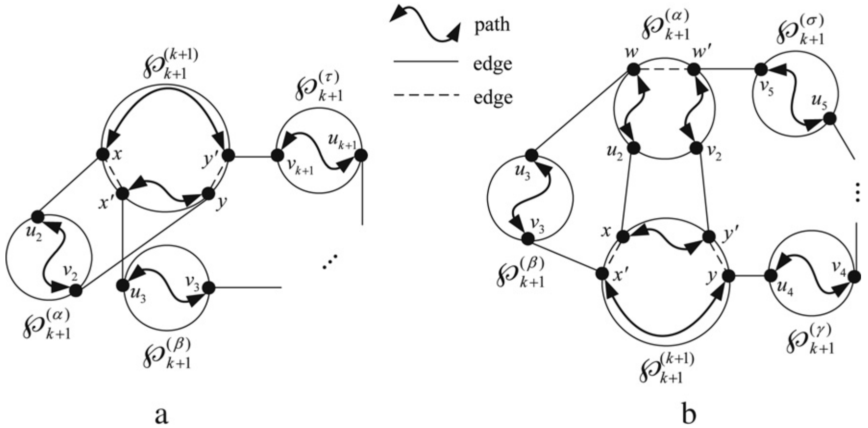

Fig. 10. A Hamiltonian cycle in $\wp_{k+1} - F$ when $|E(\wp_{k+1}^{(k+1)}) \cap F| = 2k - 5$ and $|\{\alpha, \beta\} \cap \{\gamma, \tau\}| = 1$. (a) $\alpha = \gamma$ and $\beta \neq \tau$. (b) $\alpha = \tau$ and $\beta \neq \gamma$.

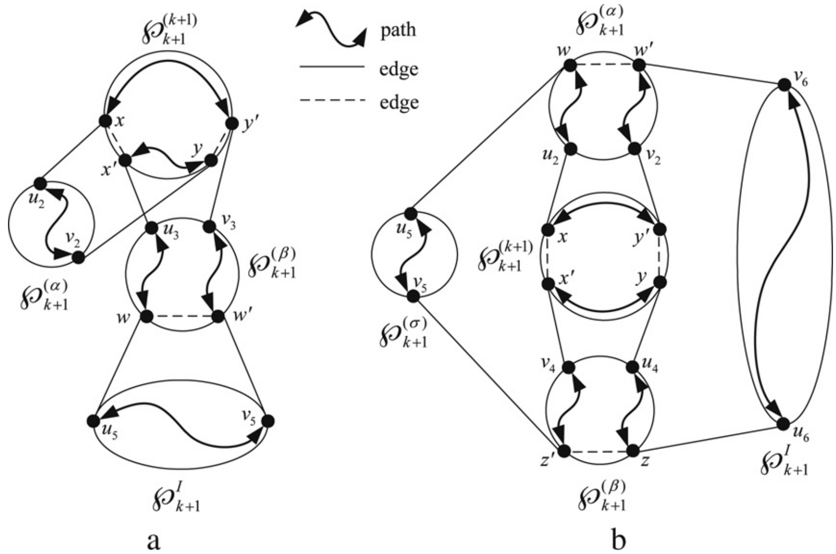

Fig. 11. A Hamiltonian cycle in $\wp_{k+1} - F$ when $|E(\wp_{k+1}^{(k+1)}) \cap F| = 2k - 5$ and $|\{\alpha, \beta\} \cap \{\gamma, \tau\}| = 2$. (a) $\alpha = \gamma$ and $\beta = \tau$. (b) $\alpha = \tau$ and $\beta = \gamma$.

The Hamiltonian cycle of Fig. 10(a) contains a $P^{(H)}_{u_2,v_2}$ in $\wp^{(\alpha)}_{k+1}$ and a $P^{(H)}_{u_3,v_{k+1}}$ in $\wp^{I}_{k+1} - F$ (assured by Lemma 3), where $I = \{1, 2, \ldots, k\} - \{\alpha\}$. The Hamiltonian cycle of Fig. 10(b) contains a $P^{(H)}_{u_3,v_3}$ in $\wp^{(\beta)}_{k+1}$ and a $P^{(H)}_{u_4,v_5}$ in $\wp^{I}_{k+1} - F$, where $I = \{1, 2, \ldots, k\} - \{\alpha, \beta\}$. There exists an edge $(w, w') \in E(\wp^{(\alpha)}_{k+1})$ with $N^{(k+1)}(w) \in V(\wp^{(\beta)}_{k+1})$ and $N^{(k+1)}(w') \in V(\wp^{(\sigma)}_{k+1})$, where $\sigma \in \{1, 2, \ldots, k\} - \{\alpha, \beta, \gamma\}$. Lemma 6 assures a $P^{(H)}_{u_2,v_2}$ in $\wp^{(\alpha)}_{k+1}$ that contains (w, w').

When $|\{\alpha, \beta\} \cap \{\gamma, \tau\}| = 2$, there exists a Hamiltonian cycle in $\wp_{k+1} - F$ as shown in Fig. 11(a) if $\alpha = \gamma$ and $\beta = \tau$, or in Fig. 11(b) if $\alpha = \tau$ and $\beta = \gamma$. The Hamiltonian cycle of Fig. 11(a) contains a $P^{(H)}_{u_2,v_2}$ in $\wp^{(\alpha)}_{k+1}$ and a $P^{(H)}_{u_5,v_5}$ in $\wp^{I}_{k+1} - F$, where $I = \{1, 2, \ldots, k\} - \{\alpha, \beta\}$. There exists an edge $(w, w') \in E(\wp^{(\beta)}_{k+1})$ with $N^{(k+1)}(w), N^{(k+1)}(w') \in V(\wp^{I}_{k+1})$, and Lemma 6 assures a $P^{(H)}_{u_3,v_3}$ in $\wp^{(\beta)}_{k+1}$ that contains (w, w').

The Hamiltonian cycle of Fig. 11(b) contains a $P^{(H)}_{u_5,v_5}$ in $\wp^{(\sigma)}_{k+1}$ and a $P^{(H)}_{u_6,v_6}$ in $\wp^{I}_{k+1} - F$, where $\sigma \in \{1, 2, \ldots, k\} - \{\alpha, \beta\}$ and $I = \{1, 2, \ldots, k\} - \{\alpha, \beta, \sigma\}$. There exist two edges $(w, w') \in E(\wp^{(\alpha)}_{k+1})$ and $(z, z') \in E(\wp^{(\beta)}_{k+1})$ with $N^{(k+1)}(w), N^{(k+1)}(z') \in V(\wp^{(\sigma)}_{k+1})$, $N^{(k+1)}(w'), N^{(k+1)}(z) \in V(\wp^{I}_{k+1})$, and $\langle N^{(k+1)}(w')\rangle_{k+1} \neq \langle N^{(k+1)}(z)\rangle_{k+1}$ if $k \geq 5$ ($\langle N^{(k+1)}(w')\rangle_{k+1} = \langle N^{(k+1)}(z)\rangle_{k+1}$ if $k = 4$). Lemma 6 assures a $P^{(H)}_{u_2,v_2}$ in $\wp^{(\alpha)}_{k+1}$ and a $P^{(H)}_{u_4,v_4}$ in $\wp^{(\beta)}_{k+1}$ that contains (w, w') and (z, z'), respectively.

4. Concluding remarks

It is both practically significant and theoretically interesting to investigate the fault tolerance of a multiprocessor system. Most of previous work adopted the random fault model, which assumed that the faults might occur anywhere without any restriction. It was shown in [22] that an n-dimensional pancake graph could tolerate up to $n - 3$ edge faults, while

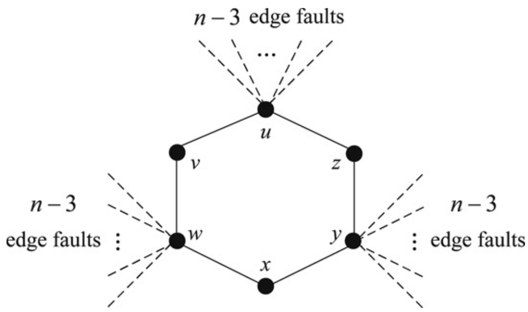

Fig. 12. A distribution of $3n - 9$ edge faults over an n-dimensional pancake graph.

retaining a fault-free Hamiltonian cycle, if the random fault model was considered. There was another fault model, namely the conditional fault model, which assumed that the fault distribution is subject to some constraints. Apparently, it is a more challenging problem to investigate the fault tolerance of a multiprocessor system under the conditional fault model.

In this paper, adopting the conditional fault model and assuming that there were two or more non-faulty edges incident to each node, we showed that an n-dimensional pancake graph contains a fault-free Hamiltonian cycle, even if there are up to $2n - 7$ edge faults, where $n \geq 4$. This is the first result on the fault tolerance of the pancake graph under the conditional fault model. There is an upper bound of $3n - 10$ on the greatest number of tolerable edge faults for the problem, as illustrated by Fig. 12, where a distribution of $3n - 9$ edge faults over an n-dimensional pancake graph is shown. It is easy to see that no fault-free Hamiltonian cycle can be found for this situation. It is an open problem to narrow down the gap between $2n - 7$ and $3n - 10$.

The routing techniques used in this paper to detour the edge faults are useful to those people who are working on the fault tolerance of the pancake graph. They may be applied to investigate the fault tolerance of the pancake graph on other problems such as pancycles [24] and connectivity under the conditional fault model. Finally, before ending this paper, it should be mentioned that the assumption of each node incident with two or more non-faulty edges is practically meaningful. When there are $2n-7$ edge faults, the probability that the assumption holds for an n-dimensional pancake graph is identical with the same probability for an n-dimensional star graph. The latter is very close to one, which was computed in [12].

Acknowledgment

The authors would like to thank the National Science Council of the Republic of China, Taiwan for financially supporting this research under Contract No. NSC 95-2221-E-239-002-.

Appendix

When $e_2 = (1234, 2134)$,

$$C_1 = \langle 1234, 3214, 2314, 4132, 1432, 2341, 3241, 1423, 4123, 2143, 3412, 4312, 2134, 3124, 1324, 4231, 2431, 1342, 3142, 2413, 4213, 1243, 3421, 4321, 1234\rangle;$$

$$C_2 = \langle 1234, 3214, 4123, 2143, 3412, 1432, 4132, 2314, 1324, 4231, 2431, 3421, 1243, 4213, 3124, 2134, 4312, 1342, 3142, 2413, 1423, 3241, 2341, 4321, 1234\rangle;$$

$$C_3 = \langle 1234, 3214, 4123, 1423, 2413, 3142, 4132, 2314, 1324, 4231, 3241, 2341, 1432, 3412, 2143, 1243, 4213, 3124, 2134, 4312, 1342, 2431, 3421, 4321, 1234\rangle.$$

When $e_2 = (1234, 3214)$,

$$C_4 = \langle 1234, 2134, 4312, 3412, 2143, 1243, 4213, 3124, 1324, 4231, 3241, 2341, 1432, 4132, 2314, 3214, 4123, 1423, 2413, 3142, 1342, 2431, 3421, 4321, 1234\rangle;$$

$$C_5 = \langle 1234, 2134, 3124, 4213, 2413, 3142, 4132, 1432, 2341, 3241, 1423, 4123, 3214, 2314, 1324, 4231, 2431, 1342, 4312, 3412, 2143, 1243, 3421, 4321, 1234\rangle;$$

$$C_6 = \langle 1234, 2134, 4312, 3412, 1432, 2341, 3241, 1423, 2413, 4213, 3124, 1324, 4231, 2431, 1342, 3142, 4132, 2314, 3214, 4123, 2143, 1243, 3421, 4321, 1234\rangle;$$

$$C_7 = \langle 1234, 2134, 4312, 1342, 3142, 2413, 4213, 3124, 1324, 4231, 2431, 3421, 1243, 2143, 3412, 1432, 4132, 2314, 3214, 4123, 1423, 3241, 2341, 4321, 1234\rangle.$$

When $e_2 = (1234, 4321)$,

$$C_8 = \langle 1234, 2134, 3124, 1324, 2314, 4132, 3142, 1342, 4312, 3412, 1432, 2341, 4321, 3421, 2431, 4231, 3241, 1423, 2413, 4213, 1243, 2143, 4123, 3214, 1234\rangle;$$

$$C_9 = \langle 1234, 2134, 3124, 4213, 1243, 2143, 4123, 1423, 2413, 3142, 4132, 1432, 3412, 4312, 1342, 2431, 3421, 4321, 2341, 3241, 4231, 1324, 2314, 3214, 1234\rangle;$$

$$C_{10} = \langle 1234, 2134, 4312, 3412, 1432, 4132, 3142, 1342, 2431, 4231, 3241, 2341, 4321, 3421, 1243, 2143, 4123, 1423, 2413, 4213, 3124, 1324, 2314, 3214, 1234\rangle;$$

$$C_{11} = \langle 1234, 2134, 4312, 3412, 2143, 1243, 3421, 4321, 2341, 1432, 4132, 2314, 1324, 3124, 4213, 2413, 3142, 1342, 2431, 4231, 3241, 1423, 4123, 3214, 1234\rangle.$$

References

[1] S.B. Akers, B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Transactions on Computers 38 (4) (1989) 555–566.
[2] S.G. Akl, Parallel Computation: Models and Methods, Prentice Hall, NJ, 1997.
[3] Y.A. Ashir, I.A. Stewart, Fault-tolerant embedding of Hamiltonian circuits in k-ary n-cube, SIAM Journal on Discrete Mathematics 15 (3) (2002) 317–328.
[4] N. Bagherzadeh, N. Nassif, S. Latifi, A routing and broadcasting scheme on faulty star graphs, IEEE Transactions on Computers 42 (11) (1993) 1398–1403.
[5] R. Bennes, S. Latifi, N. Kimura, A comparative study of job allocation and migration in the pancake network, Information Sciences 177 (2007) 2327–2335.
[6] M.Y. Chan, S.J. Lee, On the existence of Hamiltonian circuits in faulty hypercubes, SIAM Journal on Discrete Mathematics 4 (4) (1991) 511–527.
[7] J.M. Chang, J.S. Yang, Y.L. Wang, Yuwen Cheng, Panconnectivity, fault-tolerant Hamiltonicity and Hamiltonian-connectivity in alternating group graphs, Networks 44 (4) (2004) 302–310.
[8] S.A. Choudum, V. Sunitha, Augmented cubes, Networks 40 (2) (2002) 71–84.
[9] K. Day, A.E. Al-Ayyoub, Fault diameter of k-ary n-cube networks, IEEE Transactions on Parallel and Distributed Systems 8 (9) (1997) 903–907.
[10] K. Diks, A. Pele, Efficient gossiping by packets in networks with random faults, SIAM Journal on Discrete Mathematics 9 (1) (1996) 7–18.
[11] W.C. Fang, C.C. Hsu, On the fault-tolerant embedding of complete binary trees in the pancake graph interconnection network, Information Sciences 126 (2000) 191–204.
[12] J.S. Fu, Conditional fault-tolerant Hamiltonicity of star graphs, Parallel Computing 33 (2007) 488–496.
[13] L. Gardner, Z. Miller, D. Pritikin, I.H. Sudborough, One-to-many embeddings of hypercubes into Cayley graphs generated by reversals, Theory of Computing Systems 34 (2001) 399–431.
[14] W.H. Gates, C.H. Papadimitriou, Bounds for sorting by prefix reversal, Discrete Mathematics 27 (1979) 47–57.
[15] M.H. Heydari, I.H. Sudborough, On the diameter of the pancake network, Journal of Algorithms 25 (1997) 67–94.
[16] M.R. Hoseinyfarahabady, H. Sarbazi-azad, The grid-pyramid: A generalized pyramid network, The Journal of Supercomputing 37 (2006) 23–45.
[17] S.Y. Hsieh, Embedding longest fault-free paths onto star graphs with more vertex faults, Theoretical Computer Science 337 (1–3) (2005) 370–378.
[18] S.Y. Hsieh, G.H. Chen, C.W. Ho, Longest fault-free paths in star graphs with vertex faults, Theoretical Computer Science 262 (1–2) (2001) 215–227.
[19] S.Y. Hsieh, G.H. Chen, C.W. Ho, Longest fault-free paths in star graphs with edge faults, IEEE Transactions on Computers 50 (9) (2001) 960–971.
[20] S.Y. Hsieh, C.Y. Wu, Edge-fault-tolerant Hamiltonicity of locally twisted cubes under conditional edge faults, Journal of Combinatorial Optimization, doi:10.1007/s10878-008-9157-x.
[21] S.Y. Hsieh, C.W. Lee, Hamiltonicity of matching composition networks with conditional edge faults, in: Proc. of the 5th Annual Conference on Theory and Applications of Models of Computation, Xi'an, China, April 2008, pp. 160–169.
[22] C.N. Hung, H.C. Hsu, K.Y. Liang, L.H. Hsu, Ring embedding in faulty pancake graphs, Information Processing Letters 86 (2003) 271–275.
[23] H.S. Hung, J.S. Fu, G.H. Chen, Fault-free Hamiltonian cycles in crossed cubes with conditional link faults, Information Sciences 177 (24) (2007) 5664–5674.
[24] A. Kanevsky, C. Feng, On the embedding of cycles in pancake graphs, Parallel Computing 21 (1995) 923–936.
[25] S. Latifi, Combinatorial analysis of the fault-diameter of the n-cube, IEEE Transactions on Computers 42 (1) (1993) 27–33.
[26] F.T. Leighton, Introduction to Parallel Algorithms and Architecture: Arrays, Trees, Hypercubes, Morgan Kaufman, CA, 1991.
[27] C.K. Lin, H.M. Huang, L.H. Hsu, The super connectivity of the pancake graphs and the super laceability of the star graphs, Theoretical Computer Science 339 (2005) 257–271.
[28] K. Qiu, S.G. Akl, H. Meijer, On some properties and algorithms for the star and pancake interconnection networks, Journal of Parallel and Distributed Computing 22 (1994) 16–25.
[29] C.H. Tsai, Linear array and ring embeddings in conditional faulty hypercubes, Theoretical Computer Science 314 (2004) 431–443.
[30] P.Y. Tsai, Edge-fault-tolerant path/cycle embedding on three classes of Cayley graphs, Ph.D. Thesis, National Taiwan University, Taipei, Taiwan, June 2008.
[31] J. Wu, Safety levels — an efficient mechanism for achieving reliable broadcasting in hypercubes, IEEE Transactions on Computers 44 (5) (1995) 702–706.
[32] http://inrg.csie.ntu.edu.tw/~bytsai/.